

Final Exam

Math 317

August 20, 2015

Last Name: Solutions First Name: _____

Student # : _____ Instructor's Name : _____

Instructions:

No memory aids allowed. No calculators allowed. No communication devices allowed. Use the space provided on the exam. If you use the back of a page, write "see back" on the front of the page. This exam is 180 minutes long.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	17	
2	13	
3	6	
4	7	
5	8	
6	8	
7	17	
8	10	
9	8	
10	6	
Total:	100	

1. Provide a short answer in the box next to each question.

- (a) 2 points Compute $\text{div} (e^{xyz} \vec{i} + z^2 \sin(x^2 y) \vec{j} + e^{z \cos(xy)} \vec{k})$.

$$yz e^{xyz} + x^2 z^2 \cos(x^2 y) + \cos(xy) e^{z \cos(xy)}$$

- (b) 2 points Compute $\text{curl} (xz \vec{i} + 2xy \vec{j} + \ln(xyz) \vec{k})$.

$$\left\langle \frac{1}{y}, x - \frac{1}{x}, 2y \right\rangle$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & 2xy & \ln x + \ln y + \ln z \end{vmatrix} = \left\langle \frac{1}{y} - 0, x - \frac{1}{x}, 2y - 0 \right\rangle$$

- (c) 2 points It is given that at a certain time, a particle has velocity $\langle 1, 1, -1 \rangle$ and acceleration $\langle 2, 0, 1 \rangle$. Find the particle's normal component of acceleration at this time.

$$\sqrt{\frac{14}{3}}$$

$$a_T = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|} = \frac{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix}}{|\langle 1, 1, -1 \rangle|} = \frac{|\langle 1, -3, -2 \rangle|}{\sqrt{3}} = \frac{\sqrt{14}}{\sqrt{3}}$$

- (d) 3 points Evaluate $\int_C y \, dx + ye^x \, dy$ where C is the curve $\vec{r}(t) = t\vec{i} + e^t\vec{j}$, $0 \leq t \leq 1$.

$$e + \frac{1}{3}e^3 - \frac{4}{3}$$

$$\int_C y \, dx + ye^x \, dy = \int_{t=0}^1 e^t \cdot 1 + e^t \cdot e^t \cdot e^t \, dt = \int_{t=0}^1 e^t + e^{3t} \, dt$$

$x=t$ $t=0$
 $dx=dt$
 $y=e^t$
 $dy=e^t dt$

$$= \left[e^t + \frac{1}{3}e^{3t} \right]_0^1 = e + \frac{1}{3}e^3 - 1 - \frac{1}{3} = e + \frac{1}{3}e^3 - \frac{4}{3}$$

- (e) 2 points Let $f(x, y, z) = -\cos(x^2 + y^2 - z^2)$ and let $\vec{F} = \text{grad } f$. The force field \vec{F} is moving an object with mass 0.5 kg from $(0, 0, 0)$ to $(\sqrt{\pi}, \sqrt{\pi}, \sqrt{\pi})$. It is given that the object's speed at $(0, 0, 0)$ is 2 m/s . Use the Law of Conservation of Energy to find the object's speed at $(\sqrt{\pi}, \sqrt{\pi}, \sqrt{\pi})$.

$$\sqrt{12} \text{ m/s}$$

$v_0 = \text{speed at } (0, 0, 0)$

$v_1 = \text{speed at } (\sqrt{\pi}, \sqrt{\pi}, \sqrt{\pi})$

$$P(x, y, z) = -f(x, y, z) = \cos(x^2 + y^2 - z^2) \quad (\text{Potential energy})$$

$$P(0, 0, 0) + \frac{mv_0^2}{2} = P(\sqrt{\pi}, \sqrt{\pi}, \sqrt{\pi}) + \frac{mv_1^2}{2}$$

