Tips in choosing the quantities:

* Include any quantity whose rate of change is given in the question.
* Include the quantity whose rate of change you need to find.
* Do not include constant (= unchanging) quantities.

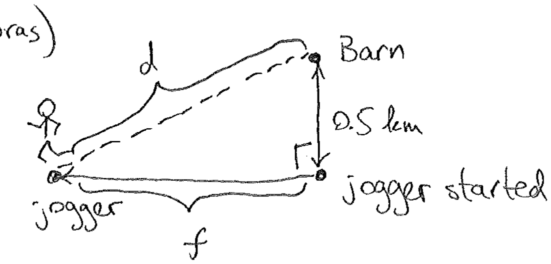
**Question 1.** In the countryside, a jogger begins his jog 500 meters south of an old barn. He heads west at a constant speed of $10_{\mathrm{km/h}}$. How fast does the distance between the jogger and the barn increase when the jogger passed 1.2 kilometers?

Quantities: $f$ - the distance the jogger ran

$d$ - the distance between the jogger and the barn

Relations: $d^2 = f^2 + 0.25$ (Pythagoras)

(or $d = \sqrt{f^2 + 0.25}$)




Given: $\boxed{f' = 10}$.

Want: $d'$ when $\boxed{f = 1.2}$.

In this case, $d^2 = 1.2^2 + 0.25 = 1.44 + 0.25 = 1.69$

so $\boxed{d = 1.3}$

Differentiate relation: $2dd' = 2ff'$

Substitute $f' = 10$, $f = 1.2$, $d = 1.3$ and solve for $d'$:

$\cancel{2} \cdot 1.3 \cdot d' = \cancel{2} \cdot 10 \cdot 1.2$

$d' = \frac{10 \cdot 1.2}{1.3} = \frac{10 \cdot 12}{13} = \frac{120}{13}$

$\boxed{d' = \frac{120}{13} \frac{\mathrm{km}}{\mathrm{h}}}$

**Question 2.** A kid blows air into a spherical balloon at a constant rate of 5 cubic inches per second.

(a) How fast does the radius of the balloon grow when the radius of the balloon is $5_{\text{in}}$.

(b) How fast does the radius of the balloon grow when the balloon contains $36\pi$ cubic inches of air?

(c) Continuing (a), at what rate does the surface area of the balloon grow?

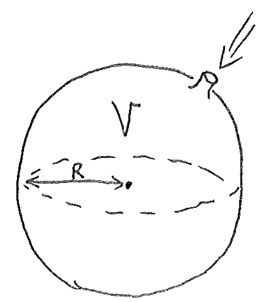
ⓐ Quantities: $R$ - the radius of the balloon

$V$ - the amount of air in the balloon

Relation: $V = \frac{4}{3}\pi R^3$

Given: $V' = 5$

Want: $R'$ when $R=5$




Differentiate relation: $V' = \frac{4}{3}\pi \cdot 3R^2 \cdot R'$

$(*)$ $V' = 4\pi R^2 R'$

Substitute $V'=5$, $R=5$, solve for $R'$:

$5 = 4\pi \cdot 5^2 R'$

$\boxed{R' = \frac{1}{20\pi} \text{ in/sec}}$

ⓑ Want: $R'$ when $V = 36\pi$.

In this case, $36\cancel{\pi} = \frac{4}{3}\cancel{\pi} R^3$

$27 = R^3$, so $R = 3$in.

Substitute $V'=5$, $R=3$ in $(*)$ and solve for $R'$:

$5 = 4\pi \cdot 3^2 R'$

$\boxed{R' = \frac{5}{36\pi} \text{ in/sec}}$

(c) Quantities: $R$ - the radius of the balloon

$S$ - the surface area of the balloon

Relation: $S = 4\pi R^2$

Want: $S'$ when $R = 5$.

