Bilinear Forms and Rings with Involution

by Uriya Aharon First

Department of Mathematics

Ph.D. Thesis

Submitted to the Senate of Bar-Ilan University

Ramat Gan, Israel *December, 2012*

This work was carried out under the supervision of Prof. Uzi Vishne
(Department of Mathematics), Bar-Ilan University

Contents

Acknowledgements

I would like to express my deep gratitude to my supervisor, Prof. Uzi Vishne, who has supported and guided me since the beginning of my M.Sc. Degree. Working with Uzi has enriched my mathematical knowledge as well as my grasp of the academic world beyond what mere words can describe.

I also greatly thank Prof. Louis H. Rowen for his useful advice and guidance, and also for his help in submitting my articles.

Many people have contributed to the contents of this work and I am grateful for their help as well. In particular, I would like to thank Michael Megrelishvili and Menny Shlossberg for their help with everything that involves topology, to Eli Matzri for his help in choosing the appropriate values in Examples 1.6.1 and 1.6.2, to David Saltman, who suggested to me the problem discussed in section 2.9, and also to Ofir Gorodetsky, Tomer Schlank and others, whose advice have contributed to the formulation and proof of Theorem 1.9.6.

I would like to thank my family and friends, especially those in Bar-Ilan University, for supporting me, stimulating my mind, and letting me share my thoughts with them.

Lastly, I would like to give special gratitude to the administrative staff of Bar-Ilan's Department od Mathematics, and in particular to the department's secretaries (and ex-secretary) Malka Bachar, Yael Madar and Lahli Bar, who pushed me forward since my arrival at Bar-Ilan as a very confused high-school student.

Introduction

From a chronological point of view, this work is written from its end to its beginning. So let us start at the end, which is the beginning.

Background and Motivation. Quadratic and hermitian forms have been investigated since the nineteenth century and have found their way into many areas of mathematics, in particular into algebra, algebraic and differential geometry and algebraic topology. The diversity of their applications eventually made them into an object of research in their own right, and many authors (e.g. Witt, Milnor, Pfister, to name just a few) have published works dedicated solely to study them. The basic problems in this area include the isometry problem (determining whether two quadratic forms are isometric), the isotropy problem (determining whether a quadratic form is isotropic) and the structure of the isometry group. All of these problems are solved to some extent over special base fields (e.g. global, local, finite, real-closed, algebraically closed, etc.; see [**65**], [**86**], [**69**] and related texts).

While classical scenarios required quadratic and hermitian forms over fields and number rings, beginning from the sixties, quadratic forms over general rings (non-commutative, with involution) were defined and investigated. This includes the works of Bak ([**6**]), Knebusch ([**55**]), Bass ([**10**]), Quebbemann, Scharlau and Schulte ([**71**]), Knus ([**56**]), Balmer ([**7**]) and others. Their combined work eventually led to the modern theory of *hermitian categories*, also called *categories with duality*, which are a purely categorical framework to work with quadratic and bilinear forms. One of the strongest results about hermitian categories roughly states that, under mild assumptions, the theory of quadratic forms over a given hermitian category can be reduced to the theory of quadratic forms over division rings with involution (e.g. see [**71**] or [**86**, Chp. 7]). The applications are numerous and include Witt's Cancelation Theorem and various structural results.

Independently, beginning also from the sixties, various authors have considered (non-symmetric) bilinear and *sesquilinear* forms over fields.[1] The isometry problem of such forms (which is equivalent to the congruence problem in GL_n), was studied by Wall ([**98**]), and his work was later used by Riemh ([**76**]) to rigorously solve the isometry problem of nondegenerate bilinear forms (over fields), where a *solution* means reduction to isometry of hermitian forms. Riehm's solution was extended almost immediately by Gabriel to degenerate forms in [**44**], and further generalizations to *sesquilinear forms* (e.g. [**75**], [**84**]) and to simultaneous isometry of two or more bilinear forms (e.g. [**88**]) have followed later. These works have many applications as well; most concern canonical representatives of isometry classes and other results about matrix theory (e.g. see [**46**], [**31**], [**101**], [**49**], [**51**], [**28**], [**93**]).

In contrast to the theory of quadratic forms, very little seems to be known about (non-symmetric) bilinear forms over rings (which are not fields). The purpose of this work is to fill some of this void.

[1] I do not assume bilinear forms to be symmetric unless this is explicitly stated.

Bilinear Forms over Fields; A Guiding Example. My M.Sc. thesis was concerned with bilinear spaces over fields. In the year after its submission, I noticed that some of its results, which seemed to need extensive usage of linear algebra, could actually be proved in a purely ring theoretic context. This observation suggested that some of the theory of bilinear forms over fields could be generalized to bilinear forms over rings, and thus my the research for my Ph.D. thesis has initiated.

Let me first demonstrate how the theory of bilinear forms over fields (which are the "easiest rings") can be treated with ring theoretic tools. Let F be a field and let (V,b) be a regular bilinear space.[2] Then b induces an anti-automorphism of $\mathrm{End}_F(V)$ given by $\sigma \mapsto \sigma^*$, where σ^* is the unique endomorphism of V satisfying

$$b(\sigma x, y) = b(x, \sigma^* y) \qquad \forall x, y \in V \ .$$

(If b were symmetric or alternating, then $*$ would have been an involution). Let W_b denote the ring $\{\sigma \in \mathrm{End}_F(V) \,|\, \sigma^{**} = \sigma\}$. Then $(W_b, *|_{W_b})$ is a ring with involution which turns out to hold a lot of information about b. For instance:

(1) There is one-to-one correspondence between representations $b = b_1 \perp \cdots \perp b_n$ and families of pairwise orthogonal idempotents $e_1, \dots, e_n$ such that $\sum_i e_i = 1$ and $e_i^* = e_i$. In particular, b is *indecomposable* (i.e. not an orthogonal sum of two non-zero bilinear forms) $\iff$ W_b does not contain non-trivial $*$-invariant idempotents.
(2) The form b is hyperbolic (i.e. there are totally isotropic subspaces V_1, V_2 with $V = V_1 \oplus V_2$) $\iff$ the involution $*|_{W_b}$ is hyperbolic (i.e. there exists an idempotent $e \in W_b$ such that $e + e^* = 1$).
(3) σ is an isometry of b $\iff$ $\sigma \in \{\tau \in W_b \,|\, \tau^*\tau = 1\}$.

In fact, one can construct a "dictionary" translating various properties of b to properties of the ring with involution $(W_b, *|_{W_b})$ (*hence the title of this dissertation*).

Let us now show how one can translate the isometry problem of bilinear forms into a *congruence problem* in $(W_b, *|_{W_b})$: The *asymmetry* of b is defined to be the unique endomorphism λ of V satisfying

$$b(x, y) = b(y, \lambda x) \qquad \forall x, y \in V \ .$$

The *conjugacy class*[3] of λ is invariant under isometry. Recall that two elements $a, b \in W_b$ are called $(*|_{W_b}-)$congruent, denoted $a \sim b$, if there exists $s \in W_b^\times$ such that $a = s^* b s$. It turns out that

(4) There is a one-to-one correspondence (depending on b) between isometry classes of regular bilinear forms whose asymmetry is conjugate to λ and the set $\{\sigma \in W_b^\times \,|\, \sigma^* = \sigma\}/\sim$ (i.e. conjugacy classes of $*$-invariant invertible elements in W_b).

This means that the isometry problem can be reduced to (1) deciding whether two bilinear forms have conjugate asymmetries (easy) and (2) the conjugacy problem in $(W_b, *|_{W_b})$.

The structure of the ring W_b is understood to some extent,[4] but not in a manner that allows easy work with involutions. In contrast to that, the ring

[2] The bilinear space (V,b) is called regular if the map $x \mapsto b(x, \_)$ from V to V^* is bijective.

[3] Two linear transformations, $f \in \mathrm{End}_F(V)$ and $g \in \mathrm{End}_F(U)$, are said to be *conjugate* if there exists an *isomorphism* $h : V \to U$ such that $h \circ f = g \circ h$. This is an equivalence relation. It is well known from linear algebra that f and g are conjugate if and only if they have the same Jordan form (or, equivalently, canonical rational form).

[4] Indeed, if λ is the *asymmetry* of b, then $W_b = \mathrm{End}_{F[x]}(V)$ where V is considered as an $F[x]$-module by letting x act as λ. As modules over $F[x]$ are well-behaved, we can determine W_b from the Jordan form of λ. However, dropping the assumption that F is a field leads to some unexpected behavior (e.g. W_b might not be artinian when F is artinian).

$\overline{W}_b := W_b/\operatorname{Jac}(W_b)$ is guaranteed to be semisimple. Furthermore, it can be shown that if $\operatorname{char} F \neq 2$, then all the properties/statements about $(W_b, *|_{W_b})$ specified above can be lifted from $\overline{W}_b$ to W_b, hence we can study b by studying semi-semisimple F-algebras with involution.

Let us exploit this to reduce isometry of bilinear forms to isometry of hermitian forms: Let β denote the involution induced by $*$ on $\overline{W}_b$. By (4) and the previous paragraph, it is enough to solve the conjugacy problem in $(\overline{W}_b, \beta)$. The ring with involution $(\overline{W}_b, \beta)$ is easily seen to be a product of rings with involution $\prod_{i=1}^{k}(W_i, \beta_i)$ with each W_i being either simple artinian or of the form $W'_i \times W'^{\mathrm{op}}_i$ with W_i simple artinian and β_i exchanging W'_i and W'_i. We may thus restrict our attention to the components (W_i, β_i). We now split into two cases. If W_i is of the form $W'_i \times W'^{\mathrm{op}}_i$, then any two β_i-invariant invertible elements are β_i-congruent (straightforward), so the congruence problem is trivial. However, if W_i is simple artinian, then by Wedderburn's Theorem, we can write $W_i \cong \mathrm{M}_{n_i}(D_i)$ for some division ring D_i (actually, D_i is a field in our case). We now invoke the following well-known theorem.

Theorem 0.1. *Let D be a f.d. division algebra over F. If $\mathrm{M}_n(D)$ admits an involution β, then D has an involution α and there exists a 1 or -1 hermitian form $h : D^n \times D^n \to D$ over (D, α) whose corresponding involution is β. That is, for all $x, y \in D^n$ and $\sigma \in \operatorname{End}_D(D^n) \cong \mathrm{M}_n(D)$, we have:*

$$h(\sigma x, y) = b(x, \sigma^{\beta} y)\ .$$

Proof. For the existence of α, see [**2**, Chp. X]. For the existence of h see [**57**, Th. 4.2]. If D is a field (as in our case), then α is just the restriction of β to $D = \operatorname{Cent}(\mathrm{M}_n(D))$. □

Let α_i, h_i be the involution and hermitian form obtained from β_i as in the theorem and let S denote the set of i-s for which W_i is simple. We now have a one-to-one correspondence between the following sets:

(i) Isometry classes of bilinear form whose asymmetry is conjugate to λ;
(ii) $\{\sigma \in W_b^{\times} \,|\, \sigma^* = \sigma\}/\sim$;
(iii) $\{\sigma \in \overline{W}_b^{\times} \,|\, \sigma^{\beta} = \sigma\}/\sim$;
(iv) $\prod_{i\in S}\{\sigma \in W_i^{\times} \,|\, \sigma^{\beta_i} = \sigma\}/\sim$;
(v) Families $\{b_i\}_{i\in S}$ such that each b_i is an n_i-dimensional 1-hermitian or -1-hermitian form over D_i, considered up to isometry.

Indeed, the correspondences (i)↔(ii) and (iv)↔(v) are just (4) above, (ii)↔(iii) and (iii)↔(iv) were explained (but not proved) in the two paragraphs before Theorem 0.1. As a corollary of the correspondence we get:

Corollary 0.2. *Isometry of regular bilinear forms over a field F of characteristic not two can be reduced to isometry of hermitian forms over f.d. division algebras over F (which are in fact fields).*

This corollary is precisely Riehm's solution ([**76**]), although he did not phrase or prove it in this manner. The advantage of the approach taken here is that *it is purely ring theoretic* and hence it might be effectively applied to bilinear forms over rings. Moreover, it turns out that various works which solve similar isometry problems of non-symmetric forms (e.g. [**75**], [**84**], [**88**] and also [**44**]) can be obtained as special cases of this general strategy.

Main Results. In this work I have generalized the previous ideas to bilinear form over rings. The effort was fruitful and the results obtained has exceeded my expectations; by a slight alternation of the definition of W_b, I was able to handle *non-regular* (e.g. degenerate) bilinear forms and, more importantly, arbitrarily large *systems of bilinear forms* (which is new even in the symmetric case). Among the results I have obtained for systems of bilinear forms over certain *good* rings (see below) *in which* 2 *is a unit* are:

(a) Witt's Cancelation Theorem (for non-symmetric non-regular systems of bilinear forms).
(b) The isometry problem can be reduced to isometry of hermitian forms over division rings (generalizing [**76**], [**75**], [**84**], [**88**] for regular non-symmetric forms, [**44**] for non-regular non-symmetric forms and [**6**], [**55**], [**71**] and related papers for regular symmetric forms).
(c) There exists a *decomposition into isotypes* (see [**84**] for definition in the non-symmetric case; see [**86**, Th. 10.8] for the symmetric case; see section 4.12 for the general case; this generalizes the references mentioned in (b)).
(d) Characterization of the indecomposable (systems of) bilinear spaces (generalizing [**93**]).
(e) If the base ring is a f.d. algebra over an algebraically closed F field and O is its isometry group, then there exists an exact sequence of F-algebraic groups $1 \to U \to O \to G \to 1$ such that U is the unipotent radical of O and G is a product of copies of $O_n(F)$, $\mathrm{GL}_m(F)$ and $\mathrm{Sp}_{2k}(F)$. (The sequence $1 \to U \to O \to G \to 1$ remains exact after taking rational points over F; this result resembles [**14**].)

There are other applications, which could not be included in this thesis due to space and time limitations, and will be given elsewhere.

Among the *good* rings are the semiprimary ring (e.g. right or left artinian rings) and, more generally, the semiperfect rings which are *pro-semiprimary*, namely isomorphic to an inverse limit of semiprimary rings. For example, any semilocal ring R for which $R = \varprojlim R/\operatorname{Jac}(R)^n$ is semiperfect and pro-semiprimary (such rings are called *complete semilocal*). In all of the results, the base module is assumed to be finitely presented, and additional mild assumptions are needed if the base ring is not semiprimary (these assumptions are satisfied by f.g. projective modules).

To put the previous results into their right context, observe that the idea of studying bilinear forms by transferring to rings with involution also appears in the literature about *symmetric* bilinear forms over rings (e.g. see [**71**], [**86**, Chp. 7], [**16**, §5]). This approach has led to the proof of most of the previous results over hermitian categories satisfying certain conditions. In particular, (a)–(d) are known to hold for *regular* (single) bilinear forms over complete discrete valuation rings. In addition, in [**16**], E. Bayer-Fluckiger and L. Fainsilber have presented a way to derive statements about non-regular bilinear forms from the regular case and have applied it to Witt's Cancelation Theorem and other results. (We will discuss [**16**] in more detail below.) Nevertheless, in contrast to the symmetric theory, the approach just described does not seem to appear in the literature about non-symmetric forms, perhaps because it is hard to say something about the structure of W_b if the base ring is not a field. (This is one of the main goals of Chapter 1.) Furthermore, all the results just mentioned assume that the module over which the form is defined is *reflexive* (see section 2.5), which is not needed in my results. To conclude, the results (a)–(e) are new mainly for non-regular or non-symmetric forms, for forms defined over *non-reflexive* modules and also for systems of bilinear forms.

Bilinear Forms over Rings. We have spoken about bilinear forms over rings without properly defining them, so let us take care of this gap. Various definitions can be found in the literature (e.g. the sesquilinear forms defined below; see the references at the opening of the introduction for more definitions), but all of them require the base ring to have an involution. (This even applies to hermitian categories in a certain sense.) Among the innovations of this work is a new definition of bilinear forms over arbitrary (non-commutative) rings (no involution is needed).

DEFINITION 0.3. *Let R be a ring. A* double R-module *is an additive group K endowed with two actions $\odot_0, \odot_1 : K \times R \to K$ such that K is a right R-module w.r.t. each of $\odot_0, \odot_1$ and $(k \odot_0 a) \odot_1 b = (k \odot_1 b) \odot_0 a$ for all $k \in K$ and $a, b \in R$. (Double R-modules are categorically equivalent to (R^{op}, R)-bimodules).*

An anti-isomorphism of a double R-module K is a map $\kappa : K \to K$ (written exponentially) such that $(k \odot_i r)^\kappa = k^\kappa \odot_{1-i} r$ for all $k \in K$, $r \in K$ and $i \in \{0, 1\}$. If in addition $\kappa^2 = \mathrm{id}_K$, then κ is called an involution.

A bilinear space *over a ring R is a triplet (M, b, K) such that M is a right R-module, K is a double R-module and $b : M \times M \to K$ is a biadditive map satisfying*

$$b(xr, y) = b(x, y) \odot_0 r \qquad \text{and} \qquad b(x, yr) = b(x, y) \odot_1 r$$

for all $x, y \in M$ and $r \in R$. If κ is an involution of K, then b is called κ-symmetric if $b(x, y) = b(y, x)^\kappa$ for all $x, y \in M$.[5]

This definition, which serves as the basis of this dissertation, includes the definitions of the references mentioned earlier and the results (a)-(e) above applied to bilinear forms in this new sense (the double R-module K can be chosen almost arbitrarily). Furthermore, in the same manner that *hermitian categories* are categorical frameworks for quadratic forms, one can define *categories with a double duality* which are categorical frameworks for our new bilinear forms. We also note that, in some sense, the new definition cannot be trivially viewed as a special case of a hermitian category (see the end of section 2.7 for details).

EXAMPLE 0.4. Let $(R, *)$ be a ring with involution and let $\lambda \in \mathrm{Cent}(R)$ such that $\lambda^* \lambda = 1$. Recall that a sesquilinear space over $(R, *)$ is a pair (M, b) such that M is a right R-module and $b : M \times M \to R$ is a biadditive map such that $b(xr, y) = r^* b(x, y)$ and $b(x, yr) = b(x, y) r$ for all $x, y \in M$ and $r \in R$. If moreover $b(y, x) = \lambda b(x, y)^*$, then b is λ-hermitian.

Make R into a double R-module by defining $r \odot_0 a = a^* r$ and $r \odot_1 a = ra$ for all $a, r \in R$. In addition let $\kappa : R \to R$ be defined by $r^\kappa = \lambda r^*$. Then κ is an involution of R, once considered as a double R-module. In addition, (M, b) is a sesquilinear space $\iff$ (M, b, R) is a bilinear form in our new sense and b is λ-hermitian $\iff$ b is κ-symmetric.

Henceforth, in order to avoid ambiguity, we will refer to sesquilinear forms as "classical bilinear forms".

EXAMPLE 0.5. The new definition allows us to work with single bilinear forms and systems of bilinear forms using the same notation. Indeed, let R be a ring and let $\{(M, b_i, K_i)\}_{i \in I}$ be a system of bilinear forms over the right R-module M. Define $K = \prod_{i \in I} K_i$ and $b : M \times M \to K$ by $b(x, y) = (b_i(x, y))_{i \in I}$. Then (M, b, K) is a bilinear form and we can treat (M, b, K) rather than the system $\{(M, b_i, K_i)\}_{i \in I}$.

[5] This definition has evolved from the (somewhat known) more primitive version, which is a combination of the definition of sesquilinear forms with some other definitions from the literature: A bilinear form over a ring with involution $(R, *)$ is a triplet (M, b, K) such that M is a right R-module, K is an (R, R)-bimodule and $b : M \times M \to K$ is a biadditive map satisfying $b(xr, y) = r^* b(x, y)$ and $b(x, yr) = b(x, y) r$. The bimodule K is also required to admit a map $\kappa : K \to K$ such that $(a \cdot k)^\kappa = k^\kappa \cdot a^*$ and $\kappa^2 = \mathrm{id}_K$. It was not until proving Theorem 0.8 below that I understood that K can be replaced with an (R^{op}, R)-bimodule and $*$ and κ can be dropped from the notation. The reason I have moved to double R-modules is because many arguments required the left R^{op}-module structure to be twisted to the right, causing ambiguity as to which right R-module structure is used.

To define *regular* bilinear forms we set the following notation: If K is a double R-module and $i \in \{0,1\}$, then K_i denotes K, considered a right R-module via $\odot_i$. Now, any bilinear space (M,b,K) over R gives rise to two maps called the *left adjoint* and *right adjoint* of b. They are defined by

$$\mathrm{Ad}_b^\ell : M \to \mathrm{Hom}_R(M,K_1), \quad (\mathrm{Ad}_b^\ell x)(y) = b(x,y) \ ,$$
$$\mathrm{Ad}_b^r : M \to \mathrm{Hom}_R(M,K_0), \quad (\mathrm{Ad}_b^r x)(y) = b(y,x) \ .$$

The form b is called right regular (right injective) if Ad_b^r is bijective (injective). Left regular and left injective forms are defined in the same manner. Right injective forms are also called *right nondegenerate*. Regularity and injectivity are not left-right symmetric properties, but if b is κ-symmetric for some involution κ of K, then the right and left versions coincide.

Observe that if a (M,b,K) is right regular, then every $\sigma \in \mathrm{End}_R(M)$ admits a unique $\sigma^* \in \mathrm{End}_R(M)$ such that

$$b(\sigma x, y) = b(x, \sigma^* y) \qquad \forall x,y \in M \ .$$

The map $*$ is easily seen to be an anti-*endo*morphism[6] of $\mathrm{End}_R(M)$ which is called the corresponding (right) anti-endomorphism of b.

Bilinear Forms and Anti-Endomorphisms. Let F be a field and let V be a f.d. F-vector space. A well-known theorem asserts that the map sending a classical regular bilinear form on V to its corresponding anti-endomorphism induces a one-to-one correspondence between classical regular bilinear forms on V, considered up to scalar multiplication, and anti-endomorphisms of $\mathrm{End}_F(V)$ fixing F. Under this correspondence, symmetric and alternating forms correspond to orthogonal and symplectic involutions, respectively (see [**57**, Chp. 1] for proof). This result, which is related to Theorem 0.1, admits various generalizations to classical bilinear forms over simple F-algebras, which play an important role in the connection between quadratic forms and involutions of central simple algebras.

The importance of this result and the necessity of a generalization of Theorem 0.1 to arbitrary division rings and involutions have raised the question of whether this correspondence generalizes to our newly defined bilinear forms. Indeed, as noted above, any *right regular* bilinear form admits a corresponding anti-endomorphism, so one would expect to have a correspondence between right regular bilinear forms defined on a right R-module M, *considered up to a suitable equivalence relation*, and anti-endomorphisms of $W := \mathrm{End}_R(M)$. This problem is studied extensively in Chapter 3 and has raised some unexpected results of a mixed nature.

Firstly, it turns out that there is a *canonical* way to assign to every anti-endomorphism α of $W = \mathrm{End}_R(M)$ a corresponding bilinear form $b_\alpha : M \times M \to K_\alpha$ satisfying

$$b_\alpha(\sigma x, y) = b_\alpha(x, \sigma^\alpha y) \qquad \forall x,y \in M \ ,$$

and b_α is κ_α-*symmetric* for some involution κ_α of K_α if α is an involution. This is remarkable since, to the best of my knowledge, there is no canonical way to construct the inverse map of the correspondence for classical bilinear forms (and moreover, the construction involves "heavy tools" as the Skolem-Noether Theorem, which are not always available). What allows this unexpected shortcut is the freedom in choosing the double R-module K_α; we do not have to identify it with a prescribed double R-module. There is also an obvious candidate for the required equivalence relation on bilinear forms: Two bilinear forms (M,b,K) and (M,b',K') are called *similar* (denoted $b \sim b'$) if there exists an isomorphism of double R-modules $f : K \to K'$ such that $b' = f \circ b$.

[6] An anti-endomorphism is an additive map which preserves the unity and reverses the order of multiplication. A bijective anti-endomorphism is call an anti-*auto*morphism.

Example 0.6. Let F be a field. Then two classical bilinear forms defined over an F-vector space V are similar $\iff$ they are the same up to scalar multiplication.

Example 0.7. Let F be a field, let V be a f.d. dimensional F-vector space and let α be an anti-endomorphism of $\mathrm{End}_F(V)$. Since $F \cong \mathrm{Cent}_F(V)$ as F-algebras, α induces an (anti-)endomorphism on F, which we keep denoting by α. It turns out that the double F-module K_α is isomorphic to the double obtained from F by defining $k \odot_0 a = a^\alpha k$ and $k \odot_1 a = ka$ for all $k, a \in F$. In particular, if $\alpha|_F = \mathrm{id}_F$, then b_α is just a classical bilinear form. Furthermore, if α is an orthogonal involution, then $\kappa_\alpha = \mathrm{id}_F$, i.e. b is symmetric, and if α is a symplectic involution, then $\kappa_\alpha = -\mathrm{id}_F$, i.e. b is anti-symmetric. Similarly, if α is an involution of the second kind, then b_α would turn out to be a λ-hermitian form over $(F, \alpha|_F)$.

In general, there is a one-to-one correspondence between *all* anti-endomorphisms of $\mathrm{End}_F(V)$ and the right regular bilinear forms on V, considered up to similarity. However, not all anti-endomorphisms correspond to classical forms.

Unfortunately, the last example does not reflect the general case. First, in general, the form b_α need not be right regular (and might even be the zero form), so the correspondence might fail! Furthermore, similarity is not a suitable equivalence relation in general. Indeed, there exists an example of a regular bilinear form b with corresponding anti-endomorphism α such that b_α is not similar to b. The latter problem can be resolved by restricting our attention to bilinear forms that are obtained from anti-endomorphisms (i.e. forms which are similar to b_α for some α). Such forms are called *generic* (and any right regular form can be swapped with its *generization*). However, the first problem is inherent and can only be solved by restricting to special cases. Among the positive results obtained are the following:

Theorem 0.8. *Let M be a right R-module and let $W = \mathrm{End}_R(M)$. Then:*

(i) *When M is finite projective, there is a one-to-one correspondence between anti-*endo*morphisms of W and generic* right *regular forms on M.*
(ii) *When M is a* generator *(of* Mod-R*), there is a one-to-one correspondence between anti-*auto*morphisms of W and generic (*right and left*) regular forms on M.*

While the previous theorem is very nice, it is quite rare that M is projective or a generator. This has led me to wonder whether I have been too eager in the sense that I have required too much of the form b_α. Indeed, the right regularity assumption is in fact superfluous. What is really needed from a bilinear form $b : M \times M \to K$ in order to have a corresponding anti-endomorphism is that for all $\sigma \in W := \mathrm{End}_R(M)$, there would exist *unique* $\sigma^* \in W$ satisfying

$$b_\alpha(\sigma x, y) = b_\alpha(x, \sigma^* y) \qquad \forall x, y \in M\ .$$

Such forms are call *right stable*.

Example 0.9. Let $b_1, b_2 : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ be the classical bilinear forms over $\mathbb{Z}$ defined by $b_1(x,y) = x^T \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right] y$ and $b_2(x,y) = x^T \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right] y$. Then b_1 and b_2 are injective (i.e. nondegenerate), but not regular. The form b_1 is right (and left) stable and its corresponding anti-endomorphism is the transpose involution on $\mathrm{M}_2(\mathbb{Z}) \cong \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^2)$. However, b_2 is not right stable since there is no $\sigma' \in \mathrm{M}_2(\mathbb{Z})$ such that $b_2(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x, y) = b_2(x, \sigma' y)$.

My doubts were eventually justified when I found an example of a module M over a ring R such that *all* anti-automorphisms of $\mathrm{End}_R(M)$ correspond to non-regular, yet stable, bilinear forms. Moreover, there exists an example of R, M, α such that b_α is degenerate and stable. These discoveries were followed by a series of positive results sharing a common flavor: If R and M can be localized such that after the localization M becomes projective or a generator, then under mild assumptions, b_α is guaranteed to be injective, provided α is an anti-*auto*morphism. The word "localization" can mean both commutative and non-commutative localization. Some of these results are summarized in the following theorem, which is derived from Theorem 3.7.19 below.

THEOREM 0.10. *Let R be a ring and let M be a faithful right R-module which is* dense *in a f.g. R-module (e.g. if M is f.g.). If at least one of the following holds, then there is a one-to-one correspondence between anti-automorphisms of* $\mathrm{End}_R(M)$ *and generic stable forms on M, considered up to similarity.*

(1) *M is torsion-free and R is a semiprime Goldie ring (e.g. a noetherian or PI domain).*
(2) *Generalizing (1): There is a* two-sided denominator set *of regular elements $S \subseteq R$ such that RS^{-1} is* right pseudo-Frobenius[7] *and M is S-torsion-free.*

The proof involves finding sufficient conditions on a module M to ensure that the endomorphism ring of $\widetilde{E}(M)$, the *rational hull* of M, is the *maximal symmetric general ring of quotients* of $\mathrm{End}(M)$. The byproducts include the following deep result about rings of quotients.

THEOREM 0.11. *Let R be a ring such that $Q^s_{\max}(R)$, the maximal symmetric general ring of quotients of R, coincides with R. Then for every* torsionless[8] *generator $M \in \text{Mod-}R$, there is a torsionless generator $G \in \text{Mod-}R$ such that $M \subseteq_d G$, every endomorphism of M extends to G and $\mathrm{End}(G) = Q^s_{\max}(\mathrm{End}(M))$. In particular, if R_R is a* cogenerator*, then any generator $M \in \text{Mod-}R$ satisfies $\mathrm{End}(M) = Q^s_{\max}(\mathrm{End}(M))$ and if R is right pseudo-Frobenius, then any faithful module $M \in \text{Mod-}R$ satisfies $\mathrm{End}(M) = Q^s_{\max}(\mathrm{End}(M))$.*

There are still many open questions regarding when b_α is regular or stable. For instance, I could not find an example of R being a noetherian domain, M being f.g. and α being bijective, such that b_α is not regular. In addition, the following conjecture is open:

CONJECTURE 0.12. *If M is* nonsingular*, then there is a one-to-one correspondence between the generic stable bilinear forms on M, considered up to similarity, and the anti-automorphisms of* $\mathrm{End}_R(M)$.

Two Applications. Part (i) of Theorem 0.8 has two nice applications. The first is an easy proof of the following result of Osborn ([**66**]).

THEOREM 0.13. *Let (W, α) be a semisimple ring with involution admitting no non-trivial α-invariant idempotents. Then (exactly) one of the following holds:*

(i) *W is a division ring.*
(ii) *$W \cong D \times D^{\mathrm{op}}$ for some division ring D and under that isomorphism α exchanges D and D^{op}.*
(iii) *$W \cong \mathrm{M}_2(F)$ for some field F and under that isomorphism α is a symplectic involution.*

Osborn's original result assumed additional conditions on (W, α) (e.g. that $2 \in W^\times$) and its proof consisted of studying the Jordan algebra induced by α. An

[7] A ring R is called right pseudo-Frobenius when all faithful right R-modules are generators. This is equivalent to R being a right cogenerator right self-injective ring, e.g. a quasi-Frobenius ring.

[8] A right R-module M is torsionless if it embeds in a product $\prod_{i\in I} R_R$ for some set I. The module R_R is a cogenerator $\iff$ all right R-modules are torsionless.

alternative easier proof goes as follows: One can quickly reduce to the case where W is simple artinian, hence there is a division ring D and a f.d. right D-vector space V such that $W \cong \mathrm{End}_D(V)$. Thus, by Theorem 0.8, $b_\alpha : V \times V \to K_\alpha$ is a regular bilinear form over D. Furthermore, it is κ_α-symmetric. The assumption that W does not have non-trivial α-invariant idempotents now is equivalent (by the "dictionary" above) to b being indecomposable. This is easily seen to force either $\dim_D V = 1$ or $\dim_D V = 2$, where in the latter case D is a field and b_α is alternating. But this implies the theorem.[9]

The second application is a partial answer to a problem suggested to me by David Saltman: Under what assumptions all or some of the following conditions are equivalent for a ring R.