$$\cos 0 + \frac{0.5 \cdot 2^2}{2} = \cos \pi + \frac{0.5}{2} v_1^2$$

$$\frac{1}{4} v_1^2 = 1 + 1 + 1 = 3$$

$$v_1 = \sqrt{12}$$

- (f)
- 4 points
- Find a potential function for the vector field

$$\vec{F} = (ye^x + yz \cos(xy)) \vec{i} + (e^x + xz \cos(xy)) \vec{j} + (e^z + \sin(xy)) \vec{k}$$

$$ye^x + z \sin(xy) + e^z$$

$$\vec{F} = \nabla f$$

$$f_x = ye^x + yz \cos(xy)$$

$$f = ye^x + z \sin(xy) + g(y, z)$$

$$f_y = e^x + xz \cos(xy) + g_y$$

$$g_y = 0$$

$$g(y, z) = h(z)$$

$$f = ye^x + z \sin(xy) + h(z)$$

$$f_z = \sin(xy) + h_z$$

$$h_z = e^z$$

$$h(z) = e^z + C$$

$$f = ye^x + z \sin(xy) + e^z + C$$

- (g)
- 2 points
- Find the work done by the force field
- \vec{F}
- of (f) in moving a particle along the curve
- $\vec{r}(t) = t^4 \vec{i} + (t^2 - t^3) \vec{j} - t^2 \vec{k}$
- ,
- $0 \leq t \leq 1$
- .

$$e^{-1} - 1$$

$$\text{work} = \int_C \vec{F} \cdot d\vec{r} \stackrel{\substack{\text{fundamental thm. of line integrals} \\ \downarrow}}{=} f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(1, 0, -1) - f(0, 0, 0)$$

$$= [0 + 0 + e^{-1}] - [0 + 0 + e^0]$$

$$= e^{-1} - 1$$

2. Consider the curve

$$\vec{r}(t) = t \vec{i} + t^2 \vec{j} + \frac{2}{3} t^3 \vec{k}.$$

(a) 3 points Find and simplify the unit tangent vector $\vec{T}(t)$.

$$\vec{r}'(t) = \langle 1, 2t, 2t^2 \rangle$$

$$|\vec{r}'(t)| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left\langle \frac{1}{1+2t^2}, \frac{2t}{1+2t^2}, \frac{2t^2}{1+2t^2} \right\rangle$$

(b) 3 points Find and simplify the curvature $\kappa(t)$.

$$\vec{r}''(t) = \langle 0, 2, 4t \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix} = \langle \overbrace{4t^2}^{4t^2}, -4t, 2 \rangle$$

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\sqrt{16t^4 + 16t^2 + 4}}{(1+2t^2)^3} = \frac{4t^2 + 2}{(2t^2 + 1)^3} = \frac{2}{(2t^2 + 1)^2}$$

- (c) 3 points Find the principal unit normal vector at $t = 1$.

$$\vec{T}'(t) = \left\langle \frac{-4t}{(1+2t^2)^2}, \frac{2(1+2t^2)-2t(4t)}{(1+2t^2)^2}, \frac{4t(1+2t^2)-2t^2(4t)}{(1+2t^2)^2} \right\rangle$$

$$\vec{T}'(1) = \left\langle -\frac{4}{9}, \frac{6-8}{9}, \frac{12-8}{9} \right\rangle = \frac{1}{9} \langle -4, -2, 4 \rangle$$

$$\begin{aligned} \vec{N}(1) &= \frac{\vec{T}'(1)}{|\vec{T}'(1)|} = \frac{\frac{1}{9} \langle -4, -2, 4 \rangle}{\frac{1}{9} \sqrt{16+4+16}} = \left\langle -\frac{4}{6}, -\frac{2}{6}, \frac{4}{6} \right\rangle = \\ &= \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle \end{aligned}$$

- (d) 2 points Find the binormal vector at $t = 1$.

