Final Exam
Math 317
August 20, 2015

Last Name: Solutions First Name: ____________

Student # : ____________ Instructor's Name : ____________

Instructions:
No memory aids allowed. No calculators allowed. No communication devices allowed. Use the space provided on the exam. If you use the back of a page, write "see back" on the front of the page. This exam is 180 minutes long.

Question	Points	Score
1	17	
2	13	
3	6	
4	7	
5	8	
6	8	
7	17	
8	10	
9	8	
10	6	
Total:	100	

1. Provide a short answer in the box next to each question.

(a) 2 points Compute div $\left(e^{xyz}\,\vec{\mathbf{i}} + z^2\sin(x^2y)\,\vec{\mathbf{j}} + e^{z\cos(xy)}\,\vec{\mathbf{k}}\right)$.

$$yze^{xyz} + x^2z^2\cos(x^2y) + \cos(xy)e^{z\cos(xy)}$$

(b) 2 points Compute curl $\left(xz\,\vec{\mathbf{i}} + 2xy\,\vec{\mathbf{j}} + \ln(xyz)\,\vec{\mathbf{k}}\right)$.

$\ln x + \ln y + \ln z$

$$\left\langle \frac{1}{y}, x - \frac{1}{x}, 2y \right\rangle$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & 2xy & \ln x + \ln y + \ln z \end{vmatrix} = \left\langle \frac{1}{y} - 0, x - \frac{1}{x}, 2y - 0 \right\rangle$$

(c) 2 points It is given that at a certain time, a particle has velocity $\langle 1, 1, -1\rangle$ and acceleration $\langle 2, 0, 1\rangle$. Find the particle's normal component of acceleration at this time.

$$\sqrt{\frac{14}{3}}$$

$$a_T = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|} = \frac{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix}}{|\langle 1, 1, -1\rangle|} = \frac{|\langle 1, -3, -2\rangle|}{\sqrt{3}} = \frac{\sqrt{14}}{\sqrt{3}}$$

(d) $\boxed{\text{3 points}}$ Evaluate $\int_C y\,dx + ye^x\,dy$ where C is the curve $\vec{\mathbf{r}}(t) = t\,\vec{\mathbf{i}} + e^t\,\vec{\mathbf{j}}$, $0 \le t \le 1$.

$$\boxed{e + \frac{1}{3}e^3 - \frac{4}{3}}$$

$$\int_C y\,dx + ye^x\,dy = \int_{t=0}^{1} e^t \cdot 1 + e^t \cdot e^t \cdot e^t\,dt = \int_{t=0}^{1} e^t + e^{3t}\,dt$$

$x = t$
$dx = dt$
$y = e^t$
$dy = e^t dt$

$$= \left[e^t + \frac{1}{3}e^{3t}\right]_0^1 = e + \frac{1}{3}e^3 - 1 - \frac{1}{3} = e + \frac{1}{3}e^3 - \frac{4}{3}$$

(e) $\boxed{\text{2 points}}$ Let $f(x, y, z) = -\cos(x^2 + y^2 - z^2)$ and let $\vec{\mathbf{F}} = \operatorname{grad} f$. The force field $\vec{\mathbf{F}}$ is moving an object with mass 0.5_{kg} from $(0, 0, 0)$ to $(\sqrt{\pi}, \sqrt{\pi}, \sqrt{\pi})$. It is given that the object's speed at $(0, 0, 0)$ is $2_{\text{m/s}}$. Use the Law of Conservation of Energy to find the object's speed at $(\sqrt{\pi}, \sqrt{\pi}, \sqrt{\pi})$.

$$\boxed{\sqrt{12}\ \text{m/s}}$$

v_0 = speed at $(0,0,0)$
v_1 = speed at $(\sqrt{\pi}, \sqrt{\pi}, \sqrt{\pi})$

$P(x,y,z) = -f(x,y,z) = \cos(x^2+y^2-z^2)$ (Potential energy)

$$P(0,0,0) + \frac{mv_0^2}{2} = P(\sqrt{\pi}, \sqrt{\pi}, \sqrt{\pi}) + \frac{mv_1^2}{2}$$