In this case, by (a), $R' = \frac{1}{20\pi}$

Differentiate Relation:

$$S' = 8\pi R R'$$

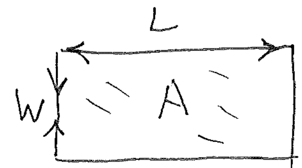
Substitute $R = 5$, $R' = \frac{1}{20\pi}$ and solve for $S'$:

$$S' = 8\pi \cdot 5 \cdot \frac{1}{20\pi}$$

$$\boxed{S' = 2 \ \text{in}^2/\text{sec}}$$

**Question 3.** The length of a rectangle is increased at a rate of $3_{\text{cm/sec}}$ while at the same time its width is decreased at a rate of $2_{\text{cm/sec}}$. At what rate does the area of the rectangle change when its length and width both equal $10_{\text{cm}}$?

Quantities: $L$ - the length of the rectangle
$W$ - the width of the rectangle
$A$ - the area of the rectangle




Relation: $A = WL$

Given: $\boxed{L' = 3}$, $\boxed{W' = -2}$

Want: $A'$ when $\boxed{L = 10}$, $\boxed{W = 10}$.

Differentiate relation: $A' = W'L + WL'$ (product rule!)

Substitute and solve for $A'$:

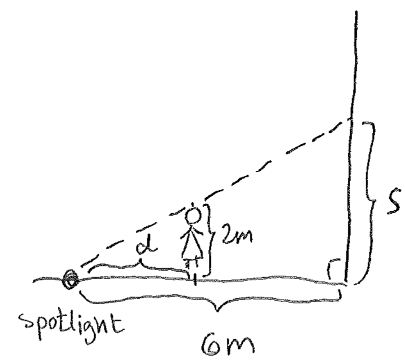
$$A' = (-2)\cdot 10 + 10\cdot 3$$

$$\boxed{A' = 10\ \frac{\text{cm}^2}{\text{sec}}}$$

**Question 4.** [1] A spotlight is placed on the ground 6 meters from a wall. A woman 2 meters tall is walking at a speed of $2_{\text{m/sec}}$ from the spotlight straight toward the wall, casting a shadow on the wall. At what rate does the height of the shadow change when the woman is 2 meters from the wall?

Quantities: $d$ – the distance between the woman and the spotlight

$s$ – the height of the shadow




Relations: $\frac{d}{6} = \frac{2}{s}$ (or $12 = ds$)

(or $\frac{s}{6} = \frac{d}{2}$ ...)

Given: $\boxed{d' = 2}$

Want: $s'$ when $\boxed{d = 4}$.

In this case, $\frac{4}{6} = \frac{2}{s}$, so $\boxed{s = 3}$.

Differentiate relation:

$$\frac{d'}{6} = -\frac{2}{s^2} \cdot s'$$

Substitute and solve for $s'$:

$$\frac{2}{6} = -\frac{2}{3^2} \cdot s'$$

$$s' = -\frac{9}{6}$$

$$\boxed{s' = -1.5 \frac{m}{sec}}$$

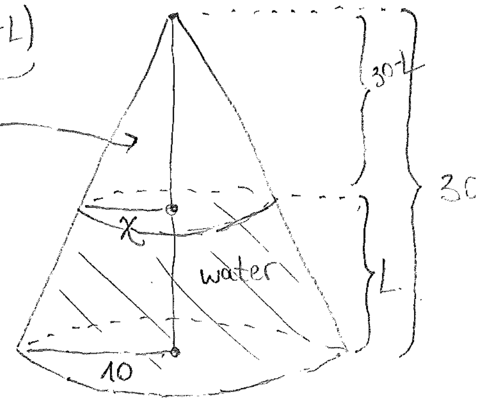
[1]Taken from a past midterm.

**Question 5.** Water are poured at a rate of $2_{cm^3/sec}$ into a test tube shaped as cone 20 centimeters wide and 30 centimeters tall (the tip of the cone points upward, the base is parallel to the ground). What is the rate of change of the water level in the test tube when the water level is 15 centimeters?