$$\vec{T}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$\begin{aligned} \vec{B}(1) &= \vec{T}(1) \times \vec{N}(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = \left\langle \frac{4}{9} + \frac{2}{9}, -\frac{4}{9} - \frac{2}{9}, -\frac{1}{9} + \frac{4}{9} \right\rangle \\ &= \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle \end{aligned}$$

- (e) 2 points Find the equation of the osculating plane (the plane containing the osculating circle) at $t = 1$.

$$\vec{r}(1) = \left\langle 1, 1, \frac{2}{3} \right\rangle$$

$$\frac{2}{3}(x-1) - \frac{2}{3}(y-1) + \frac{1}{3}\left(z - \frac{2}{3}\right) = 0$$

$$\boxed{\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}z = \frac{2}{9}}$$

3. 6 points Find the arc-length parametrization of the curve

$$\vec{r}(t) = \sin(t^2) \vec{i} + \cos(t^2) \vec{j} + \frac{1}{6}(4t+1)^{3/2} \vec{k}$$

starting from $t = 0$.

$$\vec{r}'(t) = \langle 2t \cos(t^2), -2t \sin(t^2), \frac{1}{6} \cdot \frac{3}{2} \cdot 4(4t+1)^{1/2} \rangle$$

$$\begin{aligned} |\vec{r}'(t)| &= [4t^2 \cos^2(t^2) + 4t^2 \sin^2(t^2) + 4t + 1]^{1/2} \\ &= [4t^2 + 4t + 1]^{1/2} = 2t + 1 \end{aligned}$$

$$s(t) = \int_0^t (2u+1) du = [u^2 + u]_0^t = t^2 + t$$

$$t^2 + t - s = 0$$

$$t = \frac{-1 \pm \sqrt{1+4s}}{2} \quad (\text{take "+" because } s \text{ is positive})$$

$$t = \frac{-1 + \sqrt{1+4s}}{2} = \frac{1}{2}(\sqrt{1+4s} - 1)$$

$$\begin{aligned} \vec{r}(s) &= \sin\left(\frac{1}{4}(\sqrt{1+4s} - 1)^2\right) \vec{i} + \\ &\quad \cos\left(\frac{1}{4}(\sqrt{1+4s} - 1)^2\right) \vec{j} + \\ &\quad \frac{1}{6}\left(2\sqrt{1+4s} - 1\right)^{3/2} \vec{k} \end{aligned}$$

4. Let S be the surface obtained by revolving the graph of $y = x^2$, $-1 \leq x \leq 1$, around the x axis.

(a) 2 points Give a parametrization of S .

$$\vec{r}(x, \theta) = \langle x, x^2 \cos \theta, x^2 \sin \theta \rangle$$

$$-1 \leq x \leq 1$$

$$0 \leq \theta \leq 2\pi$$

(b) 5 points Evaluate the surface integral $\iint_S \sqrt{4x^2 + 1} \, dS$.

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2x \cos \theta & 2x \sin \theta \\ 0 & -x^2 \sin \theta & x^2 \cos \theta \end{vmatrix} = \langle \overbrace{2x^3 \cos^2 \theta + 2x^3 \sin^2 \theta}^{2x^3}, -x^2 \cos \theta, -x^2 \sin \theta \rangle$$

$$|\vec{r}_x \times \vec{r}_\theta| = [4x^6 + x^4 \cos^2 \theta + x^4 \sin^2 \theta]^{\frac{1}{2}} = \sqrt{4x^6 + x^4} = x^2 \sqrt{4x^2 + 1}$$

$$\iint_S \sqrt{4x^2 + 1} \, dS = \int_{\theta=0}^{2\pi} \int_{x=-1}^1 \sqrt{4x^2 + 1} \cdot x^2 \sqrt{4x^2 + 1} \, dx \, d\theta$$

$$= 2\pi \int_{x=-1}^1 4x^4 + x^2 \, dx = 2\pi \left[\frac{4}{5} x^5 + \frac{1}{3} x^3 \right]_{-1}^1$$