$$\cos 0 + \frac{0.5 \cdot 2^2}{2} = \cos \pi + \frac{0.5}{2} v_1^2$$

$$\frac{1}{4} v_1^2 = 1 + 1 + 1 = 3$$

$$v_1 = \sqrt{12}$$

(f) $\boxed{\text{4 points}}$ Find a potential function for the vector field

$$\vec{\mathbf{F}} = (ye^x + yz\cos(xy))\,\vec{\mathbf{i}} + (e^x + xz\cos(xy))\,\vec{\mathbf{j}} + (e^z + \sin(xy))\,\vec{\mathbf{k}}$$

$$\boxed{ye^x + z\sin(xy) + e^z}$$

$\vec{F} = \vec{\nabla} f$

$f_x = ye^x + yz\cos(xy)$

$f = ye^x + z\sin(xy) + g(y,z)$

$f_y = e^x + xz\cos(xy) + g_y$

$g_y = 0$

$g(y,z) = h(z)$

$f = ye^x + z\sin(xy) + h(z)$

$f_z = \sin(xy) + h_z$

$h_z = e^z$

$h(z) = e^z + C$

$f = ye^x + z\sin(xy) + e^z + C$

(g) $\boxed{\text{2 points}}$ Find the work done by the force field $\vec{\mathbf{F}}$ of (f) in moving a particle along the curve $\vec{\mathbf{r}}(t) = t^4\,\vec{\mathbf{i}} + (t^2 - t^3)\,\vec{\mathbf{j}} - t^2\,\vec{\mathbf{k}}$, $0 \le t \le 1$.

$$\boxed{e^{-1} - 1}$$

$$\text{work} = \int_C \vec{F}\cdot d\vec{r} \overset{\text{fundamental thm. of line integrals}}{=} f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(1,0,-1) - f(0,0,0)$$

$$= [0 + 0 + e^{-1}] - [0 + 0 + e^0]$$

$$= e^{-1} - 1$$

2. Consider the curve
$$\vec{\mathbf{r}}\,(t) = t\,\vec{\mathbf{i}} + t^2\,\vec{\mathbf{j}} + \frac{2}{3}t^3\,\vec{\mathbf{k}}\,.$$

(a) $\boxed{\text{3 points}}$ Find and simplify the unit tangent vector $\vec{\mathbf{T}}(t)$.

$$\vec{r}\,'(t) = \langle 1, 2t, 2t^2 \rangle$$
$$|\vec{r}\,'(t)| = \sqrt{1+4t^2+4t^4} = 1+2t^2$$
$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left\langle \frac{1}{1+2t^2}, \frac{2t}{1+2t^2}, \frac{2t^2}{1+2t^2} \right\rangle$$

(b) $\boxed{\text{3 points}}$ Find and simplify the curvature $\kappa(t)$.

$$\vec{r}\,''(t) = \langle 0, 2, 4t \rangle$$
$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix} = \langle \overbrace{8t^2-4t^2}^{4t^2}, -4t, 2 \rangle$$

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \frac{\sqrt{16t^4+16t^2+4}}{(1+2t^2)^3} = \frac{4t^2+2}{(2t^2+1)^3} = \frac{2}{(2t^2+1)^2}$$

(c) $\boxed{\text{3 points}}$ Find the principal unit normal vector at $t = 1$.

$$\vec{T}'(t) = \left\langle \frac{-4t}{(1+2t^2)^2}, \frac{2(1+2t^2)-2t(4t)}{(1+2t^2)^2}, \frac{4t(1+2t^2)-2t^2(4t)}{(1+2t^2)^2} \right\rangle$$

$$\vec{T}'(1) = \left\langle -\frac{4}{9}, \frac{6-8}{9}, \frac{12-8}{9} \right\rangle = \frac{1}{9}\langle -4, -2, 4\rangle$$

$$\vec{N}(1) = \frac{\vec{T}'(1)}{|\vec{T}'(1)|} = \frac{\frac{1}{9}\langle -4,-2,4\rangle}{\frac{1}{9}\sqrt{16+4+16}} = \left\langle -\frac{4}{6}, -\frac{2}{6}, \frac{4}{6} \right\rangle =$$

$$= \boxed{\left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle}$$

(d) $\boxed{\text{2 points}}$ Find the binormal vector at $t = 1$.

