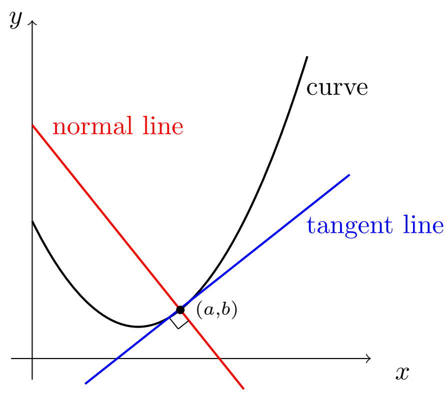
THE NORMAL LINE

Reminder. Two lines in the plane are perpendicular if and only if the product of their slopes is -1. For example, the lines

$$y = -2x + 7 \qquad \text{and} \qquad y = \frac{1}{2}x - 9$$

are perpendicular because $(-2) \cdot \frac{1}{2} = -1$.

Definition. The normal line to a curve at a point (a, b) is the line passing through (a, b) and perpendicular to the tangent line to the curve at (a, b).



The normal line can be thought of as the line perpendicular to the curve at the point (a, b).

Remark. Recall that the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$. By the reminder above we have the following conclusions:

- The slope of the normal line to $y = f(x)$ at $x = a$ is
$$-\frac{1}{f'(a)}\ .$$
- The equation of the normal line to $y = f(x)$ at $x = a$ is
$$\boxed{y = -\frac{1}{f'(a)}(x - a) + f(a)}$$

Example. Find the normal line to $y = x^2$ at the point $(2, 4)$.

Solution.

$$y' = 2x$$
$$y'(2) = 4$$

The normal line is

$$y = -\frac{1}{4}(x-2)+4$$
$$y = -\frac{1}{4}x+\frac{1}{2}+4$$
$$\boxed{y = -\frac{1}{4}x+\frac{9}{2}}$$

Example. Find the normal line to the curve

$$x\cos y + 2y = 1$$

at the point $(1,0)$.

Solution. We use implicit differentiation to find $\frac{dy}{dx}$ at the point $(1,0)$.

$$\frac{d}{dx}(x\cos y+2y) = \frac{d}{dx}1$$
$$1\cdot\cos y + x\cdot(-\sin y)y' + 2y' = 0$$

We substitute $x = 1$, $y = 0$ and solve for y'.

$$1\cdot\cos 0 + 1\cdot(-\sin 0)y' + 2y' = 0$$
$$1 + 2y' = 0$$
$$y' = -\frac{1}{2}$$
$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,0)} = -\frac{1}{2}$$

Thus, the slope of the tangent line at $(1,0)$ is $-\frac{1}{2}$. The slope of the normal line is therefore 2 (because $2\cdot(-\frac{1}{2}) = -1$) and its equation is

$$y = 2(x-1)+0$$
$$\boxed{y = 2x-2}$$