$$= 2\pi \left[\frac{4}{5} + \frac{1}{3} + \frac{4}{5} + \frac{1}{3} \right]$$

$$= 2\pi \cdot \frac{12+5+12+5}{15} = 2\pi \cdot \frac{34}{15}$$

$$= \boxed{\frac{68}{15} \pi}$$

5. 8 points Use Stoke's Theorem to evaluate the integral

$$\int_C (yz + e^{x^2})dx + (x + ye^{y^2+z^2})dy + (y + ze^{y^2+z^2})dz$$

$$\vec{F} = \langle P, Q, R \rangle$$

where C is the triangle with vertices $(2, 0, 0)$, $(0, 3, 0)$, $(0, 0, 4)$, oriented clockwise when viewed from above.

Let S be the inside of the triangle C , oriented downward. Then $C = \partial S$.

By Stokes' Theorem: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$

$$\text{curl } \vec{F} = \langle 1 + 2yze^{y^2+z^2} - 2zye^{y^2+z^2}, y - 0, 1 - z \rangle = \langle 1, y, 1 - z \rangle$$

Parametrize S : $\vec{r}(u, v) = \langle 2, 0, 0 \rangle + u\langle -2, 3, 0 \rangle + v\langle -2, 0, 4 \rangle$
 $= \langle 2 - 2u - 2v, 3u, 4v \rangle$

$$0 \leq u \leq 1, 0 \leq v \leq 1 - u$$

$$\vec{r}_u \times \vec{r}_v = \langle -2, 3, 0 \rangle \times \langle -2, 0, 4 \rangle = \langle 12, 8, 6 \rangle \rightarrow \text{points upward!}$$

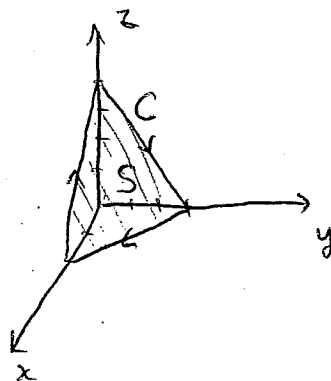
$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = - \int_{u=0}^1 \int_{v=0}^{1-u} \langle 1, 3u, 1-4v \rangle \cdot \langle 12, 8, 6 \rangle dv du$$

$$= - \int_{u=0}^1 \int_{v=0}^{1-u} 12 + 24u + 6 - 24v dv du = - \int_{u=0}^1 [18v + 24uv - 12v^2]_0^{1-u} du$$

$$= - \int_{u=0}^1 18(1-u) + 24u(1-u) - 12(1-u)^2 du$$

$$= - \int_{u=0}^1 18 - 18u + 24u - 24u^2 - 12 + 24u - 12u^2 du = - \int_{u=0}^1 6 + 30u - 36u^2 du$$

$$= - [6u + 15u^2 - 12u^3]_0^1 = - [6 + 15 - 12] = \boxed{-9}$$



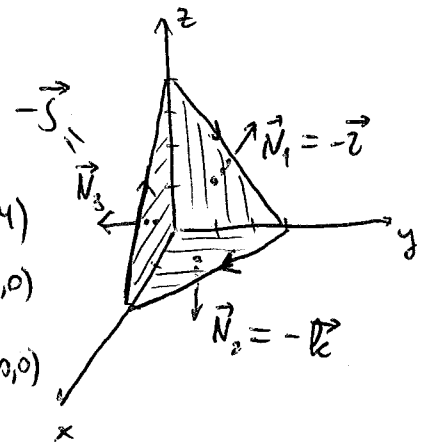
An alternative solution by one of the students who wrote the exam:

Let $S = S_1 + S_2 + S_3$ where

S_1 is the triangle w. vertices $(0,0,0), (0,3,0), (0,0,4)$

S_2 — " ————— $(0,0,0), (2,0,0), (0,3,0)$

S_3 — " ————— $(0,0,0), (0,0,4), (2,0,0)$



Oriented as in the picture.

