THE UNIVERSITY OF BRITISH COLUMBIA

MATH 317

Practice Midterm 2

August 6, 2015

TIME: 75 MINUTES

LAST NAME: Solutions FIRST NAME: ______

STUDENT # : ______ SIGNATURE: ______

This Examination paper consists of 9 pages (including this one). Make sure you have all 9.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.

MARKING:

Q1	/5
Q2	/7
Q3	/12
Q4	/16
Q5	/10
TOTAL	/0

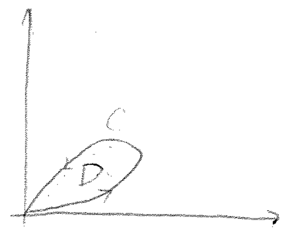
NAME OF INSTRUCTOR: Uriya First

Q1 [5 marks]

Let C be the closed curve

$$\vec{\mathbf{r}}(t) = \left\langle t - t^2, t^2 - t^3 \right\rangle, \qquad 0 \le t \le 1$$

Find the area of the domain bounded by C using Green's Theorem. You do not have to check that C is simple, closed, and positively oriented. (Hint: Find $P(x, y)$ and $Q(x, y)$ satisfying $Q_x - P_y = 1$.)



$$\text{Area}(D) = \iint_D 1\,dx\,dy \overset{\text{Green's Thm.}}{=} \int_C x\,dy$$

$$= \int_0^1 (t-t^2)(2t-3t^2)\,dt$$

$$= \int_0^1 2t^2 - 2t^3 - 3t^3 + 3t^4\,dt$$

$$= \left[\frac{2}{3}t^3 - \frac{5}{4}t^4 + \frac{3}{5}t^5\right]_0^1$$

$$= \frac{2}{3} - \frac{5}{4} + \frac{3}{5} - 0$$

$$= \frac{40 - 75 + 36}{60}$$

$$= \boxed{\frac{1}{60}}$$

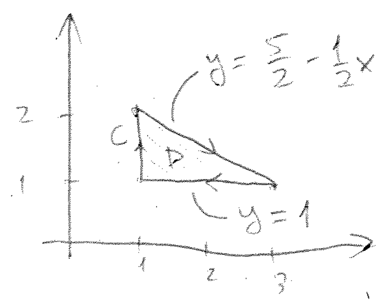
Q2 [7 marks]

"Free Style Integration": Evaluate

$$\int_C \overbrace{(y^2 + 2\arctan(x^{-1}y))}^{P}\,\mathrm{d}x + \overbrace{\ln(x^2+y^2)}^{Q}\mathrm{d}y$$

where C is the triangle with vertices $(1,1)$, $(1,2)$, $(3,1)$, oriented clockwise.

Doing this with a parameterization is awful... check $Q_x - P_y$:



$$Q_x - P_y = \frac{2x}{x^2+y^2} - 2y - \frac{2x^{-1}}{1+x^{-2}y^2}$$

$$= \frac{2x}{x^2+y^2} - 2y - \frac{2x}{x^2+y^2} = -2y \qquad \text{(much nicer...)}$$

$$\int_C P\,dx + Q\,dy \underset{\substack{\text{Green's Thm} \\ C = -\partial D}}{=} -\iint_D -2y\,dx\,dy$$

$$= \int_{x=1}^{3}\int_{y=1}^{\frac{5}{2}-\frac{1}{2}x} 2y\,dy\,dx = \int_{x=1}^{3} \left[y^2\right]_1^{\frac{5}{2}-\frac{1}{2}x} dx$$

$$= \int_{x=1}^{3} \left(\frac{5}{2} - \frac{1}{2}x\right)^2 - 1\,dx = \left[\frac{1}{3}(-2)\left(\frac{5}{2}-\frac{1}{2}x\right)^3 - x\right]_1^3$$

$$= \left[\frac{1}{3}(-2)\cdot 1^3 - 3\right] - \left[\frac{1}{3}(-2)\cdot 2^3 - 1\right]$$

$$= -\frac{2}{3} - 3 + \frac{16}{3} + 1 = \frac{-2-9+16+3}{3}$$

$$= \boxed{\frac{8}{3}}$$

Q3 [12 = 3+6+4 marks]

Let A and B be numbers and let

$$\vec{F} = \frac{yz}{x^2y^2+z^2}\,\vec{\mathbf{i}} + \frac{xz}{x^2y^2+z^2}\,\vec{\mathbf{j}} - \frac{xy}{x^2y^2+z^2}\,\vec{\mathbf{k}}$$

(a) Simplify $\operatorname{div}\vec{\mathbf{F}}$.

