THE UNIVERSITY OF BRITISH COLUMBIA

MATH 317

Midterm 2

August 6, 2015

TIME: 75 MINUTES

LAST NAME: Solution FIRST NAME: ____________

STUDENT # : ____________ SIGNATURE: ____________

This Examination paper consists of 9 pages (including this one). Make sure you have all 9.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.

MARKING:

Q1	/8
Q2	/10
Q3	/14
Q4	/10
Q5	/8
TOTAL	/50

NAME OF INSTRUCTOR: Uriya First

Q1 [8 marks]

Use Green's Theorem to evaluate

$$\int_C (\underbrace{\cos(x^2 - x) - y^3}_{P})\mathrm{d}x + (\underbrace{x^3 - y^2}_{Q})\mathrm{d}y$$

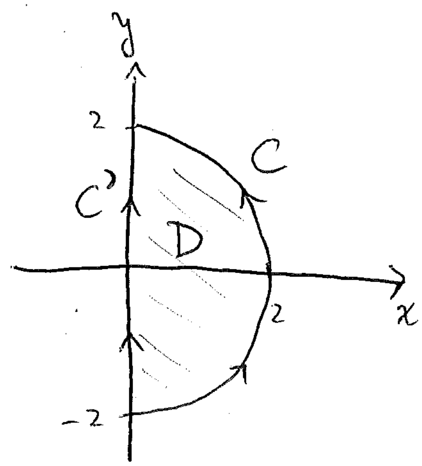
where C is the curve

$$\vec{\mathbf{r}}(t) = 2\cos t\,\vec{\mathbf{i}} + 2\sin t\,\vec{\mathbf{j}}, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

with direction increasing with t. Notice that C is <u>not closed</u>.

Let C' be $\langle 0, t\rangle$, $-2 \le t \le 2$

By Green's Theorem



$$\iint_D Q_x - P_y \, dx\,dy = \int_C P\,dx + Q\,dy - \int_{C'} P\,dx + Q\,dy$$

$$\int_C P\,dx + Q\,dy = \int_{C'} (\cos(x^2 - x) - y^3)\,dx + (x^3 - y^2)\,dy + \iint_D 3x^2 + 3y^2\,dx\,dy$$

$$= \int_{t=-2}^{2} \underbrace{(\cos(x^2-x) - y^3)\cdot 0 + (0^3 - t^2)\cdot 1}_{-t^2}\,dt + \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{2} \underbrace{3r^2 \cdot r}_{3r^3}\,dr\,d\theta$$

$$= \left[-\frac{t^3}{3}\right]_{-2}^{2} + \pi\left[\frac{3}{4}r^4\right]_0^2$$

$$= \left(-\frac{8}{3} - \frac{8}{3}\right) + \pi(12 - 0)$$

$$= \boxed{12\pi - \frac{16}{3}}$$

Q2 [10 marks]

Let $\vec{\mathbf{F}}(x,y) = P(x,y)\ \vec{\mathbf{i}} + Q(x,y)\ \vec{\mathbf{j}}$ be a vector field defined everywhere except $(-2,0)$ and $(2,0)$. Let C_1 be the circle $(x+2)^2+y^2=1$, oriented counterclockwise, and let C_2 be the circle $(x-2)^2+y^2=1$, oriented clockwise. It is given that

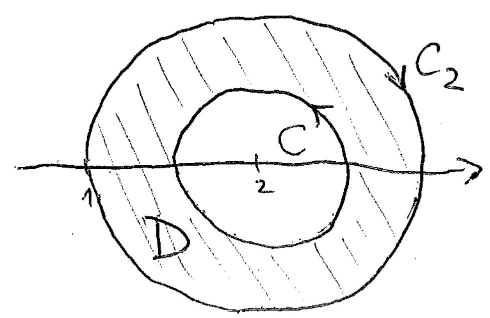
$$P_y = Q_x, \qquad \int_{C_1} \vec{F}\cdot d\vec{r} = 2, \qquad \int_{C_2} \vec{F}\cdot d\vec{r} = \pi\ .$$

Compute the following line integrals. No credit will be given for answers without an explanation. You can get partial credit by drawing all relevant curves with their correct orientation.

(a) Let C be the circle $(x-2)^2+y^2=0.25$, oriented counterclockwise.