By Stokes' Theorem $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$

$$\text{curl } \vec{F} = \langle 1 + 2yze^{y^2+z^2}, -2zye^{y^2+z^2}, y - z \rangle = \langle 1, y, 1-z \rangle$$

$$\iint_{S_1} \text{curl } \vec{F} \cdot d\vec{S} = \iint_{S_1} \langle 1, y, 1-z \rangle \cdot \langle -1, 0, 0 \rangle dS = \iint_{S_1} -1 dS = -\text{Area}(S_1) = -\frac{3 \cdot 4}{2} = -6$$

$$\iint_{S_2} \text{curl } \vec{F} \cdot d\vec{S} = \iint_{S_2} \langle 1, y, 1-z \rangle \cdot \langle 0, 0, -1 \rangle dS = \iint_{S_2} -1 dS = -\text{Area}(S_2) = -\frac{2 \cdot 3}{2} = -3$$

\uparrow
 $z=0$ on S_2

$$\iint_{S_3} \text{curl } \vec{F} \cdot d\vec{S} = \iint_{S_3} \langle 1, y, 1-z \rangle \cdot \langle 0, -1, 0 \rangle dS = \iint_{S_3} 0 dS = 0$$

\uparrow
 $y=0$ on S_3

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = -6 - 3 + 0 = \boxed{-9}$$

6. 8 points Let E be the region bounded between the paraboloids $z = 4 - x^2 - y^2$ and $z = x^2 + y^2$ and let S be the boundary of E with outward orientation. Use the Divergence Theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where

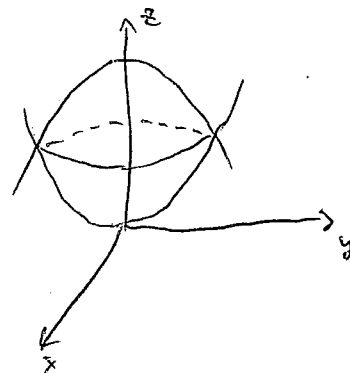
$$\vec{F} = (\sqrt{1+z^2} - xy^2) \vec{i} + (y^3 + xz^2) \vec{j} + (\sqrt{1+x^2} - zy^2) \vec{k}.$$

The paraboloids meet when

$$x^2 + y^2 = 4 - x^2 - y^2$$

$$2x^2 + 2y^2 = 4$$

$$x^2 + y^2 = 2$$



Therefore $E = \left\{ (x, y, z) \mid \begin{array}{l} x^2 + y^2 \leq 2 \\ x^2 + y^2 \leq z \leq 4 - x^2 - y^2 \end{array} \right\}$

$$\operatorname{div} \vec{F} = -y^2 + 3y^2 - y^2 = y^2$$

By the divergence theorem:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, d\text{vol} = \iint_{x^2+y^2 \leq 2} \int_{z=x^2+y^2}^{4-x^2-y^2} y^2 \, dz \, dx \, dy$$

$$= \iint_{x^2+y^2 \leq 2} y^2(4-2x^2-2y^2) \, dx \, dy = \int_0^{2\pi} \int_0^{\sqrt{2}} r^2 \sin^2 \theta (4-2r^2) r \, dr \, d\theta$$

$\begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \end{array}$

$$= \int_{\theta=0}^{2\pi} \sin^2 \theta \, d\theta \int_{r=0}^{\sqrt{2}} r^3(4-2r^2) \, dr = \int_{\theta=0}^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2\theta) \right) d\theta \int_{r=0}^{\sqrt{2}} (4r^3 - 2r^5) \, dr$$

$$= \left[\frac{1}{2} \theta - \frac{1}{4} \sin(2\theta) \right]_0^{2\pi} \left[r^4 - \frac{1}{3} r^6 \right]_0^{\sqrt{2}} = \left[\frac{1}{2} \cdot 2\pi \right] \cdot \left[4 - \frac{8}{3} \right]$$

$$= \pi \cdot \frac{4}{3} = \boxed{\frac{4}{3} \pi}$$

7. Let $\vec{r} = \langle x, y, z \rangle$, $r = |\langle x, y, z \rangle|$ and

$$\vec{F}(x, y, z) = \frac{\vec{r}}{r^3} = \left\langle \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle.$$