(1) R is Morita equivalent to a ring with involution.
(2) R is Morita equivalent to a ring with an anti-endomorphism.
(3) R is Morita equivalent to R^{op}.

The implications (1)$\Longrightarrow$(2)$\Longrightarrow$(3) are obvious. However, a well-known theorem asserts that for central simple algebras we have (3)$\Longrightarrow$(1), and Saltman has generalized this result to Azumaya algebras in [**82**].[10] Using Theorem 0.8 (and the new definition of bilinear forms in particular), I was able to show the following proposition (compare with [**82**, Th. 4.2]). Before formulating it, observe that every (R^{op}, R)-module, and in particular (R^{op}, R)-prgenerators, can be twisted into a double R-modules by considering the left R^{op}-module structure as an additional right R-module structure.

PROPOSITION 0.14. *Let R be a ring and let M be an R-progenerator. Then $\mathrm{End}_R(M)$ admits an anti-automorphism (resp. involution) if and only if there exists a regular (resp. regular and asymmetric) bilinear form (M, b, K) such that K is obtained from an (R^{op}, R)-progenerator.*

The proof consists of showing that if α is an anti-automorphism of $\mathrm{End}_R(M)$, then K_α, once considered as an (R^{op}, R)-bimodule, is a progenerator.

Proposition 0.14 means that in order to prove (3)$\Longrightarrow$(2), it is enough to show that for every (R^{op}, R)-progenerator, K, there is a regular bilinear space (M, b, K) for some R-progenerator M. This latter statement is false in general, but it is true under some finiteness assumptions on the category of f.g. projective R-modules. Such finiteness assumptions are satisfied when R is semiperfect, hence we get:

THEOREM 0.15. *If R is semiperfect, then (3)$\Longrightarrow$(2).*

I conjecture that (3)$\not\Longrightarrow$(2) in general, but I could not find any counterexample. Nevertheless, (2)$\not\Longrightarrow$(1) can be demonstrated (Example 2.9.7 below).

Basic Properties of Bilinear Forms. Another interesting consequence of the research about the connection between bilinear forms an anti-endomorphism was the realization that, over rings, being stable does not imply being injective (i.e. nondegenerate) and vice versa. This phenomenon does not happen for classical bilinear forms division rings, and that led me to reconsider the basics of the theory of bilinear forms over rings. This is the topic of Chapter 2. To formulate this

[9] The form b_α is in fact classical, i.e. it is a ± 1-hermitian form over D w.r.t. some involution. This follows from Theorem 0.1 if D is f.d. over its center. I could not find the general case in the literature, but it follows as a consequence of my work.

[10] Some mild assumption is needed for this to be true: The (R^{op}, R)-progenerator P inducing the equivalence also induces an isomorphism $\mathrm{Cent}(R) \to \mathrm{Cent}(R^{\mathrm{op}}) \cong \mathrm{Cent}(R)$. That isomorphism must be the identity. Equivalently, R and R^{op} need to be *Morita equivalent as* $\mathrm{Cent}(R)$*-algebras.*

more formally, let (M, b, K) be a bilinear space over R. Then one can consider the following properties:

(R1) b is right injective (i.e. Ad_b^r is injective).
(R2) b is right surjective (i.e. Ad_b^r is surjective).
(R3) b is right stable.

The left analogues of these properties are denoted by (L1)–(L3). It turns out that none of (R1)–(R3),(L1)–(L3) implies any of the others. Moreover, the logical implications between subsets of these properties can be explained by (R1)$\wedge$(R2)$\Longrightarrow$(R3) and its left analogue. (Here $\wedge$ denotes logical "and"; the previous logical statement means that right regular implies right stable, a fact we have noted above.)

Things become more complicated when K is assumed to have an anti-isomorphism or an involution κ. For instance, we suddenly get extra relations between (R1)–(R3) and (L1)–(L3) such as (R1)$\wedge$(R2)$\Longrightarrow$(L1). In addition, in this case we can add another member to our list of properties:

(R4) b has a unique *right κ-asymmetry*.

As might be expected, a right κ-asymmetry of b is a map $\lambda \in \mathrm{End}_R(M)$ such that $b(x, y)^\kappa = b(y, \lambda x)$. The left analogue of (R4) is denoted by (L4).

Again, while (R1)–(R4),(L1)–(L4) are equivalent for classical bilinear forms over division rings, none of these conditions implies any of others in general. As done above, I have tried to determine the logical implications between subsets of (R1)–(R4) and (L1)–(L4), but this time I did not manage to finish the project; I have proved a list of implications, which I conjecture to explain all other implications. What stops me from declaring the list as complete is the absence of several counterexamples (e.g. showing that (R4)$\wedge$(R2)$\not\Longrightarrow$(R3)). Among the remarkable (and very hard) counterexamples that were found is an example of a right regular bilinear form b admitting a *unique* right κ-asymmetry but not a left κ-asymmetry. (In this case b cannot be left regular and the asymmetry is not bijective).

One also notes that by forcing various assumptions on the ring R and the bimodule K, more logical implications can be obtained. For example, if b is a classical bilinear form over a *quasi-Frobenius*[11] ring with involution, then the conditions (R1)–(R4),(L1)–(L4) are equivalent, provided M is faithful.

Semi-Invariant Subrings. Recall the ring W_b defined above. For a right stable bilinear space (M, b, K), it was defined to be $\{\sigma \in \mathrm{End}_R(M) \,|\, \sigma^{**} = \sigma\}$, where $*$ is the corresponding anti-endomorphism of b. Studying the structure of W_b was essential to get effective results about b and has thus occupied almost half of the last year of my research. This work eventually led to the development of a new concept called *semi-invariant subrings*, which is the topic of Chapter 1.[12]

A subring R_0 of a ring R is *semi-invariant* if there exists a ring $S \supseteq R$ and a set of ring endomorphisms $\Sigma \subseteq \mathrm{End}(S)$ such that $R_0 = R^\Sigma := \{r \in R : \sigma(r) = r,\ \forall \sigma \in \Sigma\}$. A *T-semi-invariant* subring is defined in the same way, but when R and S are Hausdorff *linearly topologized rings*. If we can choose S to be R, then we get the usual notion of an *invariant subring*. (For example, W_b above is an invariant subring w.r.t. $\Sigma = \{**\}$.) While it is not obvious from the definition, semi-invariant subrings are quite common. For instance, the centralizer of any subset of R is a (T-)semi-invariant subring of R and if M is a finitely presented (abbrev.: f.p.) R-module,

[11] A ring is quasi-Frobenius (abbrev.: QF) if it is noetherian and self-injective. For example, if F is a field and G is a finite group, then FG is QF. In addition, any local artinian ring with simple socle is QF; see [**58**].

[12] It had recently came to my attention that there is already a notion of semi-invariance in the theory of invariants. The semi-invariant subrings of this dissertation, while being generalizations of *rings of invariants*, has nothing to do with semi-invariance in invariant theory.

then $\mathrm{End}(M)$ is a quotient of a (T-)semi-invariant subring of $\mathrm{M}_n(R) \times \mathrm{M}_m(R)$ for some n, m.

It turns out that various properties pass from a ring to its (T-)semi-invariant subrings. Some of these properties are summarized in the following theorems.

THEOREM 0.16. *Let R be a ring. If R is semiprimary (resp. right perfect*[13]*), then so is any semi-invariant subring of R.*

THEOREM 0.17. *Let R be a Hausdorff linearly topologized ring. If R is semiperfect and pro-semiprimary (resp. semiperfect and pro-right-perfect), then so is any T-semi-invariant subring of R.*

Regarding the notions *pro-semiprimary* and *pro-right-prefect*, a linearly topologized ring is called *pro-$\mathcal{P}$* if it is isomorphic as a topological ring to the inverse limit of discrete topological rings satisfying $\mathcal{P}$.

Beside the above application to bilinear forms, Theorems 0.16 and 0.17 also have numerous applications to semiperfect rings and *Krull-Shmidt decompositions* (see section 1.1), such as:

(1) Let R be a semiperfect pro-semiprimary ring,[14] then all f.p. R-modules admit a Krull-Schmidt decomposition (this generalizes [**92**, §6], [**19**], [**78**], [**79**]). If moreover R is right noetherian, then the endomorphism ring of any f.p. R-module is semiperfect and pro-semiprimary.
(2) Let S be a commutative semiperfect pro-semiprimary ring. Then any S-algebra R that is f.p. as an S-module is semiperfect. If moreover S is noetherian, then R is pro-semiprimary.
(3) Any representation of a monoid over a module with a semiperfect pro-semiprimary endomorphism ring has a Krull-Schmidt decomposition.

This work is described in detail in the accepted paper [**41**].

Categories With A Double Duality. Before concluding the introduction, let us return to *hermitian categories*, also called *categories with duality.* A hermitian category is a triplet $(\mathscr{H}, *, \omega)$ such that $\mathscr{H}$ is a (usually additive) category, $* : \mathscr{H} \to \mathscr{H}$ is a contravariant functor and $\omega : \mathrm{id}_{\mathscr{H}} \to **$ is a natural isomorphism satisfying a certain equation (see [**71**], [**86**, Chp. 7] or section 4.2 below). A bilinear form over $\mathscr{H}$ would consist of a pair (M, b) with $M \in \mathscr{H}$ and $b \in \mathrm{Hom}_{\mathscr{H}}(M, M^*)$. Classical bilinear forms over rings with involution can be considered as bilinear forms over an appropriate hermitian category, but the same construction cannot be adapted for our new notion of bilinear forms (see the end of section 2.7). Instead, the new bilinear forms can be understood as a bilinear forms over some *category with a double duality.* The latter is defined to be quintet $(\mathscr{A}, [0], [1], \Phi, \Psi)$ such that $\mathscr{A}$ is a category, $[0], [1] : \mathscr{A} \to \mathscr{A}$ are contravariant functors (written exponentially) and $\Phi : \mathrm{id}_{\mathscr{A}} \to [1][0]$, $\Psi : \mathrm{id}_{\mathscr{A}} \to [0][1]$ are natural transformations satisfying certain relations.[15] Bilinear forms over $\mathscr{A}$ would consist of pairs (M, b) such that $M \in \mathscr{A}$ and $b \in \mathrm{Hom}_{\mathscr{A}}(M, M^{[1]})$. The asymmetry in the definition is ostensible as the relations between Φ and Ψ induce a natural isomorphism between $\mathrm{Hom}_{\mathscr{A}}(M, M^{[0]})$ and $\mathrm{Hom}_{\mathscr{A}}(M, M^{[1]})$.

EXAMPLE 0.18. (i) Let R be a ring and let K be a double R-module. For every $M \in \text{Mod-}R$ and $i \in \{0, 1\}$, define $M^{[i]} := \mathrm{Hom}_R(M, K_{1-i})$ (recall that K_{1-i} stands for K considered as right R-module w.r.t. $\odot_{1-i}$). We make $M^{[i]}$ into a right

[13] A ring R is right perfect if it is semilocal and $\mathrm{Jac}(R)$ is right T-nilpotent. See [**9**] or section 1.2 for more equivalent definitions.

[14] and even more generally, a *quasi-π_∞-regular* ring, e.g. an inverse limit of π_∞-regular rings.

[15] Caution: since [0] and [1] are written exponentially, [1][0] actually means $[0] \circ [1]$ (since $M^{[1][0]} = (M^{[1]})^{[0]}$ for $M \in \mathscr{A}$).

R-module by letting $(f \cdot r)m = (fm) \odot_i r$ for all $f \in M^{[i]}$, $m \in M$ and $r \in R$. Now define $\Phi_M : M \to M^{[1][0]}$ and $\Psi_M : M \to M^{[0][1]}$ by $(\Phi_M x)f = f(x)$ and $(\Psi_M x)g = g(x)$ for all $x \in M$, $f \in M^{[1]}$ and $g \in M^{[0]}$. Then $(\text{Mod-}R, [0], [1], \Phi, \Psi)$ is a category with a double duality. A bilinear form (M, b, K) over R corresponds to the bilinear form (M, Ad^r_b) over Mod-R.

(ii) $(\mathscr{H}, *, \omega)$ is a hermitian category $\iff$ $(\mathscr{H}, *, *, \omega, \omega)$ is a category with a double duality *and* ω is a natural *isomorphism*.

One of the deepest (and most difficult) results of this work is the following.

THEOREM 0.19. *Let* $(\mathscr{A}, [0], [1], \Phi, \Psi)$ *be a category with a double duality. Then there exists a hermitian category* $(\mathscr{H}, *, \omega)$ *such that the category of* arbitrary *bilinear forms over* $\mathscr{A}$ *is equivalent to the category of* regular symmetric *bilinear forms over* $\mathscr{H}$*. The category* $\mathscr{H}$ *is the category of* Kronecker modules *over* $\mathscr{A}$*.*

Roughly speaking, Theorem 0.19 asserts that the theory of bilinear forms over a given category with duality is equivalent to the theory of *regular symmetric* bilinear forms over another category with duality (see sections 4.3–4.5 for an extensive discussion). Moreover, it explains why it is even possible at all to reduce the theory of non-symmetric forms to the theory of regular symmetric forms. In fact, the results (a)–(e) stated above, and also work of Riehm and his predecessors, can be shown to "factor" via the equivalence of Theorem 0.19.

I should note that a result having a similar flavor was obtained by E. Bayer-Fluckiger and L. Fainsilber in [**16**]. They have used a different construction to show that the category of arbitrary *symmetric* bilinear forms over a given hermitian category is equivalent to the category of regular symmetric bilinear forms over another hermitian category. In addition, very recently, I was introduced with the current (and still unpublished) work of D. Moldovan. In his Ph.D. dissertation ([**64**]; submitted in 2012; done under the supervision of E. Bayer-Fluckiger), Moldovan proved a version of Theorem 0.19 for hermitian categories and has used it to deduce various results, including special cases of (a), (b) and (c) above. (For example, he obtained Witt's Cancelation Theorem for classical bilinear forms over algebras of finite type over discrete valuation rings.) Both [**16**] and [**64**] require that all objects in the given hermitian category are reflexive (i.e. that $\omega : \mathrm{id}_{\mathscr{H}} \to **$ is a natural *isomorphism*, rather than just a natural *transformation*). This is not needed in Theorem 0.19, though. *(Note: E. Bayer-Fluckiger, D. Moldovan and I eventually combined our results and submitted them as a joint work; see* [**11**]*.)*

One should also point out that Theorem 0.19 can be effectively applied to systems of bilinear forms; see section 4.5.

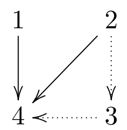
Notes to the Reader

The Chapters

This work consists of five chapters. Chapter 0, entitled "Preliminaries", surveys some known results and definitions from ring theory and category theory that are used throughout this work. It is meant to make this text more negotiable to non-experts. (Nevertheless, some familiarity with elementary non-commutative ring theory, category theory and topology are still assumed.) Experts (and also non-experts) may skip this chapter and consult it upon need. The basic knowledge needed to read Chapter 0 can be mostly found in the first chapters of *Ring Theory* by L. Rowen ([**80**]).

Chapter 1, entitled "Semi-Invariant Subrings", discusses the (new) theory of semi-invariant subrings and its various applications (except its applications to bilinear forms). It serves as the ring theoretic infrastructure of the work. Chapter 2, entitled "Bilinear Forms Over Rings", defines and studies the basic properties of bilinear forms over rings. Categories with a double duality, which are a categorical framework for bilinear forms, are also defined and discussed. Chapter 3, entitled "Bilinear Forms and Anti-Endomorphisms", studies the connection between bilinear forms and anti-endomorphisms. Finally, Chapter 4, entitled "Isometry and Decomposition", is devoted to proving strong results about isometry of bilinear forms and the structure of their isometry group (e.g. results (a)–(e) of the Introduction).

The dependency between Chapters 1–4 is illustrated in the following diagram:



An arrow means strong dependency (i.e. do not attempt to read the destination of the arrow before reading most of its source) and a dotted arrow means weak dependency (it is enough to read a small part of the source before reading the destination).

Notation and Conventions

Rings: Unless specified otherwise, all rings are assumed to have a unity and ring homomorphisms are required to preserve it. Subrings are assumed to have the same unity as the ring containing them. Given a ring R, denote its set of invertible elements by $R^\times$, its Jacobson radical by $\mathrm{Jac}(R)$, its set of idempotents by $\mathrm{E}(R)$ and its center by $\mathrm{Cent}(R)$. The $n \times n$ matrices over R are denoted by $\mathrm{M}_n(R)$. We let $\mathrm{End}(R)$ (resp. $\mathrm{Aut}(R)$) denote the set of ring homomorphisms (resp. isomorphisms) from R to itself. If $X \subseteq R$ is any set, then its right (left) annihilator in R is denoted by $\mathrm{ann}^r_R X$ ($\mathrm{ann}^\ell_R X$). The subscript R will be dropped when understood from the context. Throughout, a *semisimple* ring means a *semisimple artinian* ring.

Whenever referring to a ring property admitting right and left versions (e.g. being noetherian) without specifying whether it is left or right, we mean both versions. (For example, a "noetherian ring" means "right and left noetherian ring"). This rule applies to non-ring-theoretic properties as well.

Modules: The category of right (left) R-modules is denoted by Mod-R (R-Mod). For $M \in$ Mod-R, we let $E(M)$ ($\widetilde{E}(M)$) denote the injective envelope (rational hull) of M. We write $N \leq M$ to denote that N is a submodule of M. We also write $N \subseteq_e M$ if N is essential in M and $N \subseteq_d M$ if N is dense in M.

In case M can be considered as a module over more than one ring, we use M_R (resp. ${}_RM$) to denote "M, considered as a right (resp. left) R-module". In particular, R_R (resp. ${}_RR$) means "R, considered as a right (resp. left) R-module".

Inverse Limits of Rings: By saying that $\{R_i, f_{ij}\}$ is an *inverse system of rings* indexed by I, we mean that: (1) I is a *directed set* (i.e. a partially ordered set such that for all $i, j \in I$ there is $k \in I$ with $i, j \leq k$), (2) R_i ($i \in I$) are rings and $f_{ij} : R_j \to R_i$ ($i \leq j$) are ring homomorphisms and (3) $f_{ii} = \mathrm{id}_{R_i}$ and $f_{ij}f_{jk} = f_{ik}$ for all $i \leq j \leq k$ in I. When the maps $\{f_{ij}\}$ are obvious or of little interest, we will drop them from the notation, writing $\{R_i\}_{i \in I}$ instead. The *inverse limit* of $\{R_i, f_{ij}\}$ will be denoted by $\varprojlim \{R_i\}_{i \in I}$. It can be understood as the set of I-tuples $(a_i)_{i \in I} \in \prod_{i \in I} R_i$ such that $f_{ij}(a_j) = a_i$ for all $i \leq j$ in I.

Miscellaneous: The natural numbers $\mathbb{N}$ are not assumed to include 0. For a prime number p, $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) denotes the p-adic integers (resp. numbers) and $\mathbb{Z}_{\langle p \rangle}$ denotes $S^{-1}\mathbb{Z}$ with $S = \mathbb{Z} \setminus p\mathbb{Z}$.

List of Abbreviations

Abbreviation	Meaning
abbrev.	abbreviated/abbreviation
f.d.	finite dimensional
f.g.	finitely generated
f.p.	finitely presented
l.h.s.	left hand side
r.h.s.	right hand side
resp.	respectively
s.t.	such that
w.l.o.g.	without loss of generality
w.r.t.	with respect to
Th. (Thms.)	theorem(s)
Prp. (Prps.)	proposition(s)
Lm.	lemma
Cr.	corollary
Ex.	example
Exer.	exercise
Ch. (Chs.)	chapter(s)
ACC	ascending chain condition
DCC	descending chain condition
PI	polynomial identity (ring)
QF	quasi-Frobenius
PF	pseudo-Frobenius
QI	quasi-injective
AR-property	Artin-Rees property
TAR-property	topological Artin-Rees property
LT	linearly topologized
FPRT	finite projective representation type
c.w.d.d.	category with a double duality

List of Notations and Symbols

Symbol	Description	Page
Mod-R (R-Mod)	category of right (left) R-modules	—
$\mathrm{Jac}(R)$	Jacobson radical of a ring R	—
$R^\times$	units of a ring R	—
$\mathrm{E}(R)$	set of idempotents of a ring R	—
$\mathrm{M}_n(R)$	ring of $n \times n$ matrices over R	—
$Q^r_{\mathrm{max}}(R)$, $Q^\ell_{\mathrm{max}}(R)$	maximal right/left ring of quotients of R	22
$Q^s_{\mathrm{max}}(R)$	maximal symmetric ring of quotients of R	128
$\mathbb{N}$	natural numbers (without zero)	—
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	integers, rational numbers, real numbers, complex numbers	—
$\mathbb{Z}_p$ ($\mathbb{Q}_p$)	p-adic integers (numbers)	—
$\mathbb{Z}_{\langle p \rangle}$	$\mathbb{Z}$ localized (but not completed) at $p\mathbb{Z}$	—
$E(M)$	injective envelope of a module M	—
$\widetilde{E}(M)$	rational hull of a module M	20
$\mathrm{u.\,dim}\, M$	uniform dimension of a module M	16
$\mathscr{LTR}_2$	category of Hausdorff LT rings	34
$\mathcal{I}_R$	set of open ideals in a topological ring R	34
$\mathrm{Hom_c}$ ($\mathrm{End_c}$, $\mathrm{Aut_c}$)	continuous homomorphisms (endo-, auto-)	34
τ^M_n	—	44
$\mathscr{P}_{\mathrm{disc}}$, $\mathscr{P}_{\mathrm{top}}$, $\mathscr{P}_{\mathrm{mor}}$, $\mathscr{P}_{\mathrm{sp}}$	—	42
$M^{[0]}$ ($M^{[1]}$)	0-dual (1-dual) of a module M	57
DMod-R	category of double R-modules	56
Ad^r_b (Ad^ℓ_b)	right (left) adjoint map of a bilinear form b	57
$\mathrm{Bil}_K(A, B)$	set of bilinear pairings $b : A \times B \to K$	83
$I_{A,B}$	—	60
Φ_M, Ψ_M	—	59
u_κ	—	62
$\mathrm{End}^-(R)$, $\mathrm{Aut}^-(R)$	set of anti-endomorphisms/anti-automorphisms of a ring R	101
b_α, K_α	the bilinear form/double-module associated with an anti-endomorphism α	102
$\otimes_\alpha$	—	102
$\mathrm{Kr}(\mathscr{A})$	category of Kronecker modules over $\mathscr{A}$	149
$Z(b)$	Kronecker module of a bilinear form b	150

CHAPTER 0

Preliminaries

This chapter presents some definitions and well-known facts from ring theory, category theory and the theory of topological rings that are used throughout the text. Its purpose is to make this dissertation more negotiable to non-expects and hence it is not mandatory. In fact, the reader can skip this chapter entirely and return to it upon need. Note that sections 0.7–0.10 are needed only for section 3.7, so the reader can postpone reading them.

For the benefit of the reader, below is a list of the definitions and topics discussed in this chapter.

Section 0.1: Topological Groups and Rings. Topological Groups, Topological Rings, Local Bases.

Section 0.2: Natural Transformations. Natural Transformations, Natural Isomorphisms.

Section 0.3: Additive Categories. Preadditive Categories, Additive Categories, Additive Functors.

Section 0.4: More Category Theory. Faithful and Full Functors, Subcategories, Full Subcategories, Generators, Cogenerators, Equivalence of Categories.

Section 0.5: Morita Equivalence. Morita Equivalence, Progenerators, Full Idempotents, Morita Context, Morita's Theorems.

Section 0.6: Quasi-Frobenius Rings and Related Notions. Self-Injective Rings, Quasi-Frobenius Rings, Frobenius Algebras, Kasch Rings, Cogenerator Rings, Pseudo-Frobenius Rings.

Section 0.7: Uniform Dimension. Essential Submodules and Essential Extensions, Injective Hulls, Uniform Dimension.

Section 0.8: Classical Rings of Fractions. Non-Commutative Localization, Right Denominator Sets, Ore Rings and Ore Domains, Classical Rings of Fractions, Goldie Rings, Goldie's Theorem.

Section 0.9: Rational Extensions. Dense Submodules and Rational Extensions, Rational Hulls, Singular Radical, Nonsingular Submodules, Nonsingular Rings.

Section 0.10: General Rings of Quotients. General Rings of Quotients, Maximal Rings of Quotients, Theorems of Johnson and Gabriel.

0.1. Topological Groups and Rings

In this subsection we give the basics of topological groups, rings and modules. For a detailed discussion and proofs see [**99**].

A *topological group* consists of a group G endowed with a topology such that the maps

$$m:\ \begin{array}{ccc} G\times G & \to & G \\ (x,y) & \mapsto & xy \end{array} \qquad\qquad i:\ \begin{array}{ccc} G & \to & G \\ x & \mapsto & x^{-1} \end{array}$$

are continuous. We also say that the topology on G, denoted τ_G, is a *group topology.* If we let N_x stand for the set of neighborhoods[1] of x, then these conditions are equivalent to

(1) For all $U \in N_{xy}$ there exists $V \in N_x$, $W \in N_y$ such that $VW \subseteq U$;
(2) for all $U \in N_x$ there exists $V \in N_{x^{-1}}$ such that $V^{-1} \subseteq U$

where $x, y \in G$. The sets N_x and N_y can be replaced with bases of neighborhoods[2] at x and y, respectively. Morphisms of topological groups are defined to be continuous group homomorphisms.

Example 0.1.1. (i) Any group can be made into a topological group by endowing it with the discrete topology.

(ii) $(\mathbb{R}, +)$, $(\mathbb{R}^\times, \cdot)$, $(\mathbb{Q}_p, +)$, $(\mathbb{Z}_p, +)$ are topological groups.

(iii) $\mathrm{GL}_n(\mathbb{R})$ is a topological group w.r.t. the topology induced from $\mathrm{M}_n(\mathbb{R})$ (an n^2-dimensional Euclidean space).

(iv) If G is a topological group and H is a normal subgroup, then G/H is a topological group w.r.t. the quotient topology. The latter is defined as follows: a subset $U \subseteq G/H$ is open if and only if its preimage in G is is open. This makes the standard epimorphism $G \to G/H$ in a continuous group homomorphism.

(v) Let G be an infinite group. Then G is not a topological group w.r.t. the cofinite topology (despite the fact that $x \mapsto x^{-1}$ is a homeomorphism).

Proposition 0.1.2. *Let G be a topological group with unity e. Then:*

(i) *For all $x \in G$, $N_x = xN_e$. Here, xN_e stands for $\{xU \,|\, U \in N_x\}$.*
(ii) *For any subsect $X \subseteq G$, the closure of X, $\overline{X}$, is given by $\bigcap_{U \in N_e} XU$.*

In (ii), N_e can be replaced with any basis of neighborhoods of e.

The previous proposition implies that the topology on G can be recovered from N_e, or any basis of neighborhoods of e. Such a basis is called *a local basis of* G. (In general, a local basis basis at x means a basis of neighborhoods of x.) The following theorem provides necessary and sufficient conditions on a set $\mathcal{B} \subseteq P(G)$ to be a local basis of G w.r.t. some (uniquely determined) group topology. This theorem is extremely useful in constructing examples, since one can specify the local basis rather then describing τ_G.

Theorem 0.1.3. *Let G be a group with unity e and let $\mathcal{B}$ be a nonempty collection of subsets of G containing e. Then $\mathcal{B}$ is a local basis of some group topology (which is then uniquely determined) if and only if the following conditions are satisfied:*

(0) *For all $U, V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ with $W \subseteq U \cap V$.*[3]
(1) *For all $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V^2 \subseteq U$.*
(2) *For all $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V^{-1} \subseteq U$.*
(3) *For all $U \in \mathcal{B}$ and $x \in G$, there exists $V \in \mathcal{B}$ such that $xVx^{-1} \subseteq U$.*

Proof (sketch). Take τ_G to be the collection of sets $X \subseteq G$ such that for all $x \in X$, there exists $U \in \mathcal{B}$ with $xU \subseteq X$. The rest is routine. □

Example 0.1.4. (i) The set $\mathcal{B} = \{\mathbb{Z}_p, p\mathbb{Z}_p, p^2\mathbb{Z}_p, \dots\}$ is a local basis for $(\mathbb{Q}_p, +)$.

(ii) G is a discrete topological group if and only if $\{\{e\}\}$ is a local basis.

---

[1] A neighborhood of x is defined to be a set U containing an open set U_0 such that $x \in U_0$. Neighborhoods are not assumed to be open unless this is stated explicitly.

[2] A basis of neighborhoods of x is a set $\mathcal{B}$ consisting of neighborhoods of x such that every neighborhood of x contains an element of $\mathcal{B}$. Equivalently, this means $\mathcal{B}$ is a basis for the filter of neighborhoods of x.

[3] This is equivalent to saying that $\mathcal{B}$ is a filter base, as it must be.

(iii) Let G be a group and let $\mathcal{K}$ be a family of normal subgroups of G which is closed to finite intersection (e.g. the set $\mathcal{B}$ of (i)). Then $\mathcal{K}$ is a local basis for a group topology on G. Indeed, conditions (0)–(3) of the previous theorem are easily seen to hold (take $V = U$ in (1)–(3)). In this case $\mathcal{K}$ also turns out to consist of open sets and the cosets of elements of $\mathcal{K}$ form a basis for the topology on G.

Continuity of group homomorphism can also be characterized using local bases.

Proposition 0.1.5. *Let G, G' be topological groups with local bases $\mathcal{B}$, $\mathcal{B}'$ and let $f : G \to G'$ be a group homomorphism. Then f is continuous $\iff$ for all $U' \in \mathcal{B}'$, there exists $U \in \mathcal{B}$ with $f(U) \subseteq U'$ $\iff$ for all $U' \in \mathcal{B}'$, $f^{-1}(U') \in N_e$.*

Before we move to topological rings, we note the following remarkable result, due to Pontryagin, which asserts that the separation axioms T_0 and $\mathrm{T}_{3\frac{1}{2}}$ coincide for topological groups.[4] Observe that by Proposition 0.1.2, a topological group G satisfies T_0 if and only if $\bigcap_{U\in\mathcal{B}} U = \{e\}$ for some (and hence any) local basis $\mathcal{B}$. In particular, such groups are Hausdorff.

Theorem 0.1.6 (Pontryagin). *For topological groups,* $\mathrm{T}_0 \implies \mathrm{T}_{3\frac{1}{2}}$.

A *topological ring* consists of a ring R endowed with a topology such that the maps

$$a: \begin{array}{ccc} R\times R & \to & R \\ (x,y) & \mapsto & x+y \end{array} \qquad\qquad m: \begin{array}{ccc} R\times R & \to & R \\ (x,y) & \mapsto & xy \end{array}$$

are continuous. We also say that the topology on R, denoted τ_R, is a *ring topology.* These assumptions imply that the map $x \mapsto (-x) : R \to R$ is continuous since $-x = (-1)\cdot x$ (but this requirement should be added if R is not assumed to have a unity). Therefore, $(R, +)$ is a topological group, hence all the terminology and most of the previous results apply to R. In particular, the topology on R can be determined by specifying a local basis $\mathcal{B}$ and the closure of any subset $X \subseteq R$ is given by

$$\overline{X} = \bigcap_{U\in\mathcal{B}} (X + U) \ .$$

Example 0.1.7. (i) Any ring can be made into a topological ring by endowing it with the discrete topology.

(ii) $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are topological rings w.r.t. their standard topologies.