$$\operatorname{div}\vec{F} = \frac{-yz\cdot 2y^2x}{(x^2y^2+z^2)^2} + \frac{-xz\cdot 2x^2y}{(x^2y^2+z^2)^2} + \frac{-(-xy)\cdot 2z}{(x^2y^2+z^2)^2}$$

$$= \frac{-2x^3yz - 2xy^3z + 2xyz}{(x^2y^2+z^2)^2}$$

(b) Simplify $\operatorname{curl}\vec{\mathbf{F}}$.

$$\operatorname{curl}\vec{F} = \left[-\frac{x(x^2y^2+z^2)-xy(2x^2y)}{(x^2y^2+z^2)^2} - \frac{x(x^2y^2+z^2)-xz(2z)}{(x^2y^2+z^2)^2}\right]\vec{i}$$

$$+\left[\frac{y(x^2y^2+z^2)-yz(2z)}{(x^2y^2+z^2)^2} + \frac{y(x^2y^2+z^2)-xy(2xy^2)}{(x^2y^2+z^2)^2}\right]\vec{j}$$

$$+\left[\frac{z(x^2y^2+z^2)-xz(2xy^2)}{(x^2y^2+z^2)^2} - \frac{z(x^2y^2+z^2)-yz(2x^2y)}{(x^2y^2+z^2)^2}\right]\vec{k}$$

$$= \left[\frac{-x^3y^2-xz^2+2x^3y^3-x^3y^2-xz^2+2xz^2}{(x^2y^2+z^2)^2}\right]\vec{i}$$

$$+\left[\frac{x^2y^3+yz^2-2yz^2+x^2y^3+yz^2-2x^2y^3}{(x^2y^2+z^2)^2}\right]\vec{j}$$

$$+\left[\frac{x^2y^2z+z^3-2x^2y^2z-x^2y^2z-z^3+2x^2y^2z}{(x^2y^2+z^2)^2}\right]\vec{k}$$

$$= \vec{0}$$

(c) Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the circle $\vec{\mathbf{r}}(t) = \langle 2\cos t, 1, 2\sin t\rangle$, $0 \le t \le 2\pi$ with direction increasing with t.

C is a closed curve and curl $\vec{F} = 0$.
However, $\vec{F}$ is not defined everywhere (and more precisely, the domain of $\vec{F}$ is not simply connected), so we cannot conclude that $\vec{F}$ is conservative.
Instead, we compute the integral directly.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{2\pi} \left\langle \frac{2\sin t}{4\cos^2 t + 4\sin^2 t}, \frac{4\sin t\cos t}{4\cos^2 t + 4\sin^2 t}, \frac{-2\cos t}{4\cos^2 t + 4\sin^2 t} \right\rangle \cdot \langle -2\sin t, 0, 2\cos t\rangle \, dt$$

$$= \int_{t=0}^{2\pi} \frac{1}{4}\left(-4\sin^2 t + 0 - 4\cos^2 t\right) dt$$

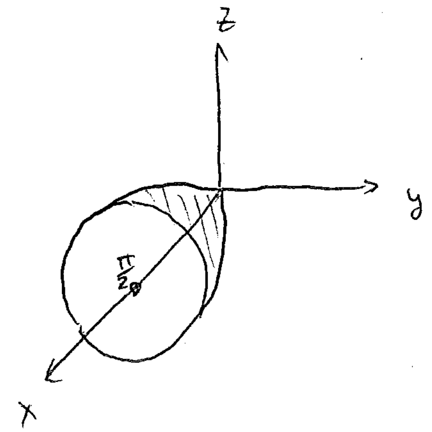
$$= \int_{t=0}^{2\pi} -1 \, dt = \boxed{-2\pi}$$

(So $\vec{F}$ is indeed not conservative...)

Q4 [16 = 2+6+8 marks]

The graph of the function $y = \sin x$ revolves around the x axis. Let S be the part of the resulting surface lying between the planes $x = 0$ and $x = \frac{\pi}{2}$.

(a) Find a parameterization of S.



$$\vec{r}(x,\theta) = \langle x, \sin x\cos\theta, \sin x\sin\theta\rangle$$

$$0 \le x \le \pi/2$$

$$0 \le \theta \le 2\pi$$

(b) Find the equation of the tangent plane to S at $(\frac{\pi}{3}, -\frac{3}{4}, -\frac{\sqrt{3}}{4})$.