$$\vec{T}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \end{vmatrix} = \left\langle \frac{4}{9}+\frac{2}{9}, -\frac{4}{9}-\frac{2}{9}, -\frac{1}{9}+\frac{4}{9} \right\rangle$$

$$= \boxed{\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle}$$

(e) $\boxed{\text{2 points}}$ Find the <u>equation</u> of the osculating plane (the plane containing the osculating circle) at $t = 1$.

$$\vec{r}(1) = \left\langle 1, 1, \frac{2}{3} \right\rangle$$

$$\frac{2}{3}(x-1) - \frac{2}{3}(y-1) + \frac{1}{3}\left(z - \frac{2}{3}\right) = 0$$

$$\boxed{\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}z = \frac{2}{9}}$$

3. $\boxed{\text{6 points}}$ Find the arc-length parametrization of the curve

$$\vec{\mathbf{r}}\,(t) = \sin(t^2)\,\vec{\mathbf{i}} + \cos(t^2)\,\vec{\mathbf{j}} + \frac{1}{6}(4t+1)^{3/2}\,\vec{\mathbf{k}}$$

starting from $t = 0$.

$$\vec{r}\,'(t) = \left\langle 2t\cos(t^2), -2t\sin(t^2), \frac{1}{6}\cdot\frac{3}{2}\cdot 4(4t+1)^{\frac{1}{2}} \right\rangle$$

$$|\vec{r}\,'(t)| = \left[4t^2\cos^2(t^2) + 4t^2\sin^2(t^2) + 4t + 1\right]^{1/2}$$

$$= \left[4t^2 + 4t + 1\right]^{1/2} = 2t+1$$

$$s(t) = \int_0^t (2u+1)\,du = \left[u^2+u\right]_0^t = t^2+t$$

$$t^2 + t - s = 0$$

$$t = \frac{-1 \pm \sqrt{1+4s}}{2} \qquad (\text{take "+" because } s \text{ is positive})$$

$$t = \frac{-1+\sqrt{1+4s}}{2} = \frac{1}{2}\left(\sqrt{1+4s} - 1\right)$$

$$\vec{r}(s) = \sin\left(\frac{1}{4}\left(\sqrt{1+4s}-1\right)^2\right)\vec{i} +$$
$$\cos\left(\frac{1}{4}\left(\sqrt{1+4s}-1\right)^2\right)\vec{j} +$$
$$\frac{1}{6}\left(2\sqrt{1+4s}-1\right)^{3/2}\vec{k}$$

4. Let S be the surface obtained by revolving the graph of $y = x^2$, $-1 \leq x \leq 1$, around the x axis.

(a) $\boxed{\text{2 points}}$ Give a parametrization of S.

$$\vec{r}(x,\theta) = \langle x, x^2\cos\theta, x^2\sin\theta \rangle$$

$$-1 \leq x \leq 1$$
$$0 \leq \theta \leq 2\pi$$

(b) $\boxed{\text{5 points}}$ Evaluate the surface integral $\iint_S \sqrt{4x^2+1}\,\mathrm{d}S$.

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2x\cos\theta & 2x\sin\theta \\ 0 & -x^2\sin\theta & x^2\cos\theta \end{vmatrix} = \langle \overbrace{2x^3\cos^2\theta + 2x^3\sin^2\theta}^{2x^3}, -x^2\cos\theta, -x^2\sin\theta \rangle$$

$$|\vec{r}_x \times \vec{r}_\theta| = \left[4x^6 + x^4\cos^2\theta + x^4\sin^2\theta\right]^{1/2} = \sqrt{4x^6+x^4} = x^2\sqrt{4x^2+1}$$

$$\iint_S \sqrt{4x^2+1}\,dS = \int_{\theta=0}^{2\pi}\int_{x=-1}^{1} \sqrt{4x^2+1}\cdot x^2\sqrt{4x^2+1}\,dx\,d\theta$$

$$= 2\pi \int_{x=-1}^{1} 4x^4 + x^2\,dx = 2\pi\left[\frac{4}{5}x^5 + \frac{1}{3}x^3\right]_{-1}^{1}$$

