The University of British Columbia

MATH 317
Midterm 1
21 July 2015

Time: 75 minutes

LAST NAME: Solution FIRST NAME: ____________

STUDENT # : ____________ SIGNATURE: ____________

This Examination paper consists of 8 pages (including this one). Make sure you have all 8.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.

MARKING:

Q1	/16
Q2	/5
Q3	/8
Q4	/9
Q5	/12
TOTAL	/50

Name of Instructor: Uriya First

Q1 [16 marks]

A helium balloon weighing 0.02kg is released from ground level on a windy day. It is given that the balloon's position after t seconds ($0 \le t \le 5$) is

$$\vec{\mathbf{r}}\,(t) = \frac{\sqrt{2}}{3} t^3\,\vec{\mathbf{i}} + (2t - t^2)\,\vec{\mathbf{j}} + (2t + t^2)\,\vec{\mathbf{k}}\,.$$

(the coordinates are measured in meters).

(a) At what time does the balloon's velocity parallel to the plane $2y + z = 2$?

$$\vec{r}\,'(t) = \langle \sqrt{2}\,t^2, 2-2t, 2+2t \rangle$$

$$0 = \vec{r}\,'(t) \cdot \langle 0, 2, 1 \rangle = 2(2-2t) + 2 + 2t = 4 - 4t + 2 + 2t$$

$$2t = 6$$

$$\boxed{t = 3_{sec}}$$

(b) Find the force $\vec{F}(t)$ that the balloon feels at time t.

$$\vec{a}(t) = \vec{r}\,''(t) = \langle 2\sqrt{2}\,t, -2, 2 \rangle$$

$$\vec{F}(t) = m\vec{a}(t) = 0.02 \langle 2\sqrt{2}\,t, -2, 2 \rangle$$

$$= \boxed{\langle 0.04\sqrt{2}\,t, -0.04, 0.04 \rangle}$$

(c) Find the curvature of the balloon's path at $t = 1$.

$$\vec{r}\,'(1) = \langle \sqrt{2}, 0, 4 \rangle$$

$$\vec{r}\,''(1) = \langle 2\sqrt{2}, -2, 2 \rangle$$

$$|\vec{r}\,'(1)| = \sqrt{2 + 0 + 16} = \sqrt{18} = 3\sqrt{2}$$

$$\vec{r}\,'(1) \times \vec{r}\,''(1) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \sqrt{2} & 0 & 4 \\ 2\sqrt{2} & -2 & 2 \end{vmatrix} = \langle 8, \overbrace{8\sqrt{2} - 2\sqrt{2}}^{6\sqrt{2}}, -2\sqrt{2} \rangle$$

$$K(1) = \frac{|\vec{r}\,'(1) \times \vec{r}\,''(1)|}{|\vec{r}\,'(1)|^3} = \frac{\sqrt{64 + 2 \cdot 36 + 8}}{3^3 \sqrt{2^3}} = \frac{\sqrt{144}}{3^3\sqrt{8}} = \frac{12}{3^3 2\sqrt{2}} = \boxed{\frac{2}{9\sqrt{2}}}$$

(d) Find the distance that the balloon has traveled after t seconds.

$$|\vec{r}\,'(t)| = \left[(\sqrt{2}\,t^2)^2 + (2-2t)^2 + (2+2t)^2\right]^{1/2}$$

$$= \left[2t^4 + 4 - 8t + 4t^2 + 4 + 8t + 4t^2\right]^{1/2}$$

$$= \left[2t^4 + 8t^2 + 8\right]^{1/2}$$

$$= \sqrt{2}\cdot\sqrt{t^4 + 4t^2 + 4}$$

$$= \sqrt{2}\,(t^2+2)$$

$$s = \int_0^t \sqrt{2}\,(u^2+2)\,du$$

$$= \sqrt{2}\left[\frac{u^3}{3} + 2u\right]_0^t$$

$$= \sqrt{2}\left[\frac{t^3}{3} + 2t\right] - 0$$

$$= \boxed{\frac{\sqrt{2}}{3}t^3 + 2\sqrt{2}\,t}$$

Q2 [5 marks]

Let C be the intersection of the surfaces $3x + xy = z^2$ and $y^2 + 4z^2 = 4$. Find a parameterization of C.