(iii) If R is a ring and I is an ideal of R, then R/I is a topological ring once endowed with the quotient topology. The standard map $R \to R/I$ is then a homomorphism of topological rings.

The following theorem is an analogue of Theorem 0.1.3 for rings.

Theorem 0.1.8. *Let R be a ring and let $\mathcal{B}$ be a nonempty collection of subsets of R containing 0. Then $\mathcal{B}$ is a local basis of some ring topology (which is then uniquely determined) if and only if the following conditions are satisfied:*

(0) *For all $U, V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ with $W \subseteq U \cap V$.*
(1) *For all $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V + V \subseteq U$.*
(2) *For all $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V \cdot V \subseteq U$.*
(3) *For all $U \in \mathcal{B}$ and $x \in R$, there exists $V \in \mathcal{B}$ such that $Vx, xV \subseteq U$.*

---

[4] A topological space (X, τ) satisfies T_0 if for all distinct $x, y \in X$ there is $U \in \tau$ such that $|\{x, y\} \cap U| = 1$. The space (X, τ) satisfies $\mathrm{T}_{3\frac{1}{2}}$ if it is Hausdorff (i.e., T_2) and for any closed set $A \subseteq X$ and $a \in X$ with $a \notin A$, there exists a continuous function $f : X \to [0, 1]$ with $f(A) = 0$ and $f(a) = 1$.

EXAMPLE 0.1.9. Let R be a ring and let $\mathcal{B}$ be a nonempty set of ideals that is closed under intersection. Then conditions (0)–(3) of the previous theorem are satisfied (take $V = U$ in (1)–(3)), hence R admits a unique ring topology with local basis $\mathcal{B}$. This topology turns out to be spanned by the cosets of the ideals in $\mathcal{B}$.

Given a topological ring R, a *topological right R-module* is a right R-module M endowed with a topology, τ_M, such that $(M, +)$ is a topological group and the map

$$\begin{array}{ccc} M \times R & \to & M \\ (m, r) & \mapsto & mr \end{array}$$

is continuous. In this case we say that the topology on M is an *R-module topology.*

PROPOSITION 0.1.10. *Let R be a topological ring and let M be a right R-module endowed with some* group *topology. Let $\mathcal{B}_R$ be a local basis for R and let $\mathcal{B}_M$ be a local basis for M. Then M is a topological R-module if and only if*

(1) *For all $U \in \mathcal{B}_M$, there are $J \in \mathcal{B}_R$ and $V \in \mathcal{B}_M$ such that $VJ \subseteq U$.*
(2) *For all $U \in \mathcal{B}_M$ and $r \in R$, there is $V \in \mathcal{B}_M$ such that $Vr \subseteq U$.*
(3) *For all $U \in \mathcal{B}_M$ and $m \in M$, there is $J \in \mathcal{B}_R$ such that $mJ \subseteq U$.*

THEOREM 0.1.11. *Let R be a topological ring with local basis $\mathcal{B}_R$ and let M be a right R-module. Let $\mathcal{B}$ be a nonempty collection of subsets of M containing 0_M. Then $\mathcal{B}$ is a local basis of some R-module topology (which is then uniquely determined) if and only if the following conditions are satisfied:*

(0) *For all $U, V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ with $W \subseteq U \cap V$.*
(1) *For all $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V + V \subseteq U$.*
(3) *For all $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ and $J \in \mathcal{B}_R$ such that $VJ \subseteq U$.*
(3) *For all $U \in \mathcal{B}$ and $r \in R$, there exists $V \in \mathcal{B}$ such that $rV \subseteq U$.*
(4) *For all $U \in \mathcal{B}$ and $m \in M$, there exists $J \in \mathcal{B}_R$ such that $mJ \subseteq U$.*

0.2. Natural Transformations

In the following three sections we present some definitions from category theory. For an extensive discussion, see [**42**]. The reader should be familiar with the definition of a category and a functor before proceeding.

Let $\mathscr{A}, \mathscr{B}$ be categories and let $F, G : \mathscr{A} \to \mathscr{B}$ be two (covariant) functors. A *natural transformation* from F to G is a collection of maps $\{t_A\}_{A \in \mathscr{A}}$ such that $t_A \in \mathrm{Hom}_{\mathscr{B}}(FA, GA)$ and for every $A, B \in \mathscr{A}$ and $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ we have

$$t_B \circ Ff = Gf \circ t_A \ .$$

That is, the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle t_A}\downarrow & & \downarrow{\scriptstyle t_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

We then write

$$t : F \to G \ .$$

In case F, G are *contravariant* functors, a natural transformation is a collection of maps $\{t_A\}_{A \in \mathscr{A}}$ with $t_A \in \mathrm{Hom}_{\mathscr{B}}(FA, GA)$ such that for every $A, B \in \mathscr{A}$ and $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ we have

$$t_A \circ Ff = Gf \circ t_B \ .$$

That is, the following diagram commutes:

$$\begin{array}{ccc} FA & \xleftarrow{Ff} & FB \\ {\scriptstyle t_A}\downarrow & & \downarrow{\scriptstyle t_B} \\ GA & \xleftarrow{Gf} & GB \end{array}$$

Natural transformations t for which t_A is an isomorphism for all $A \in \mathscr{A}$ are called *natural isomorphisms*. In this case, t^{-1} (i.e. $\{t_A^{-1}\}_{A\in\mathscr{A}}$) is a natural isomorphism from G to F (check!).

Note: Natural transformations (isomorphisms) are also called morphisms (isomorphisms) of functors.

Example 0.2.1. Let $\mathscr{A}, \mathscr{B}$ be categories and let $F : \mathscr{A} \to \mathscr{B}$ be any functor. For every $A \in \mathscr{A}$, define $t_A \in \mathrm{Hom}_{\mathscr{B}}(FA, FA)$ to be id_{FA}. Then t is a natural isomorphism from F to itself. Indeed, for all $A, B \in \mathscr{A}$ and $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, we have $t_{FB} \circ Ff = Ff \circ t_{FA}$ and $t_{FA} = \mathrm{id}_{FA}$ is an isomorphism.

Example 0.2.2. Let F be a field and let Mod-F be the category of F-vector spaces. Then the map $V \mapsto V^* := \mathrm{Hom}_F(V, F)$ induces a contravariant functor from Mod-F to itself. Thus, $** : \text{Mod-}F \to \text{Mod-}F$ is a covariant functor. For every $V \in \text{Mod-}F$, let $\omega_V : V \to V^{**}$ be the standard embedding of V in V^{**} given by $(\omega_V x)f = f(x)$ (where $x \in V$ and $f \in V^*$). Then

$$\omega : \mathrm{id}_{\text{Mod-}F} \to **$$

i.e., ω is a natural transformation from the identity functor $\mathrm{id}_{\text{Mod-}F}$ on Mod-F to $**$. Indeed, for all $U, V \in \text{Mod-}F$ and $f \in \mathrm{Hom}_F(U, V)$, we have

$$\omega_V \circ f = f^{**} \circ \omega_U \ .$$

However, ω is not a natural isomorphism since ω_V is not bijective for inifinite dimensional V. Nevertheless, if we replace Mod-F with the category of f.d. F-vector spaces, then ω becomes a natural isomorphism.

Example 0.2.3. Let F be a field admitting an automorphism $\sigma \neq \mathrm{id}_F$. For every $V \in \text{Mod-}F$, let V^σ denote the F-vector space obtained from V by replacing the operation of F on V by $\diamond_\sigma : V \times F \to V$, defined by $v \diamond_\sigma a = v \cdot \sigma(a)$. Define a functor $G : \text{Mod-}F \to \text{Mod-}F$ by $GV = V^\sigma$ and $Gf = f$ for all $V \in \text{Mod-}F$ and any morphism f in Mod-F. Then G is a covariant functor and $GV \cong \mathrm{id}_{\text{Mod-}F} V = V$ for every $V \in \text{Mod-}F$. Nevertheless, there is no *natural isomorphism* from G to $\mathrm{id}_{\text{Mod-}F}$.

Indeed, assume by contradiction that $t : G \to \mathrm{id}_{\text{Mod-}F}$ is a natural isomorphism. Let V be a 1-dimensional vector space and let $0 \neq v \in V$. Then $t_V : V^\sigma \to V$ is an isomorphism, hence $t_V(v) = v \cdot a$ for some $a \in F^\times$. Now, let $b \in F$ be such that $\sigma(b) \neq b$ and let $f : V \to V$ be defined by $f(x) = x \cdot b$. By the naturalness of t, we must have

$$t_V \circ f = t_V \circ Gf = \mathrm{id}_{\text{Mod-}F} f \circ t_V = f \circ t_V \ .$$

But this means that

$$\begin{aligned} v \cdot (a\sigma^{-1}(b)) &= t_V(v) \cdot \sigma^{-1}(b) = t_V(v \diamond_\sigma \sigma^{-1}(b)) = t_V(vb) \\ &= t_V(fv) = f(t_V v) = f(v \cdot a) = v \cdot (ab) \end{aligned}$$

which in turn implies $a\sigma^{-1}(b) = ab$, hence $\sigma(b) = b$, a contradiction.

Natural transformations can also be defined for bifunctors, and more generally, multi-functors. The latter are, roughly, functors taking several variables (such as $\mathrm{Hom}(\_, \_)$). Rather than spelling out all the definitions, let us exhibit an explicit example.

EXAMPLE 0.2.4. Let K be a field. Consider the bifunctors F and G from Mod-$K \times$ Mod-$K \to$ Mod-K defined by

$$F(U, V) = \mathrm{Hom}_F(U, V^*)$$
$$G(V, U) = \mathrm{Hom}_F(V, U^*) \ .$$

If $U, U', V, V' \in$ Mod-K and $f \in \mathrm{Hom}_K(U', U)$, $g \in \mathrm{Hom}_K(V', V)$, then $F(f, g)$ and $G(f, g)$ are defined by

$$F(f, g) = g^* \circ \_ \circ f \in \mathrm{Hom}_K(\mathrm{Hom}_K(U, V^*), \mathrm{Hom}_K(U', V'^*))$$
$$G(f, g) = f^* \circ \_ \circ g \in \mathrm{Hom}_K(\mathrm{Hom}_K(V, U^*), \mathrm{Hom}_K(V', U'^*)) \ .$$

In particular, F and G are contravariant in both variables. For every $U, V \in$ Mod-K, define $I_{U,V} : F(U, V) \to G(U, V)$ by

$$I_{U,V}(h) = h^* \circ \omega_V \in \mathrm{Hom}(V, U^*) = G(U, V) \qquad \forall\ h \in F(U, V) = \mathrm{Hom}_K(U, V^*)$$

where ω_V is as in Example 0.2.2. Then

$$I : F \to G \ .$$

This holds since for all f, g, U, U', V, V' as above, we have

$$I_{U',V'} \circ F(f, g) = G(f, g) \circ I_{U,V} \ .$$

Moreover, I is actually a natural isomorphism. The details are left as an exercise to the reader.

At certain times, we will say that a given homomorphism between two objects is *natural*. This is merely an abbreviation for the following two claims:

(1) The way to obtain the objects in question is functorial (i.e. it also sends morphisms to morphisms in a way that respects composition).
(2) The map defined is a natural transformation between the two functors of (1).

For example, let R be a ring. By saying that a right R-module M is *naturally isomorphic* to $\mathrm{Hom}_R(R_R, M)$ as abelian groups we mean that (1) both maps $M \mapsto M$ and $M \mapsto \mathrm{Hom}_R(R_R, M)$ give rise to (covariant) functors from Mod-R to the category of abelian groups and (2) there is a natural isomorphism between these functors. In contrast to that, using the notation of Example 0.2.3, there is *no natural isomorphism* between V and V^σ, despite the fact that V is always isomorphic to V^σ.

0.3. Additive Categories

A category $\mathscr{A}$ is called *preadditive* if for all $A, A' \in \mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(A, A')$ is endowed with an (additive) abelian group structure such that the composition action is biadditive. That is, for all $A, A', A'' \in \mathscr{A}$, $f, g \in \mathrm{Hom}_{\mathscr{A}}(A, A')$ and $f', g' \in \mathrm{Hom}_{\mathscr{A}}(A', A'')$ we have

$$(f' + g') \circ f = f' \circ f + g' \circ f$$
$$f' \circ (f + g) = f' \circ f + f' \circ g \ .$$

In this case, for all $A \in \mathscr{A}$, $\mathrm{End}_{\mathscr{A}}(A)$ is a ring. In addition, we can speak about the *zero morphism* between two objects.

A category $\mathscr{A}$ is additive if it satisfies the following conditions:

(1) $\mathscr{A}$ has a *zero object*.
(2) $\mathscr{A}$ is preadditive.
(3) Finite *biproducts* exist in $\mathscr{A}$.

The definition of a zero object appears below, but I prefer not to give here the definition of a biproduct and rather refer the reader to any good book about category theory (e.g. [**42**]). The flavor of what a zero object and a biproduct are can be seen by looking at the category of right R-modules. In this category, the zero module 0 is the zero object and the biproduct of modules $A_1, \dots, A_t$ is just the direct sum $A_1 \oplus \cdots \oplus A_t$. The symbols 0 and $A_1 \oplus \cdots \oplus A_t$ will be used to denote the zero object and the biproduct in any additive category.

Definition 0.3.1. *Let $\mathscr{C}$ be a category. A zero object in $\mathscr{C}$ is an object $0 \in \mathscr{C}$ such that for all $C \in \mathscr{C}$*

$$|\operatorname{Hom}_{\mathscr{C}}(C, 0)| = |\operatorname{Hom}_{\mathscr{C}}(0, C)| = 1 \ .$$

In this case, 0 is uniquely determined up to isomorphism, so up to that rank freedom, we can speak about the *zero object of $\mathscr{C}$.*

Example 0.3.2. (i) The categories of R-modules and f.g. R-modules are additive categories.

(ii) Let $n \in \mathbb{N}$. The category $\mathscr{C}$ of vector spaces over a field F of dimension n or less is preadditive (since $\operatorname{Hom}_F(U, V)$ is an abelian group for all $U, V \in \mathscr{C}$ and the composition is biadditive), but it is not additive. Indeed, $\mathscr{C}$ has a zero object, but not all finite biproducts exist (since the biproduct of two n-dimensional vector spaces would be a $2n$-dimensional vector space).

(iii) The category of (non-abelian) groups $\mathscr{G}$ is not preadditive.

If $\mathscr{A}$ is an additive category, then the abelian group structure on $\operatorname{Hom}_{\mathscr{A}}(A, A')$ can be recovered from $\mathscr{A}$ by purely categorical means. In particular, the Hom-sets in $\mathscr{A}$ admit only one abelian group structure making $\mathscr{A}$ into an additive category. (This is false for preadditive categories, though). In addition, the object 0 and the operation $\oplus$ satisfy many expected properties such as

(i) $A \oplus 0$ is *naturally* isomorphic to A.
(ii) $\operatorname{Hom}_{\mathscr{A}}(A, B \oplus B')$ is *naturally* isomorphic to $\operatorname{Hom}_{\mathscr{A}}(A, B) \oplus \operatorname{Hom}_{\mathscr{A}}(A, B')$ as abelian groups.
(iii) $\operatorname{Hom}_{\mathscr{A}}(A \oplus A', B)$ is *naturally* isomorphic to $\operatorname{Hom}_{\mathscr{A}}(A, B) \oplus \operatorname{Hom}_{\mathscr{A}}(A', B)$ as abelian groups.

In particular, the latter two statements imply

$$\operatorname{Hom}_{\mathscr{A}}(\oplus_{i=1}^t A_i, \oplus_{j=1}^s B_j) \cong \prod_{i=1}^t \prod_{j=1}^s \operatorname{Hom}_{\mathscr{A}}(A_i, B_j) \ .$$

It is customary to write the r.h.s. of this isomorphism in matrix form, namely

$$\operatorname{Hom}_{\mathscr{A}}(\oplus_{i=1}^t A_i, \oplus_{j=1}^s B_j) \cong \begin{bmatrix} \operatorname{Hom}_{\mathscr{A}}(A_1, B_1) & \dots & \operatorname{Hom}_{\mathscr{A}}(A_t, B_1) \\ \vdots & \ddots & \vdots \\ \operatorname{Hom}_{\mathscr{A}}(A_1, B_s) & \dots & \operatorname{Hom}_{\mathscr{A}}(A_t, B_s) \end{bmatrix} .$$

In particular, any morphism $f \in \operatorname{Hom}_{\mathscr{A}}(\oplus_{i=1}^t A_i, \oplus_{j=1}^s B_j)$ can be represented by an $s \times t$ matrix (f_{ji}) with $f_{ji} \in \operatorname{Hom}_{\mathscr{A}}(A_i, B_j)$. In this representation, composition of morphisms becomes matrix multiplication. Formally speaking, f_{ji} can be extracted from f by the formula $f_{ji} = p_j \circ f \circ e_i$ where p_j is the *projection* $\oplus_{k=1}^s B_k \to B_j$ and e_i is the *embedding* $A_i \to \oplus_{k=1}^t A_k$. The matrix representation is commonly used to describe morphisms between biproducts and it will be used repeatedly in this text.

Example 0.3.3. Keeping the above notation, the matrix representation of the zero morphism from $\oplus_{i=1}^{t}A$ to $\oplus_{j=1}^{s}B$ is

$$\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

where the 0 in the (j,i) place is the zero morphisms from A_i to B_j. In case $t = s$ and $A_i = B_i$ for all i, the matrix representation of the identity morphism $\mathrm{id} : \oplus_{i=1}^{t}A_i \to \oplus_{j=1}^{s}B_j = \oplus_{i=1}^{t}A_i$ is

$$\begin{bmatrix} \mathrm{id}_{A_1} & & \\ & \ddots & \\ & & \mathrm{id}_{A_t} \end{bmatrix}$$

(there are zero morphisms outside the diagonal).

A functor F (covariant or contravariant) between *preadditive* categories $\mathscr{A}$ and $\mathscr{B}$ is called *additive* if it respects the additive group structure on the Hom-sets. That is, for all $A, A' \in \mathscr{A}$, the map

$$F : \mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(FA, FA')$$

is an abelian group homomorphism.

An *additive* (covariant or contravariant) functor F between *additive* categories $\mathscr{A}$ and $\mathscr{B}$ is a functor sending biproducts to biproducts. That is, for all $A, A' \in \mathscr{A}$, $F(A \oplus A')$ is the biproduct of FA and FA' or, equivalently, $F(A \oplus A')$ is naturally isomorphic to $FA \oplus FA'$. In this case, it is customary to identify $F(A \oplus A')$ with $FA \oplus FA'$. Additive functors between additive categories are also additive when considered as functors between preadditive categories.

Example 0.3.4. Let R be a ring and let M be any R-module. Let F be the functor from Mod-R to itself sending every object to M and any morphism to id_M. Then F is not additive if $M \neq 0$ (since, roughly, $F(A \oplus B) = M$ is not naturally isomorphic to $FA \oplus FB = M \oplus M$). However, if $M = 0$, then F is additive.

Example 0.3.5. Let F be a field and let $* : \text{Mod-}F \to \text{Mod-}F$ be the contravariant functor defined by $V^* = \mathrm{Hom}_F(V, F)$. Then $*$ is an additive (contravariant) functor. The identification between $(U \oplus V)^*$ and $U^* \oplus V^*$ is given by $f \mapsto (f|_U, f|_V)$. The reader should try to verify that this isomorphism is natural.

0.4. More Category Theory

0.4.1. Faithful and Full Functors. Throughout, $\mathscr{A}$ and $\mathscr{B}$ are categories. A functor $F : \mathscr{A} \to \mathscr{B}$ is called *faithful* (resp. *full*) if for all $A, A' \in \mathscr{A}$, the map:

$$F : \mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(FA, FA')$$

is injective (resp. surjective). Contravariant faithful and full functors are defined in the same manner with $\mathrm{Hom}_{\mathscr{B}}(FA, FA')$ replaced by $\mathrm{Hom}_{\mathscr{B}}(FA', FA)$.

Example 0.4.1. Assume $\mathscr{B}$ consists of a single object and a single morphism. Then there is precisely one functor from $\mathscr{A}$ to $\mathscr{B}$ and it is full. It is faithful if and only if $\mathrm{Hom}_{\mathscr{A}}(A, A')$ contains exactly one element for all $A, A' \in \mathscr{A}$.

Example 0.4.2. Let F be a field and let $* : \text{Mod-}F \to \text{Mod-}F$ be the contravariant functor defined by $V^* = \mathrm{Hom}_F(V, F)$. Then $*$ is faithful since the map $* : \mathrm{Hom}_F(U, V) \to \mathrm{Hom}_F(V^*, U^*)$ is injective for all $U, V \in \text{Mod-}F$. However, $*$ is not full since the previous map is not bijective for $U = F$ and $V = \bigoplus_{\alpha_0} F$.

EXAMPLE 0.4.3. If $F : \mathscr{A} \to \mathscr{B}$ is an additive functor between preadditive categories, then F is faithful if and only if for any morphism f in $\mathscr{A}$, $Ff = 0$ implies $f = 0$.

A *subcategory* of $\mathscr{A}$ is a category $\mathscr{A}_0$ such that:

(1) $A \in \mathscr{A}_0$ implies $A \in \mathscr{A}$.
(2) $A, A' \in \mathscr{A}_0$ and $f \in \mathrm{Hom}_{\mathscr{A}_0}(A, A')$ implies $f \in \mathrm{Hom}_{\mathscr{A}}(A, A')$.
(3) If $A \in \mathscr{A}$, then the identity morphism of A in $\mathscr{A}_0$ is the identity morphism of A in $\mathscr{A}$.[5]

In this case, the functor $\mathrm{id}_{\mathscr{A}_0}$ can be considered as a functor from $\mathscr{A}_0$ to $\mathscr{A}$. The subcategory $\mathscr{A}_0$ is called *full* if the functor $\mathrm{id}_{\mathscr{A}_0} : \mathscr{A}_0 \to \mathscr{A}$ is full. This is equivalent to

$$\mathrm{Hom}_{\mathscr{A}_0}(A, A') = \mathrm{Hom}_{\mathscr{A}}(A, A')$$

for all $A, A' \in \mathscr{A}_0$.

EXAMPLE 0.4.4. (i) Let R be a ring. The category of f.g. right R-modules is a full subcategory of Mod-R, the category of all right R-modules.

(ii) Let $\mathscr{C}$ be the category whose objects are the objects of Mod-R and whose morphisms are the isomorphisms of Mod-R, i.e. $\mathrm{Hom}_{\mathscr{C}}(M, N)$ is the set of R-module isomorphisms from M to N. Then $\mathscr{C}$ is a subcategory of Mod-R and it is not full despite the fact that its objects are the objects of Mod-R.

0.4.2. Generators and Cogenerators. Let $\mathscr{A}$ be category. An object G is called a *generator* (of $\mathscr{A}$) if for all $A, A' \in \mathscr{A}$ and distinct $f, g \in \mathrm{Hom}_{\mathscr{A}}(A, A')$, there exists $h \in \mathrm{Hom}(G, A)$ such that

$$f \circ h \neq g \circ h \ .$$

This is equivalent to saying that the functor $\mathrm{Hom}_{\mathscr{A}}(G, \_)$ (from $\mathscr{A}$ to the category of sets) is faithful (check!). A *cogenerator* (of $\mathscr{A}$) is the dual notion of a generator (i.e. a generator in the opposite category). Explicitly, $U \in \mathscr{A}$ is a cogenerator if for all $A, A' \in \mathscr{A}$ and distinct $f, g \in \mathrm{Hom}_{\mathscr{A}}(A, A')$, there exists $h \in \mathrm{Hom}(A', U)$ such that

$$h \circ f \neq h \circ g \ .$$

Alternatively, U is a cogenerator if the (contravariant) functor $\mathrm{Hom}_{\mathscr{A}}(\_, U)$ is faithful.

EXAMPLE 0.4.5. Let $\mathscr{A}$ be the category of abelian groups (or $\mathbb{Z}$-modules).

(i) $\mathbb{Z}$ is a generator of $\mathscr{A}$. Indeed, let A, A', f, g be as above. Then there exists $x \in A$ such that $f(x) \neq g(x)$. Define $h : \mathbb{Z} \to A$ by $h(n) = x \cdot n$ and observe that $f \circ h \neq g \circ h$ since $(f \circ h)(1) \neq (g \circ h)(1)$.

(ii) For all $A \in \mathscr{A}$, $\mathbb{Z} \oplus A$ is a generator of $\mathscr{A}$. This is shown by a slight adjustment of the argument of (i).

(iii) $\mathbb{Q}$ is not a generator of $\mathscr{A}$. Indeed, consider $\mathrm{id}_{\mathbb{Z}/2}, 0_{\mathbb{Z}/2} \in \mathrm{Hom}_{\mathscr{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$. Then there is no $h : \mathbb{Q} \to \mathbb{Z}/2$ such that $\mathrm{id}_{\mathbb{Z}/2} \circ h \neq 0_{\mathbb{Z}/2} \circ h$.

(iv) $\mathbb{Q}/\mathbb{Z}$ is a cogenerator of $\mathscr{A}$; see [**58**, Ex. 19.11].

(v) $\mathbb{Z}$ is not a generator of $\mathscr{A}$. For instance, consider $\mathrm{id}_{\mathbb{Q}}, 0_{\mathbb{Q}} \in \mathrm{Hom}_{\mathscr{A}}(\mathbb{Q}, \mathbb{Q})$. Then there is no $h : \mathbb{Q} \to \mathbb{Z}$ such that $h \circ \mathrm{id}_{\mathbb{Q}} \neq h \circ 0_{\mathbb{Q}}$.

The following proposition generalizes part (ii) of the previous example.

PROPOSITION 0.4.6. *Let $G \in \mathscr{A}$ be a generator. Then any object G' admitting an epic morphism $G' \to G$ is also a generator. Dually, let $U \in \mathscr{A}$ be a cogenerator. Then any object U' admitting a monic morphism $U \to U'$ is also a cogenerator.*[6]

---

[5] This condition is sometimes dropped from the definition.

[6] Recall that a morphism $f : A \to A'$ is *monic* if for all $g_1, g_2 : B \to A$, $f \circ g_1 = f \circ g_2 \implies g_1 = g_2$ and *epic* if for all $h_1, h_2 : A' \to B$, $h_1 \circ f = h_2 \circ f \implies h_1 = h_2$.

Generators and cogenerators of Mod-R (also called *R-generators* and *R-cogenerators*) admit several equivalent definitions which are summarized in the following propositions. Some of the equivalent conditions apply to any *Grothenieck category* (e.g. Mod-R). We also note that condition (b) in both propositions characterizes generators and cogenerators in all preadditive categories.

PROPOSITION 0.4.7. *Let R be a ring and $G \in$ Mod-R. The following are equivalent:*

(a) *G is a generator.*
(b) *For all $A, B \in$ Mod-R and $0 \neq f \in \mathrm{Hom}_R(A, B)$, there exists $g \in \mathrm{Hom}_R(G, A)$ such that $f \circ g \neq 0$.*
(c) *Any right R-module is an epimorphic image of $\bigoplus_{i\in I} G$ for some set I.*
(d) *R_R is an epimorphic image of $\bigoplus_{i\in I} G$ for some set I.*
(e) *R_R is a summand of G^n for some $n \in \mathbb{N}$.*

PROPOSITION 0.4.8. *Let R be a ring and $U \in$ Mod-R. The following are equivalent:*

(a) *U is a cogenerator.*
(b) *For all $A, B \in$ Mod-R and $0 \neq f \in \mathrm{Hom}_R(A, B)$, there exists $g \in \mathrm{Hom}_R(B, U)$ such that $g \circ f \neq 0$.*
(c) *For any $A \in$ Mod-R and $0 \neq x \in A$, there exists $f \in \mathrm{Hom}_R(A, U)$ with $f(x) \neq 0$.*
(d) *Any right R-module embeds in $\prod_{i\in I} U$ for some set I.*
(e) *U contains a copy of the* injective hull *of any simple right R-module.*

0.4.3. Equivalence of Categories. Two categories $\mathscr{A}$ and $\mathscr{B}$ are called *equivalent* if there exist two functors $F : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{B} \to \mathscr{A}$ and two natural isomorphisms $\delta : GF \to \mathrm{id}_{\mathscr{A}}$ and $\varepsilon : FG \to \mathrm{id}_{\mathscr{B}}$. In this case, $(F, G, \delta, \varepsilon)$ is called an equivalence from $\mathscr{A}$ to $\mathscr{B}$. We also say that F induces an equivalence of categories from $\mathscr{A}$ to $\mathscr{B}$ (however, F does not determine G, ε, δ).

An equivalence between $\mathscr{A}$ and $\mathscr{B}$ roughly means that, modulo isomorphism of objects, the two categories are the same. In particular, categorical statements about objects can be transferred from $\mathscr{A}$ to $\mathscr{B}$ and back. More explicitly, if $F : \mathscr{A} \to \mathscr{B}$ induces an equivalence and $\mathcal{P}$ is a property of objects that is phrased in a purely categorical manner, then an object $A \in \mathscr{A}$ has $\mathcal{P}$ if and only if FA has $\mathcal{P}$. For example, A is projective (resp.: injective, a generator, a cogenerator, a zero object, etc.) if and only if FA is. Furthermore, categorical properties of categories hold for $\mathscr{A}$ if and only if they hold for $\mathscr{B}$. For instance, $\mathscr{A}$ is additive (resp. abelian) if and only if $\mathscr{B}$ is and in this case the the functors that induce the equivalence are additive (resp. exact).[7]

We should also note that if F induces a duality from $\mathscr{A}$ to $\mathscr{B}$, then so is any functor F' that is isomorphic to F (i.e. a functor for which there is a natural isomorphism $t : F \to F'$).

EXAMPLE 0.4.9. Let $\mathscr{A}$ be a category with one object and one morphism and let $\mathscr{B}$ be a nonempty category such that there is precisely one morphism between any two objects in $\mathscr{B}$. Then $\mathscr{A}$ is equivalent to $\mathscr{B}$. Indeed, let $A \in \mathscr{A}$ be the only object of A and let $B \in \mathscr{B}$. Define $F : \mathscr{A} \to \mathscr{B}$ by $FA = B$ and $F\,\mathrm{id}_A = \mathrm{id}_B$ and let G be the only functor from $\mathscr{B}$ to $\mathscr{A}$. Then $GF = \mathrm{id}_{\mathscr{A}}$ and in particular, $\mathrm{id} : GF \to \mathrm{id}_{\mathscr{A}}$ is a natural isomorphism. Now define $\varepsilon : FG \to \mathrm{id}_{\mathscr{B}}$ by letting $\varepsilon_{B'}$ to be the unique element in $\mathrm{Hom}_{\mathscr{B}}(FGB', \mathrm{id}_{\mathscr{B}} B') = \mathrm{Hom}_{\mathscr{B}}(B, B')$. Then ε is a natural isomorphism (check!), hence $\mathscr{A}$ is equivalent to $\mathscr{B}$. However, $\mathscr{A}$ is not isomorphic to $\mathscr{B}$ unless $\mathscr{B}$ also contains one object.

[7] Caution: Being preadditive is not a categorical property.

EXAMPLE 0.4.10. Morita equivalence, described below, is an example of an equivalence between categories.

If F induces an equivalence from $\mathscr{A}$ to $\mathscr{B}$ then F is faithful, full and for any $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ with $FA \cong B$ (take $A = GB$). The converse is also true, provided one accepts a strong enough version of the axiom of choice (which applies to *classes* rather than sets). The conditions just specified make it easy to check whether a functor induces an equivalence. However, for most applications, the full description of the equivalence is required.

0.5. Morita Equivalence

This section briefly presents the basis of Morita theory, which classifies equivalences between module categories. Throughout, Mod-(R, S) denotes the category of (R, S)-bimodules. For a detailed discussion and proofs, see [**58**, §18] or [**80**, §4.1].