$\int_C \vec{\mathbf{F}}\cdot d\vec{r} = \boxed{-\pi}$

By Green's Theorem:



$$0 = \iint_D \underbrace{Q_x - P_y}_{=0}\,dx\,dy = \int_{-C_2-C} \vec{F}\cdot d\vec{r}$$

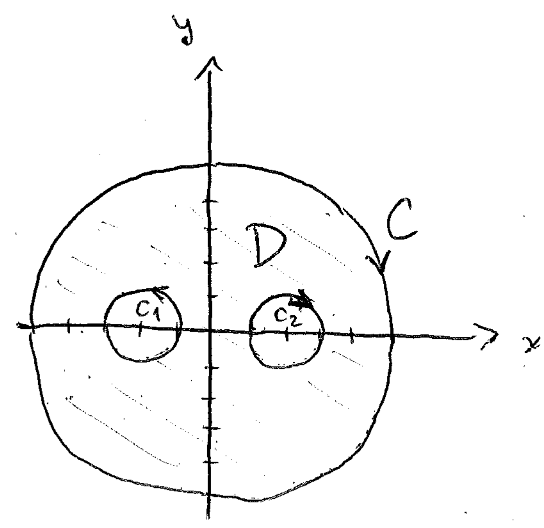
$$= -\int_{C_2}\dots - \int_C \dots = -\pi - \int_C \dots$$

So $\int_C \vec{F}\cdot d\vec{r} = -\pi$

(b) Let C be the circle $x^2+y^2=25$, oriented clockwise.

$\int_C \vec{\mathbf{F}}\cdot d\vec{r} = \boxed{\pi - 2}$

By Green's Theorem:



$$0 = \iint_D Q_x - P_y\,dx\,dy = \int_{-C_1+C_2-C} \vec{F}\cdot d\vec{r}$$

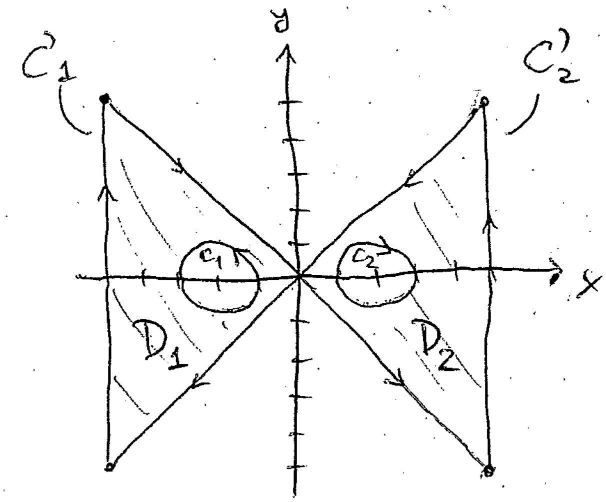
$$= -\int_{C_1}\dots + \int_{C_2}\dots - \int_C \dots$$

$$= -2 + \pi - \int_C \dots$$

So $\int_C \dots = \pi - 2$

(c) Let C be the curve consisting of the four oriented line segments: $(-5,5)$ to $(5,-5)$, $(5,-5)$ to $(5,5)$, $(5,5)$ to $(-5,-5)$, and $(-5,-5)$ to $(-5,5)$. (Notice that C is not simple.)

$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \boxed{-2-\pi}$



$$0 = \iint_{D_1} Q_x - P_y \, dx\,dy = -\int_{C_1'} \ldots - \int_{C_1} \ldots = -\int_{C_1'} -2$$

$$\Rightarrow \int_{C_1'} \ldots = -2$$

$$0 = \iint_{D_2} Q_x - P_y \, dx\,dy = \int_{C_2'} \ldots + \int_{C_2} \ldots = \int_{C_2'} + \pi$$

$$\Rightarrow \int_{C_2'} \ldots = -\pi$$

<u>SO</u>: $\int_C = \int_{C_1} + \int_{C_2} = -2-\pi$ $\quad C = C_1' + C_2'$

Q3 [14 marks]

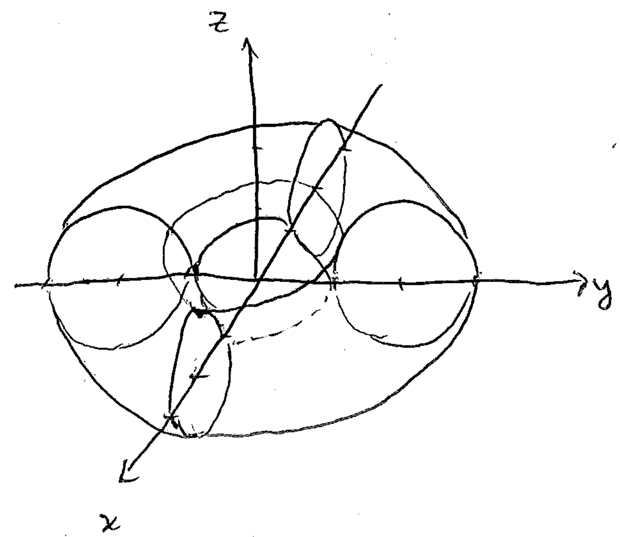
Let S be the surface parameterized by

$$\vec{\mathbf{r}}(u,v) = \cos u(\cos v + 2)\,\vec{\mathbf{i}} + \sin u(\cos v + 2)\,\vec{\mathbf{j}} + \sin v\,\vec{\mathbf{k}}$$

where $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

(a) Draw S. (Hint: Try substituting $u = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.)