$$= 2\pi\left[\frac{4}{5} + \frac{1}{3} + \frac{4}{5} + \frac{1}{3}\right]$$

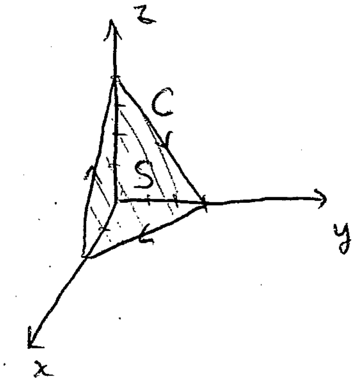
$$= 2\pi \cdot \frac{12+5+12+5}{15} = 2\pi \cdot \frac{34}{15}$$

$$= \boxed{\frac{68}{15}\pi}$$

5. $\boxed{\text{8 points}}$ Use Stoke's Theorem to evaluate the integral

$$\int_C \overbrace{(yz+e^{x^2})}^{P}dx + \overbrace{(x+ye^{y^2+z^2})}^{Q}dy + \overbrace{(y+ze^{y^2+z^2})}^{R}dz \qquad \vec{F} = \langle P, Q, R\rangle$$

where C is the triangle with vertices $(2,0,0)$, $(0,3,0)$, $(0,0,4)$, oriented clockwise when viewed from above.



Let S be the inside of the triangle C, oriented downward. Then $C = \partial S$.

By Stokes' Theorem: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$

$$\operatorname{curl}\vec{F} = \langle 1 + 2yze^{y^2+z^2} - 2zye^{y^2+z^2},\ y-0,\ 1-z\rangle = \langle 1, y, 1-z\rangle$$

Parametrize S: $\vec{r}(u,v) = \langle 2,0,0\rangle + u\langle -2,3,0\rangle + v\langle -2,0,4\rangle$

$$= \langle 2-2u-2v,\ 3u,\ 4v\rangle$$

$$0 \le u \le 1,\quad 0 \le v \le 1-u$$

$\vec{r}_u \times \vec{r}_v = \langle -2,3,0\rangle \times \langle -2,0,4\rangle = \langle 12, 8, 6\rangle \rightarrow$ points upward!

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = -\int_{u=0}^{1}\int_{v=0}^{1-u} \langle 1, 3u, 1-4v\rangle\cdot\langle 12,8,6\rangle\, dv\, du$$

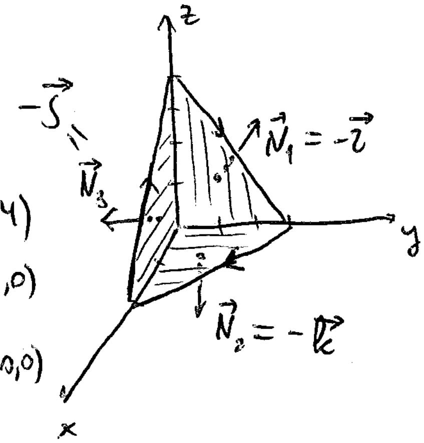
$$= -\int_{u=0}^{1}\int_{v=0}^{1-u} 12+24u+6-24v\, dv\, du = -\int_{u=0}^{1}\left[18v+24uv-12v^2\right]_0^{1-u} du$$

$$= -\int_{u=0}^{1} 18(1-u)+24u(1-u)-12(1-u)^2\, du$$

$$= -\int_{u=0}^{1} 18-18u+24u-24u^2-12+24u-12u^2\, du = -\int_{u=0}^{1} 6+30u-36u^2\, du$$

$$= -\left[6u+15u^2-12u^3\right]_0^1 = -\left[6+15-12\right] = \boxed{-9}$$

An alternative solution by one of the students who wrote the exam:



Let $S = S_1 + S_2 + S_3$ where

S_1 is the triangle w. vertices $(0,0,0),(0,3,0),(0,0,4)$

S_2 — " — $(0,0,0),(2,0,0),(0,3,0)$

S_3 — " — $(0,0,0),(0,0,4),(2,0,0)$

Oriented as in the picture.