Find x, θ with $\vec{r}(x,\theta) = (\frac{\pi}{3}, -\frac{3}{4}, -\frac{\sqrt{3}}{4})$ $\quad 0 \le \theta \le 2\pi$

$$\left.\begin{array}{l} x = \frac{\pi}{3}, \quad \sin x\cos\theta = -\frac{3}{4} \Rightarrow \frac{\sqrt{3}}{2}\cos\theta = -\frac{3}{4} \Rightarrow \cos\theta = -\frac{\sqrt{3}}{2} \\ \qquad\quad \sin x \sin\theta = -\frac{\sqrt{3}}{4} \Rightarrow \frac{\sqrt{3}}{2}\sin\theta = -\frac{\sqrt{3}}{4} \Rightarrow \sin\theta = -\frac{1}{2} \end{array}\right\} \Rightarrow \theta = \frac{7}{6}\pi$$

$$\vec{r}_x = \langle 1, \cos x\cos\theta, \cos x\sin\theta\rangle = \langle 1, -\frac{\sqrt{3}}{4}, -\frac{1}{4}\rangle$$

$$\vec{r}_\theta = \langle 0, -\sin x\sin\theta, \sin x\cos\theta\rangle \underset{x=\frac{\pi}{3},\ \theta=\frac{7}{6}\pi}{=} \langle 0, \frac{\sqrt{3}}{4}, -\frac{3}{4}\rangle$$

$$\vec{r}_x(\tfrac{\pi}{3}, \tfrac{7}{6}\pi) \times \vec{r}_\theta(\tfrac{\pi}{3}, \tfrac{7}{6}\pi) = \left\langle \frac{3\sqrt{3}}{16} + \frac{\sqrt{3}}{16}, \frac{3}{4}, \frac{\sqrt{3}}{4}\right\rangle = \left\langle \frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{\sqrt{3}}{4}\right\rangle$$

$$\frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right) + \frac{3}{4}\left(y + \frac{3}{4}\right) + \frac{\sqrt{3}}{4}\left(z + \frac{\sqrt{3}}{4}\right) = 0$$

$$\boxed{\frac{\sqrt{3}}{4}x + \frac{3}{4}y + \frac{\sqrt{3}}{4}z = \frac{\pi}{4\sqrt{3}} - \frac{3}{4}}$$

(c) It is given that the temperature of a substance with conductivity $K = \frac{1}{2}$ at the point (x, y, z) is $u(x, y, z) = 2y^2 + 3z^2$. Find the flow of heat ~~into~~ on S, inward.

The vector field of the heat flow is

$$\vec{F} = -\frac{1}{2}\vec{\nabla}u = -\frac{1}{2}\langle 0, 4y, 6z\rangle = \langle 0, -2y, -3z\rangle$$

$$\vec{r}_x \times \vec{r}_\theta = \langle \underbrace{\cos x \sin x \cos^2\theta + \cos x \sin x \sin^2\theta}_{\cos x \sin x}, -\sin x \cos\theta, -\sin x \sin\theta\rangle$$

↳ points toward the inside of S!

$$\text{heat flow} = \iint_S \vec{F}\cdot d\vec{S} = \int_{x=0}^{\pi/2}\int_{\theta=0}^{2\pi} \vec{F}(\vec{r}(x,\theta))\cdot(\vec{r}_x \times \vec{r}_\theta)\, d\theta\, dx$$

$$= \int_{x=0}^{\pi/2}\int_{\theta=0}^{2\pi} \langle 0, -2\sin x\cos\theta, -3\sin x \sin\theta\rangle \cdot \langle \cos x \sin x, -\sin x\cos\theta, -\sin x \sin\theta\rangle\, d\theta\, dx$$

$$= \int_{x=0}^{\pi/2}\int_{\theta=0}^{2\pi} 2\sin^2 x\cos^2\theta + 3\sin^2 x \sin^2\theta \, d\theta\, dx$$

$$= \int_{x=0}^{\pi/2} \sin^2 x \int_{\theta=0}^{2\pi} 2\cos^2\theta + 3\sin^2\theta\, d\theta\, dx = \int_{x=0}^{\pi/2}\sin^2 x\, dx \int_{\theta=0}^{2\pi} 2+\sin^2\theta\, d\theta$$

$$= \int_{x=0}^{\pi/2} \frac{1}{2} - \frac{1}{2}\cos 2x\, dx \cdot \int_{\theta=0}^{2\pi} \frac{5}{2} - \frac{1}{2}\cos 2\theta\, d\theta$$

$$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2} \cdot \left[\frac{5}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \left[\frac{1}{2}\cdot\frac{\pi}{2} - 0\right]\cdot\left[\frac{5}{2}\cdot 2\pi - 0\right]$$

$$= \boxed{\frac{5}{4}\pi^2}$$

Q5 [10 marks]

Complete the following parameterizations:

(a) The part of the hyperboloid $z = \sqrt{1 + 2x^2 + 2y^2}$ lying under the plane $z = y + 3$ can be parameterized as $\vec{\mathbf{r}}\,(u, v) = u\,\vec{\mathbf{i}} + v\,\vec{\mathbf{j}} + \sqrt{1 + 2u^2 + 2v^2}$ where $A(u - B)^2 + C(v - D)^2 \leq 1$.