DEFINITION 0.5.1. *Two rings R and S are said to be* Morita equivalent *if the categories* Mod-R *and* Mod-S *are equivalent.*

Many ring theoretic properties of the ring R can be phrased as categorical statements on Mod-R and are thus guaranteed to pass to any ring which is Morita equivalent to R. For example, the properties right noetherian, right artinian, semisimple, right (semi)hereditary (and also: right nonsingular, $\operatorname{u.dim} R_R < \infty$, right self-injective, quasi-Frobenius, which are defined in the following sections) are categorical properties and are thus preserved under Morita equivalence. In general, ring theoretic properties that are preserved under Morita equivalence (even if not categorical) are called *Morita invariant*. By the end of this section we shall have exhibited several more Morita invariant properties.

It turns out that there is a very explicit way to decide whether two rings R and S are Morita equivalent and, moreover, one can characterize (up to functor isomorphism) all the equivalences between Mod-R and Mod-S. We shall now give the details.

DEFINITION 0.5.2. *Let R be a ring. A right R-module P is called a* progenerator *(or R-progenerator) if P is f.g., projective and a generator (of* Mod-R*).*

Note that by Proposition 0.4.7, a right R-module P is a progenerator if and only if there is $n \in \mathbb{N}$ such that P is a summand of R^n and R_R is a summand of P^n.

EXAMPLE 0.5.3. (i) A f.g. projective module over a *commutative* ring R is a progenerator if and only if it is faithful. This is due to Azumaya.

(ii) If P is a finite projective right R-module, then $R_R \oplus P$ is a progenerator.

(iii) For any $P \in$ Mod-R and $n \in \mathbb{N}$, P^n is a progenerator if and only if P is a progenerator.

(iv) Let R be a semisimple ring. Then an R-module is a progenerator if and only if it is faithful.

(v) Let F be a field and let T_n be the ring of $n \times n$ upper-triangular matrices over F. Let P be the right ideal of T_n consisting of matrices with all rows being zero except possibly the top one. Then P is projective and faithful, but it is not a generator.

The following proposition characterizes the summands of R_R which are R-progenerators.

PROPOSITION 0.5.4. *Call an idempotent e of a ring R* full *if $ReR = R$. Then an idempotent $e \in \mathrm{E}(R)$ is full $\iff$ eR_R is an R-progenerator $\iff$ ${}_R Re$ is an R-progenerator.*

Definition 0.5.5. *A* Morita context *consists of a sextet* $(R, P, S, Q; \alpha, \beta)$ *such that* R *and* S *are rings,*

$$\begin{aligned} &P \in \text{Mod-}(S, R),\\ &Q \in \text{Mod-}(R, S),\\ &\alpha \in \text{Hom}_{\text{Mod-}(R,R)}(Q \otimes_S P, R),\\ &\beta \in \text{Hom}_{\text{Mod-}(S,S)}(P \otimes_R Q, S), \end{aligned}$$

and

$$\alpha(q \otimes p) \cdot q' = q \cdot \beta(p \otimes q'), \qquad \beta(p \otimes q) \cdot p' = p \otimes \alpha(q \otimes p')$$

for all $p, p' \in P$ *and* $q, q' \in Q$.

Example 0.5.6. Let P be any right R-module and let $S = \text{End}(P_R)$. Then P be be considered as an (S, R)-bimodule. Furthermore, $Q := P^* = \text{Hom}_R(P, R_R)$ can be considered as an (R, S)-bimodule by letting

$$(r \cdot q \cdot s)(p) = r \cdot q(s(p)) \qquad \forall\, r \in R, q \in Q, s \in S, p \in P\ .$$

Now define $\alpha : Q \otimes_S P \to R$ and $\beta : P \otimes_R Q \to S$ by

$$\alpha(q \otimes p) = q(p) \qquad \text{and} \qquad \beta(p \otimes q) = [p' \mapsto p \cdot q(p')]\ .$$

Then $(R, P, S, Q; \alpha, \beta)$ is a Morita context (check!) called the *Morita context associated with* P.

Proposition 0.5.7. *Let* $(R, P, S, Q; \alpha, \beta)$ *be a Morita context. The following are equivalent:*

(a) α *and* β *are onto.*
(b) α *and* β *are bijective.*
(c) P_R *is a progenerator.*

In this case, $Q \cong \text{Hom}_R(P_R, R_R) \cong \text{Hom}_S({}_SP, {}_SS)$ *as* (R, S)*-bimodules,* $P \cong \text{Hom}_R({}_RQ, {}_RR) \cong \text{Hom}_S(Q_S, S_S)$ *as* (S, R)*-bimodules,* $R \cong \text{End}_S({}_SP) \cong \text{End}_S(Q_S)$ *as rings and* $S \cong \text{End}_R({}_RQ) \cong \text{End}_R(P_R)$ *as rings. In particular, under suitable identifications,* $(R, P, S, Q; \alpha, \beta)$ *is the Morita context associated with* P.

We can now formulate Morita's three theorems about equivalence of module categories.

Theorem 0.5.8 (Morita I). *Let* R *be a ring, let* P *be an* R*-progenerator and let* $(R, P, Q, S; \alpha, \beta)$ *be the Morita context associated with* P*. Then:*

(i) *The functors* $\_ \otimes_R Q : \text{Mod-}R \to \text{Mod-}S$ *and* $\_ \otimes_S P : \text{Mod-}S \to \text{Mod-}R$ *induce an equivalence of categories.*
(ii) *The functors* $P \otimes_R \_ : R\text{-Mod} \to S\text{-Mod}$ *and* $Q \otimes_S \_P : S\text{-Mod} \to Q\text{-Mod}$ *induce an equivalence of categories.*

Proof (sketch). Let $M \in \text{Mod-}R$ and $N \in \text{Mod-}S$. We only define the natural isomorphisms $\delta : M \otimes_R Q \otimes_S P \to M$ and $\varepsilon : N \otimes_S P \otimes_R Q \to N$ which are needed to show (i). They are given by $M \otimes_R (Q \otimes_S P) \xrightarrow{\text{id}_M \otimes \alpha} M \otimes_R {}_RR_R \cong M$ and $N \otimes_S (P \otimes_R Q) \xrightarrow{\text{id}_N \otimes \beta} N \otimes_S {}_SS_S \cong N$. □

Theorem 0.5.9 (Morita II). *Let* R *and* S *be rings and let* $(F, G, \delta, \varepsilon)$ *be an equivalence of categories from* Mod-R *to* Mod-S*. Let* $Q = F(R_R)$ *and* $P = G(S_S)$*. Then:*

(i) *There is an* (S, R)*-bimodule structure on* P *and an* (R, S)*-bimodule structure on* Q.
(ii) P_R*,* ${}_RQ$*,* Q_S*,* ${}_SP$ *are progenerators.*
(iii) *There are isomorphism of functors* $F \cong \_ \otimes_R Q$ *and* $G \cong \_ \otimes_S P$.

(iv) *Using the previous isomorphisms, consider* $\delta_R : GF(R_R) \to R_R$ *and* $\varepsilon_S : FG(S_S) \to S_S$ *as maps* $Q \otimes_R P(\otimes_R R) \to R$ *and* $P \otimes_S Q(\otimes_S S) \to S$*. Then* $(R, P, S, Q; \delta_R, \varepsilon_S)$ *is the Morita context associated with* P_R*.*

In particular, up to suitable natural identifications, the equivalence $(F, G, \varepsilon, \delta)$ *is the equivalence obtained from* P *as in Theorem 0.5.8.*

To state Morita's Third Theorem, we define an (S, R)-*progenerator* to be an (S, R)-bimodule P such that P_R is a progenerator and $S = \mathrm{End}_R(P)$. (This is equivalent to ${}_SP$ being a progenerator and $R = \mathrm{End}_S(P)$.)

Theorem 0.5.10 (Morita III). *Let* R *and* S *be rings. There is a one-to-one correspondence between isomorphism classes of equivalences* Mod-$R \to$ Mod-S *and isomorphism classes of* (S, R)*-progenerators. Furthermore, composition of such equivalences corresponds to the tensor product of the corresponding progenerators.*

When phrased explicitly, the last part of the previous theorem means that if R_1, R_2, R_3 are rings and there are equivalences of categories Mod-$R_1 \to$ Mod-R_2 and Mod-$R_2 \to$ Mod-R_3 corresponding to an (R_1, R_2)-progenerator P_1 and an (R_2, R_3)-progenerator P_2 respectively, then the composition of the equivalences corresponds to the (R_1, R_3)-progenerator $P_1 \otimes_{R_2} P_2$. In particular, the bimodule $P_1 \otimes_{R_2} P_2$ *is* an (R_1, R_3)-progenerator.

As an immediate consequence of Morita's theorems, we get:

Corollary 0.5.11. *Let* R *and* S *be a rings. Then* Mod-R *is equivalent to* Mod-S $\iff$ *there exists a* right R*-progenerator* P *such that* $S \cong \mathrm{End}_R(P)$ $\iff$ *there exists a* left R*-progenerator* Q *such that* $S \cong \mathrm{End}_R(Q)$ $\iff$ R-Mod *is equivalent to* S-Mod.

In particular, we see that Morita equivalence is a left-right symmetric property. Combining this with Proposition 0.5.4, yields:

Corollary 0.5.12. *Let* R *and* S *be rings. Then* R *is Morita equivalent to* S *if and only if there is* $n \in \mathbb{N}$ *and a* full *idempotent* $e \in \mathrm{M}_n(R)$ *such that* $S \cong e\mathrm{M}_n(R)e$.

Proof (sketch). If $S \cong e\mathrm{M}_n(R)e$, then S is Morita equivalent to $\mathrm{M}_n(R)$ (by Proposition 0.5.4 and Morita's First Theorem). As $\mathrm{M}_n(R)$ is clearly equivalent to R (since $\mathrm{M}_n(R) \cong \mathrm{End}_R(R^n)$), R is Morita equivalent to S. Conversely, if R is Morita equivalent to S, then there is a right R-progenerator P such that $S \cong \mathrm{End}_R(P)$. Let P' be an R-module such that $P \oplus P' \cong R^n$ and let e denote the projection from R^n to P with kernel P'. Then $\mathrm{End}_R(P) \cong e\,\mathrm{End}_R(R^n)e = e\mathrm{M}_n(R)e$. One can show that e is full and this finishes the proof. □

The last corollary means that if we want to check that a ring theoretic property $\mathcal{P}$ is Morita invariant it is enough to verify that:

(1) R has $\mathcal{P} \implies \mathrm{M}_n(R)$ has $\mathcal{P}$ for all $n \in \mathbb{N}$.
(2) R has $\mathcal{P} \implies eRe$ has $\mathcal{P}$ for any full idempotent $e \in \mathrm{E}(R)$.

(We should note that many ring theoretic properties pass to eRe even without assuming e is full.) In particular, the properties prime, semiprime, simple, seimlocal, semiperfect and semiprimary can be shown to be Morita invariant in this way.

0.6. Quasi-Frobenius Rings and Related Notions

This section presents a short survey about quasi-Frobenius and pseudo-Frobenius rings. Its purpose is mainly to present the various equivalent definitions and some examples. For more details and proofs see [**58**, Chs. 6–7] and also [**54**].

Recall that a ring R is called right self-injective if the right R-module R_R is injective. Combining this assumption with the ascending chain condition (ACC) yields the definition of a quasi-Frobenius ring.

Definition 0.6.1. *A ring R is called* quasi-Frobenius *(abbrev.: QF) if R is right self-injective and right noetherian.*

The following (very hard) theorem, which is the combined work of several authors, presents some alternative definitions of QF rings. In particular, it shows that QF is a left-right symmetric property which is preserved under Morita equivalence (by conditions (e) and (f) below).

Theorem 0.6.2. *Let R be a ring. The following conditions are equivalent:*

(a) *R is QF (i.e. right noetherian and right self-injective).*
(b) *R is left noetherian and right self-injective.*
(c) *R is artinian and self-injective.*
(d) *R is right noetherian and satisfies $\operatorname{ann}^r \operatorname{ann}^\ell A = A$ for any right ideal $A \leq R_R$ and $\operatorname{ann}^\ell \operatorname{ann}^r B = B$ for any left ideal $B \leq R_R$. In particular, the maps ann^r and ann^ℓ define an anti-isomorphism of lattices between the right ideals and the left ideals of R.*
(e) *Every injective right R-module is projective.*
(f) *Every projective right R-module is injective.*

Example 0.6.3. (i) Every artinian ring with a simple socle is QF (this follows from Theorem 0.6.10 below). For example, if F is a field, then $F[x]/\langle x^n\rangle$ is QF.
(ii) Any semisimple ring is QF.
(iii) $R_1 \times \cdots \times R_t$ is QF $\iff$ each R_i is QF.
(iv) If R is a Dedekind domain, then R/I is QF for all $0 \neq I \trianglelefteq R$.
(v) If G is a finite group and R is QF, then the group ring RG is QF as well.

Among the important examples of QF rings are Frobenius algebras, which are defined as follows.

Definition 0.6.4. *Let F be a field. A* Frobenius algebra *over F is a f.d. algebra A admitting an F-linear map $t : A \to F$ such that the bilinear form $b : A \times A \to F$ defined by $b(x, y) = t(xy)$ is nondegenerate.*

Proposition 0.6.5. *Any Frobenius algebra over a field F is QF.*

Proof (sketch). The set $\operatorname{Hom}_F(A, F)$ has an (A, A)-bimodule structure (see part (ii) of the next example) and the map $x \mapsto t(x \cdot \_)$ induces an isomorphism of right A-modules $A \to \operatorname{Hom}_F(A, F)$. Since the r.h.s. is well-known to be an injective A-module, A_A is injective. □

Example 0.6.6. Let K be a field.
(i) Let G be a finite group. Then KG is a Frobenius algebra; define $t : KG \to K$ by $t(\sum_g a_g g) = a_{1_G}$.
(ii) Let A be a f.d. K-algebra and let $A' = \operatorname{Hom}_K(A, K)$. Then A' can be made into an (A, A)-bimodule by letting $(a \cdot f)(b) = f(ba)$ and $(f \cdot a)(b) = f(ab)$ for all $f \in A'$ and $a, b \in A$. Let $B = \{\left[\begin{smallmatrix} a & f \\ 0 & a \end{smallmatrix}\right] \mid a \in A, f \in A'\}$. Then B is a Frobenius algebra. Indeed, let $t : B \to K$ be defined by $t(\left[\begin{smallmatrix} a & f \\ 0 & a \end{smallmatrix}\right]) = f(a)$. This example demonstrates that any f.d. algebra is an epimorphic image of a Frobenius algebra.

In order to proceed, recall that a ring R is called *right Kasch* if R_R contains a copy of every simple right R-module. In addition, for all $M \in$ Mod-R, let $M^* := \operatorname{Hom}_R(M, R)$ and observe that M^* can be considered as a left R-module by setting

$$(r \cdot f)(m) = r \cdot f(m) \qquad \forall f \in M^*,\, m \in M,\, r \in R$$

The map $* : \text{Mod-}R \to R\text{-Mod}$ is a contravariant functor and similarly, abusing the notation, we get a contravariant functor $* : R\text{-Mod} \to \text{Mod-}R$ which is given by $M^* := \text{Hom}_R(M, R)$ but with M being a left R-module. For every left or right module M, there is a natural homomorphism $\omega_M : M \to M^{**}$ given by $(\omega_M x)f = f(x)$. The module M is called *reflexive* if ω_M is a bijection.

PROPOSITION 0.6.7. *Assume R is QF. Then:*

(i) *Any right R-module embeds in a free module.*
(ii) *Any f.g. R-module is reflexive (i.e. the map $\omega_M : M \to M^{**}$ is an isomorphism).*
(iii) *An R-module M is f.g. if and only if M^* is f.g.*

COROLLARY 0.6.8. *If R is QF, then R_R is a cogenerator and R is right (and also left) Kasch.*

PROOF. That R_R is a cogenerator follows from Proposition 0.4.8(d) and part (i) of the previous proposition. In addition, Proposition 0.4.8(e) implies that any cogenerator contains a copy of any simple right R-module, hence R_R is right Kasch. □

The following theorems provide additional characterizations of QF rings.

THEOREM 0.6.9 (Dieudonne). *An artinian ring R is QF if and only if for any (left or right) simple R-module M, M^* is simple or the zero module. In this case M^* is actually a simple R-module.*

THEOREM 0.6.10. *An artinian ring R is QF if and only if it is Kasch and for every primitive idempotent $e \in \text{E}(R)$, the socle of eR and Re is simple.*

REMARK 0.6.11. Some of the results just stated follow from the fact that if R is QF, then $*$ induces a *duality* between the categories of f.g. right R-modules and f.g. left R-modules. See [**58**] for more details.

The previous results imply that a QF ring R is a *right cogenerator ring*, i.e. R_R is a right cogenerator of Mod-R. This leads to the following definition.

DEFINITION 0.6.12. *A* right pseudo-Frobenius *(abbrev.: PF) ring is a right self-injective right cogenerator ring.*

The following theorem, again due to several authors, provides equivalent definitions. In contrast to being QF, being PF is not a right-left symmetric property (this was open for some while, though). We also note that it is also common to define right PF using condition (e) below.

THEOREM 0.6.13. *Let R be a ring. Then the following are equivalent:*

(a) *R is right PF (i.e. R is right self-injective and right cogenerator).*
(b) *R is right self-injective and right Kasch.*
(c) *R is right self-injective and* $\text{ann}^r \, \text{ann}^\ell A = A$ *for any right ideal $A \le R_R$.*
(d) *R is right self-injective, semiperfect (or semilocal) and* $\text{soc}(R_R) \subseteq_e R_R$.
(e) *Any faithful right R-module is a generator.*
(f) *R_R is a cogenerator and R is left Kasch.*

0.7. Uniform Dimension

In this section, we recall the definition and basic properties of uniform dimension. This theory, due to A. Goldie, is discussed in detail in [**58**, §6].

We begin by recalling the basics of essential extensions and injective hulls. Throughout, R is a ring.

Let M be a right R-module. A submodule $N \leq M$ is said to be *essential* in M if $N \cap N' \neq 0$ for any $0 \neq N' \leq M$. This is equivalent to saying that for any $0 \neq m \in M$, there is $r \in R$ such that $0 \neq mr \in N$. In this case, we also say that M is an *essential extension* of N.

The following facts are easy to prove. They will be used freely henceforth.

Proposition 0.7.1. *Let M, N, K be right R-module.*

(i) *If $M \leq N \leq K$, then $M \subseteq_e K \iff M \subseteq_e N$ and $N \subseteq_e K$.*
(ii) *If $M, N \subseteq_e K$, then $M \cap N \subseteq_e K$.*
(iii) *Let $f \in \mathrm{Hom}_R(M, N)$ and assume $K \subseteq_e N$. Then $f^{-1}(K) \subseteq_e M$.*

Every right module M admits a maximal essential extension, denoted $E(M)$; the maximality means that $E(M)$ does not have non-trivial essential extensions. The extension $M \hookrightarrow E(M)$ is unique up to isomorphism in the sense that if $M \hookrightarrow M'$ is another maximal essential extension, then there is a module isomorphism $f : M' \to E(M)$ such that $f|_M = \mathrm{id}_M$ (i.e. $E(M) \cong M'$ as *extensions of* M). It turns out that $E(M)$ is injective and can also be characterized as

- the *smallest* injective module containing M, *or*
- the only injective essential extension of M.

Thus, $E(M)$ is usually called the *injective hull* or *injective envelope* of M. Note that while $E(M)$ is uniquely determined to isomorphism, the map sending M to $E(M)$ is *not* functorial. This is despite the fact that any homomorphism between two right R-modules $M \to M'$ extends to a (not necessarily unique) homomorphism $E(M) \to E(M')$ (this follows from the injectivity of $E(M')$).

The fact that homomorphisms between modules extend to their injective hulls has the following useful consequence.

Proposition 0.7.2. *If M and M' are two R-modules and $f : M \to M'$ is a monomorphism, then any homomorphism $\hat{f} : E(M) \to E(M')$ extending f is also a monomorphism. Furthermore, if $f(M) \subseteq_e M'$, then $\hat{f}$ is an isomorphism. In particular, if $M \subseteq M'$, then $E(M)$ can be understood as a submodule of $E(M')$ and if $M \subseteq_e M'$, then $E(M) = E(M')$.*

Proof. The assumptions imply $\ker \hat{f} \cap M = 0$. Since $M \subseteq_e E(M)$, this means $\ker \hat{f} = 0$, hence $\hat{f}$ is a monomorphism. If $f(M) \subseteq_e M'$, then $f(M) \subseteq_e E(M')$ (because $M' \subseteq_e E(M')$). Thus, $\hat{f}(E(M)) \subseteq_e E(M')$ (since the l.h.s. contains $f(M)$). As $E(M)$ is injective, the embedding $\hat{f}$ must split, so there is $N \leq E(M')$ with $E(M') = N \oplus \hat{f}(E(M))$. But then $N \cap \hat{f}(E(M)) = 0$, so we must have $N = 0$ (since $\hat{f}(E(M)) \subseteq_e E(M')$) and this means $\hat{f}$ is surjective. □

Definition 0.7.3. *Let M be a right R-module. The* uniform dimension *of M, denoted* $\mathrm{u.dim}\, M$*, is defined to be the maximal $n \in \mathbb{N} \cup \{0\}$ (or ∞) such that M contains a direct sum of n nonzero right R-modules.*

Example 0.7.4. (i) $\mathrm{u.dim}\, M = 0$ if and only if $M = 0$.

(ii) Assume $M \neq 0$. Then $\mathrm{u.dim}\, M = 1$ if and only if for any two submodules $0 \neq N, N' \leq M$, we have $N \cap N' \neq 0$ (otherwise, $N \oplus N' \leq M$, implying $\mathrm{u.dim}\, M \geq 2$). Such modules are called *uniform*. Uniform modules have the property that any nonzero submodule is essential. For example, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ are uniform $\mathbb{Z}$-modules.

(iii) The uniform dimension of the $\mathbb{Z}$-module $\mathbb{Z}/6$ is 2. This holds since $\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$.

(iv) If R is a field, then the unform dimension of a f.g. R-module is its dimension (so uniform dimension can be considered as a generalization of the dimension).

(v) Generalizing (iv): If M is a f.g. semisimple module, then $\mathrm{u.dim}\, M = \mathrm{length}(M)$. (However, this fails for non-semisimple modules).

(vi) Also generalizing (iv): Assume R is an integral domain and let F be its fraction field. The uniform dimension of a *torsion-free* R-module M is just $\dim_F(M \otimes_R F)$.

(vii) Let F be a field and let $R = F\langle x, y\rangle$ (R is the free F-algebra generated by two non-commuting indeterminates $\{x, y\}$). Then $\mathrm{u.dim}\, R_R = \infty$. Indeed, $R_R \supseteq \bigoplus_{n=0}^{\infty} x^n yR$.

Remark 0.7.5. Caution: $\mathrm{u.dim}\, R_R$ might be different from $\mathrm{u.dim}\, {}_RR$

The following proposition is very useful for determining what is the uniform dimension.

Proposition 0.7.6. *Let M be a right R-module and let $n \in \mathbb{N}$. Then:*

(i) $\mathrm{u.dim}\, M = n \iff$ *there are* uniform *submodules $A_1, \dots, A_n \leq M$ such that $A_1 \oplus \cdots \oplus A_n \subseteq_e M$.*
(ii) $\mathrm{u.dim}\, M = \infty \iff$ *there are nonzero submodules $A_1, A_2, \dots \leq M$ such that $A_1 \oplus A_2 \oplus \ldots \subseteq M$.*

We finish this section by stating several more facts.

Proposition 0.7.7. *Let M, N be R-modules and let $M' \leq M$. Then:*

(i) $\mathrm{u.dim}\, M' \leq \mathrm{u.dim}\, M$. *Equality holds when $M' \subseteq_e M$. In particular,* $\dim M = \mathrm{u.dim}\, E(M)$.
(ii) *If* $\mathrm{u.dim}\, M' = \mathrm{u.dim}\, M < \infty$*, then $M' \subseteq_e M$.*
(iii) $\mathrm{u.dim}\, M \leq \mathrm{u.dim}\, M' + \mathrm{u.dim}(M/M')$ *(with the standard conventions about adding ∞).*
(iv) $\mathrm{u.dim}(M \oplus N) = \mathrm{u.dim}\, M + \mathrm{u.dim}\, N$ *(with the standard conventions about adding ∞).*

0.8. Classical Rings of Fractions

This section briefly surveys the theory of classical non-commutative localization, which is due to O. Ore, A. Goldie and others. For an extensive discussion and proofs see [**58**, §10–11] or [**80**, §3.1].

Let R be a ring and let S be a submonoid of $(R, \cdot)$. A *classical right ring of fractions of R* w.r.t. S is a ring R' together with a ring homomorphism $\varphi : R \to R'$ such that the following properties are satisfied:

(1) $\varphi(s)$ is invertible in R' for all $s \in S$.
(2) Every element of R' can be written as $\varphi(r)\varphi(s)^{-1}$ for some $r \in R$ and $s \in S$.
(3) $\ker \varphi = \{r \in R \,|\, \exists s \in S : rs = 0\}$

In this case, the ring extension $\varphi : R \to R'$ is uniquely determined to isomorphism (of extension of R) and we write $R' = RS^{-1}$. (For *classical left rings of fractions*, the notation is $S^{-1}R$). The map φ is often omitted from the notation and rs^{-1} is used to denote $\varphi(r)\varphi(s)^{-1}$. Observe that if R is commutative, then RS^{-1} is precisely the usual localization of R at S. However, in contrary to the commutative case, the ring RS^{-1} need not exist. Sufficient and necessary conditions for its existence are provided in the following theorem.

Theorem 0.8.1. *Let R be a ring and let S be a multiplicative submonoid of R. The ring RS^{-1} exists if and only if the following conditions are satisfied:*

(1) *For all $s \in S$ and $r \in R$, $sR \cap rS \neq \phi$.*

(2) *For all* $s \in S$ *and* $r \in R$ *such that* $sr = 0$*, there exists* $s' \in S$ *such that* $rs' = 0$.

Proof (sketch). We will briefly describe an explicit construction of RS^{-1}. The elements of RS^{-1} will be the set of pairs $(r,s) \in R \times S$ considered modulo the following equivalence relation: $(r,s) \sim (r',s')$ if there exists $u, u' \in R$ such that

$$su = s'u' \in S \qquad \text{and} \qquad ru = r'u' \ .$$

The equivalence class of (r,s) will be denoted by rs^{-1}. (Using this notation, the equivalence relation just means $rs^{-1} = (ru)(su)^{-1} = (r'u')(s'u')^{-1} = r's'^{-1}$).

To define addition, note that every two fractions $r_1 s_1^{-1}, r_2 s_2^{-1} \in RS^{-1}$ can be changed to have the same denominator. Indeed, by condition (1), $s_1 S \cap s_2 R \neq \phi$, so there are $s \in S$ and $r \in R$ such that $s_1 s = s_2 r$ (this implies that $s_2 r \in S$). We now have

$$r_1 s_1^{-1} = (r_1 s)(s_1 s)^{-1}$$
$$r_2 s_2^{-1} = (r_2 r)(s_2 r)^{-1} = (r_2 r)(s_1 s)^{-1} \ .$$

The sum of $r_1 s_1^{-1}$ and $r_2 s_2^{-1}$ is thus defined to be $(r_1 s + r_2 r)(s_1 s)^{-1}$.

The definition of product of $r_1 s_1^{-1}, r_2 s_2^{-1}$ uses a similar idea. By (1), there are $r \in R$ and $s \in S$ such that $s_1 r = r_2 s$ (this implies $s_1 r \in S$). We now define

$$(r_1 s_1^{-1})(r_2 s_2^{-1}) = (r_1 r)(s_2 s)^{-1} \ .$$

The reason for this is that $s_1 r = r_2 s$ means that $rs^{-1} = s_1^{-1} r_2$ (if the r.h.s. was defined) and then $(r_1 s_1^{-1})(r_2 s_2^{-1})$ should be $r_1(s_1^{-1} r_2) s_2^{-1} = r_1 r s^{-1} s_2^{-1} = (r_1 r)(s_2 s)^{-1}$.

We leave it to the reader to check that the addition and multiplication are well-defined and make RS^{-1} into a ring whose unity and zero elemnts are $1_R 1_R^{-1}$ and $0_R 1_R^{-1}$, respectively.

We now define $\varphi : R \to RS^{-1}$ by $\varphi(r) = r 1_R^{-1}$. Then $r \in \ker \varphi$ if and only if $r 1_R^{-1} = 0_R 1_R^{-1}$. That is, there are $u, u' \in R$ such that $1_R u = 1_R u' \in S$ and $ru = 0u'$. These conditions are equivalent to $u = u' \in S$ and $ru = 0$, hence $\ker \varphi = \{r \in R \,|\, \exists s \in S : rs = 0\}$. □

Definition 0.8.2. *A multiplicative submonoid of a ring* R *which satisfies conditions (1) and (2) of Theorem 0.8.1 is called a* right denominator set. *(Left denominator sets are defined in a similar manner.)*

Proposition 0.8.3. *If* S *is a right and left denominator set in* R*, then the rings* RS^{-1} *and* $S^{-1}R$ *are isomorphic as extension of* R*.*

Example 0.8.4. Any multiplicative submonoid of R which is contained in $\mathrm{Cent}(R)$ is a right and left denominator set (check!).

Recall that an element $r \in R$ is called *regular* if $\mathrm{ann}^r r = 0$ and $\mathrm{ann}^\ell r = 0$. It is easy to check that if S is a right denominator set in R, then the map $R \to RS^{-1}$ is injective if and only if S consists of regular elements. If the set of *all* regular elements is a right denominator set, then R is called *right Ore*. In this case, we let $Q^r_{\mathrm{cl}}(R)$ denote the ring obtained by localizing at this set. The ring $Q^r_{\mathrm{cl}}(R)$, when it exists, is called the *classical right ring of fractions of* R. Its left analogue is denoted by $Q^\ell_{\mathrm{cl}}(R)$. When both $Q^r_{\mathrm{cl}}(R)$ and $Q^\ell_{\mathrm{cl}}(R)$ exist, they coincide.

Example 0.8.5. (i) If all regular elements in R are invertible, then $Q^r_{\mathrm{cl}}(R)$ and $Q^\ell_{\mathrm{cl}}(R)$ exist and coincide with R. Such rings are called *classical*. For example, any right or left artinian ring R is classical. It also turns out that the rings $Q^r_{\mathrm{cl}}(R)$ and $Q^\ell_{\mathrm{cl}}(R)$ are classical (when they exist), so $Q^r_{\mathrm{cl}}(Q^r_{\mathrm{cl}}(R)) = Q^\ell_{\mathrm{cl}}(Q^r_{\mathrm{cl}}(R)) = Q^r_{\mathrm{cl}}(R)$ and a similar statement holds for $Q^\ell_{\mathrm{cl}}(R)$.

(ii) Any commutative ring R is right and left Ore. In this case, the ring $Q^r_{\mathrm{cl}}(R) = Q^\ell_{\mathrm{cl}}(R)$ is sometimes called the *total ring of fractions* of R. In particular, if R an integral domain, then $Q^r_{\mathrm{cl}}(R)$ is the faction field of R.

(iii) By definition, a domain R is right Ore when $R\setminus\{0\}$ is a right denominator set. A straightforward argument shows that this is equivalent to $aR\cap bR \neq \{0\}$ for all $0 \neq a, b \in R$.

(iv) Let F be a field. The ring $R = F\langle x, y\rangle$ is a domain, but it is not right nor left Ore since $xR \cap yR = 0$ and $Rx \cap Ry = 0$. In particular, $Q^r_{\mathrm{cl}}(R)$ and $Q^\ell_{\mathrm{cl}}(R)$ do not exist.