Quantities: L – the level of water in the tube

W – the amount of water in the tube

Relations: $$W = \underbrace{\frac{1}{3}\pi 10^2 \cdot 30}_{\text{volume of tube}} - \underbrace{\frac{1}{3}\pi\left(\frac{1}{3}(30-L)\right)^2(30-L)}_{\text{volume of}}$$




$$W = 1000\pi - \frac{1}{27}\pi(30-L)^3$$

$$\frac{x}{30-L} = \frac{10}{30}$$

$$x = \frac{1}{3}(30-L)$$

Given: $W' = 2$

Want: $L'$ when $L = 15$

Differentiate relation:

$$W' = -\frac{1}{27}\pi \cdot 3(30-L)^2 \cdot (-L')$$

Substitute $W' = 2$, $L = 15$. and solve for $L'$:

$$2 = -\frac{1}{27}\pi \cdot 3(30-15)^2(-L')$$

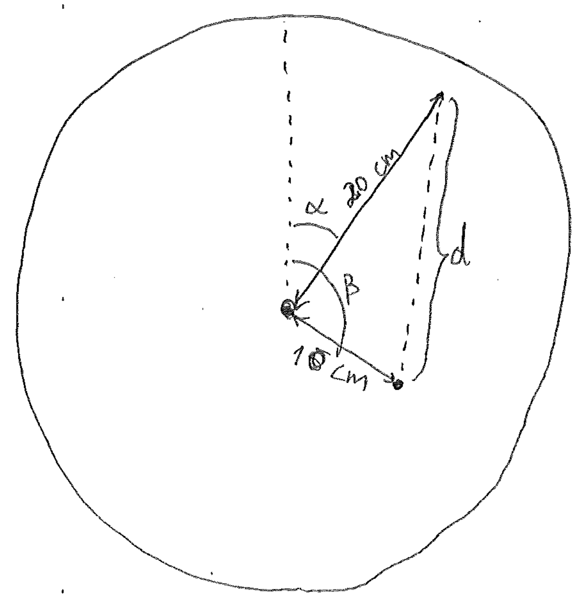
$$2 = \frac{\pi}{9} \cdot 15^2 L'$$

$$L' = \frac{2}{15^2} \cdot \frac{9}{\pi} = \frac{2}{5^2 \cdot 3^2} \cdot \frac{9}{\pi} = \frac{2}{5^2\pi}$$

$$\boxed{L' = \frac{2}{25\pi} \ \mathrm{cm/sec}}$$

**Question 6.** A clock has two hands, indicating the hours and the minutes respectively. The hour hand is $10_{cm}$ long while the minutes hand is $20_{cm}$ long. Suppose that the hour is 3:00. What is the rate of change of the distance between the tips of the clock's hands? Specify your answer in centimeters per hour.
(Suggestion: Use the law of cosines.)

Quantities: $\alpha$ - see picture
$\beta$ - see picture
$d$ - the distance between the tips of the clock's hands




Relations: By the Law of Cosines:

$$d^2 = 20^2 + 10^2 - 2\cdot 10\cdot 20\cos(\beta-\alpha)$$

or $$d = \sqrt{500 - 400\cos(\beta-\alpha)}$$

Given: $\boxed{\alpha' = 2\pi}$ (because the minutes hand completes one round $= 2\pi$ rad every hour)

$\boxed{\beta' = \frac{2\pi}{12} = \frac{\pi}{6}}$ (because the hours hand completes one round every 12 hrs.).

Want: $d'$ when $\boxed{\alpha = 0}$, $\boxed{\beta = \frac{\pi}{2}}$ ("the hour is 3:00")

Differentiate relation: $$d' = \frac{1}{2\sqrt{500-400\cos(\beta-\alpha)}}\cdot\left(+400\sin(\beta-\alpha)\right)\cdot(\beta'-\alpha')$$

Substitute and solve for $d'$:

$$d' = \frac{1}{2\sqrt{500-400\cos(\frac{\pi}{2}-0)}}\cdot\left(+400\sin\left(\frac{\pi}{2}-0\right)\right)\left(\frac{\pi}{6}-2\pi\right) = \frac{1}{2\sqrt{500}}\cdot 400\left(\frac{\pi}{6}-\frac{12\pi}{6}\right)$$

$$= \frac{400}{20\sqrt{5}}\cdot\left(-\frac{11\pi}{6}\right) = -\frac{20\cdot 11\pi}{6\sqrt{5}} = \boxed{-\frac{110\pi}{3\sqrt{5}}\ \text{cm}/\text{h}}$$