- (a) 5 points Let S denote the sphere $x^2 + y^2 + z^2 = (\frac{1}{10})^2$ with positive orientation. Evaluate the flux integral $\iint_S \vec{F} \cdot d\vec{S}$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N} dS = \iint_S \frac{\vec{r}}{r^3} \cdot \frac{\vec{r}}{r} dS$$

$$= \iint_S \frac{r^2}{r^4} dS = \iint_S \frac{1}{r^2} dS$$

$$\begin{array}{l} \boxed{r = \frac{1}{10} \text{ on } S} \rightarrow \iint_S \frac{1}{(\frac{1}{10})^2} dS = 100 \cdot \text{Area}(S) \end{array}$$

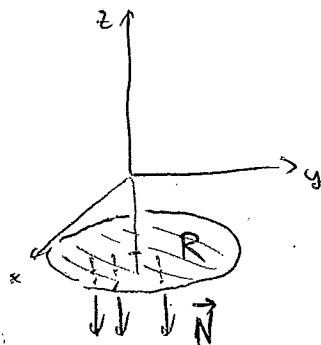
$$= 100 \cdot \left(\frac{1}{10}\right)^2 4\pi = \boxed{4\pi}$$

- (b) 6 points Let R be the part of the plane $z = -1$ lying inside the cylinder $x^2 + y^2 = 1$. Evaluate the flux of \vec{F} through R downwards.

Parameterize R : $\vec{F}(x,y) = \langle x, y, -1 \rangle$
 $x^2 + y^2 \leq 1$

$$\vec{N} = -\vec{k} = \langle 0, 0, -1 \rangle$$

$$|\vec{F}_x \times \vec{F}_y| = |\langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle| = |\langle 0, 0, 1 \rangle| = 1$$



$$\iint_R \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot \vec{N} dS = \iint_{x^2+y^2 \leq 1} \langle \dots, \dots, \frac{-1}{(x^2+y^2+1)^{3/2}} \rangle \cdot \langle 0, 0, -1 \rangle \cdot 1 dx dy$$

$$= \iint_{x^2+y^2 \leq 1} \frac{1}{(x^2+y^2+1)^{3/2}} dx dy = \int_{\theta=0}^{2\pi} \int_{r=0}^1 \frac{r}{(r^2+1)^{3/2}} dr d\theta$$

$x = r \cos \theta$
 $y = r \sin \theta$

$$= 2\pi \int_{r=0}^1 \frac{r dr}{(r^2+1)^{3/2}} = 2\pi \cdot \frac{1}{2} \int_{u=1}^2 \frac{du}{u^{3/2}} = \pi \left[-2u^{-1/2} \right]_1^2$$

$u = r^2 + 1$
 $du = 2r dr$

$$= \pi \left[-\frac{2}{\sqrt{2}} + 2 \right] = \boxed{(2 - \sqrt{2})\pi}$$

- (c) 3 points Simplify $\operatorname{div} \vec{F} = \operatorname{div}(\vec{r}/r^3)$.

$$\begin{aligned}
 \operatorname{div} \frac{\vec{r}}{r^3} &= r^{-3} \cdot \operatorname{div} \vec{r} + \operatorname{grad} r^{-3} \cdot \vec{r} \\
 &= 3r^{-3} + (-3r^{-4} \operatorname{grad} r) \cdot \vec{r} \\
 &= 3r^{-3} + (-3r^{-4}) \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle \cdot \langle x, y, z \rangle \\
 &= 3r^{-3} - 3r^{-4} \cdot \frac{x^2 + y^2 + z^2}{r} \\
 &= 3r^{-3} - 3r^{-4} \cdot \frac{r^2}{r} = \boxed{0}
 \end{aligned}$$

- (d) 3 points Let S' be the part of the paraboloid $z = 1 - 2x^2 - 2y^2$ lying above the plane $z = -1$, oriented toward the positive z -axis. Use your previous answers and the Divergence Theorem to evaluate $\iint_{S'} \vec{F} \cdot d\vec{S}$.