$$y^2 + 4z^2 = 4 \Rightarrow \frac{y^2}{4} + z^2 = 1$$

take: $y = 2\cos t$, $z = \sin t$, $0 \le t \le 2\pi$

$$3x + xy = z^2 \Rightarrow x(3+y) = z^2$$

$$x = \frac{z^2}{3+y} = \frac{(\sin t)^2}{3 + 2\cos t}$$

$$\boxed{\vec{r}(t) = \left\langle \frac{(\sin t)^2}{3+2\cos t}, 2\cos t, \sin t \right\rangle \quad 0 \le t \le 2\pi}$$

Q3 [8 marks]

Below are 4 plots of (possibly scaled) vector fields. Clearly mark each plot with the letter corresponding to the vector field it represents. Note that 2 letters will not be used.

A: $\vec{F}(x,y) = y\vec{\mathbf{i}} + x\vec{\mathbf{j}}$

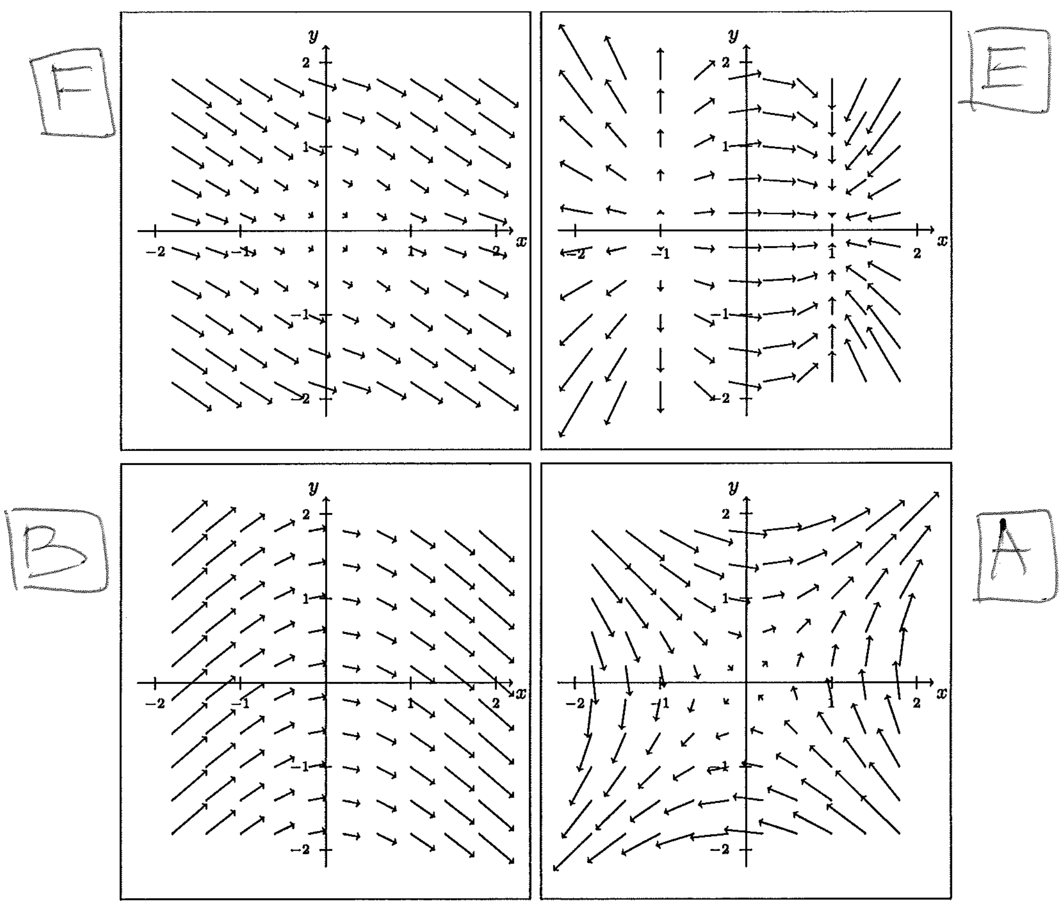
B: $\vec{F}(x,y) = \sqrt{x^2+1}\,\vec{\mathbf{i}} - x\vec{\mathbf{j}}$