The following two theorems, which are due to Goldie, ensure that certain rings are right Ore and have fairly nice classical ring of fractions.

THEOREM 0.8.6 (Goldie). *Let R be a* domain. *Then R is right Ore* $\iff$ $\mathrm{u.\,dim}\, R_R < \infty \iff \mathrm{u.\,dim}\, R_R = 1 \iff$ *$aR \cap bR \neq 0$ for any $0 \neq a, b \in R$* $\iff$ *$Q^r_{\mathrm{cl}}(R)$ exists and it is a division ring.*

EXAMPLE 0.8.7. (i) It turns out that any right noetherian domain and any PI domain[8] is right Ore. (Indeed, a right noetherian ring R must have $\mathrm{u.\,dim}\, R_R < \infty$. The PI case follows from a result of Jategaonkar asserting that if R is a domain and $0 \neq a, b \in R$ satisfy $aR \cap bR = 0$, then the ring spanned by a, b and $\mathrm{Cent}(R)$ is a free $\mathrm{Cent}(R)$-algebra with two (non-commuting) generators and hence R cannot be PI.)

(ii) Let R be a right Ore *domain*, let $\sigma : R \to R$ be an injective ring automorphism an let $\delta : R \to R$ be a derivation. Then the twisted polynomial ring $R[x;\sigma]$ and the differential polynomial ring $R[x;\delta]$ are also right Ore domains.[9]

DEFINITION 0.8.8. *A ring R is called* right Goldie *if R has ACC on right annihilators and* $\mathrm{u.\,dim}\, R_R < \infty$.

EXAMPLE 0.8.9. Any right noetherian ring is right Goldie.

THEOREM 0.8.10 (Goldie). *Let R be a ring. Then the following conditions are equivalent.*

(a) *R is a semiprime right Goldie ring.*
(b) *$Q^r_{\mathrm{cl}}(R)$ exists and it is a semisimple ring.*

REMARK 0.8.11. Although right noetherian rings are right Goldie, there are right noetherian rings which are not right Ore.

Let us go back to the general case of a ring R and right denominator set S. We finish this section by noting that right R-modules can also be localized at S. Indeed, for any right R-module M, one can construct an RS^{-1}-module MS^{-1} by mimicking the construction in the proof of Theorem 0.8.1. That is, the elements of MS^{-1} are pairs $(m, s) \in M \times S$, considered up to the following equivalence relation: $(m, s) \sim (m', s')$ if there are $u, u' \in R$ such that $su = su' \in S$ and $mu = mu'$. We let ms^{-1} stand for the equivalence class of (m, s). The rest of the details are left to the reader. The following theorem summarizes some of the properties of MS^{-1}.

THEOREM 0.8.12. *Let R be a ring, let S be a right denominator set and let M be a right R-module. Then:*

[8] A ring R is called a *PI ring* (which stands for *polynomial identity ring*) if there exists a nonzero polynomial $f(x_1, \dots, x_r) \in \mathbb{Z}\langle x_1, x_2, \dots\rangle$ whose (nonzero) coefficients are either 1 or -1 such that $f(r_1, \dots, r_t) = 0$ for all $r_1, \dots, r_t \in R$.

[9] The ring $R[x;\sigma]$ is defined to be the ring of formal finite sums $\sum_i r_i x^i$ with $r_i \in R$ subject to the relation $xr = \sigma(r)x$ for all $r \in R$. The ring $R[x;\delta]$ is defined in the same manner except the relation which is $xr = rx + \delta(r)$.

(i) *MS^{-1} is a right RS^{-1}-module.*
(ii) *The map $M \to MS^{-1}$ defined by $m \mapsto m1_R^{-1}$ is an R-module homomorphism with kernel $\{m \in M \mid \exists s \in S : ms = 0\}$.*
(iii) *The map $M \mapsto MS^{-1}$ from* Mod-R *to* Mod-RS^{-1} *is functorial; for $f \in \mathrm{Hom}_R(M,N)$, define $fS^{-1} \in \mathrm{Hom}_{RS^{-1}}(MS^{-1}, NS^{-1})$ by*

$$(fS^{-1})(ms^{-1}) = (fm)s^{-1} \qquad \forall\ ms^{-1} \in MS^{-1}\ .$$

(iv) *The functor $M \mapsto MS^{-1}$ is exact.*
(v) *There is a natural isomorphism between $M \otimes_R RS^{-1}$ and MS^{-1}. It is given by $m \otimes_R (rs^{-1}) \mapsto (mr)s^{-1}$.*
(vi) *The ring RS^{-1} is flat as a right R-module.*
(vii) *If the map $M \to MS^{-1}$ of (ii) is injective, then*

$$\mathrm{u.dim}\, M_R = \mathrm{u.dim}\, MS^{-1}_R = \mathrm{u.dim}\, MS^{-1}_{RS^{-1}}\ .$$

0.9. Rational Extensions

This section is devoted to rational extensions and merely serves as preparation for the next section about general rings of quotients. As this is somewhat related, we also consider nonsingular modules at the end of the section. For more details and proofs, see [**58**, §7–8].

Throughout, R is a ring. Let M be a right R-module. A submodule $N \leq M$ is *dense* in M, denoted $N \subseteq_d M$, if for all $x, y \in M$ with $x \neq 0$, there is $r \in R$ such that $xr \neq 0$ and $yr \in N$. In this case, we also say that M is a *rational extension* of N. Observe that if we take $x = y$ in the definition, we get the definition of an essential submodule. Thus

$$N \subseteq_d M \quad \Longrightarrow \quad N \subseteq_e M\ .$$

The converse is not true, though.

The definition of density can be also be phrased using the following notation: For all $y \in M$, let

$$y^{-1}N = \{r \in R \,:\, yr \in N\}\ .$$

Then $y^{-1}N$ is a right ideal and $N \subseteq_d M$ if and only if $x \cdot y^{-1}N \neq 0$ for all $x, y \in M$ with $x \neq 0$.

Example 0.9.1. (i) When considered as $\mathbb{Z}$-modules, $\mathbb{Z} \subseteq_d \mathbb{Q}$ (straightforward).

(ii) More generally, if R is a domain and M is a torsion-free right R-module, then $N \subseteq_d M$ if and only if $N \subseteq_e M$. Indeed, assume that $N \subseteq_e M$ and $x, y \in M$ with $x \neq 0$. If $y = 0$, then $x1_R \neq 0$ and $y1_R \in N$. Otherwise, there is $r \in R$ such that $0 \neq yr \in N$. This implies $r \neq 0$, so $xr \neq 0$ since M is torsion-free.

(iii) Let p be a prime number and consider the $\mathbb{Z}$-modules $M = \mathbb{Z}/p^2\mathbb{Z}$ and $N = p\mathbb{Z}/p^2\mathbb{Z}$. Then $N \subseteq_e M$, but $N \nsubseteq_d M$. To see the latter, take $x = p + p^2\mathbb{Z}$ and $y = 1 + p^2\mathbb{Z}$. It is easy to see that if $yn \in N$ for some $n \in \mathbb{Z}$, then $xn = 0$, hence $N \nsubseteq_d M$.

(iv) Let J be a *two-sided* ideal of R. Consider J as a submodule of R_R and observe that for all $y \in R$, $y^{-1}J \supseteq J$. Thus, $x \cdot y^{-1}J = 0$ implies $xJ = 0$, i.e. $x \in \mathrm{ann}^\ell J$. Thus, if $\mathrm{ann}^\ell J = 0$, then $J_R \subseteq_d R_R$. The converse is also true, for if $xJ = 0$, then $x \cdot 1_R^{-1}J = xJ = 0$. We thus conclude that $\mathrm{ann}^\ell J = 0 \iff J_R \subseteq_d R_R$.

(v) As a consequence of (iv) we get: If $a \in \mathrm{Cent}(R)$ and $\mathrm{ann}\, a = 0$, then $aR_R \subseteq_d R_R$.

(vi) Let $0 \neq z \in R$. Then $\mathrm{ann}^r z \nsubseteq_d R_R$. Indeed, $z \cdot 1_R^{-1}(\mathrm{ann}^r z) = z \cdot \mathrm{ann}^r z = 0$.

The following proposition presents equivalent definitions for density.

PROPOSITION 0.9.2. *Let M be a right R-module admitting a submodule $N \le M$. The following are equivalent:*

(a) $N \subseteq_d M$.
(b) $\mathrm{Hom}(M/N, E(M)) = 0$.
(c) *For any submodule $N \le N' \le M$, $\mathrm{Hom}(N'/N, M) = 0$.*

PROPOSITION 0.9.3. *Let M, N, K be right R-modules.*

(i) *If $M \le N \le K$, then $M \subseteq_d K \iff M \subseteq_d N$ and $N \subseteq_d K$.*
(ii) *If $M, N \subseteq_d K$, then $M \cap N \subseteq_d K$.*
(iii) *Let $f \in \mathrm{Hom}_R(M, N)$ and assume $K \subseteq_d M$, then $f^{-1}(K) \subseteq_d M$.*
(iv) *If $N \subseteq_d M$ and $y \in M$, then $y^{-1}N \subseteq_d R_R$ (this is a special case of (iii)).*

PROOF. Parts (i)–(iii) are routine. To see (iv), define $f \in \mathrm{Hom}_R(R_R, M)$ by $f(r) = yr$. Then $y^{-1}N = f^{-1}(N)$, hence $y^{-1}N \subseteq_d R_R$ by (iii). □

Let M be a right R-module. The module M is called *rationally closed* if it has no proper rational extension. It turns out that every module M is dense in some rationally closed module, denoted $\widetilde{E}(M)$. The module $\widetilde{E}(M)$ is called the *rational hull* of M and it is unique in sense that if M' is another rationally closed rational extension of M, then there is an isomorphism $f : \widetilde{E}(M) \to M'$ such that $f|_M = \mathrm{id}_M$. Moreover, in contrast to to injective hulls, the isomorphism f is uniquely determined. The module $\widetilde{E}(M)$ can be identified with the following submodule of $E(M)$:

$$\widetilde{E}(M) = \{x \in E(M) \,|\, \forall h \in \mathrm{End}_R(E(M)) : h(M) = 0 \Longrightarrow h(x) = 0\} \ . \tag{1}$$

In fact, this is the only way to embed $\widetilde{E}(M)$ in $E(M)$ such that M is fixed.

We should point out that homomorphisms between modules need not extend to their rational hulls.

EXAMPLE 0.9.4. (i) If M is injective or rationally closed, then $\widetilde{E}(M) = M$.

(ii) $\widetilde{E}(\mathbb{Z}_\mathbb{Z}) = \mathbb{Q}_\mathbb{Z}$. This can be checked by showing that $E(\mathbb{Z}_\mathbb{Z}) = \mathbb{Q}_\mathbb{Z}$ and then using the fact that $\mathbb{Z} \subseteq_d \mathbb{Q}$.

(iii) Consider the $\mathbb{Z}$-module $\mathbb{Z}/p$ (p is a prime number). Then $\widetilde{E}(\mathbb{Z}/p) = \mathbb{Z}/p$, but $E(\mathbb{Z}/p) \cong \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. In particular, $\widetilde{E}(\mathbb{Z}/p) \ncong E(\mathbb{Z}/p)$. This can be shown using (1). Also note that the homomorphism $\mathbb{Z} \to \mathbb{Z}/p$ give by $n \mapsto n + p\mathbb{Z}$ cannot be extended to a homomorphism from $\widetilde{E}(\mathbb{Z}) = \mathbb{Q}$ to $\widetilde{E}(\mathbb{Z}/p) = \mathbb{Z}/p$.

(iv) Let M be a f.g. right R-module that contains a copy of any simple right R-module. Then $\widetilde{E}(M) = M$. Indeed, assume by contradiction that $M \subseteq_d M'$ with M' strictly bigger than M. Without loss of generality, M' is also finitely generated. Thus, M is contained in a maximal submodule M'' of M'. Since M'/M'' is simple, it embeds in M. Thus, $\mathrm{Hom}_R(M'/M, M) \neq 0$, implying $\mathrm{Hom}(M'/M, M') \neq 0$, which contradicts Proposition 0.9.2(b).

(v) As a special case of (iv), we get: If R is *right Kasch* (see section 0.6), then $\widetilde{E}(R_R) = R_R$.

There is an important family of modules for which essential and rational extensions are the same thing. These are the nonsingular modules, which are defined as follows.

DEFINITION 0.9.5. *For any right R-module M, define the* singular radical *of M by*

$$\mathcal{Z}(M) = \{m \in M \ : \ \mathrm{ann}_R\, m \subseteq_e R_R\} \ .$$

Then $\mathcal{Z}(M)$ is a submodule of M and M is called nonsingular *if $\mathcal{Z}(M) = 0$. The ring R is called* right nonsingular *if the module R_R is nonsingular.*

EXAMPLE 0.9.6. (i) A right $\mathbb{Z}$-module is nonsingular if and only if it is torsion-free. (More generally, this holds for any right Ore domain.)

(ii) Any simple ring is right and left nonsingular. This follows from the fact that $\mathcal{Z}(R_R)$ is always a proper ideal of R.

PROPOSITION 0.9.7. *Let $N \leq M$ be right R-modules. Then:*

(i) *M is nonsingular $\implies$ N is nonsingular. The converse holds when $N \subseteq_e M$.*
(ii) *If M is nonsingular, then $\widetilde{E}(M) = E(M)$.*
(iii) *If at least one of M, N is nonsingular, then $N \subseteq_d M \iff N \subseteq_e M$.*

0.10. General Rings of Quotients

In this last section we recall the basics of general rings of quotients. The results stated in this section are the combined work of several authors, including Utumi, Osofsky, Johnson, Gabriel and others. For an extensive discussion see [**58**, §13] and also [**80**, §3.4].

Let R be a ring. A *general right ring of quotients* (or just a *right quotient ring*) of R is a ring Q containing R such that $R_R \subseteq_d Q_R$. As with rational extensions, it turns out that any ring R admits a unique *maximal right quotient ring*, denoted $Q^r_{\max}(R)$. This is stated in the following (highly non-trivial) theorem.

THEOREM 0.10.1. *Let R be a ring. Then there exists a right quotient ring of R, denoted $Q^r_{\max}(R)$, such that:*

(i) *For any right quotient ring of R, Q, there exists a unique ring homomorphism $Q \to Q^r_{\max}(R)$ that fixes R.*
(ii) *Any proper ring extension of $Q^r_{\max}(R)$ is not a right quotient ring of R.*
(iii) *$Q^r_{\max}(R)_R \cong \widetilde{E}(R_R)$.*

PROOF (SKETCH). There are two common ways to construct $Q^r_{\max}(R)$. We will briefly present both of them, but not prove that they satisfy (i)–(iii).

The first way is very short but less explicit. Let $I := E(R_R)$ and $H = \mathrm{End}(I_R)$. Then I is a left H-module. Define $Q^r_{\max}(R) = \mathrm{End}({}_H I)$ and observe that R embeds in $Q^r_{\max}(R)$ by $r \mapsto [i \mapsto ir] \in \mathrm{End}({}_H I)$ (here i is an element of I). It can be shown that $Q^r_{\max}(R)$ satisfies the conditions (i)–(iii).

The second way is more explicit, but more tedious. Consider pairs (A, f) such that A is a dense right ideal of R and $f \in \mathrm{Hom}_R(A, R_R)$. We define an equivalence relation on the set of such pairs by $(A, f) \sim (B, g) \iff f|_{A\cap B} = g|_{A\cap B}$. Let $[A, f]$ stand for the equivalence class of (A, f). We define $Q^r_{\max}(R)$ to be the set of equivalence classes $[A, f]$ and embed R in $Q^r_{\max}(R)$ by sending $r \in R$ to $[R, x \mapsto rx] \in Q^r_{\max}(R)$. Addition and multiplication in $Q^r_{\max}(R)$ are defined as follows: Let $[A, f], [B, g] \in Q^r_{\max}(R)$. Then by Proposition 0.9.3(ii)–(iii), $A \cap B$ and $g^{-1}(A) \cap B$ are dense in R_R. Using this, we define

$$[A, f] + [B, g] = [A \cap B, f|_{A\cap B} + g|_{A\cap B}] \ ,$$
$$[A, f] \cdot [B, g] = [g^{-1}(A) \cap B, f \circ g] \ .$$

We can now explain why (iii) holds. Let $m \in \widetilde{E}(M)$ and consider the map $f_m : R_R \to \widetilde{E}(M)$ given by $f_m(r) = mr$. Then by Proposition 0.9.3(iii), $f_m^{-1}(M) \subseteq_d R_R$. Thus, $[f_m^{-1}(M), f|_{f_m^{-1}(M)}] \in Q^r_{\max}(R)$. It is not hard to see that the map sending $m \in \widetilde{E}(M)$ to $[f_m^{-1}(M), f|_{f_m^{-1}(M)}]$ is an injective homomorphism of right R-modules, which also turns out to be surjective, thus implying (iii). □

The ring $Q^r_{\max}(R)$ can also be characterized (up to isomorphism of extensions of R) as:

(1) the only ring $Q \supseteq R$ such that $Q_R \cong \widetilde{E}(R)_R$;
(2) the only right quotient ring of R, Q, satisfying $Q^r_{\max}(Q) = Q$.

If Q satisfies any of these conditions then there exists a unique isomorphism $Q \to Q^r_{\max}(R)$ fixing R pointwise. Note that both conditions allow an "easy" verification that a given ring extension of R is $Q^r_{\max}(R)$. Condition (2) also implies the following nice corollary.

COROLLARY 0.10.2. *Any ring automorphism of R admits a unique extension into an automorphism of $Q^r_{\max}(R)$.*

PROOF (SKETCH). Let σ be an automorphism of R. Then $Q^r_{\max}(R)$ can be considered a ring extension of R via $\sigma : R \to Q^r_{\max}(R)$ (rather than via $\mathrm{id}_R : R \to Q^r_{\max}(R)$). To avoid ambiguity, let us denote the extension of R obtained in this manner by $Q^r_{\max}(R)^\sigma$ (so $Q^r_{\max}(R)^\sigma = Q^r_{\max}(R)$ as rings, but R embeds in $Q^r_{\max}(R)^\sigma$ via σ). It is routine to verify that $R_R \subseteq_d Q^r_{\max}(R)^\sigma_R$, hence $Q^r_{\max}(R)^\sigma$ is a right quotient ring of R. In addition, since $Q^r_{\max}(R)^\sigma \cong Q^r_{\max}(R)$ as rings, $Q^r_{\max}(Q^r_{\max}(R)^\sigma) = Q^r_{\max}(R)^\sigma$. Thus, by (2) above, $Q^r_{\max}(R)^\sigma$ is also a maximal right quotient ring of R, so there is a unique isomorphism of extensions of R, from $Q^r_{\max}(R)$ to $Q^r_{\max}(R)^\sigma$. When understood as a map from $Q^r_{\max}(R)$ to itself, this isomorphism is an automorphism of $Q^r_{\max}(R)$ extending σ. □

One can also define *left quotient rings* and discuss the maximal left quotient ring of R, denoted $Q^\ell_{\max}(R)$. Note that in contrast to *classical rings of factions*, the left and right maximal rings of quotients might be non-isomorphic as extensions of R.

EXAMPLE 0.10.3. Let F be a field and let T_n be the ring of $n \times n$ upper-triangular matrices. Then $Q^\ell_{\max}(T_n) = Q^r_{\max}(T_n) = \mathrm{M}_n(F)$ (with T_n identified as a subring of $\mathrm{M}_n(F)$ in the standard way). To see this, it is enough to check that $\mathrm{M}_n(F)$ is a right and left quotient ring of T_n (which we leave to the reader) and that $Q^r_{\max}(\mathrm{M}_n(F)) = Q^\ell_{\max}(\mathrm{M}_n(F)) = \mathrm{M}_n(F)$, which follows from the fact that $\mathrm{M}_n(F)$ is left and right self injective and thus cannot have essential (not to say rational) extensions.

See [**58**, §13] for more explicit computations of maximal rings of quotients.

REMARK 0.10.4. If $Q^r_{\mathrm{cl}}(R)$ exists, then it is a right quotient ring of R and hence admits a unique embedding into $Q^r_{\max}(R)$. However, the latter can be strictly larger; for instance, in the previous example we have $Q^r_{\mathrm{cl}}(T_n) = T_n$ (since T_n is artinian), but $Q^r_{\max}(T_n) = \mathrm{M}_n(F)$. Nevertheless, if $Q^r_{\max}(Q^r_{\mathrm{cl}}(R)) = Q^r_{\mathrm{cl}}(R)$, then by (2) above $Q^r_{\mathrm{cl}}(R) = Q^r_{\max}(R)$. In particular, if R is a semiprime right Goldie ring, then $Q^r_{\mathrm{cl}}(R) = Q^r_{\max}(R)$ (by Theorem 0.8.10).

We finish with two strong structural results about $Q^r_{\max}(R)$, which are due to Johnson and Gabriel.

THEOREM 0.10.5 (Johnson). *$Q^r_{\max}(R)$ is von-Neumann regular*[10] $\iff$ *R is right nonsingular. In this case $Q^r_{\max}(R)$ is right self-injective.*

THEOREM 0.10.6 (Gabriel). *$Q^r_{\max}(R)$ is semisimple $\iff$ R is right nonsingular and* $\mathrm{u.dim}\, R < \infty$.

REMARK 0.10.7. The assumption that R is nonsingular and $\mathrm{u.dim}\, R < \infty$ implies that R is right Goldie. The converse holds when R is semiprime.

[10] A ring R is von-Neumann regular if for all $x \in R$ there is $y \in R$ such that $xyx = x$. For example, the endomorphism ring of an arbitrary vector space is von-Neumann regular.

CHAPTER 1

Semi-Invariant Subrings

Call a subring R_0 of a ring R *semi-invariant* if R_0 is the ring of invariants in R under some set of ring endomorphisms of some ring containing R. In this chapter, we study semi-invariant subrings of semiperfect rings and present applications to various areas such as Krull-Schmidt decompositions and representations of rings and monoids. The results of this chapter will form the the ring-theoretic infrastructure to Chapter 4.

Parts of this chapter can also be found in [**41**].

1.1. Preface

Let R be a ring and let $J = \mathrm{Jac}(R)$. The ring R is *semilocal* if R/J is semisimple. If in addition J is idempotent lifting, then R is called *semiperfect*. For a detailed discussion on semiperfect rings, see [**80**, §2.7] and [**9**]. Semiperfect rings play an important role in representation theory and module theory because of the Krull-Schmidt Theorem. Recall that an object A in an additive category $\mathscr{A}$ is said to have a *Krull-Schmidt decomposition* if it is a sum of (non-zero) indecomposable objects and any two such decompositions are the same up to isomorphism and reordering.

Theorem 1.1.1 (Krull-Schmidt, for Categories). *Let $\mathscr{A}$ be an additive category in which all idempotents split (e.g. an abelian category) and let $A \in \mathscr{A}$. If $\mathrm{End}_{\mathscr{A}}(A)$ is semiperfect, then A has a Krull-Schmidt decomposition and the endomorphism ring of any indecomposable summand of A is local.*

Generalizations of this theorem and counterexamples of some natural variations have been widely studied (e.g. [**35**],[**8**],[**3**],[**33**] and also [**32**]) and there has been considerable interest in finding rings over which all finitely presented modules have a Krull-Schmidt decomposition (e.g. [**92**, §6],[**19**],[**78**],[**79**],[**96**]; Theorem 1.8.3(iii) below generalizes all these references except the last).

Example 1.1.2. Semiperfect rings naturally occur upon taking completions:

(1) Let R be a semilocal ring and let $J = \mathrm{Jac}(R)$. Then the J-adic completion of R, $\varprojlim\{R/J^n\}_{n\in\mathbb{N}}$, is well known to be semiperfect. If the natural map $R \to \varprojlim\{R/J^n\}_{n\in\mathbb{N}}$ is an isomorphism, then R is called *complete semilocal*. Such rings (especially noetherian or with Jacobson radical f.g. as a right ideal) appear in various areas (e.g. [**63**], [**48**], [**92**, §6], [**79**]).

(2) Let R be a noetherian integral domain, let A be an R-algebra that is finitely generated as an R-module and let $P \in \mathrm{Spec}(R)$. Then the completion of A at P is semiperfect (and noetherian). (See [**72**, §6]; This assertion can also be shown using the results of this chapter).

Let R be any ring and let $R_0 \subseteq R$ be a subring.

(a) Call R_0 a *semi-invariant* subring if there is a ring $S \supseteq R$ and a set $\Sigma \subseteq \mathrm{End}(S)$ such that $R_0 = R^{\Sigma} := \{r \in R : \sigma(r) = r\ \forall \sigma \in \Sigma\}$ (elements of Σ are not required to be injective nor surjective). The *invariant* subrings of R are the subrings for which we can choose $S = R$.

(b) Call R_0 a *semi-centralizer* subring if there is a ring $S \supseteq R$ and a set $X \subseteq S$ such that $R_0 = \mathrm{Cent}_R(X) := \{r \in R : rx = xr \ \forall x \in X\}$. If we can choose $S = R$, then R_0 is a *centralizer* subring.

(c) Recall that R_0 is *rationally closed* in R if $R^\times \cap R_0 = R_0^\times$. That is, elements of R_0 that are invertible in R are also invertible in R_0.

Semi-centralizer and semi-invariant subrings are clearly rationally closed. The latter were studied (for semilocal R) in [**23**] and invariant subrings (w.r.t. an arbitrary set) were considered in [**19**]. However, the notion of semi-invariant subrings appears to be new.

The purpose of this chapter is to study semi-invariant subrings of semiperfect rings where our motivation comes from the Krull-Schmidt theorem and the following observations, verified in sections 1.3:

(1) For any ring R, a subring of R is semi-invariant if and only if it is semi-centralizer. In particular, all centralizers of subsets of R are semi-invariant subrings.
(2) If $R \subseteq S$ are rings and M is a right S-module, then $\mathrm{End}(M_S)$ is a semi-invariant subring of $\mathrm{End}(M_R)$.
(3) If M is a finitely presented right R-module, then $\mathrm{End}(M_R)$ is a quotient of a semi-invariant subring of $\mathrm{M}_n(R)$ for some n.

While in general semi-invariant subrings of semiperfect rings need not be semiperfect (see Examples 1.6.1-1.6.3 below), we show that this is true for special families of semiperfect rings, e.g. for semiprimary and right perfect rings (Theorem 1.4.6; see section 1.2 for definitions). In addition, if the ring in question is *pro-semiprimary*, i.e. an inverse limit of semiprimary rings (e.g. the rings of Example 1.1.2), then its *T-semi-invariant* subrings (e.g. centralizer subrings; see section 1.5 for definition) are semiperfect. This actually holds under milder assumptions regarding whether the ring can be endowed with a "good" topology; see Theorems 1.5.10 and 1.5.15.

Our results together with the previous observations and the Krull-Schmidt Theorem lead to numerous applications including:

(1) The center and any maximal commutative subring of a semiprimary (resp. right perfect, semiperfect and pro-semiprimary) ring is semiprimary (resp. right perfect, etc.).
(2) If R is a semiperfect pro-semiprimary ring, then all f.p. modules over R have a semiperfect endomorphism ring and hence admit a Krull-Schmidt decomposition. If moreover R is right noetherian, then the endomorphism ring of a f.g. right R-modules is pro-semiprimary. (This generalizes Swan ([**92**, §6]), Bjork ([**19**]) and Rowen ([**78**], [**79**]) and also relates to works of Vámos ([**96**]), Facchini and Herbera ([**34**]); see Remark 1.8.4 for more details.)
(3) If S is a commutative semiperfect pro-semiprimary ring and R is an S-algebra that is *Hausdorff* (see Section 1.8) and f.p. as an S-module, then R is semiperfect. If moreover S is noetherian, then the Hausdorff assumption is superfluous and R is pro-semiprimary, hence the assertions of (2) apply. (The first statement is known to hold under mild assumptions for *Henselian* rings; see [**96**, Lm. 12].)
(4) If ρ is a representation of a ring or a monoid over a module with a semiperfect pro-semiprimary endomorphism ring, then ρ has a Krull-Schmidt decomposition.
(5) Let $R \subseteq S$ be rings and let M be a right S-module. If $\mathrm{End}(M_R)$ is semiprimary (resp. right perfect), then so is $\mathrm{End}(M_S)$. In particular, M has a Krull-Schmidt decomposition over S. (Compare with [**34**, Pr. 2.7].)

Additional applications concern bilinear forms (Chapter 4 below) and getting a "Jordan Decomposition" for endomorphisms of modules with semiperfect pro-semiprimary endomorphism ring. We also conjecture that (3) holds for non-commutative S under mild assumptions (see section 1.10).

Other interesting byproducts of our work are the fact that a pro-semiprimary ring is an inverse limit of some of its semiprimary quotients and Theorem 1.9.6 below. (The former assertion fails when replacing semiprimary with right artinian; see Example 1.9.11 and the comment before it).

Remark 1.1.3. It is still open whether all semiperfect pro-semiprimary rings are complete semilocal. However, this is true for noetherian rings; see section 1.9.

Section 1.2 contains definitions and well-known facts required for the exposition. Section 1.3 presents the basics of semi-invariant subrings; we present five equivalent characterizations of them and show that they naturally appear in various situations. As all our characterizations use the existence of *some* ambient ring, we ask whether there is a definition avoiding this. In section 1.4, we prove that various ring properties pass to semi-invariant subrings, e.g. being semiprimary and being right perfect. Section 1.5 develops the theory of T-semi-invariant subrings. The discussion leads to a proof that several properties, such as being pro-semiprimary and semiperfect, are inherited by T-semi-invariant subrings. Section 1.6 presents counterexamples; we show that semi-invariant subrings of semiperefect rings need not be semiperfect, even when the ambient ring is pro-semiprimary. In addition, we show that in general none of the properties discussed in sections 1.4 and 1.5 pass to rationally closed subrings. The latter implies that there are non-semi-invariant rationally closed subrings. In sections 1.7 and 1.8 we present applications of our results (most applications were briefly described above) and in section 1.9 we specialize them to *strictly pro-right-artinian* rings (e.g. noetherian pro-semiprimary rings), which are better behaved. Section 1.10 describes some issues that are still open. The addendum is concerned with providing conditions implying that the topologies $\{\tau_n^M\}_{n=1}^{\infty}$ defined in section 1.8 coincide.

1.2. Preliminaries

This section recalls some definitions and known facts that will be used throughout this chapter. Some of the less known facts include proofs for the sake of completion. If no reference is specified, proofs can be found in [**80**], [**9**] or [**58**].

Let R be a semilocal ring. The ring R is called semiprimary (right perfect) if $\mathrm{Jac}(R)$ is nilpotent (right T-nilpotent[1]). Since any nil ideal is idempotent lifting, right perfect rings are clearly semiperfect.

Proposition 1.2.1 (Bass' Theorem P, partial). *Let R be a ring. Then R is left (right) perfect $\iff$ every left (right) R-module has a projective cover $\iff$ R has DCC on principal* right *(*left*) ideals.*

Proposition 1.2.2. *Let R be a ring. Then R is semiperfect $\iff$ every right (left) f.g. R-module has a projective cover $\iff$ there are orthogonal idempotents $e_1, \dots, e_r \in R$ such that $\sum_{i=1}^{r} e_i = 1$ and e_iRe_i is local for all i.*

Proposition 1.2.3. *(i) Being semiprimary (resp.: right perfect, semiperfect, semilocal, pro-semiprimary) is preserved under Morita equivalence.*

(ii) Let R be a ring and $e \in \mathrm{End}(R)$. Then R is semiprimary (resp.: right perfect, semiperfect, semilocal) if and only if eRe and $(1-e)R(1-e)$ are.