Let E be the region bounded between R and S' , except the inside of S .

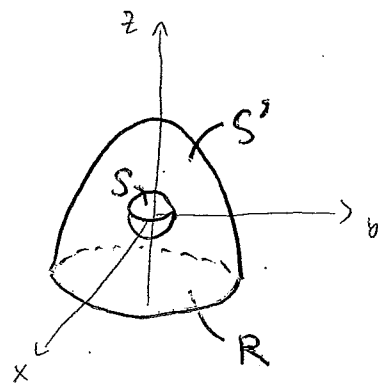
$$\partial E = S' + R - S$$

By the Divergence Theorem

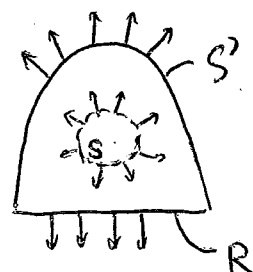
$$0 = \iiint_E \operatorname{div} \vec{F} \, dV = \iint_{S'} \vec{F} \cdot d\vec{S} + \iint_R \vec{F} \cdot d\vec{S} - \iint_S \vec{F} \cdot d\vec{S}$$

Therefore

$$\begin{aligned}
 \iint_{S'} \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot d\vec{S} - \iint_R \vec{F} \cdot d\vec{S} \\
 &= 4\pi - (2 - \sqrt{2})\pi = \boxed{(2 + \sqrt{2})\pi}
 \end{aligned}$$



view from the side



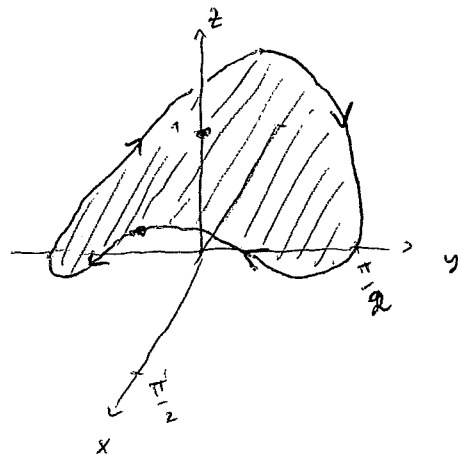
8. For each of the following closed oriented curves, find an oriented surface S whose boundary is the given oriented curve. You must describe the surface in words, draw it, give a parametrization, and indicate the orientation (e.g. by drawing the normal vector field, or by writing "upward", "downward", etc.). If the surface is made of several pieces, give a parametrization and an orientation for each of the pieces.

- (a) 4 points C is the intersection of the cylinder $x^2 + y^2 = \frac{\pi^2}{4}$ with the surface $z = \cos y$, oriented clockwise when viewed from above.

S is the part of $z = \cos y$ inside the cylinder $x^2 + y^2 = \frac{\pi^2}{4}$, oriented downward

$$\vec{r}(x, y) = \langle x, y, \cos y \rangle$$

$$x^2 + y^2 \leq \frac{\pi^2}{4}$$



- (b) 6 points C consists of the five oriented line segments: $(0, 0, 0)$ to $(0, 1, 0)$, $(0, 1, 0)$ to $(0, 0, 1)$, $(0, 0, 1)$ to $(1, 1, 0)$, $(1, 1, 0)$ to $(1, 0, 0)$, $(1, 0, 0)$ to $(0, 0, 0)$.