By Stokes' Theorem $\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$

$$\operatorname{curl}\vec{F} = \langle 1 + 2yze^{y^2+z^2} - 2zye^{y^2+z^2},\, y-0,\, 1-z\rangle = \langle 1, y, 1-z\rangle$$

$$\iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_1} \langle 1,y,1-z\rangle\cdot\langle -1,0,0\rangle\, dS = \iint_{S_1} -1\, dS = -\text{Area}(S_1) = -\frac{3\cdot 4}{2} = -6$$

$$\iint_{S_2} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_2} \langle 1,y,1-z\rangle\cdot\langle 0,0,-1\rangle\, dS \underset{\substack{\uparrow\\ z=0 \text{ on } S_2}}{=} \iint_{S_2} -1+0\, dS = -\text{Area}(S_2) = -\frac{2\cdot 3}{2} = -3$$

$$\iint_{S_3} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_3} \langle 1,y,1-z\rangle\cdot\langle 0,-1,0\rangle\, dS = \iint_{S_3} 0\, dS = 0$$

$\uparrow$ $y=0$ on S_3

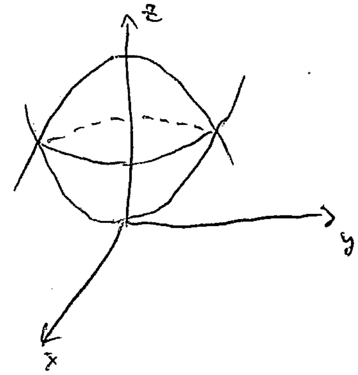
$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = -6 - 3 + 0 = \boxed{-9}$$

6. $\boxed{\text{8 points}}$ Let E be the region bounded between the paraboloids $z = 4 - x^2 - y^2$ and $z = x^2+y^2$ and let S be the boundary of E with outward orientation. Use the Divergence Theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where

$$\vec{\mathbf{F}} = (\sqrt{1+z^2} - xy^2)\,\vec{\mathbf{i}} + (y^3 + xz^2)\,\vec{\mathbf{j}} + (\sqrt{1+x^2} - zy^2)\,\vec{\mathbf{k}}\,.$$

The paraboloids meet when

$$x^2+y^2 = 4-x^2-y^2$$
$$2x^2+2y^2 = 4$$
$$x^2+y^2 = 2$$



Therefore $E = \left\{(x,y,z) \;\middle|\; \begin{array}{l} x^2+y^2 \le 2 \\ x^2+y^2 \le z \le 4-x^2-y^2 \end{array}\right\}$

$\operatorname{div}\vec{F} = -y^2 + 3y^2 - y^2 = y^2$

By the divergence theorem:

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dvol = \iint_{x^2+y^2\le 2} \int_{z=x^2+y^2}^{4-x^2-y^2} y^2 \, dz\, dx\, dy$$

$$= \iint_{x^2+y^2\le 2} y^2(4-2x^2-2y^2)\,dx\,dy \underset{\substack{\uparrow \\ x=r\cos\theta \\ y=r\sin\theta}}{=} \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}} r^2\sin^2\theta\,(4-2r^2)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \sin^2\theta\, d\theta \int_{r=0}^{\sqrt{2}} r^3(4-2r^2)\,dr = \int_{\theta=0}^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(2\theta)\, d\theta \int_{r=0}^{\sqrt{2}} 4r^3 - 2r^5\, dr$$

$$= \left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{2\pi}\left[r^4 - \frac{1}{3}r^6\right]_0^{\sqrt{2}} = \left[\frac{1}{2}\cdot 2\pi\right]\left[4-\frac{8}{3}\right]$$

$$= \pi\cdot\frac{4}{3} = \boxed{\frac{4}{3}\pi}$$

7. Let $\vec{\mathbf{r}} = \langle x, y, z\rangle$, $r = |\langle x, y, z\rangle|$ and

$$\vec{\mathbf{F}}(x,y,z) = \frac{\vec{\mathbf{r}}}{r^3} = \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle .$$

(a) $\boxed{\text{5 points}}$ Let S denote the sphere $x^2+y^2+z^2 = (\frac{1}{10})^2$ with positive orientation. Evaluate the flux integral $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N}\, dS = \iint_S \frac{\vec{r}}{r^3} \cdot \frac{\vec{r}}{r}\, dS$$