[1] An ideal $I \trianglelefteq R$ is left T-nilpotent if for any sequence $x_1, x_2, x_3, \dots \in I$, the sequence $x_1, x_2x_1, x_3x_2x_1, \dots$ eventually vanishes.

Proof. All statements regarding semiprimary, right perfect, semiperfect and semilocal rings are well known. The other statements follow from the next lemma. □

Lemma 1.2.4. *Let $\{R_i, f_{ij}\}$ be an inverse system of rings and let $R = \varprojlim \{R_i\}$. Denote by f_i the natural map from R to R_i. Then:*

(i) *For all $n \in \mathbb{N}$, $\varprojlim \{\mathrm{M}_n(R_i)\}_{i\in I} \cong \mathrm{M}_n(\varprojlim \{R_i\}) = \mathrm{M}_n(R)$.*
(ii) *For all $e \in \mathrm{E}(R)$, $\varprojlim \{e_i R_i e_i\}_{i\in I} \cong eRe$ where $e_i = f_i(e)$.*

Proof. This is straightforward. □

Part (ii) of Proposition 1.2.3 does not hold for pro-semiprimary rings. For instance, take $R = \{\left[\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_p,\ v \in \bigoplus_{i=1}^{\infty} \mathbb{Z}_p\}$ (where $\mathbb{Z}_p$ are the p-adic integers) and let e be the matrix unit e_{11}.

Theorem 1.2.5 (Levitski). *Let R be a right noethrian ring. Then any nil subring of R is nilpotent.*

Let R be a ring. An element $a \in R$ is called *right π-regular* (in R) if the right ideal chain $aR \supseteq a^2R \supseteq a^3R \supseteq \dots$ stabilizes.[2] If a is both left and right π-regular we will say it is π-regular. A ring all of whose elements are right π-regular is called *π-regular*. It was shown by Dischinger in [**27**] that the latter property is actually left-right symmetric.

Since π-regularity is not preserved under Morita equivalence (see [**81**]), it is convenient to introduce the following notion: A ring R is called π_∞-regular[3] if $\mathrm{M}_n(R)$ is π-regular for all n.

Proposition 1.2.6. *(i) Let R be a π-regular (π_∞-regular) ring and $e \in \mathrm{E}(R)$. Then eRe is π-regular (π_∞-regular).*

(ii) π_∞-regularity is preserved under Morita equivalence.

Proof. (i) Assume R is π-regular, let $e \in R$ and let $a = eae \in eRe$. By definition, there is $b \in R$ and $n \in \mathbb{N}$ such that $a^n = a^{n+1}b$. Multiplying by e on the right, we get $a^n = a^{n+1}ebe$, hence $a^n(eRe) = a^{n+1}(eRe)$.

Assume R is π_∞-regular and let $e \in R$. Let I denote identity matrix in $\mathrm{M}_n(R)$. Then $(eI)\mathrm{M}_n(R)(eI) = \mathrm{M}_n(eRe)$. By the previous argument, the left hand side is π-regular; hence we are through.

(ii) We only need to check that $\mathrm{M}_n(R)$ is π_∞-regular for all $n \in \mathbb{N}$, which is obvious from the definition, and that eRe is π_∞-regular, which follows from (i). □

Proposition 1.2.7. *Let R be a ring and let N denote its prime radical (i.e. the intersection of all prime ideals). Then R is π-regular (π_∞-regular) if and only if R/N is.*

Proof. See [**80**, §2.7]. (The argument is easily generalized to π_∞-regular rings.) □

Remark 1.2.8. Any PI semilocal ring with nil Jacobson radical is π_∞-regular (see [**78**, Apx.]). However, there are semilocal rings with nil Jacobson radical that are not π-regular, see [**81**].

Remark 1.2.9. We have the following implications:

$$\text{right artinian} \Longrightarrow \text{semiprimary} \Longrightarrow \text{left/right perfect} \stackrel{(1.2.1)}{\Longrightarrow} \pi_\infty\text{-regular} \Longrightarrow \pi\text{-regular}$$

However, all these notions coincide for right noetherian rings. Indeed, assume R is π-regular and right noetherian and let $J = \mathrm{Jac}(R)$. Then J is nil (see

[2] This notion of π-regularity is sometimes called strong π-regularity.
[3] This property is sometimes called completely π-regular.

Lemma 1.4.4(i)), hence Theorem 1.2.5 implies $J^n = 0$ for some $n \in \mathbb{N}$. By Lemma 1.4.4(ii) below, R is semiperfect and in particular R/J is semisimple. As R is right noetherian, the right R/J-modules $\{J^{i-1}/J^i\}_{i=1}^n$ are f.g., hence their length as right R-modules is finite. It follows that R_R has a finite length, so R is right artinian.

Throughout, we will implicitly use the next lemma. Notice that it implies that being semiprimary (resp.: right perfect, semiperfect, semilocal) passes to quotients.

LEMMA 1.2.10. *Let R be a semilocal ring. Then any epimorphism of rings $\varphi : R \to S$ satisfies $\varphi(\mathrm{Jac}(R)) = \mathrm{Jac}(S)$.*

PROOF. $\varphi(\mathrm{Jac}(R))$ is an ideal of $\varphi(R) = S$ and $1+\varphi(\mathrm{Jac}(R)) = \varphi(1+\mathrm{Jac}(R)) \subseteq \varphi(R^\times) \subseteq S^\times$, hence $\varphi(\mathrm{Jac}(R)) \subseteq \mathrm{Jac}(S)$. On the other hand, $S/\varphi(\mathrm{Jac}(R))$ is a quotient of $R/\mathrm{Jac}(R)$ which is semisimple. Therefore, $S/\varphi(\mathrm{Jac}(R))$ is semisimple, implying $\varphi(\mathrm{Jac}(R)) \supseteq \mathrm{Jac}(S)$. □

1.3. Semi-Invariant Subrings

This section presents the basic properties of semi-invariant subrings. We begin by showing that for any ring, the semi-invariant subrings are precisely the semi-centralizer subrings.

PROPOSITION 1.3.1. *Let $R_0 \subseteq R$ be rings. The following are equivalent:*

(a) *There is a ring $S \supseteq R$ and a set $\Sigma \subseteq \mathrm{End}(S)$ such that $R_0 = R^\Sigma$.*
(b) *There is a ring $S \supseteq R$ and a subset $X \subseteq S$ such that $R_0 = \mathrm{Cent}_R(X)$.*
(c) *There is a ring $S \supseteq R$ and $\sigma \in \mathrm{Aut}(S)$ such that $\sigma^2 = \mathrm{id}$ and $R_0 = R^{\{\sigma\}}$.*
(d) *There is a ring $S \supseteq R$ and an* inner *automorphism $\sigma \in \mathrm{Aut}(S)$ such that $\sigma^2 = \mathrm{id}$ and $R_0 = R^{\{\sigma\}}$.*
(e) *There are rings $\{S_i\}_{i\in I}$ and ring homomorphisms $\psi_i^{(1)}, \psi_i^{(2)} : R \to S_i$ such that $R_0 = \{r \in R : \psi_i^{(1)}(r) = \psi_i^{(2)}(r),\ \forall i \in I\}$.*

Note that condition (e) implies that the family of semi-invariant subrings is closed under intersection.

PROOF. We prove (a)$\Longrightarrow$(e)$\Longrightarrow$(c)$\Longrightarrow$(d)$\Longrightarrow$(b)$\Longrightarrow$(a).

(a)$\Longrightarrow$(e): Take $I = \Sigma$ and define $S_\sigma = S$, $\psi_\sigma^{(1)} = \sigma$ and $\psi_\sigma^{(2)} = \mathrm{id}_R$.

(e)$\Longrightarrow$(c): Let $\{S_i, \psi_i^{(1)}, \psi_i^{(2)}\}_{i\in I}$ be given. Without loss of generality, we may assume there is $i_0 \in I$ such that $S_{i_0} = R$ and $\psi_{i_0}^{(1)} = \psi_{i_0}^{(2)} = \mathrm{id}_R$. Define $S = \prod_{(i,j)\in I\times\{1,2\}} S_{ij}$ where $S_{ij} = S_i$ and let $\Psi : R \to S$ be given by

$$\Psi(r) = \left(\psi_i^{(j)}(r)\right)_{(i,j)\in I\times\{1,2\}} \in S\ .$$

The existence of i_0 above implies Ψ is injective. Let $\sigma \in \mathrm{Aut}(S)$ be the automorphism exchanging the $(i,1)$ and $(i,2)$ components of S for all $i \in I$. Then one easily checks that $\sigma^2 = \mathrm{id}$ and $\Psi(R)^{\{\sigma\}} = \Psi(R_0)$. We finish by identifying R with $\Psi(R)$.

(c)$\Longrightarrow$(d): Let S, σ be given and let $S' = S[x;\sigma]$ denote the ring of σ-twisted polynomials with (left) coefficients in S. Observe that $(x^2-1) \in \mathrm{Cent}(S')$ (since $\sigma^2 = \mathrm{id}$), hence $S'' := S'/\left\langle x^2-1\right\rangle$ is a free left S-module with basis $\{\overline{1}, \overline{x}\}$ (where $\overline{a}$ is the image of $a \in S'$ in S''). Let $\tau \in \mathrm{Aut}(S'')$ be conjugation by $\overline{x}$. Then $\tau^2 = \mathrm{id}$ and $R^{\{\tau\}} = \{r \in R : \overline{x}r = r\overline{x}\} = \{r \in R : \sigma(r) = r\} = R^{\{\sigma\}} = R_0$.

(d)$\Longrightarrow$(b): This is a clear.

(b)$\Longrightarrow$(a): Let S, X be given. Let $S' = S(\!(t)\!)$ be the ring of formal Laurent series $\sum_{n=k}^\infty a_n t^n$ $(k \in \mathbb{Z})$ with coefficients in S. The elements of S' commuting with X are precisely the elements that commute with $t^{-1}+X$ (as t^{-1} is central in S'). However, it is easily seen that all elements in $t^{-1}+X$ are invertible. For all $x \in X$, let $\sigma_x \in \mathrm{End}(S')$ be the inner automorphism of S' given by conjugation with $t^{-1}+x$ and let $\Sigma = \{\sigma_x \mid x \in X\}$. Then $R^\Sigma = \mathrm{Cent}_R(t^{-1}+X) = \mathrm{Cent}_R(X) = R_0$. $\square$

COROLLARY 1.3.2. *Let R, W be rings and let $\varphi : R \to W$ be a ring homomorphism. Assume $W_0 \subseteq W$ is a semi-invariant subring of W. Then $\varphi^{-1}(W_0)$ is a semi-invariant subring of R.*

PROOF. By Proposition 1.3.1(e), there are rings $\{S_i\}_{i\in I}$ and ring homomorphisms $\psi_i^{(1)}, \psi_i^{(2)} : W \to S_i$ such that $W_0 = \{r \in R : \psi_i^{(1)}(r) = \psi_i^{(2)}(r)\ \forall i \in I\}$. Define $\varphi_i^{(n)} = \psi_i^{(n)} \circ \varphi$ and note that $\varphi^{-1}(W_0) = \{r \in R : \varphi(r) \in W_0\} = \{r \in R : \psi_i^{(1)}\varphi(r) = \psi_i^{(2)}\varphi(r)\ \forall i \in I\} = \{r \in R : \varphi_i^{(1)}(r) = \varphi_i^{(2)}(r)\ \forall i \in I\}$. $\square$

The equivalent conditions of Proposition 1.3.1 require the existence of some ambient ring. This leads to the following question:

QUESTION 1. *Is there an intrinsic definition of semi-invariant subrings?*

Informally, we ask for a definition that would make it easy to show that a given (rationally closed) subring is *not* semi-invariant.

The next proposition is useful for producing examples of semi-invariant subrings.

PROPOSITION 1.3.3. *Let $R \subseteq S$ be rings and let K be a central subfield of S. Then $R \cap K$ is a semi-invariant subring of R.*

PROOF. Let $S' = S \otimes_K S$ and define $\varphi_1, \varphi_2 : S \to S'$ by $\varphi_1(s) = s \otimes 1$ and $\varphi_2(s) = 1 \otimes s$. As K is a central subfield, it is easy to check that $\{s \in S : \varphi_1(s) = \varphi_2(s)\} = K$, hence $\{s \in R : \varphi_1(s) = \varphi_2(s)\} = R \cap K$. We are done by Proposition 1.3.1(e). $\square$

COROLLARY 1.3.4. *Let K be a field. Then the semi-invariant subrings of K are precisely its subfields.*

PROOF. Any semi-invariant subring $R \subseteq K$ satisfies $R^\times = R \cap K^\times = R \setminus \{0\}$, hence it is a field. The converse follows from Proposition 1.3.3. $\square$

REMARK 1.3.5. If K/L is an algebraic field extension, then L is an invariant subring of K if and only if K/L is Galois.

We finish this section by introducing two cases where semi-invariant subrings naturally appear.

PROPOSITION 1.3.6. *Let $R \subseteq S$ be rings and let M be a right S-module. Then $\mathrm{End}(M_S)$ is a semi-invariant subring of $\mathrm{End}(M_R)$.*

PROOF. There is a homomorphism $\varphi : S^{\mathrm{op}} \to \mathrm{End}(M_{\mathbb{Z}})$ given by $\varphi(s^{\mathrm{op}})(m) = ms$ for all $m \in M$. It is straightforward to check that $\mathrm{End}(M_S) = \mathrm{Cent}_{\mathrm{End}(M_{\mathbb{Z}})}(\mathrm{im}\,\varphi)$. As $\mathrm{End}(M_S) \subseteq \mathrm{End}(M_R)$, it follows that $\mathrm{End}(M_S) = \mathrm{Cent}_{\mathrm{End}(M_R)}(\mathrm{im}\,\varphi)$, hence $\mathrm{End}(M_S)$ is a semi-centralizer subring of $\mathrm{End}(M_R)$. $\square$

PROPOSITION 1.3.7. *Let $\mathscr{A}$ be an abelian category and let $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence in $\mathscr{A}$ such that for any $c \in \mathrm{End}(C)$ there are $b \in \mathrm{End}(B)$ and $a \in \mathrm{End}(A)$ with $cg = gb$ and $bf = fa$ (e.g.: if both A and B are projective, or if B is projective and f is injective). Then $\mathrm{End}(C)$ is isomorphic to a quotient of:*

(i) *a semi-invariant subring of $\mathrm{End}(A) \times \mathrm{End}(B)$;*
(ii) *an invariant and a centralizer subring of $\mathrm{End}(A \oplus B)$, provided f is injective.*

PROOF. Let $B_0 = \operatorname{im} f = \ker g$. Define R to be the subring of $\operatorname{End}(B)$ consisting of maps $b \in \operatorname{End}(B)$ for which there is $a \in \operatorname{End}(A)$ with $bf = fa$. Then for all $b \in R$, $b(B_0) = b(\operatorname{im} f) = \operatorname{im}(fa) \subseteq \operatorname{im} f = B_0$. Therefore, there is *unique* $c \in \operatorname{End}(C)$ such that $cg = gb$. The map sending b to c is easily seen to be a ring homomorphism from R to $\operatorname{End}(C)$ and the assumptions imply it is onto. Therefore, $\operatorname{End}(C)$ is a quotient of R.

Let $S = \operatorname{End}(A \oplus B)$. We represent elements of S as matrices $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$ with $x \in \operatorname{End}(A), y \in \operatorname{Hom}(B, A), z \in \operatorname{Hom}(A, B), w \in \operatorname{End}(B)$. Let D denote the diagonal matrices in S (i.e. $\operatorname{End}(A) \times \operatorname{End}(B)$) and let $W = \operatorname{Cent}_S(\left[\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right])$. Then for $a \in \operatorname{End}(A)$ and $b \in \operatorname{End}(B)$, $fa = bf$ if and only if $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] \in W$. Define a ring homomorphism $\varphi : D \to \operatorname{End}(B)$ by $\varphi(\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]) = y$. Then $\varphi(\operatorname{Cent}_D(\left[\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right])) = \varphi(D \cap W) = R$. It follows that $\operatorname{End}(C)$ is a quotient of R, which is a quotient of $D \cap W$, which is a semi-centralizer subring of $D = \operatorname{End}(A) \times \operatorname{End}(B)$. This settles (i). To see (ii), notice that if f is injective, then W consists of upper-triangular matrices, hence φ can be extended to W, which is a centralizer and an invariant subring of S since $W = \operatorname{Cent}_S(\left[\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right])$ and $\left[\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right] \in S^\times$. □

1.4. Properties Inherited by Semi-Invariant Subrings

In this section, we prove that being semiprimary (right perfect, semiperfect-and-π_∞-regular, semiperfect-and-π-regular) passes to semi-invariant subrings. We also present a supplementary result for algebras over fields.

Our first step is introducing an equivalent condition for π-regularity of elements of a ring.

LEMMA 1.4.1. *Let R be a ring and let $a \in R$ be a π-regular element. Define:*

$$A = \bigcap_{k=1}^{\infty} a^k R, \qquad B = \bigcup_{k=1}^{\infty} \operatorname{ann}^r a^k,$$
$$A' = \bigcap_{k=1}^{\infty} R a^k, \qquad B' = \bigcup_{k=1}^{\infty} \operatorname{ann}^\ell a^k.$$

Then there is $e \in \mathrm{E}(R)$ such that $A = eR$, $B = fR$, $A' = Re$ and $B' = Rf$ where $f := 1 - e$. In particular, $R_R = A \oplus B$ and ${}_R R = A' \oplus B'$.

PROOF. Let $n \in \mathbb{N}$ be such that $a^n R = a^k R$ and $Ra^n = Ra^k$ for all $k \geq n$. Notice that this implies $\operatorname{ann}^r a^n = \operatorname{ann}^r a^k$ and $\operatorname{ann}^\ell a^n = \operatorname{ann}^r a^k$ for all $k \geq n$.

We begin by showing $R_R = A \oplus B$. That ${}_R R = A' \oplus B'$ follows by symmetry. The argument is similar to the proof of Fitting's Lemma (see [**80**, §2.9]): Let $r \in R$. Then $a^n r \in a^n R = a^{2n} R$, hence there is $s \in R$ with $a^n r = a^{2n} s$. Observe that $a^n(r - a^n s) = 0$ and $a^n s \in a^n R$, so $r = a^n s + (r - a^n s) \in A + B$. Now suppose $r \in A \cap B$. Then $r = a^n s$ for some $s \in R$. However, $r \in B = \operatorname{ann}^r a^n$ implies $s \in \operatorname{ann}^r a^{2n} = \operatorname{ann}^r a^n$, so $r = a^n s = 0$.

Since $R_R = A \oplus B$, there is $e \in R$ such that $e \in A$ and $f := 1 - e \in B$. It is well known that in this case $e^2 = e$, $A = eR$ and $B = fR$. This implies $B' = \operatorname{ann}^\ell a^n = \operatorname{ann}^\ell a^n R = \operatorname{ann}^\ell eR = Rf$ and $A' = Ra^n \subseteq \operatorname{ann}^\ell \operatorname{ann}^r Ra^n = \operatorname{ann}^\ell \operatorname{ann}^r a^n = \operatorname{ann}^\ell fR = Re$. As $W = A' \oplus B' = Re \oplus Rf$ we must have $A' = Re$. □

PROPOSITION 1.4.2. *Let R be ring and $a \in R$. Then a is π-regular $\iff$ there is $e \in \mathrm{E}(R)$ such that*

(A) *$a = eae + faf$ where $f := 1 - e$.*
(B) *eae is invertible in eRe.*
(C) *faf is nilpotent.*

In this case, the idempotent e is uniquely determined by a.

PROOF. Assume a is π-regular and let e, f, A, B, A', B', n be as in Lemma 1.4.1. Then $ae \in aeR = aA = a(a^n R) = a^{n+1} R = A = eR$ and $af \in aB = a \operatorname{ann}^r a^n \subseteq$

$\mathrm{ann}^r a^n = B = fR$. Therefore, $ae = eae$ and $af = faf$, hence $a = ae + af = eae + faf$. This implies $a^k = (eae)^k + (faf)^k$ for all $k \in \mathbb{N}$. As $a^n \in eR$, we have $a^n = ea^n$, hence $(eae)^n + (faf)^n = a^n = e(eae)^n + e(faf)^n = (eae)^n$ which implies $(faf)^n = 0$. In particular, for all $k \geq n$, $a^k = (eae)^k + (faf)^k = (eae)^k$. Since $e \in a^n R$ there is $x \in R$ such that $e = a^n x = (eae)^n x$. Multiplying by e on the right yields $e = (eae)((eae)^{n-1}xe)$, hence eae is right invertible in eRe. By symmetry, eae is left also left invertible in eRe, hence we conclude that e satisfies (A)–(C).

Now assume there is $e \in \mathrm{E}(R)$ satisfying (A)–(C) and let b be the inverse of a in eRe. Then $a^k = (eae)^k + (faf)^k$ for all $k \in \mathbb{N}$. Condition (C) now implies there is $n \in \mathbb{N}$ such that $a^k = (eae)^k$ for all $k \geq n$. Therefore, for all $k \geq n$, $a^n = (eae)^n = (eae)^k b^{k-n} = a^k b^{k-n} \in a^k R$ impying $a^n R = a^k R$. By symmetry, $Ra^n = Ra^k$ for all $k \geq n$, so a is π-regular.

Finally, assume both $e, e' \in \mathrm{E}(R)$ satisfy conditions (A)–(C) and let $f = 1 - e$, $f' = 1 - e'$. By the previous paragraph a is π-regular, hence Lemma 1.4.1 implies $R = A \oplus B$ where $A = \bigcap_{k=1}^{\infty} a^k R$ and $B = \bigcup_{k=1}^{\infty} \mathrm{ann}^r a^k R$. Let b be the inverse of eae in eRe and let $n \in \mathbb{N}$ be such that $(faf)^n = 0$. Then $e = (eae)^k b^k = a^k b^k \in a^k R$ for all $k \geq n$, hence $e \in A$, and $a^n f = (eae)^n f = 0$, hence $f \in B$. Similarly, $e' \in A$ and $f' \in B$. It follows that $e, e' \in A$ and $f, f' \in B$. Since $1 = e + f = e' + f'$ and $R = A \oplus B$, we must have $e = e'$. □

Let R, a be as in Proposition 1.4.2. Henceforth, we call the unique idempotent e satisfying conditions (A)–(C) the *associated idempotent* of a (in R).

COROLLARY 1.4.3. *(i) Let R be a ring, $R_0 \subseteq R$ a semi-invariant subring and let $a \in R_0$ be π-regular in R. Then a is π-reuglar in R_0.*

(ii) A semi-invariant subring of a π-regular (π_∞-regular) ring is π-regular (π_∞-regular).

PROOF. (i) Let $S \supseteq R$ and $\Sigma \subseteq \mathrm{End}(S)$ be such that $R_0 = R^\Sigma$, and let $a \in R_0$ be π-regular in R. Let e be the associated idempotent of a in R. Then e is clearly the associated idempotent of a in S (hence a is π-regular in S). However, it is straightforward to check that $\sigma(e)$ satisfies conditions (A)–(C) (in S) for all $\sigma \in \Sigma$ (since $\sigma(a) = a$), so the uniqueness of e forces $e \in S^\Sigma \cap R = R^\Sigma = R_0$. Therefore, a is π-regular in R_0.

(ii) The π-regular case is clear from (i). The π_∞-regular case follows when one notes that if R_0 is a semi-invariant subring of R, then $\mathrm{M}_n(R_0)$ is a semi-invariant subring of $\mathrm{M}_n(R)$ for all $n \in \mathbb{N}$. □

LEMMA 1.4.4. *Let R be a π-regular ring. Then:*

(i) $\mathrm{Jac}(R)$ *is nil.*
(ii) *R is semiperfect $\iff$ R does not contain an infinite set of orthogonal idempotents.*

PROOF. For $a \in R$, let e_a denote the associated idempotent of a and let $f_a = 1 - e_a$.

(i) Let $a \in \mathrm{Jac}(R)$ and let b be the inverse of $e_a a e_a$ in $e_a R e_a$. Then $e_a = b(e_a a e_a) \in \mathrm{Jac}(R)$, hence $e_a = 0$, implying $a = f_a a f_a$ is nilpotent.

(ii) That R is semiperfect clearly implies R does not contain an infinite set of orthogonal idempotents, so assume the converse. Let $a \in R$. Observe that if $e_a = 0$ then a is nilpotent and if $e_a = 1$ then a is invertible. Therefore, if $e_a \in \{0, 1\}$ for all $a \in R$, then R is local and in particular, semiperfect.

Assume there is $a \in R$ with $e := e_a \notin \{0, 1\}$. We now apply an inductive argument to deduce that eRe and $(1-e)R(1-e)$ are semiperfect, thus proving R is semiperfect (by Proposition 1.2.3). The induction process must stop because otherwise there is a sequence of idempotents $\{e_k\}_{k=0}^{\infty} \subseteq R$ such that $e_k \in e_{k-1} R e_{k-1}$

and $e_k \notin \{0, e_{k-1}\}$. This implies $\{e_{k-1} - e_k\}_{k=1}^{\infty}$ is an infinite set of non-zero orthogonal idempotents, which cannot exist by our assumptions. □

LEMMA 1.4.5. *Let $R_0 \subseteq R$ be rings. If R is semiperfect and both R_0 and R are π-regular, then R_0 is semiperfect and* $\operatorname{Jac}(R_0)^n \subseteq \operatorname{Jac}(R)$ *for some $n \in \mathbb{N}$. If in addition R is semiprimary (right perfect), then so is R_0.*

PROOF. By Lemma 1.4.4(ii), R does not contain an infinite set of orthogonal idempotents. Therefore, this also applies to R_0, so the same lemma implies R_0 is semiperfect. Let φ denote the natural projection from R to $R/\operatorname{Jac}(R)$. By Lemma 1.4.4(i), $\varphi(\operatorname{Jac}(R_0))$ is nil. Therefore, by Theorem 1.2.5 (applied to $\varphi(R)$, which is semisimple), $\varphi(\operatorname{Jac}(R_0))$ is nilpotent, hence there is $n \in \mathbb{N}$ such that $\operatorname{Jac}(R_0)^n \subseteq \operatorname{Jac}(R)$. If moreover R is semiprimary (right perfect), then $\operatorname{Jac}(R)$ is nilpotent (right T-nilpotent). The inclusion $\operatorname{Jac}(R_0)^n \subseteq \operatorname{Jac}(R)$ then implies $\operatorname{Jac}(R_0)$ is nilpotent (right T-nilpotent), so R_0 is semiprimary (right perfect). □

THEOREM 1.4.6. *Let R be a ring and let R_0 be a semi-invariant subring of R. If R is semiprimary (resp.: right perfect, semiperfect and π_∞-regular, semiperfect and π-regular), then so is R_0. In addition, there is $n \in \mathbb{N}$ such that* $\operatorname{Jac}(R_0)^n \subseteq \operatorname{Jac}(R)$.

PROOF. Recall that being right perfect implies being π-regular by Proposition 1.2.1. Given that, the theorem follows from Corollary 1.4.3 and Lemma 1.4.5. □

COROLLARY 1.4.7. *Let $R \subseteq S$ be rings and let M be a right S-module. If* $\operatorname{End}(M_R)$ *is semiprimary (resp.: right perfect, semiperfect and π_∞-regular, semiperfect and π-regular), then so is* $\operatorname{End}(M_S)$ *and there exists $n \in \mathbb{N}$ s.t.* $\operatorname{Jac}(\operatorname{End}(M_S))^n \subseteq \operatorname{Jac}(\operatorname{End}(M_R))$.

PROOF. This follows from Theorem 1.4.6 and Proposition 1.3.6. □

REMARK 1.4.8. Camps and Dicks proved in [**23**] that a *rationally closed* subring of a semilocal ring is semilocal, thus implying the semilocal analogues of Theorem 1.4.6 and Corollary 1.4.7, excluding the part regarding the Jacobson radical (which indeed fails in this case; see Example 1.6.7). In fact, the semilocal analogue of Corollary 1.4.7 was noticed in [**34**, Pr. 2.7]. However, we cannot use this analogue with the Krull-Schmidt Theorem (as we do in section 1.7 with our results) because modules with semilocal endomorphism ring need not have a Krull-Schmidt decomposition, as shown in [**35**] and [**8**].

Nevertheless, as there are plenty of weaker Krull-Schmidt theorems for modules that do not require $\operatorname{End}(M_R)$ to be semiperfect (mainly due to Facchini et al.; e.g. [**8**], [**33**]), it might be that if M, R, S are as in Corollary 1.4.7 and $\operatorname{End}(M_R)$ is merely semiperfect, then M has a Krull-Schmidt decomposition over S (despite the fact $\operatorname{End}(M_S)$ need not be semiperfect). To the best knowledge of our knowledge, this topic is still open.

We finish this section with a supplementary result for algebras.

PROPOSITION 1.4.9. *Let $R \subseteq S$ be rings and $\Sigma \subseteq \operatorname{End}(S)$. Assume there is a division ring $D \subseteq R$ such that $\sigma(D) \subseteq D$ for all $\sigma \in \Sigma$. Then* $\dim_{D^\Sigma} R^\Sigma \leq \dim_D R$.

PROOF. Consider the left D-vector space $V = DR^\Sigma$. Let $\{v_i\}_{i\in I} \subseteq R^\Sigma$ be a left D-basis for V. We claim $\{v_i\}_{i\in I}$ is a left D^Σ-basis for R^Σ. Indeed, let $v \in R^\Sigma$. Then there are unique $\{d_i\}_{i\in I} \subseteq D$ (where almost all are 0) such that $v = \sum_i d_i v_i$. However, for all $\sigma \in \Sigma$, $v = \sigma(v) = \sum_i \sigma(d_i) v_i$ so $\sigma(d_i) = d_i$ for all $i \in I$. It follows that $v \in \sum_{i\in I} D^\Sigma v_i$. Therefore, $\dim_{D^\Sigma} R^\Sigma = \dim_D V \leq \dim_D R$. □

REMARK 1.4.10. An *invariant* subring of a f.d. algebra need not be left nor right artinian, even when invariants are taken w.r.t. to the action of a finite cyclic group. This was demonstrated by Bjork in [**19**, §2]. In particular, the assumption $\sigma(D) \subseteq D$ for all $\sigma \in \Sigma$ in Proposition 1.4.9 is essential. However, Bjork also proved that if Σ is a *finite group* acting on a f.d. algebra over a *perfect* field, then the invariant subring (w.r.t. Σ) is artinian; See [**19**, Th. 2.4]. For a detailed discussion on when a subring of an artinian ring is artinian, see [**19**] and [**20**].

1.5. T-Semi-Invariant Subrings

In this section, we specialize the notions of semi-invariance and and π-regularity to certain topological rings. As a result we obtain a topological analogue of Theorem 1.4.6 (Theorem 1.5.10), which is used to prove that *T-semi-invariant* subrings of semiperfect pro-semiprimary rings are semiperfect and pro-semiprimary (Theorem 1.5.15). Note that once restricted to *discrete* topological rings, some of the results of this section reduce to results from the previous sections. However, the latter are not superfluous since we will rely on them. For a general reference about topological rings, see [**99**].

DEFINITION 1.5.1. *A topological ring R is called* linearly topologized *(abbreviated: LT) if it admits a local basis (i.e. a basis of neighborhoods of* 0*) consisting of two-sided ideals. In this case the topology on R is called* linear.

Let us set some general notation: For a topological ring R, we let $\mathcal{I}_R$ denote its set of open ideals. Then R is LT if and only if $\mathcal{I}_R$ is a local basis. We use $\mathrm{Hom}_{\mathrm{c}}$ ($\mathrm{End}_{\mathrm{c}}$) to denote continuous homomorphisms (endomorphisms). The category of *Hausdorff* linearly topologized rings will be denoted by $\mathscr{LTR}_2$, where $\mathrm{Hom}_{\mathscr{LTR}_2}(A,B) = \mathrm{Hom}_{\mathrm{c}}(A,B)$ for all $A, B \in \mathscr{LTR}_2$.[4] A subring of a topological ring is assumed to have the induced topology. In particular, if $R \in \mathscr{LTR}_2$ then so is any subring of R. The following facts will be used freely throughout the paper. For proofs, see [**99**, §3].