The circle $(\cos v+2)\vec{i}+\sin v\,\vec{k}$
(or $(x-2)^2+z^2=1$, $y=0$)
is revolving around the z-axis



(b) Below is a list of parameterized surfaces. Clearly mark those which also parameterizes S.

A: $\vec{\mathbf{r}}(u,v) = \langle -\sin v(2-\cos u), \cos v(2-\cos u), \sin u\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

B: $\vec{\mathbf{r}}(u,v) = \left\langle v\cos u, v\sin u, \sqrt{1-(v-2)^2}\right\rangle$, $0 \le u \le 2\pi$, $1 \le v \le 3$.

C: $\vec{\mathbf{r}}(u,v) = \langle \cos(2u)(\cos v - 2), \sin(2u)(\cos v - 2), \sin v\rangle$, $0 \le u \le \pi$, $0 \le v \le 2\pi$.

D: $\vec{\mathbf{r}}(u,v) = \left\langle \sin u(2+\sqrt{1-v^2}), \cos u(2+\sqrt{1-v^2}), v\right\rangle$, $0 \le u \le 2\pi$, $-1 \le v \le 1$.

E: $\vec{\mathbf{r}}(u,v) = \left\langle \sqrt{1-u^2}(2-\sin(2v)), 2u - u\sin(2v), \cos(2v)\right\rangle$, $-1 \le u \le 1$, $0 \le v \le \pi$.

A describes the circle $(2-\cos u)\vec{i}+\sin u\,\vec{k}$ revolving around the z-axis
$\Rightarrow$ same as S!

C describes the circle $(-2+\cos v)\vec{i}+\sin v\,\vec{k}$ revolving around the z-axis
$\Rightarrow$ same as S!

B, D, E only describe "half" of S.
(each describes a different "half")

(c) Find the area of S.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u(\cos v+2) & \cos u(\cos v+2) & 0 \\ -\cos u \sin v & -\sin u \sin v & \cos v \end{vmatrix}$$

$$= \left\langle \cos u \cos v(\cos v+2),\ \sin u \cos v(\cos v+2),\ \underbrace{\sin^2 u \sin v(\cos v+2) + \cos^2 u \sin v(\cos v+2)}_{\sin v(\cos v+2)} \right\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \left[\cos^2 u \cos^2 v(\cos v+2)^2 + \sin^2 u \cos^2 v(\cos v+2)^2 + \sin^2 v(\cos v+2)^2\right]^{1/2}$$

$$= \left[\cos^2 v(\cos v+2)^2 + \sin^2 v(\cos v+2)^2\right]^{1/2}$$

$$= \left[(\cos v+2)^2\right]^{1/2} = \cos v+2$$

$$\text{Area}(S) = \int_{u=0}^{2\pi} \int_{v=0}^{2\pi} \cos v+2 \, dv\, du$$

$$= 2\pi \left[\sin v + 2v\right]_0^{2\pi}$$

$$= 2\pi \cdot [4\pi]$$

$$= \boxed{8\pi^2}$$

Q4 [10 marks]

Let $f(u)$ be a function defined for any number u, and let $f' = \frac{\partial f}{\partial u}$ and $f'' = \frac{\partial^2}{\partial u^2} f$. Consider the vector field

$$\vec{\mathbf{F}}(x, y, z) = f(r)\vec{\mathbf{r}}$$

where $\vec{\mathbf{r}} = \langle x, y, z \rangle$ and $r = |\vec{\mathbf{r}}|$. In the following questions, express you final answer in terms of $\vec{\mathbf{r}}$, r, f, f' and f''.

(a) Simplify grad $f(r)$. (Hint: Use the chain rule.)

$$\text{grad } f(r) = \left\langle \frac{\partial}{\partial x} f(r), \frac{\partial}{\partial y} f(r), \frac{\partial}{\partial z} f(r) \right\rangle \qquad \text{(chain rule)}$$

$$= \left\langle f'(r)\frac{\partial r}{\partial x}, f'(r)\frac{\partial r}{\partial y}, f'(r)\frac{\partial r}{\partial z} \right\rangle$$

$$= f'(r)\left\langle \frac{2x}{2\sqrt{x^2+y^2+z^2}}, \frac{2y}{2\sqrt{x^2+y^2+z^2}}, \frac{2z}{2\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$= f'(r) \cdot \frac{\vec{r}}{r} = \boxed{f'(r)r^{-1}\vec{r}}$$

(b) Simplify div $\vec{\mathbf{F}}$.