$$= \iint_S \frac{r^2}{r^4}\, dS = \iint_S \frac{1}{r^2}\, dS$$

$$= \iint_S \frac{1}{(1/10)^2}\, dS = 100 \cdot \text{Area}(S)$$

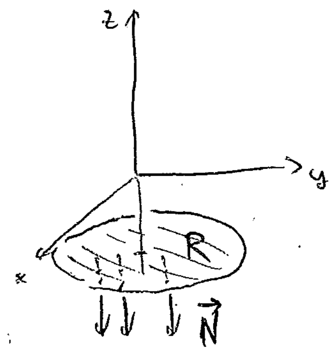
$r = \frac{1}{10}$ on S

$$= 100 \cdot \left(\frac{1}{10}\right)^2 4\pi = \boxed{4\pi}$$

(b) $\boxed{\text{6 points}}$ Let R be the part of the plane $z = -1$ lying inside the cylinder $x^2+y^2 = 1$. Evaluate the flux of $\vec{\mathbf{F}}$ though R downwards.

Parameterize R: $\vec{r}(x,y) = \langle x, y, -1\rangle$

$x^2+y^2 \le 1$



$\vec{N} = -\vec{k} = \langle 0,0,-1\rangle$

$|\vec{r}_x \times \vec{r}_y| = |\langle 1,0,0\rangle \times \langle 0,1,0\rangle| = |\langle 0,0,1\rangle| = 1$

$$\iint_R \vec{F}\cdot d\vec{S} = \iint_R \vec{F}\cdot\vec{N}\,dS = \iint_{x^2+y^2\le 1} \left\langle \text{---}, \text{---}, \frac{-1}{(x^2+y^2+1)^{3/2}}\right\rangle \cdot \langle 0,0,-1\rangle \cdot 1\,dx\,dy$$

$$= \iint_{x^2+y^2\le 1} \frac{1}{(x^2+y^2+1)^{3/2}}\,dx\,dy \underset{\substack{\uparrow \\ x=r\cos\theta \\ y=r\sin\theta}}{=} \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \frac{r}{(r^2+1)^{3/2}}\,dr\,d\theta$$

$$= 2\pi \int_{r=0}^{1} \frac{r\,dr}{(r^2+1)^{3/2}} \underset{\substack{\uparrow \\ u=r^2+1 \\ du=2r\,dr}}{=} 2\pi\cdot\frac{1}{2}\int_{u=1}^{2}\frac{du}{u^{3/2}} = \pi\left[-2u^{-\frac{1}{2}}\right]_1^2$$

$$= \pi\left[-\frac{2}{\sqrt{2}} + 2\right] = \boxed{(2-\sqrt{2})\pi}$$

(c) $\boxed{\text{3 points}}$ Simplify $\operatorname{div}\vec{\mathbf{F}} = \operatorname{div}(\vec{\mathbf{r}}/r^3)$.

$$\operatorname{div}\frac{\vec{r}}{r^3} = r^{-3}\cdot\operatorname{div}\vec{r} + \operatorname{grad} r^{-3}\cdot\vec{r}$$

$$= 3r^{-3} + \left(-3r^{-4}\operatorname{grad} r\right)\cdot\vec{r}$$

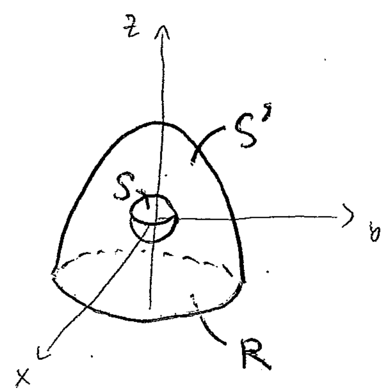
$$= 3r^{-3} + (-3r^{-4})\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right\rangle\cdot\langle x, y, z\rangle$$

$$= 3r^{-3} - 3r^{-4}\cdot\frac{x^2+y^2+z^2}{r}$$

$$= 3r^{-3} - 3r^{-4}\cdot\frac{r^2}{r} = \boxed{0}$$

(d) $\boxed{\text{3 points}}$ Let S' be the part of the paraboloid $z = 1 - 2x^2 - 2y^2$ lying above the plane $z = -1$, oriented toward the positive z-axis. Use your previous answers and the Divergence Theorem to evaluate $\iint_{S'} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}$.

Let E be the region bounded between R and S', except the inside of S.