(1) Let $(G,+)$ be an abelian topological group and let $\mathcal{B}$ be a local basis of G. Then for any subset $X \subseteq G$, $\overline{X} = \bigcap_{U\in\mathcal{B}}(X+U)$.
(2) Under the previous assumptions, G is Hausdorff $\iff$ $\overline{\{0\}} = \bigcap_{U\in\mathcal{B}} U = \{0\}$.
(3) Given a ring R and a filter base of ideals $\mathcal{B}$, there exists a unique ring topology on R with local basis $\mathcal{B}$. This topology makes R into an LT ring.

EXAMPLE 1.5.2. (i) Any ring assigned with the discrete topology is LT.

(ii) $\mathbb{Z}_p$ (with the p-adic topology) is LT but $\mathbb{Q}_p$ is not.

(iii) Let R be an LT ring and let $n \in \mathbb{N}$. We make $\mathrm{M}_n(R)$ into an LT ring by assigning it the unique ring topology with local basis $\{\mathrm{M}_n(I) \mid I \in \mathcal{I}_R\}$.

(iv) If R is LT and $e \in \mathrm{E}(R)$, then eRe is LT w.r.t. the induced topology.

(v) Let $\{R_i\}_{i\in I}$ be LT rings. Then $\prod_{i\in I} R_i$ is LT w.r.t. the product topology.

(vi) Let R be an inverse limit of LT rings $\{R_i\}_{i\in I}$. Embed R in $\prod_{i\in I} R_i$ and give it the topology induced from the product topology on $\prod_{i\in I} R_i$. Then by (v) R is LT.[5]

(vii) If R is LT and $J \trianglelefteq R$, then R/J with the quotient topology is LT. Indeed, $\{I/J \mid J \subseteq I \in \mathcal{I}_R\}$ is a local basis for that topology. The ring R/J is Hausdorff if and only if J is closed, and discrete if and only if J is open.

---

[4] The subscript "2" in $\mathscr{LTR}_2$ stands for the second separation axiom T_2 (i.e. being Hausdorff).

[5] With this topology R is the inverse limit of $\{R_i\}_{i\in I}$ in category of topological rings, i.e. it admits the required universal property.

The last example implies that $\mathscr{LTR}_2$ is closed to products, inverse limits and forming matrix rings (with the appropriate topologies). We will say that a property $\mathcal{Q}$ of LT rings is preserved under Morita equivalence if whenever $R \in \mathscr{LTR}_2$ has $\mathcal{Q}$, then so does $\mathrm{M}_n(R)$ and eRe for $e \in \mathrm{E}(R)$ s.t. eR is a progenerator.[6]

Definition 1.5.3. *Let $R \in \mathscr{LTR}_2$. A subring $R_0 \subseteq R$ is called a* T-semi-invariant *subring if there is $R \subseteq S \in \mathscr{LTR}_2$ and a set $\Sigma \subseteq \mathrm{End}_c(S)$ such that $R_0 = R^{\Sigma}$. The subring R_0 is called a* T-semi-centralizer *subring if there is $R \subseteq S \in \mathscr{LTR}_2$ and a set $X \subseteq S$ such that $R_0 = \mathrm{Cent}_R(X)$.*

A T-semi-invariant subring is always closed. In addition, there is an analogue of Proposition 1.3.1 for T-semi-invariant rings.

Proposition 1.5.4. *Let R_0 be a subring of $R \in \mathscr{LTR}_2$. The following are equivalent:*

(a) *There is $R \subseteq S \in \mathscr{LTR}_2$ and a set $\Sigma \subseteq \mathrm{End}_c(S)$ such that $R_0 = R^{\Sigma}$.*
(b) *There is $R \subseteq S \in \mathscr{LTR}_2$ and a subset $X \subseteq S$ such that $R_0 = \mathrm{Cent}_R(X)$.*
(c) *There is $R \subseteq S \in \mathscr{LTR}_2$ and $\sigma \in \mathrm{Aut}_c(S)$ with $\sigma^2 = \mathrm{id}$ and $R_0 = R^{\{\sigma\}}$.*
(d) *There is $R \subseteq S \in \mathscr{LTR}_2$ and an* inner *automorphism $\sigma \in \mathrm{Aut}_c(S)$ such that $\sigma^2 = \mathrm{id}$ and $R_0 = R^{\{\sigma\}}$.*
(e) *There are LT Hausdorff rings $\{S_i\}_{i\in I}$ and continuous homomorphisms $\psi_i^{(1)}, \psi_i^{(2)} : R \to S_i$ such that $R_0 = \{r \in R : \psi_i^{(1)}(r) = \psi_i^{(2)}(r)\ \forall i \in I\}$.*

Proof. This is essentially the proof of Proposition 1.3.1, but we need to endow the rings constructed throughout the proof with topologies making them into LT Hasudorff rings that contain R as a topological ring. This is briefly done below; the details are left for the reader.

(b)$\Longrightarrow$(a): Give $S(\!(t)\!)$ the unique ring topology with local basis $\{I(\!(t)\!) \mid I \in \mathcal{I}_S\}$, where $I(\!(t)\!)$ denotes the set of polynomials with coefficients in I.

(e)$\Longrightarrow$(c): Assign to $S = \prod_{(i,j)\in I\times\{1,2\}} S_{ij}$ the product topology.

(c)$\Longrightarrow$(d): Observe that $\mathcal{B} = \{I \cap \sigma(I) \mid I \in \mathcal{I}_S\}$ is a local basis of S and $\sigma(J) = J$ for all $J \in \mathcal{B}$. Assign $S' = S[x;\sigma]$ the unique ring topology with local basis $\{J[x;\sigma] \mid J \in \mathcal{B}\}$, where $J[x;\sigma]$ denotes the set of polynomials with (left) coefficients in J, and give $S'' = S'/\langle x^2 - 1\rangle$ the quotient topology. □

We now generalize the notion of π-regularity for topological rings. Our definition is inspired by Proposition 1.4.2.

Definition 1.5.5. *Let $R \in \mathscr{LTR}_2$ and $a \in R$. The element a is called* quasi-π-regular *in R if there is an idempotent $e \in \mathrm{E}(R)$ such that:*

(A) *$a = eae + faf$ where $f := 1 - e$.*
(B) *eae is invertible in eRe.*
(C′) *$(faf)^n \xrightarrow{n\to\infty} 0$ (w.r.t. the topology on R).*

Call R quasi-π-regular *if all its elements are quasi-π-regular.*

Since we only consider LT rings, condition (C′) means that for any $I \in \mathcal{I}_R$ there is $n \in \mathbb{N}$ such that $(faf)^n \in I$. This implies that quasi-π-regularity coincide with π-regularity for discrete topological rings (take $I = \{0\}$) and that if a is quasi-π-regular in R then $a + I$ is π-regular in R/I for all $I \in \mathcal{I}_R$. In particular, if R is quasi-π-regular then R/I is π-regular. We will call the idempotent e satisfying conditions (A),(B) and (C′) the *associated idempotent* of a. The following lemma shows that it is unique.

[6] Caution: There is a notion of Morita equivalence for (right) LT rings, but we will not use it in this dissertation; see [**47**] and related articles. (The ring-theoretic Morita equivalence implies the topological Morita equivalence, but not vice versa).

LEMMA 1.5.6. *Let $R \in \mathscr{LTR}_2$ and $a \in R$ a quasi-π-regular element. Then the idempotent e satisfying conditions (A),(B) and (C′) is uniquely determined by a.*

PROOF. Assume both e and e' satisfy conditions (A), (B), (C′) and let $I \in \mathcal{I}_R$. Then $e+I$ and $e'+I$ are associated idempotents of $a+I$ in R/I, hence $e+I = e'+I$, or equivalently $e - e' \in I$. It follows that $e - e' \in \bigcap_{I\in\mathcal{I}_R} I = \{0\}$ (since R is Hausdorff), so $e = e'$. □

REMARK 1.5.7. (i) In the assumptions of the previous lemma, it is also possible to show that $eR = \bigcap_{n=1}^{\infty} a^n R$ and $(1-e)R = \{r \in R : a^n r \xrightarrow{n\to\infty} 0\}$.

(ii) If we do not restrict to LT Hausdorff rings, the associated idempotent need not be unique. For example, in $\mathbb{Q}_p$ both 0 and 1 are associated idempotents of p. (It is not known if Lemma 1.5.6 holds under the assumption that R is *right LT*, i.e. has a local basis of right ideals.)

(iii) If one assigns a semiperfect ring R with $\bigcap_{n=1}^{\infty} \mathrm{Jac}(R)^n = \{0\}$ the $\mathrm{Jac}(R)$-adic topology, then R becomes a Hausdorff LT ring, and for any $a \in R$, there is an idempotent e satisfying conditions (B) and (C′) (but such e need not be unique even when R is simple). However, condition (A) might be impossible to satisfy for some a-s. Indeed, the ring R constructed in Example 1.6.1 below, which is isomorphic to $\mathrm{M}_4(\mathbb{Z}_{\langle 3\rangle})$, is a semiperfect ring having no ring topology making it into a quasi-π-regular Hausdorff LT ring. As $\mathbb{Z}_{\langle 3\rangle}$ is quasi-π-regular w.r.t. the 3-adic topology (since it is local), it follows that quasi-π-regularity is not preserved under Morita equivalence. (This also follows from the comment before Proposition 1.2.6.)

It light of the last remark, it is convenient to call an LT Hausdorff ring R *quasi-π_∞-regular* if $\mathrm{M}_n(R)$ is quasi-π-regular for all n. This property is preserved under Morita equivalence and turns out to be related with *Henselianity* (see Section 1.8). However, to avoid cumbersome notation, we will not mention it in this section. All statements henceforth can be easily seen to hold when replacing (quasi-)π-regular with (quasi-)π_∞-regular.

COROLLARY 1.5.8. *(i) Let $R \in \mathscr{LTR}_2$, let R_0 be a T-semi-invariant subring of R and let $a \in R_0$ be quasi-π-regular in R. Then a is quasi-π-reuglar in R_0.*

(ii) A T-semi-invariant subring of a quasi-π-regular ring is quasi-π-regular.

PROOF. This is similar to the proof of Corollary 1.4.3. □

LEMMA 1.5.9. *Let $R \in \mathscr{LTR}_2$ be quasi-π-regular. Then:*

- (i) *For all $a \in \mathrm{Jac}(R)$, $a^n \xrightarrow{n\to\infty} 0$. (That is, $\mathrm{Jac}(R)$ is "topologically nil").*
- (ii) *R is semiperfect $\iff$ R does not contain an infinite set of orthogonal idempotents.*

PROOF. (i) Let $I \in \mathcal{I}_R$. Then $a + I \in (\mathrm{Jac}(R) + I)/I \subseteq \mathrm{Jac}(R/I)$, so by Lemma 1.4.4(i) applied to R/I (which is π-regular), there is $n \in \mathbb{N}$ such that $a^n \in I$.

(ii) We only show the non-trivial implication. For $a \in R$, let e_a denote associated idempotent of a. Note that $e_a = 1$ implies $a \in R^\times$ and $e_a = 0$ implies $a^n \xrightarrow{n\to\infty} 0$.

Assume $e_a \in \{0, 1\}$ for all $a \in R$. We claim R is local. This is clear if $R = \{0\}$. Otherwise, let $a \in R$ and assume by contradiction that $e_a = e_{1-a} = 0$. Let $R \neq I \in \mathcal{I}_R$ (here we need $R \neq \{0\}$). Then there is $n \in \mathbb{N}$ such that $a^n, (1-a)^n \in I$, implying $(1-a^n)^n = (1-a)^n(1+a+\cdots+a^{n-1})^n \in I$. We can write $1 = (1-a^n)^n + a^n h(a)$ for some $h(x) \in \mathbb{Z}[x]$, thus getting $1 \in I$, in contradiction to the assumption $I \neq R$. Therefore, one of e_a, e_{1-a} is 1, hence one of a, $1-a$ is invertible.

Now assume there is $a \in R$ with $e := e_a \notin \{0, 1\}$. Then we can induct on eRe and $(1-e)R(1-e)$ as in the proof of Lemma 1.4.4(ii). However, we need to verify that eRe is quasi-π-regular (w.r.t. the induced topology). Let $b \in eRe$. It enough to show $e_b \in eRe$, i.e. $e_b = ee_be$. As $R \in \mathscr{LTR}_2$, this is equivalent to $e_b + I = ee_be + I$ for all $I \in \mathcal{I}_R$. Indeed, since R/I is π-regular, so is $e(R/I)e$ (by Proposition 1.2.6(i)), hence $b + I$ has an associated idempotent $\varepsilon \in e(R/I)e$. However, it easy to see that ε is also the associated idempotent of $b + I$ in R/I, so necessarily $\varepsilon = e_b + I$. As $\varepsilon = (e + I)\varepsilon(e + I)$, it follows that $e_b + I = ee_be + I$. □

We can now state and prove a T-semi-invariant analogue of Theorem 1.4.6.

THEOREM 1.5.10. *Let R_0 be a T-semi-invariant subring of a semiperfect and quasi-π-regular ring $R \in \mathscr{LTR}_2$. Then R_0 is semiperfect and quasi-π-regular and there is $n \in \mathbb{N}$ such that* $\mathrm{Jac}(R_0)^n \subseteq \mathrm{Jac}(R)$.

PROOF. That R_0 is quasi-π-regular and semiperfect follows form Corollary 1.5.8 and Lemma 1.5.9(ii). Now let $I \in \mathcal{I}_R$. Then both R/I and $(R_0 + I)/I$ are semiperfect and π-regular (since $(R_0+I)/I \cong R_0/(R_0 \cap I)$ and $R_0 \cap I$ is open in R_0). Therefore, by Lemma 1.4.5, there is $n_I \in \mathbb{N}$ such that $\mathrm{Jac}((R_0 + I)/I)^{n_I} \subseteq \mathrm{Jac}(R/I)^{n_I}$. As $\mathrm{Jac}(R/I) = (\mathrm{Jac}(R) + I)/I$, this implies $\mathrm{Jac}(R_0)^{n_I} \subseteq \mathrm{Jac}(R) + I$. However, $(R/I)/(\mathrm{Jac}(R/I)) \cong R/(\mathrm{Jac}(R) + I)$ is a quotient of $R/\mathrm{Jac}(R)$ which is semisimple, hence the index of nilpotence of any of is subsets is bounded (when finite) by $\mathrm{length}(R/\mathrm{Jac}(R))$.[7] Therefore, there is $n \in \mathbb{N}$ such that for all $I \in \mathcal{I}_R$, $\mathrm{Jac}(R_0)^n \subseteq \mathrm{Jac}(R) + I$ or equivalently, $\mathrm{Jac}(R_0)^n \subseteq \bigcap_{I \in \mathcal{I}_R}(\mathrm{Jac}(R) + I) = \overline{\mathrm{Jac}(R)}$. Thus, we are done by the following lemma. □

LEMMA 1.5.11. *Let $R \in \mathscr{LTR}_2$ be quasi-π-regular. Then $R^\times$ and* $\mathrm{Jac}(R)$ *are closed.*

PROOF. Let $a \in \overline{R^\times}$ and let e be its associated idempotent. Then for any $I \in \mathcal{I}_R$ there is $a_I \in R^\times$ such that $a - a_I \in I$. Clearly $e + I$ is the associated idempotent of $a + I = a_I + I$ in R/I. However, $a_I + I \in (R/I)^\times$ and thus $1 + I$ is its associated idempotent. It follows that $e + I = 1 + I$ for all $I \in \mathcal{I}_R$, hence $e = 1$ and $a \in R^\times$.

Now assume $a \in \overline{\mathrm{Jac}(R)}$. It is enough to show that for all $b \in R$, $1 + ab \in R^\times$. Let $I \in \mathcal{I}_R$ and let $a_I \in \mathrm{Jac}(R)$ be such that $a - a_I \in I$. Then $1 + a_Ib \in R^\times$ and $(1 + ab) - (1 + a_Ib) \in I$. Therefore, $1 + ab \in \bigcap_{I \in \mathcal{I}_R}(R^\times + I) = \overline{R^\times} = R^\times$. □

REMARK 1.5.12. (i) The assumption that R is quasi-π-regular in the last lemma is essential; see Example 1.9.2 (take $n = 0$). In addition, $\mathrm{Jac}(R)^2$ need not be closed even when R is quasi-π-regular; see Example 1.9.11.

(ii) If R is quasi-π-regular and semiperfect, then $R^\times$ and $\mathrm{Jac}(R)$ are also open. Indeed, by the last lemma $\mathrm{Jac}(R) = \overline{\mathrm{Jac}(R)} = \bigcap_{I \in \mathcal{I}_R}(\mathrm{Jac}(R) + I)$, hence $\mathrm{Jac}(R)$ is an intersection of open ideals. Since $R/\mathrm{Jac}(R)$ is artinian, $\mathrm{Jac}(R)$ is the intersection of finitely many such ideals, thus open. The set $R^\times$ is open since it is a union of cosets of $\mathrm{Jac}(R)$.

In order to apply Theorem 1.5.10 to pro-semiprimary rings, we need to recall some facts about complete topological rings. While the exact definition (see [**99**, §7-8]) is of little use to us, we will need the following results. Let $R \in \mathscr{LTR}_2$, then:

(1) R is complete if and only if R is isomorphic to an inverse limit of an inverse system of discrete topological rings $\{R_i\}_{i \in I}$. In this case, if φ_i is the natural map from R to R_i, then $\{\ker \varphi_i \mid i \in I\}$ is a local basis of R.

[7] Actually, the index of nilpotence is bounded in any right noetherian ring R. Indeed, the prime radical of R, denoted N, is nilpotent and R/N is a semiprime Goldie ring. Therefore, by Goldie's Theorem, R/N embeds in a in a semisimple ring and thus has a bounded index of nilpotence.

(2) If R is complete and $\mathcal{B}$ is a local basis consisting of ideals, then $R \cong \varprojlim \{R/I\}_{I\in\mathcal{B}}$. (Note that R/I is discrete for all $I \in \mathcal{B}$.)

We will also use the fact that a closed subring of a complete ring is complete. (This can be verified directly for rings in $\mathscr{LTR}_2$ using the previous facts.)

We now specialize the definition of pro-semiprimary rings given in section 1.1 to topological rings. For a ring property $\mathcal{P}$, a topological ring R will be called *pro-$\mathcal{P}$* if R is isomorphic as a topological ring to the inverse limit of an inverse system of discrete rings satisfying $\mathcal{P}$. If in addition the natural map from R to each of these rings is onto[8], then R will be called *strictly pro-$\mathcal{P}$*. Clearly any pro-$\mathcal{P}$ ring is complete and lies in $\mathscr{LTR}_2$. An LT ring R is strictly pro-$\mathcal{P}$ if and only if it is complete and admits a local basis of ideals $\mathcal{B}$ such that R/I has $\mathcal{P}$ for all $I \in \mathcal{B}$. Notice that if $\mathcal{P}$ is preserved under Morita equivalence, then so does being pro-$\mathcal{P}$ and being strictly pro-$\mathcal{P}$ (because the isomorphisms in Lemma 1.2.4 are also topological isomorphisms).

Remark 1.5.13. Any inverse limit of (non-topological) rings satisfying $\mathcal{P}$ can be endowed with a linear ring topology making it into a pro-$\mathcal{P}$ ring, but this topology usually depends on the inverse system used to construct the ring. However, when $\mathcal{P}$ = semiprimary and the ring is right noetherian, the topology is uniquely determined and always coincide with the Jacobson topology! See section 1.9.

Recalling Remark 1.2.9, the following lemma implies that pro-semiprimary rings are quasi-π-regular.

Lemma 1.5.14. *Let $\{R_i, f_{ij}\}$ be an I-indexed inverse system of π-regular rings and let $R = \varprojlim \{R_i\}_{i\in I}$. Then R is quasi-π-regular.*

Proof. We identify R with the set of compatible I-tuples in $\prod_{i\in I} R_i$ (i.e. tuples $(x_i)_{i\in I}$ satisfying $f_{ij}(x_j) = x_i$ for all $i \le j$ in I). Let $a = (a_i)_{i\in I} \in R$ and let $e_i \in \mathrm{E}(R_i)$ be the associated idempotent of a_i in R_i. The uniqueness of e_i implies that $e = (e_i)_{i\in I}$ is compatible and hence lie in R. We claim that e is the associated idempotent of a in R. Conditions (A) and (C$'$) are straightforward, so we only check (B): Let b_i be the inverse of $e_i a_i e_i$ in $e_i R e_i$. Then for all $i \le j$ in I, $f_{ij}(b_j)$ is also an inverse of $e_i a_i e_i$ in $e_i R e_i$, hence $f_{ij}(b_j) = b_i$. Therefore, $b := (b_i)_{i\in I}$ is compatible and lies in R. Clearly $b = ebe$ and $b(eae) = (eae)b = e$ (since this holds in each coordinate), so condition (B) is satisfied. We thus conclude that R is quasi-π-regular. □

The converse of Lemma 1.5.9 is almost true; if $R \in \mathscr{LTR}_2$ is quasi-π-regular, then R is dense in a pro-π-regular ring, namely $\varprojlim \{R/I\}_{I\in\mathcal{I}_R}$. The following theorem implies that T-semi-invariant subrings of semiperfect pro-semiprimary rings are semiperfect and pro-semiprimary (w.r.t. the induced topology).

Theorem 1.5.15. *Assume $R = \varprojlim \{R_i\}_{i\in I}$ where each R_i is π-regular. Denote by J_i the kernel of the natural map $R \to R_i$ and let R_0 be a T-semi-invariant subring of R. Then:*

(i) *R_0 is quasi-π-regular and $R_0 = \varprojlim \{R_0/(J_i \cap R_0)\}_{i\in I}$.*

(ii) *If R does not contain an infinite set of orthogonal idempotents, then R_0 is semiperfect and there is $n \in \mathbb{N}$ such that* $\mathrm{Jac}(R_0)^n \subseteq \mathrm{Jac}(R)$.

(iii) *For all $i \in I$, $R_0/(J_i \cap R_0)$ is π-regular. If moreover R_i is semiprimary (right perfect, semiperfect), then so is $R_0/(J_i \cap R_0)$. In particular, if R is pro-semiprimary (pro-right-perfect, pro-π-regular-and-semiperfect), then so is R_0.*

[8] This is not trivial since the maps in the inverse system are not assumed to be onto.

PROOF. By Lemma 1.5.14, R is quasi-π-regular, so the first assertion of (i) is Corollary 1.5.8(ii). As for the second assertion, R is complete and R_0 is closed in R, hence R_0 is complete. Since $\{R_0 \cap J_i \,|\, i \in I\}$ is a local basis of R_0, $R_0 = \varprojlim \{R_0/(J_i \cap R_0)\}_{i\in I}$. (ii) follows from Lemma 1.5.9(ii) and Theorem 1.5.10. As for (iii), $R_0/(J_i \cap R_0)$ is π-regular as a quotient of a quasi-π-regular ring with an open ideal. The rest follows from Lemma 1.4.5 (applied to $R_0/(J_i \cap R_0)$ identified as a subring of R_i). □

Let $\mathcal{P} \in$ {semiprimary, right-perfect, π-regular-and-semiperfect, π-regular}. Then Theorem 1.5.15 implies that pro-$\mathcal{P}$ rings are *strictly* pro-$\mathcal{P}$ (take $R_0 = R$). In fact, we can prove an even stronger result:

COROLLARY 1.5.16. *In the previous notation, the inverse limit of a small category of pro-$\mathcal{P}$ rings is strictly pro-$\mathcal{P}$.*

PROOF. Let $\mathscr{C}$ be a small category of pro-$\mathcal{P}$ rings and let $\{R_i\}_{i\in I}$ be the objects of $\mathscr{C}$. Then $R = \varprojlim \mathscr{C}$ can be identified with the set of I-tuples $(x_i)_{i\in I} \in \prod_{i\in I} R_i$ such that $f(x_j) = x_i$ for all $i, j \in I$ and $f \in \mathrm{Hom}_{\mathscr{C}}(R_j, R_i)$. Clearly $S := \prod_{i\in I} R_i$ is pro-$\mathcal{P}$. If we can prove that R is a T-semivariant subring of S, then we are through by Theorem 1.5.15. Indeed, let π_i denote the projection from S to R_i. For all $i, j \in I$ and $f \in \mathrm{Hom}_{\mathscr{C}}(R_j, R_i)$ define $\varphi_f^{(1)}, \varphi_f^{(2)} : S \to R_i$ by $\varphi_f^{(1)} = \pi_i$, $\varphi_f^{(2)} = f \circ \pi_j$. Then $R = \{x \in S : \varphi_f^{(1)}(x) = \varphi_f^{(2)}(x)\, \forall f\}$, hence R is a T-semi-invariant subring of S by Proposition 1.5.4(e). □

In some sense, Corollary 1.5.16 includes Theorem 1.4.6 and part of Theorem 1.5.15 because a T-semi-invariant subring can be understood as the inverse limit of a category with two objects. (Indeed, if $R \subseteq S \in \mathscr{LTR}_2$ and Σ is a submonoid of $\mathrm{End_c}(S)$, then take $\mathrm{Ob}(\mathscr{C}) = \{R, S\}$ with $\mathrm{End}_{\mathscr{C}}(S) = \Sigma$, $\mathrm{End}_{\mathscr{C}}(R) = \{\mathrm{id}_R\}$, $\mathrm{Hom}_{\mathscr{C}}(S, R) = \phi$ and $\mathrm{Hom}_{\mathscr{C}}(R, S) = \{i\}$ where $i : R \to S$ is the inclusion map.)

COROLLARY 1.5.17. *If R is pro-semiprimary, then for any $J \in \mathcal{I}_R$ there is $n \in \mathbb{N}$ such that $\mathrm{Jac}(R)^n \subseteq J$. In particular, $\bigcap_{n=1}^{\infty} \mathrm{Jac}(R)^n = \{0\}$.*

PROOF. Assume $R = \varprojlim \{R_i\}_{i\in I}$ with each R_i semiprimary and let J_i be as in Theorem 1.5.15. Since $\{J_i \,|\, i \in I\}$ is a local basis, there is $i \in I$ such that $J_i \subseteq J$. By Theorem 1.5.15(iii), R/J_i is semiprimary, hence there is $n \in \mathbb{N}$ such that $\mathrm{Jac}(R/J_i)^n = 0$. As $\mathrm{Jac}(R/J_i) \supseteq (\mathrm{Jac}(R) + J_i)/J_i$, we get $\mathrm{Jac}(R)^n \subseteq J_i \subseteq J$. □

REMARK 1.5.18. We will show in Proposition 1.8.7 that *Henselian* rank-1 valuation rings are quasi-π_∞-regular. In particular, non-complete such rings (e.g. the $\mathbb{Q}$-algebraic elements in $\mathbb{Z}_p$) are examples of non-complete quasi-π_∞-regular rings.

In addition, we suspect that the following are also explicit examples of non-complete quasi-π_∞-regular rings: (1) the ring of power series $\sum_{i=0}^{\infty} a_i t^i \in \mathbb{Z}_p[[t]]$ with $a_i \to 0$ endowed with the t-adic topology (such rings are common in rigid geometry); (2) the ring in the comment after Lemma 1.2.4 w.r.t. its Jacobson topology.

1.6. Counterexamples

This section consists of counterexamples. In particular, we show that:

1. If R is a semiperfect ring and $\Sigma \subseteq \mathrm{End}(R)$, then R^Σ need not be semiperfect even when Σ is a finite group and even when Σ consists of a single automorphism. Similarly, if $X \subseteq R$ is a set, then $\mathrm{Cent}_R(X)$ need not be semiperfect even when X consists of a single element.
2. The semiperfect analogue of Corollary 1.4.7 is not true in general.

(3) A semi-invariant subring of a semiperfect pro-semiprimary ring need not be semiperfect even when closed (in contrast to T-semi-invariant subrings).
(4) Rationally closed subrings of a f.d. algebra need not be semiperfect. In particular, Theorem 1.4.6 does not generalize to rationally closed subrings.
(5) No two of the families of semi-invariant, invariant, centralizer and rationally closed subrings coincide in general.

We note that (1) is also true if we replace semiperfect with artinian. This was treated at the end of section 1.4.

We begin with demonstrating (1). Our examples use Azumaya algebras and we refer the reader to [**83**] for definition and details.

Example 1.6.1. Let S be a discrete valuation ring with maximal ideal πS, residue field $k = S/\pi S$ and fraction field F, and let A be an Azumaya algebra over S. Recall that this implies $A/\pi A$ is a central simple k-algebra and $\mathrm{Jac}(A) = \pi A$. Assume the following holds:

(a) $D = F \otimes_S A$ is a division ring.
(b) $A/\pi A$ has zero divisors.

In addition, assume there is a set $X \subseteq A^\times$ generating A as an S-algebra (such X always exists). Note that conditions (a) and (b) imply that A is not semiperfect because A contains no non-trivial idempotents while $A/\pi A = A/\mathrm{Jac}(A)$ does contain such idempotents, hence $\mathrm{Jac}(A)$ is not idempotent lifting.

Define $R = A \otimes_S A^{\mathrm{op}}$ and let $\Sigma = \{\sigma_x\}_{x\in X}$ where σ_x is conjugation by $1 \otimes x^{\mathrm{op}}$. Then R is an S-Azumaya algebra which is an S-order inside $D \otimes_F D^{\mathrm{op}} \cong M_r(F)$. It is well-known that this implies $R \cong \mathrm{End}(P_S)$ for some faithful finite projective S-module P (in fact, P is free since S is local). Therefore, R is Morita equivalent to S, hence semiperfect. On the other hand, $R^\Sigma = \mathrm{Cent}_R(\{1 \otimes x^{\mathrm{op}} \,|\, x \in X\}) = \mathrm{Cent}_R(S \otimes A^{\mathrm{op}}) = A \otimes S \cong A$, so R^Σ is not semiperfect.

An explicit choice for S, A, F, D is $S = \mathbb{Z}_{\langle 3\rangle}$ ($\pi = 3$), $F = \mathbb{Q}$, $D = (-1,-1)_{\mathbb{Q}} = \mathbb{Q}[i,j \,|\, ij = -ji, i^2 = j^2 = -1]$ and $A = S[i,j]$. If we take $X = \{i,j\}$, then Σ will consist of two inner automorphisms which are easily seen to generate an automorphism group isomorphic to $(\mathbb{Z}/2) \times (\mathbb{Z}/2)$.

Example 1.6.2. Let S, π, A, F, D satisfy conditions (a),(b) of Example 1.6.1. In addition, assume there is a cyclic Galois extension K/F such that:

(c) K/F is totally ramified at π.
(d) $K \otimes_F D$ splits (i.e. $K \otimes_F D \cong M_t(K)$).

Write $\mathrm{Gal}(K/F) = \langle\sigma\rangle$.

Let T denote the integral closure of S in K. Then $\sigma(T) = T$. We claim that $T \otimes_S A$ is semiperfect, but $(T \otimes_S A)^{\{\sigma\otimes 1\}}$ is not. Indeed, $T^{\{\sigma\}} = T \cap F = S$, so $(T \otimes A)^{\{\sigma\otimes 1\}} = S \otimes A \cong A$ which is not semiperfect as explained in the previous example. On the other hand, $T \otimes A$ is a T-Azumaya algebra and a T-order in $K \otimes D \cong M_t(K)$. Again, this implies $T \otimes A$ is Morita equivalent to T. But T is local because K/F is totally ramified at π, therefore $T \otimes A$ is semiperfect.