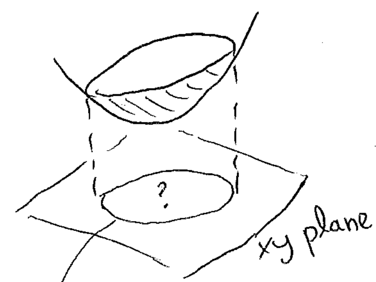
$A = \boxed{\frac{2}{17}}$

$B = \boxed{0}$

$C = \boxed{\frac{1}{17}}$

$D = \boxed{3}$

$$y+3 = z = \sqrt{1+2x^2+2y^2}$$
$$y^2+6y+9 = 1+2x^2+2y^2$$
$$2x^2+y^2-6y = 8$$
$$2x^2+y^2-6y+9 = 17$$
$$2x^2+(y-3)^2 = 17$$
$$\frac{2}{17}x^2 + \frac{1}{17}(y-3)^2 = 1$$

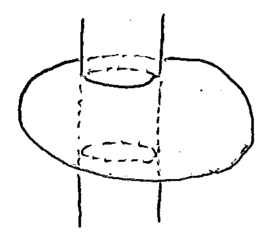


(b) The part of the cylinder $x^2 + y^2 = 4$ lying inside the ellipsoid $\frac{x^2}{40} + \frac{(y-1)^2}{20} + \frac{(z-1)^2}{80} = 1$ can be parameterized as $\vec{\mathbf{r}}\,(\theta, r) = A\cos\theta\,\vec{\mathbf{i}} + A\sin\theta\,\vec{\mathbf{j}} + r\,\vec{\mathbf{k}}$ where $0 \leq \theta \leq 2\pi$ and $f(\theta) \leq r \leq g(\theta)$.

$A = \boxed{2}$ (assume $A \geq 0$)

$f(\theta) = \boxed{1 - \sqrt{80 - 8\cos^2\theta - 4(2\sin\theta - 1)^2}}$

$g(\theta) = \boxed{1 + \sqrt{80 - 8\cos^2\theta - 4(2\sin\theta - 1)^2}}$



$$4 = x^2+y^2 = A^2\cos^2\theta + A^2\sin^2\theta = A^2 \Longrightarrow \boxed{A=2} \quad (A \geq 0)$$

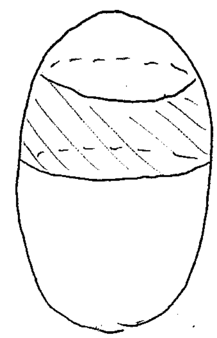
$$\frac{x^2}{40} + \frac{(y-1)^2}{20} + \frac{(z-1)^2}{80} = 1$$
$$(z-1)^2 = 80\left(1 - \frac{x^2}{40} - \frac{(y-1)^2}{20}\right) = 80 - 2x^2 - 4(y-1)^2$$
$$z - 1 = \pm\sqrt{80 - 2x^2 - 4(y-1)^2}$$
$$z = 1 \pm \sqrt{80 - 2x^2 - 4(y-1)^2} = 1 \pm \sqrt{80 - 8\cos^2\theta - 4(2\sin\theta - 1)^2}$$

(c) The part of the ellipsoid $x^2 + y^2 + \frac{1}{6}z^2 = 1$ lying between the planes $z = \sqrt{2}$ and $z = \sqrt{3}$ can be parameterized as $\vec{\mathbf{r}}(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, A\cos\phi)$ where $0 \le \theta \le 2\pi$ and $B \le \phi \le C$.

$A = \boxed{\sqrt{6}}$ (assume $A \ge 0$)

$B = \boxed{\frac{\pi}{4}}$

$C = \boxed{\arccos\frac{1}{\sqrt{3}}}$



$$1 = x^2+y^2+\frac{1}{6}z^2 = \sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \frac{1}{6}A^2\cos^2\phi$$
$$= \sin^2\phi + \frac{1}{6}A^2\cos^2\phi = 1 + \left(\frac{1}{6}A^2 - 1\right)\cos^2\phi$$
$$\Rightarrow \boxed{A = \sqrt{6}} \quad (A \geq 0)$$

$$\sqrt{2} \le A\cos\phi \le \sqrt{3}$$

$$\frac{1}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{6}} \le \cos\phi \le \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$$

$$\frac{\pi}{4} = \arccos\frac{1}{\sqrt{2}} \le \phi \le \arccos\frac{1}{\sqrt{3}}$$