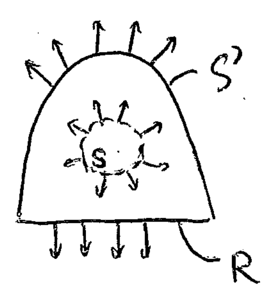


$$\partial E = S' + R - S$$

By the Divergence Theorem

$$0 = \iiint_E \overset{0}{\operatorname{div}\vec{F}}\, dvol = \iint_{S'} \cdots + \iint_R \cdots - \iint_S \cdots$$

View from the side



Therefore

$$\iint_{S'} \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot d\vec{S} - \iint_R \vec{F}\cdot d\vec{S}$$

$$= 4\pi - (2-\sqrt{2})\pi = \boxed{(2+\sqrt{2})\pi}$$

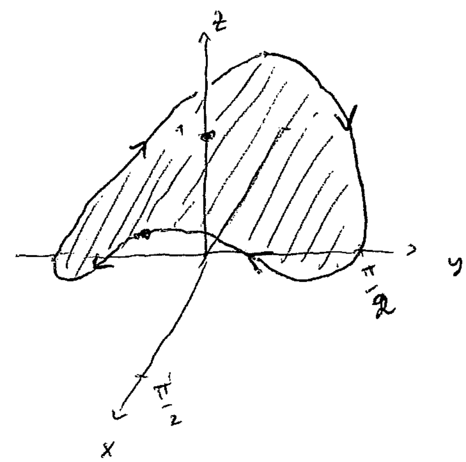
8. For each of the following closed oriented curves, find an oriented surface S whose boundary is the given oriented curve. You must <u>describe the surface in words</u>, <u>draw it</u>, <u>give a parametrization</u>, and <u>indicate the orientation</u> (e.g. by drawing the normal vector field, or by writing "upward", "downward", etc.). If the surface is made of several pieces, give a parametrization and an orientation for each of the pieces.

(a) $\boxed{\text{4 points}}$ C is the intersection of the cylinder $x^2+y^2=\frac{\pi^2}{4}$ with the surface $z=\cos y$, oriented clockwise when viewed from above.

S is the part of $z=\cos y$ inside the cylinder $x^2+y^2=\frac{\pi^2}{4}$, oriented downward

$$\vec{r}(x,y)=\langle x,y,\cos y\rangle$$

$$x^2+y^2\leq\frac{\pi^2}{4}$$



(b) $\boxed{\text{6 points}}$ C consists of the five oriented line segments: $(0,0,0)$ to $(0,1,0)$, $(0,1,0)$ to $(0,0,1)$, $(0,0,1)$ to $(1,1,0)$, $(1,1,0)$ to $(1,0,0)$, $(1,0,0)$ to $(0,0,0)$.

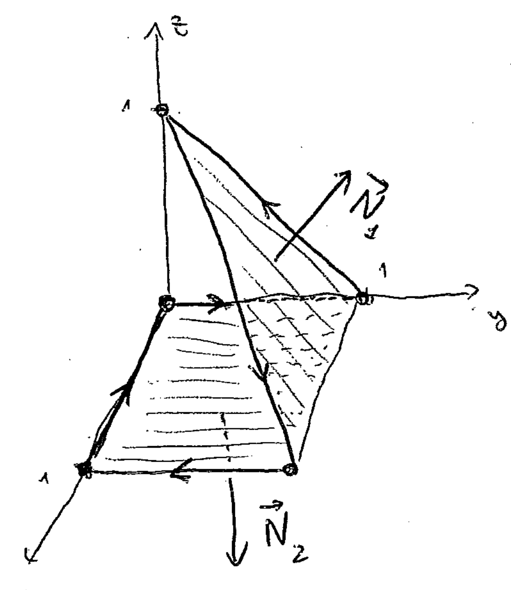
$S=S_1+S_2$

S_1: is the triangle with vertices $(0,1,0),(0,0,1),(1,1,0)$, oriented upward.

$$\vec{r}(u,v)=\langle 0,0,1\rangle+u\langle 0,1,-1\rangle+v\langle 1,1,-1\rangle$$
$$=\langle v,u+v,1-u-v\rangle$$
$$0\leq u\leq 1,\quad 0\leq v\leq 1-u$$

S_2: is the square with vertices $(0,0,0),(0,1,0),(1,1,0),(1,0,0)$, oriented downward.

$$\vec{r}(u,v)=\langle u,v,0\rangle \qquad 0\leq u\leq 1$$
$$0\leq v\leq 1$$



9. Determine whether the following vector fields are conservative in the domain $\{(x,y) \neq (0,0)\}$. No marks will be given for answers without a correct explanation.