C: $\vec{F}(x,y) = (y+1)\vec{\mathbf{i}} + (x+1)\vec{\mathbf{j}}$

D: $\vec{F}(x,y) = (x-y)\vec{\mathbf{i}} + e^{y}\vec{\mathbf{j}}$

E: $\vec{F}(x,y) = 2\cos(\frac{\pi}{2}x)\vec{\mathbf{i}} - xy\vec{\mathbf{j}}$

F: $\vec{F}(x,y) = \sqrt{x^2+y^2}\,\vec{\mathbf{i}} - \sqrt{|xy|}\,\vec{\mathbf{j}}$



Q4 [9 marks]

Let

$$\vec{F}(x,y) = (e^x + e^{x+2y})\,\vec{\mathbf{i}} + (2e^{2y} + 2e^{x+2y})\,\vec{\mathbf{j}}$$

(a) Show that $\vec{F}$ is conservative and find a potential function.

We find f with $\vec{F} = \vec{\nabla} f$

$$f_x = e^x + e^{x+2y}$$

$$f = \int e^x + e^{x+2y}\,dx + C(y) = e^x + e^{x+2y} + C(y)$$

$$2e^{2y} + \cancel{2e^{x+2y}} = f_y = \cancel{2e^{x+2y}} + C'(y)$$

$$C'(y) = 2e^{2y}$$

$$C(y) = \int 2e^{2y}\,dy + D = e^{2y} + D \quad (\text{take } D=0)$$

$$\boxed{f(x,y) = e^x + e^{x+2y} + e^{2y}}$$

(b) Find $\int_C \vec{F}(x,y) \cdot d\vec{\mathbf{r}}$ where C is the the parameterized curve $\vec{\mathbf{r}}(t) = \langle \sin(\pi t), \tan(\pi t^2) \rangle$ $(-\frac{1}{2} \le t \le \frac{1}{2})$ with direction increasing with t.

$$\vec{r}(-\tfrac{1}{2}) = \langle \sin(-\tfrac{\pi}{2}), \tan(\tfrac{\pi}{4}) \rangle = \langle -1, 1 \rangle$$

$$\vec{r}(\tfrac{1}{2}) = \langle \sin(\tfrac{\pi}{2}), \tan(\tfrac{\pi}{4}) \rangle = \langle 1, 1 \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(-1,1)$$

$$= [e^1 + e^3 + e^2] - [e^{-1} + e^1 + e^2]$$

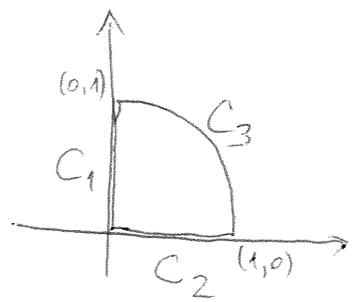
$$= \boxed{e^3 - e^{-1}}$$

Q5 [12 marks]

Let C be the closed plane curve consisting of the line segment from $(0,1)$ to $(0,0)$, the line segment from $(0,0)$ to $(1,0)$, and the quarter of the unit circle $x^2+y^2=1$ starting at $(1,0)$ and ending at $(0,1)$. Find

$$\int_C xe^y \, ds \, .$$

Parameterize $C = C_1 + C_2 + C_3$:



C_1: $\langle 0, t\rangle$, $0 \le t \le 1$

C_2: $\langle t, 0\rangle$, $0 \le t \le 1$

C_3: $\langle \cos t, \sin t\rangle$, $0 \le t \le \frac{\pi}{2}$

$$\int_{C_1} xe^y \, ds = \int_0^1 0 \cdot \underbrace{e^t \, |\langle 0,1\rangle|}_{=0} \, dt = 0$$

$$\int_{C_2} xe^y \, ds = \int_0^1 t \underbrace{e^0 \, |\langle 1,0\rangle|}_{=1} \, dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\int_{C_3} xe^y \, ds = \int_0^{\pi/2} \cos t \, e^{\sin t} \overbrace{|\langle -\sin t, \cos t\rangle|}^{1} \, dt$$

$$= \int_0^{\pi/2} e^{\sin t} \cos t \, dt \qquad \left(\begin{array}{l} u = \sin t \\ du = \cos t \, dt \end{array}\right)$$

$$= \int_0^1 e^u \, du = \left[e^u\right]_0^1 = e^1 - e^0 = e - 1$$

$$\int_C xe^y \, ds = \int_{C_1} \ldots + \int_{C_2} \ldots + \int_{C_3} \ldots = 0 + \frac{1}{2} + (e-1) = \boxed{e - \frac{1}{2}}$$

Extra space for work.