$$\text{div } \vec{F} = \text{div}(f(r)\vec{r}) \overset{\text{product rule}}{=} f(r) \cdot \text{div } \vec{r} + (\text{grad } f(r)) \cdot \vec{r}$$

$$= f(r)(1+1+1) + f'(r)r^{-1}\vec{r} \cdot \vec{r}$$

$$= 3f(r) + f'(r)r^{-1}\overbrace{(x^2+y^2+z^2)}^{r^2}$$

$$= \boxed{3f(r) + f'(r)r}$$

(c) Simplify curl $\vec{\mathbf{F}}$.

$$\operatorname{curl}\vec{F} = \operatorname{curl}(f(r)\vec{r}) \overset{\text{product rule}}{=} f(r)\operatorname{curl}\vec{r} + \operatorname{grad} f(r) \times \vec{r}$$

$$= f(r)\cdot\vec{0} + f'(r)r^{-1}\underbrace{\vec{r}\times\vec{r}}_{\vec{0}} = \boxed{\vec{0}}$$

(d) Simplify $\nabla^2(f(r))$.

$$\nabla^2(f(r)) = \operatorname{div}(\operatorname{grad} f(r)) = \operatorname{div}\left(f'(r)r^{-1}\vec{r}\right)$$

Write $g(u) = f'(u)u^{-1}$

Then $g'(u) = f''(u)u^{-1} - f'(u)u^{-2}$

Now:

$$\operatorname{div}(f'(r)r^{-1}\vec{r}) = \operatorname{div}(g(r)\vec{r}) \overset{\text{part (b)}}{=}$$

$$= 3g(r) + g'(r)r = 3f'(r)r^{-1} + (f''(r)r^{-1} - f'(r)r^{-2})r$$

$$= 3f'(r)r^{-1} - f'(r)r^{-1} + f''(r)$$

$$= \boxed{2f'(r)r^{-1} + f''(r)}$$

Q5 [8 marks]

Let S be the part of the cylinder $y^2+z^2=4$ lying between the planes $x=3+\frac{1}{2}y+\frac{1}{2}z$ and $x=-\frac{1}{2}z$.

(a) Complete the following parametrization: The surface S can be parameterized as $\vec{\mathbf{r}}(u,v) = v\,\vec{\mathbf{i}} + 2\cos u\,\vec{\mathbf{j}} + 2\sin u\,\vec{\mathbf{k}}$ where $0 \le u \le 2\pi$ and $f(u) \le v \le g(u)$ with

$f(u) = \boxed{-\sin u}$

$g(u) = \boxed{3+\cos u+\sin u}$

$$-\frac{1}{2}\cdot 2\sin u = -\frac{1}{2}z \le x \le 3+\frac{1}{2}y+\frac{1}{2}z = 3+\frac{1}{2}2\cos u+\frac{1}{2}2\sin u$$

$$f(u) = -\sin u \qquad\qquad 3+\cos u+\sin u = g(u)$$

(b) Compute the flux integral $\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}$ where $\vec{\mathbf{F}}(x,y,z) = e^{xyz}\,\vec{\mathbf{i}} + \vec{\mathbf{k}}$ and S is oriented such that its normal vector points toward the $\boxed{\text{inside}}$ the cylinder $y^2+z^2=4$. (You may use the identities $\sin^2\alpha = \frac{1-\cos(2\alpha)}{2}$ and $\cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$.)

$$\vec{r}_u = \langle 0, -2\sin u, 2\cos u\rangle$$
$$\vec{r}_v = \langle 1, 0, 0\rangle$$
$$\vec{r}_u \times \vec{r}_v = \langle 0, 2\cos u, 2\sin u\rangle \longrightarrow \text{points } \boxed{\text{outside}} \text{ of the cylinder!}$$

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_{\substack{0\le u\le 2\pi \\ -\sin u \le v \le 3+\cos u+\sin u}} \langle e^{xyz}, 0, 1\rangle \cdot \langle 0, 2\cos u, 2\sin u\rangle \, du\, dv$$

$$= \int_{u=0}^{2\pi}\int_{v=-\sin u}^{3+\sin u+\cos u} -2\sin u \, dv\, du = \int_{u=0}^{2\pi} -2\sin u\,(3+2\sin u+\cos u)\, du$$

$$= \int_{u=0}^{2\pi} -6\sin u - 4\sin^2 u - 2\sin u\cos u \, du = \int_{u=0}^{2\pi} -6\sin u - 2 + 2\cos(2u) - \sin(2u)\, du$$

$$= \Big[6\cos u - 2u + \sin(2u) + \tfrac{1}{2}\cos(2u)\Big]_0^{2\pi}$$

$$= -2\cdot 2\pi = \boxed{-4\pi}$$