(a) $\boxed{\text{4 points}}$ $\vec{\mathbf{F}}(x,y) = \underbrace{\frac{x}{x^2+y^2}}_{=P}\,\vec{\mathbf{i}} + \underbrace{\frac{y}{x^2+y^2}}_{=Q}\,\vec{\mathbf{j}}$

[We have $Q_x = P_y$, but the domain is not simply connected.]

We find f with $\vec{F} = \vec{\nabla} f$:

$$f_x = \frac{x}{x^2+y^2}$$

$$f = \int \frac{x\,dx}{x^2+y^2} = \int \frac{\frac{1}{2}du}{u} =$$

$u = x^2+y^2$
$du = 2x\,dx$

$$= \frac{1}{2}\ln u + C(y) = \frac{1}{2}\ln(x^2+y^2) + C(y)$$

$$f_y = \frac{1}{2}\cdot\frac{2y}{x^2+y^2} + C'(y) = \frac{y}{x^2+y^2} + C'(y)$$

$$C'(y) = 0$$

$$C(y) = D$$

$$f = \frac{1}{2}\ln(x^2+y^2) + D$$

Since $\vec{F} = \vec{\nabla} f$,

$\vec{F}$ is conservative!

(b) $\boxed{\text{4 points}}$ $\vec{\mathbf{F}}(x,y) = \underbrace{\frac{-y}{x^2+y^2}}_{=P}\,\vec{\mathbf{i}} + \underbrace{\frac{x}{x^2+y^2}}_{=Q}\,\vec{\mathbf{j}}$

[We have $Q_x = P_y$, but the domain is not simply connected.]

Let C be the curve $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \underbrace{\left\langle \frac{-\sin t}{1}, \frac{\cos t}{1} \right\rangle}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\langle -\sin t, \cos t\rangle}_{\vec{r}\,'(t)}\, dt = \int_0^{2\pi} 1\, dt = 2\pi$$

The integral of $\vec{F}$ along the closed curve C is not 0

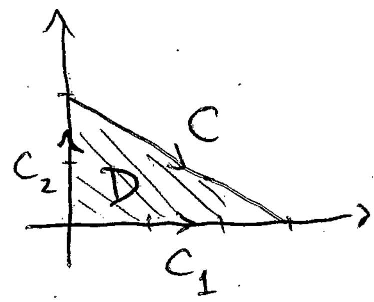
$\Longrightarrow$ $\vec{F}$ is not conservative!

10. $\boxed{\text{6 points}}$ Evaluate

$$\int_C (\pi + y + y\cos(x^2y^2))\mathrm{d}x + (\pi^3 + 3x + x\cos(x^2y^2))\mathrm{d}y$$

where C is the line segment from $(0, 2)$ to $(3, 0)$.
(Hint: Some of the terms simplify when $x = 0$ or $y = 0$.)

$$Q_x - P_y = 3 + \cos(x^2y^2) - x\sin(x^2y^2)\cdot 2xy^2$$
$$- 1 - \cos(x^2y^2) + y\sin(x^2y^2)\cdot 2yx^2 = 2$$



Let: C_1: $\langle t, 0\rangle$, $0 \le t \le 3$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 0$

C_2: $\langle 0, t\rangle$, $0 \le t \le 2$, $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 1$

D = the inside of the triangle with vertices $(0,2), (0,3), (0,0)$

By Green's Theorem $\iint_D Q_x - P_y \, dxdy = -\int_C - \int_{C_2} + \int_{C_1}$

$$\int_C = -\iint_D \overbrace{Q_x - P_y}^{2} \, dxdy - \int_{C_2} \overbrace{Pdx}^{0} + Qdy + \int_{C_1} Pdx + \overbrace{Qdy}^{0}$$

$$= -2\,\text{Area}(D) - \int_{t=0}^{2} (\pi^3 + 0)\, dt + \int_{t=0}^{3} (\pi + 0)\, dt$$

$$= -2 \cdot \frac{3 \cdot 2}{2} - 2\pi^3 + 3\pi$$

$$= \boxed{-6 - 2\pi^3 + 3\pi}$$