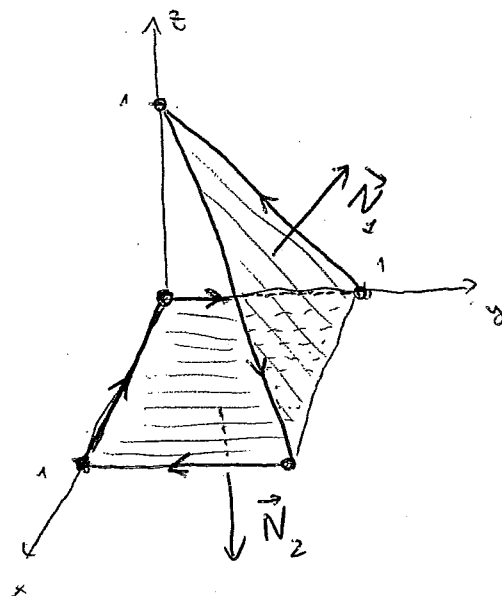
$$S = S_1 + S_2$$

S_1 is the triangle with vertices $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, oriented upward.

$$\begin{aligned} \vec{r}(u, v) &= \langle 0, 0, 1 \rangle + u \langle 0, 1, -1 \rangle + v \langle 1, 1, -1 \rangle \\ &= \langle v, u+v, 1-u-v \rangle \\ 0 &\leq u \leq 1, \quad 0 \leq v \leq 1-u \end{aligned}$$

S_2 is the square with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(1, 0, 0)$, oriented downward.

$$\begin{aligned} \vec{r}(u, v) &= \langle u, v, 0 \rangle & 0 \leq u \leq 1 \\ & & 0 \leq v \leq 1 \end{aligned}$$



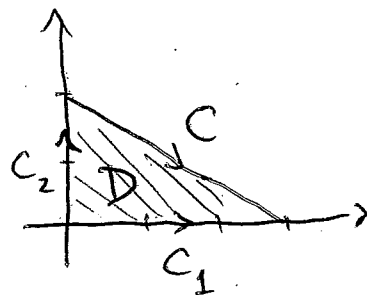
10. 6 points Evaluate

$$\int_C (\pi + y + y \cos(x^2 y^2)) dx + (\pi^3 + 3x + x \cos(x^2 y^2)) dy$$

where C is the line segment from $(0, 2)$ to $(3, 0)$.(Hint: Some of the terms simplify when $x = 0$ or $y = 0$.)

$$Q_x - P_y = 3 + \cos(x^2 y^2) - x \sin(x^2 y^2) \cdot 2xy^2$$

$$-1 - \cos(x^2 y^2) + y \sin(x^2 y^2) \cdot 2yx^2 = 2$$



Let: $C_1: \langle t, 0 \rangle, 0 \leq t \leq 3, \frac{dx}{dt} = 1, \frac{dy}{dt} = 0$

$C_2: \langle 0, t \rangle, 0 \leq t \leq 2, \frac{dx}{dt} = 0, \frac{dy}{dt} = 1$

D = the inside of the triangle with vertices $(0, 2), (0, 3), (3, 0)$

By Green's Theorem $\iint_D Q_x - P_y \, dx \, dy = - \int_C \vec{F} \cdot d\vec{r} = - \int_{C_2} + \int_{C_1}$

$$\int_C = - \iint_D \overbrace{Q_x - P_y}^2 \, dx \, dy - \int_{C_2} \overbrace{P}^0 dx + \overbrace{Q}^0 dy + \int_{C_1} \overbrace{P}^0 dx + \overbrace{Q}^0 dy$$

$$= -2 \text{Area}(D) - \int_{t=0}^2 (\pi^3 + 0) \, dt + \int_{t=0}^3 (\pi + 0) \, dt$$

$$= -2 \cdot \frac{3 \cdot 2}{2} - 2\pi^3 + 3\pi$$

$$= \boxed{-6 - 2\pi^3 + 3\pi}$$