If we take S, A, F, D as in the previous example, then $T = S[\sqrt{-3}]$, $K = \mathbb{Q}[\sqrt{-3}]$ will satisfy (c) and (d). Indeed, dimension constraints imply $K \otimes D$ is either a division ring or $M_2(K)$, but $\sqrt{-3} + i + j + ij \in K \otimes D$ has reduced norm zero so the latter option must hold.

Example 1.6.3. Start with a semiperfect ring R and $\sigma \in \mathrm{End}(R)$ such that $R^{\{\sigma\}}$ is not semiperfect (e.g. those of Example 1.6.2). Let $R' = R[[t;\sigma]]$ be the ring of σ-twisted formal power series with left coefficients in R (i.e. $\sigma(r)t = tr$ for all $r \in R$) and let $x = 1 + t \in (R')^\times$. We claim R' is semiperfect, but $\mathrm{Cent}_{R'}(x)$ is not. Indeed, $\mathrm{Cent}_{R'}(x) = \mathrm{Cent}_{R'}(t) = R^{\{\sigma\}}[[t]]$. We are finished by applying the following proposition for $R[[t;\sigma]]$ and $R^{\{\sigma\}}[[t]]$.

PROPOSITION 1.6.4. *For any ring W and $\tau \in \mathrm{End}(W)$, W is semiperfect if and only if $W[[t;\tau]]$ is.*

PROOF. Let $V = W[[t;\tau]]$ and let $J = \mathrm{Jac}(W) + Vt \trianglelefteq V$. Then $V/J \cong W/\mathrm{Jac}(W)$. Since the latter ring has zero Jacobson radical, $J \supseteq \mathrm{Jac}(V)$. However, $1 + J \subseteq V^\times$ implies $J \subseteq \mathrm{Jac}(V)$, thus we get $\mathrm{Jac}(V) = J$. The isomorphism $V/J \cong W/\mathrm{Jac}(W)$ now implies that V is semilocal $\iff$ W is semilocal. We finish by observing that Vt is idempotent lifting (this immediate as $V = W \oplus Vt$), hence J is idempotent lifting in $V \iff J/Vt$ is idempotent lifting in $V/Vt \iff \mathrm{Jac}(W)$ is idempotent lifting in W. □

We now show (2), relying on the previous examples.

EXAMPLE 1.6.5. Let R be a semiperfect ring and let $X \subseteq R$ be such that $\mathrm{Cent}_R(X)$ is not semiperfect (the existence of such R and X was shown in previous examples). Let $Y = \{y_a \mid a \in X\}$ be a set of formal variables and let $S = R\langle Y\rangle$ be the ring of non-commutative polynomials in Y over R (Y commutes with R). We can make R into a right S-module by considering the standard right action of R onto itself and extending it to S by defining $r \cdot y_a = ar$ for all $a \in X$. Let M denote the right S-module obtained thusly. Identify R with $\mathrm{End}(M_R) = \mathrm{End}(R_R)$ via $r \mapsto (m \mapsto rm) \in \mathrm{End}(R_R)$. It is straightforward to check that $\mathrm{End}(M_S)$ now corresponds to $\mathrm{Cent}_R(X)$. Therefore, $\mathrm{End}(M_R) \cong R$ is semiperfect but $\mathrm{End}(M_S) \cong \mathrm{Cent}_R(X)$ is not semiperfect.

The next example demonstrates (3).

EXAMPLE 1.6.6. Let p, q be distinct primes. Endow $R = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Q}$ with the product topology (the topology on $\mathbb{Q}$ is the discrete topology). Then R is clearly semiperfect and pro-semiprimary. Define $K = \{(a,a,a) \mid a \in \mathbb{Q}\}$ and let $R_0 = R \cap K$. Then R_0 is a semi-invariant subring of R by Proposition 1.3.3 (take $S = \mathbb{Q}_p \times \mathbb{Q}_q \times \mathbb{Q}$) and it is routine to check R_0 is closed. However, R_0 is not semiperfect. Indeed, it is isomorphic to $T = M^{-1}\mathbb{Z}$ where $M = \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$. The ring T is not semiperfect because it has no non-trivial idempotents while $T/\mathrm{Jac}(T) \cong T/pqT \cong T/pT \times T/qT$ has such. As a result, $\mathrm{Jac}(T)$ cannot be idempotent lifting.

The following example shows that rationally closed subrings of a f.d. algebra need not be semiperfect. As f.d. algebras are semiprimary, this shows that Theorem 1.4.6 fails for rationally closed subrings.

EXAMPLE 1.6.7. Let $K = \mathbb{Q}(x)$ and let $R = K \times K \times K$. Define

$$S' = \{f/g \mid f, g \in \mathbb{Q}[x], g(0) \neq 0, g(1) \neq 0\}$$

and observe that S' is not semiperfect since it is a domain but $S'/\mathrm{Jac}(S')$ has non-trivial idempotents. (Indeed, $S'/\mathrm{Jac}(S') = S'/\langle x(x-1)\rangle \cong S'/\langle x\rangle \times S'/\langle x-1\rangle \cong \mathbb{Q} \times \mathbb{Q}$.) Define $\varphi : S' \to R$ to be the $\mathbb{Q}$-algebra homomorphism obtained by sending x to $a := (0,1,x) \in R$ and let $S = \operatorname{im}\varphi$. It is easy to verify that φ is well-defined and injective, hence S is not semiperfect. However, S is rationally closed in R. To see this, let $q(x) \in S'$ and assume $q(a) \in R^\times$. Then $q(0), q(1) \neq 0$. This implies $q(x) \in (S')^\times$, hence $q(a) = \varphi(q(x)) \in S^\times$.

We finish by demonstrating (5). The subring S of the last example cannot be semi-invariant, for otherwise we would get a contradiction to Theorem 1.4.6. In particular, S is not an invariant subring nor a centralizer subring. Next, let $R = \mathbb{Q}[\sqrt[3]{2}, \sqrt{3}]$. Then the only centralizer subring of R is R itself, the invariant subrings of R are R and $\mathbb{Q}[\sqrt[3]{2}]$ (Remark 1.3.5) and the semi-invariant subrings of R are the four subfields of R (Corollary 1.3.4). In particular, R admits a semi-invariant non-invariant subring and an invariant non-centralizer subring.

1.7. Applications

This section presents applications of the previous results. In order to avoid cumbersome phrasing, we introduce the following families of ring-theoretic properties:

$$\mathscr{P}_{\mathrm{disc}} = \left\{ \begin{array}{c} \text{semiprimary, right perfect, left perfect, } \pi_\infty\text{-regular and semiperfect,} \\ \pi_\infty\text{-regular, } \pi\text{-regular and semiperfect, } \pi\text{-regular} \end{array} \right\}$$

$$\mathscr{P}_{\mathrm{top}} = \left\{ \begin{array}{c|c} \text{pro-}\mathcal{P},\ \text{pro-}\mathcal{P} \text{ and semiperfect,} & \mathcal{P} \in \mathscr{P}_{\mathrm{disc}} \\ \text{quasi-}\mathcal{Q},\ \text{quasi-}\mathcal{Q} \text{ and semiperfect} & \mathcal{Q} \in \{\pi_\infty\text{-regular, } \pi\text{-regular}\} \end{array} \right\}$$

(For example, "quasi-π_∞-regular and semiperfect" lies in $\mathscr{P}_{\mathrm{top}}$.) Note that the properties in $\mathscr{P}_{\mathrm{disc}}$ apply to rings while the properties in $\mathscr{P}_{\mathrm{top}}$ apply to LT rings. Nevertheless, we will sometimes address non-topological rings as satisfying one of the properties of $\mathscr{P}_{\mathrm{top}}$, meaning that they satisfy it w.r.t. *some* linear ring topology. We also define $\mathscr{P}_{\mathrm{mor}}$ (resp. $\mathscr{P}_{\mathrm{sp}}$) to be the set of properties in $\mathscr{P}_{\mathrm{disc}} \cup \mathscr{P}_{\mathrm{top}}$ which are preserved under Morita equivalence (resp. imply that the ring is semiperfect). Recall that a property in $\mathscr{P}_{\mathrm{top}}$ is preserved under Morita equivalence if this holds in the sense of Section 1.5 (and not in the sense of [**47**]). For example, "π-regular" and "quasi-π-regular" do not lie in $\mathscr{P}_{\mathrm{mor}}$ nor in $\mathscr{P}_{\mathrm{sp}}$, "pro-semiprimary and semiperfect" lies in both $\mathscr{P}_{\mathrm{sp}}$ and $\mathscr{P}_{\mathrm{mor}}$, and "pro-semiprimary" lies in $\mathscr{P}_{\mathrm{mor}}$, but not in $\mathscr{P}_{\mathrm{sp}}$.

THEOREM 1.7.1. *Let R be a ring and R_0 a subring.*

(i) *If R has $\mathcal{P} \in \mathscr{P}_{\mathrm{disc}}$ and R_0 is semi-invariant, then R_0 has $\mathcal{P}$.*
(ii) *If $R \in \mathscr{LTR}_2$ has $\mathcal{P} \in \mathscr{P}_{\mathrm{top}}$ and R_0 is T-semi-invariant, then R_0 has $\mathcal{P}$ (w.r.t. the induced topology).*
(iii) *In both (i) and (ii), if $\mathcal{P} \in \mathscr{P}_{\mathrm{sp}}$, then $\mathrm{Jac}(R_0)^n \subseteq \mathrm{Jac}(R)$ for some $n \in \mathbb{N}$.*

Our first application follows from the fact that a centralizer subring is always (T-)semi-invariant:

COROLLARY 1.7.2. *Let $\mathcal{P} \in \mathscr{P}_{\mathrm{disc}} \cup \mathscr{P}_{\mathrm{top}}$ and let R be a ring satisfying $\mathcal{P}$. Then $\mathrm{Cent}(R)$ and any maximal commutative subring of R satisfy $\mathcal{P}$.*

PROOF. $\mathrm{Cent}(R)$ is the centralizer of R and a maximal commutative subring of R is itself's centralizer. Now apply Theorem 1.7.1. □

Surprisingly, the author could not find in the literature results that are similar to the previous corollary, except the fact that the center of a right artinian ring is semiprimary. (This follows from a classical result of Jacobson, stating that the endomorphism ring of any module of finite length is semiprimary, together with the fact that the center of a ring R is isomorphic to $\mathrm{End}({}_RR_R)$.)

The next applications concern endomorphism rings of finitely presented modules. We will only treat here the non-topological properties (i.e. $\mathscr{P}_{\mathrm{disc}}$). The topological analogues of the results to follow require additional notation and are thus postponed to the next section.

THEOREM 1.7.3. *Let R be a ring satisfying $\mathcal{P} \in \mathscr{P}_{\mathrm{disc}} \cap \mathscr{P}_{\mathrm{mor}}$ and let M be a finitely presented right R-module. Then $\mathrm{End}(M_R)$ satisfies $\mathcal{P}$.*

PROOF. There is an exact sequence $R^n \to R^m \to M \to 0$ with $n, m \in \mathbb{N}$. Since R^n, R^m are projective, we may apply Proposition 1.3.7 to deduce that $\mathrm{End}(M)$ is a quotient of a semi-invariant subring of $\mathrm{End}(R^n) \times \mathrm{End}(R^m) \cong \mathrm{M}_n(R) \times \mathrm{M}_m(R)$. The latter has $\mathcal{P}$ because any $\mathcal{P} \in \mathscr{P}_{\rm disc} \cap \mathscr{P}_{\rm mor}$ is preserved under Morita equivalence and under taking finite products. Since all ring properties in $\mathscr{P}_{\rm disc}$ pass to quotients, we are done by Theorem 1.7.1. □

Corollary 1.7.4. *Let $\varphi : S \to R$ be a ring homomorphism. Consider R as a right S-module via φ and assume it is finitely presented. Then if S satisfies $\mathcal{P} \in \mathscr{P}_{\rm disc} \cap \mathscr{P}_{\rm mor}$, so does R.*

Proof. By Theorem 1.7.3, $\mathrm{End}(R_S)$ has $\mathcal{P}$. Therefore, by Corollary 1.4.7, $R \cong \mathrm{End}(R_R)$ has $\mathcal{P}$. □

Remark 1.7.5. Theorem 1.7.3 actually follows from results of Bjork, who proved the semiprimary case and part of the left/right perfect cases ([**19**, Thms. 4.1-4.2]), and Rowen, who proved the left/right perfect and the semiperfect-and-π_∞-regular cases ([**78**, Cr. 11 and Th. 8(iii)]). Our approach suggests a single simplified proof to all the cases. Note that we cannot replace "finitely presented" with "finitely generated" in Theorem 1.7.3; in [**20**, Ex. 2.1], Bjork presents a right artinian ring with a cyclic left module having a non-semilocal endomorphism ring.

By arguing as in the proof of Theorem 1.7.3, one can also obtain:

Theorem 1.7.6. *Let $0 \to A \to B \to C \to 0$ be an exact sequence in an abelian category $\mathscr{A}$ and assume B is projective.*

(i) *If* $\mathrm{End}(A)$ *and* $\mathrm{End}(B)$ *has* $\mathcal{P} \in \mathscr{P}_{\rm disc}$, *then* $\mathrm{End}(C)$ *has* $\mathcal{P}$.
(ii) *If* $\mathrm{End}(A \oplus B)$ *has* $\mathcal{P} \in \mathscr{P}_{\rm sp}$, *then* $\mathrm{End}(C)$ *is semiperfect.*

Next, we turn to representations over modules with "good" endomorphism rings. By a representation of a monoid (ring) G over a right R-module M, we mean a mononid (ring) homomorphism $\rho : G \to \mathrm{End}(M)$ (so G acts on M via ρ).

Corollary 1.7.7. *Let R be a ring and let ρ be a representation of a monoid (or a ring) G over a right R-module M. Assume that one of the following holds:*

(i) $\mathrm{End}(M)$ *has* $\mathcal{P} \in \mathscr{P}_{\rm disc} \cup \mathscr{P}_{\rm top}$.
(ii) *There is a sub-monoid (subring)* $H \subseteq G$ *such that* $\mathrm{End}(\rho|_H)$ *has* $\mathcal{P} \in \mathscr{P}_{\rm disc}$.
(iii) $\mathrm{End}(M)$ *is LT and Hausdorff and there is a sub-monoid (or a subring)* $H \subseteq G$ *such that* $\mathrm{End}(\rho|_H)$ *has* $\mathcal{P} \in \mathscr{P}_{\rm top}$ *w.r.t. the induced topology.*

Then $\mathrm{End}(\rho)$ *has* $\mathcal{P}$. *Moreover, if* $\mathcal{P} \in \mathscr{P}_{\rm sp}$, *then* ρ *has a Krull-Schmidt decomposition* $\rho \cong \rho_1 \oplus \cdots \oplus \rho_t$ *and* $\mathrm{End}(\rho_i)$ *is local and has* $\mathcal{P}$ *for all* $1 \le i \le t$.

Proof. (i) follows from (ii) and (iii) if we take H to be the trivial monoid (or the prime subring of G, if G is a ring). To see (ii) (resp. (iii)), notice that $\mathrm{End}(\rho) = \mathrm{Cent}_{\mathrm{End}(\rho|_H)}(\rho(G))$. Therefore, $\mathrm{End}(\rho)$ is a semi-invariant (resp. T-semi-invariant) subring of $\mathrm{End}(\rho|_H)$, hence by Theorem 1.7.1, $\mathrm{End}(\rho)$ has $\mathcal{P}$.

Now, if $\mathcal{P} \in \mathscr{P}_{\rm sp}$, then $\mathrm{End}(\rho)$ is semiperfect. The Krull-Schmidt Theorem then implies ρ has a Krull-Schmidt decomposition $\rho \cong \rho_1 \oplus \cdots \oplus \rho_t$ and $\mathrm{End}(\rho_i)$ is local for all i. We finish by noting that $\mathrm{End}(\rho_i) \cong e\,\mathrm{End}(\rho)e$ for some $e \in \mathrm{E}(\mathrm{End}(\rho))$ and hence $\mathrm{End}(\rho_i)$ has $\mathcal{P}$ (since for any ring R and $e \in \mathrm{E}(R)$, R has $\mathcal{P}$ implies eRe has $\mathcal{P}$). □

Assume R is a ring and M is a right R-module such that $\mathrm{End}(M_R)$ is semiperfect and quasi-π-regular (see Theorem 1.8.3 below for cases when this happens). Then the endomorphisms of M have a "Jordan decomposition" in the following sense: If $f \in \mathrm{End}(M_R)$, then we can consider M as a right $R[x]$-module by letting x act as f. Clearly $\mathrm{End}(M_{R[x]}) = \mathrm{Cent}_{\mathrm{End}(M_R)}(f)$, so by Theorem 1.5.10, $\mathrm{End}(M_{R[x]})$ is semiperfect. Therefore, $M_{R[x]}$ has a Krull-Schmidt decomposition $M = M_1 \oplus \cdots \oplus M_t$. (Notice that each M_i is an f-invariant submodule of M). This decomposition plays the role of a Jordan decomposition for f, since the isomorphism classes of $M_1, \dots, M_t$ (as $R[x]$-modules) determine the conjugation class of f. In particular, studying endomorphisms of M can be done by classifying LE-modules over $R[x]$.

Finally, the results of this chapter can be applied in a rather different manner to bilinear forms. This will be done in detail in Chapter 4, but we are in a good position to describe the general idea: Let $*$ be an anti-endomorphism of a ring R (i.e. an additive, unity-preserving map that reverses order of multiplication). Then $\sigma = *^2$ is an endomorphism of R and $*$ becomes an involution on the invariant subring $R^{\{\sigma\}}$. As some claims on $(R, *)$ can be reduced to claims on $(R^{\{\sigma\}}, *|_{R^{\{\sigma\}}})$, our results become a useful tool for studying the former. Recalling that bilinear (resp. sesquilinear) forms correspond to certain anti-endomorphisms and quadratic (resp. hermitian) forms correspond to involutions (see [**57**, Ch. I]), these ideas, taken much further, can be used to reduce the isomorphism problem of bilinear forms to the isomorphism problem of hermitian forms. This was actually done (using other methods) for bilinear forms over fields by Riehm ([**76**]), who later generalized this with Shrader-Frechette to sesquilinear forms over semisimple algebras ([**75**]). We can improve these results for bilinear (sesquilinear) forms over various semiperfect pro-semiprimary rings (e.g. f.g. algebras over $\mathbb{Z}_p$). This approach is described in Chapter 4.

1.8. Modules over Linearly Topologized Rings

In this section we extend Theorem 1.7.3 and other applications to LT rings. This is done by properly topologizing modules and endomorphisms rings of modules over LT rings.

Let R be an LT ring and let M be a right R-module. Then M can be made into a topological R-module by taking $\{x + MJ \mid J \in \mathcal{I}_R\}$ as a basis of neighborhoods of $x \in M$. (That M is indeed a topological module follows from [**99**, Th. 3.6].) Notice that any homomorphism of modules is continuous w.r.t. this topology. Furthermore, $\mathrm{End}(M)$ can be linearly topologized by taking $\{\mathrm{Hom}_R(M, MJ) \mid J \in \mathcal{I}_R\}$ as a local basis.[9] We will refer to the topologies just defined on M and $\mathrm{End}(M)$ as their *natural topologies*. In general, that R is Hausdorff does not imply M or $\mathrm{End}(M)$ are Hausdorff. (E.g., for any distinct primes $p, q \in \mathbb{Z}$, the $\mathbb{Z}$-module $\mathbb{Z}/q$ is not Hausdorff w.r.t. the p-adic topology on $\mathbb{Z}$.) Observe that $\overline{\{0_{\mathrm{End}(M)}\}} = \bigcap_{J \in \mathcal{I}_R} \mathrm{Hom}(M, MJ) = \mathrm{Hom}(M, \bigcap_{J \in \mathcal{I}_R} MJ) = \mathrm{Hom}(M, \overline{\{0_M\}})$, so M is Hausdorff implies $\mathrm{End}(M)$ is Hausdorff.

Now let $E_\bullet$ be a finite resolution of M, i.e. $E_\bullet$ consists of an exact sequence $E_{n-1} \to \dots \to E_0 \to E_{-1} = M \to 0$.[10] The maps $E_i \to E_{i-1}$ will be denoted by d_i. We say that $E_\bullet$ has the *lifting property* if any $f_{-1} \in \mathrm{End}(M)$ can be extended to a chain complex homomorphism $f_\bullet : E_\bullet \to E_\bullet$. (Recall that $f_\bullet$ consists of a sequence $\{f_i\}_{i=-1}^{n-1}$ such that $f_i \in \mathrm{End}(E_i)$ and $d_i f_i = f_{i-1} d_i$ for all i.) In other words, $E_\bullet$ has the lifting property if and only if the following commutative diagram can be completed for every $f_{-1} \in \mathrm{End}(M)$.

$$\begin{array}{ccccccccc}
E_{n-1} & \longrightarrow & \dots & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & M \\
\downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} \\
E_{n-1} & \longrightarrow & \dots & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & M
\end{array}$$

[9] This topology is the uniform convergence topology (w.r.t. the natural uniform structure of M). If M is f.g. then this topology coincides with the pointwise convergence topology (i.e. the topology induced from the the product topology on M^M).

[10] We do not require the map $E_{n-1} \to E_{n-2}$ to be injective.

For example, any projective resolution has the lifting property. We define a linear ring topology τ_E on $\mathrm{End}(M)$ as follows: For all $J \in \mathcal{I}_R$, define $\mathrm{B}(J, E)$ to be the set of maps $f_{-1} \in \mathrm{End}(M, MJ)$ that extend to a chain complex homomorphism $f_\bullet : E_\bullet \to E_\bullet$ such that $\mathrm{im}\, f_i \subseteq E_i J$ for all $-1 \leq i < n$. The lifting property implies $\mathrm{B}(J, E) \trianglelefteq \mathrm{End}(M)$ and it is clear that $\mathcal{B}_E := \{\mathrm{B}(J, E) \,|\, J \in \mathcal{I}_R\}$ is a filter base. Therefore, there is a unique ring topology on $\mathrm{End}(M)$, denoted τ_E, having $\mathcal{B}_E$ as a local basis.

It turns out that if $E_\bullet$ is a *projective* resolution, then τ_E only depends on the length of E, i.e. the number n. Indeed, if $P_\bullet, P'_\bullet$ are two *projective* resolutions of length n of M, then the map $\mathrm{id}_M : M \to M$ gives rise to chain complex homomorphisms $\alpha_\bullet : P_\bullet \to P'_\bullet$ and $\beta_\bullet : P'_\bullet \to P_\bullet$ with $\alpha_{-1} = \beta_{-1} = \mathrm{id}_M$. Now, if $J \in \mathcal{I}_R$ and $f_{-1} \in \mathrm{B}(J, P)$, then there is $f_\bullet : P_\bullet \to P_\bullet$ such that $\mathrm{im}\, f_i \subseteq P_i J$ for all i. Define $f'_\bullet = \alpha_\bullet f_\bullet \beta_\bullet$. Then $\mathrm{im}\, f'_i \subseteq \alpha_i(P_i J) \subseteq P'_i J$ for all i and $f'_{-1} = \mathrm{id}_M f_{-1} \mathrm{id}_M = f_{-1}$, so $f_{-1} \in \mathrm{B}(J, P')$. By symmetry, we get $\mathrm{B}(J, P) = \mathrm{B}(J, P')$ for all $J \in \mathcal{I}_R$, hence $\tau_P = \tau_{P'}$.

The topology of $\mathrm{End}(M)$ obtained from a projective resolution of length n will be denoted by τ_n^M and the closure of the zero ideal in that topology will be denoted by I_n^M. Note that $\tau_1^M \subseteq \tau_2^M \subseteq \dots$ and that τ_1^M is the natural topology on $\mathrm{End}(M)$ (i.e. the topology induced from the local basis $\{\mathrm{Hom}(M, MJ) \,|\, J \in \mathcal{I}_R\}$). (Indeed, if $P_\bullet :\ P_0 \to M \to 0$ is a projective resolution of length 1, then any $f \in \mathrm{Hom}(M, MJ)$ can be lifted to $f_0 : P_0 \to P_0 J$ because the map $P_0 J \to MJ$ is onto, hence $\mathrm{B}(P, J) = \mathrm{Hom}(M, MJ)$.) More generally, for any resolution $E_\bullet$ of M, τ_E contains the natural topology on $\mathrm{End}(M)$. Therefore, if M is Hausdorff, then τ_E is Hausdorff. In the addendum, we provide sufficient conditions for $\tau_1^M, \tau_2^M, \dots$ to coincide.

With this terminology, we can generalize Proposition 1.3.7:

PROPOSITION 1.8.1. *Let R be an LT ring and let $E :\ A \to B \to C \to 0$ be an exact sequence of right R-modules satisfying the lifting property (w.r.t. C) and such that A and B are Hausdorff. Assign $\mathrm{End}(A)$ and $\mathrm{End}(B)$ the natural topology and endow $\mathrm{End}(C)$ with τ_E. Then $\mathrm{End}(C)$ is isomorphic as a topological ring to a quotient of a T-semi-invariant subring of $\mathrm{End}(A) \times \mathrm{End}(B)$.*

PROOF. We use the notation of the proof of Proposition 1.3.7. By that proof, $\mathrm{End}(C)$ is isomorphic to a quotient of $\mathrm{Cent}_D(\left[\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right])$. It is easy to check that the embedding $D \hookrightarrow S$ is a topological embedding, hence $\mathrm{End}(C)$ is isomorphic to a quotient of a T-semi-invariant subring of D. That the quotient topology on $\mathrm{End}(C)$ is indeed τ_E is routine. □

We are now in position to generalize previous results.

LEMMA 1.8.2. *Let $R \in \mathscr{LTR}_2$ and $\mathcal{P} \in \mathscr{P}_{\mathrm{disc}}$. Then R is pro-$\mathcal{P}$ if any only if R is complete and R/I has $\mathcal{P}$ for all $I \in \mathcal{I}_R$.*

PROOF. If R is complete and R/I has $\mathcal{P}$ for all $I \in \mathcal{I}_R$, then $R \cong \varprojlim \{R/I\}_{I \in \mathcal{I}_R}$, so R is pro-$\mathcal{P}$. On the other hand, if R is pro-$\mathcal{P}$, then it is complete. In addition, it is strictly pro-$\mathcal{P}$ (Corollary 1.5.16), hence there is a local basis of ideals $\mathcal{B}$ such that R/I has $\mathcal{P}$ for all $I \in \mathcal{B}$. Now, let $I \in \mathcal{I}_R$. Then there is $I_0 \in \mathcal{B}$ contained in I. Therefore, R/I is a quotient of R/I_0. As the latter has $\mathcal{P}$, so does R/I. □

Recall that a topological ring is first countable if it admits a countable local basis. If R is pro-$\mathcal{P}$, then this is equivalent to saying that R is the inverse limit of countably many discrete rings satisfying $\mathcal{P}$.

THEOREM 1.8.3. *Let $R \in \mathscr{LTR}_2$ be a ring and let M be a f.p. right R-module.*

(i) *If R is first countable and satisfies $\mathcal{P} \in \mathscr{P}_{\text{top}} \cap \mathscr{P}_{\text{mor}}$, then* $\text{End}(M)/I_2^M$ *satisfies $\mathcal{P}$ when* $\text{End}(M)$ *is endowed with τ_2^M. In particular, if M is Hausdorff, then* $\text{End}(M)$ *has $\mathcal{P}$.*
(ii) *Assume R is quasi-π_∞-regular and let $i \in \{1,2\}$. Then* $\text{End}(M)/I_i^M$ *is quasi-π_∞-regular when* $\text{End}(M)$ *is endowed with τ_i^M. In particular, if M is Hausdorff, then* $\text{End}(M)$ *is quasi-π_∞-regular w.r.t. τ_1^M.*
(iii) *If R is semiperfect and quasi-π_∞-regular, then* $\text{End}(M)$ *is semiperfect.*

Proof. (i) The argument in the proof of Theorem 1.7.3 shows that $\text{End}(M)$ is a quotient of an LT Hausdorff ring satisfying $\mathcal{P}$, which we denote by W (use Proposition 1.8.1 instead of Proposition 1.3.7). I_2^M is a closed ideal of $\text{End}(M)$ and therefore $\text{End}(M)/I_2^M$ is a quotient of W by a closed ideal. We finish by claiming that for any closed ideal $I \trianglelefteq W$, W/I satisfies $\mathcal{P}$. We will only check the case $\mathcal{P} = \text{pro-}\mathcal{Q}$ for $\mathcal{Q} \in \mathscr{P}_{\text{disc}}$. The other cases are straightforward or follow from the pro-$\mathcal{Q}$ case. Indeed, any open ideal of W/I is of the form J/I for some $J \in \mathcal{I}_W$, hence by Lemma 1.8.2, $(W/I)/(J/I) \cong W/J$ satisfies $\mathcal{Q}$. In addition, that R is first countable implies W is first countable, hence by the Birkhoff-Kakutani Theorem, W is metrizable. By [**22**, p. 163], a Hausdorff quotient of a complete metric ring is complete, hence W/I is complete. Therefore, by Lemma 1.8.2 (applied to W/I), W/I is pro-$\mathcal{Q}$.

(ii) The case $i = 2$ follows from the argument of (i) since being π-regular passes to quotients by closed ideals (the first countable assumption is not needed). As for $i = 1$, since I_1^M is closed in τ_1^M, it is also closed in τ_2^M. Therefore, $\text{End}(M)/I_1^M$ is quasi-π_∞-regular when M is equipped with τ_2^M. We are done by observing that if a ring is quasi-π_∞-regular w.r.t. a given topology, then it is quasi-π_∞-regular w.r.t. any linear Hausdorff sub-topology.

(iii) By (i) $\text{End}(M)$ is a quotient of a semiperfect ring, namely W. □

Remark 1.8.4. Part (iii) of Theorem 1.8.3 was proved in [**79**] for complete semilocal rings with Jacobson radical f.g. as a right ideal and in [**78**] for semiperfect π_∞-regular rings. Both conditions are included in being semiperfect and quasi-π_∞-regular. In addition, Vámos proved in [**96**, Lms. 13-14] that all *finitely generated* or torsion-free of finite rank modules rank over a Henselian integral domain[11] have semiperfect endomorphism ring. Results of similar flavor were also obtained in [**34**], where it is shown that the endomorphism ring of a f.p. (resp. f.g.) module over a semilocal (resp. commutative semilocal) ring is semilocal.

Rowen proved in [**79**] that the endomorphism ring of every f.p. right module M over a complete semilocal ring R with a Jacobson radical f.g. as a right ideal is complete w.r.t. its Jacobson topology ([**79**, Prp. A]), but he proves that $\text{End}(M)$ is complete semilocal only when R is right noetherian ([**79**, Th. B]). Using the previous theorem, we can weaken the right noetherian assumption, thus obtaining the following corollary.

Corollary 1.8.5. *Let R be a complete semilocal ring with Jacobson radical f.g. as a right ideal. Then the endomorphism ring of every f.p. right R-module is complete semilocal.*

Proof. By [**79**, Prp. A], the endomorphism ring is complete w.r.t. to its Jacobson topology and by Theorem 1.8.3(iii) it is semiperfect. □

Corollary 1.8.6. *Let S be a commutative LT ring and let R be an S-algebra s.t. R is f.p. and Hausdorff as an S-module. Then:*

[11] A commutative ring R is called *Henselian* if R is local and *Hensel's Lemma* applies to R.

(i) *If S is quasi-π_∞-regular, then R is quasi-π_∞-regular (w.r.t. to* some *linear ring topology). If moreover S is semiperfect, then so is R.*
(ii) *If S satisfies $\mathcal{P} \in \mathscr{P}_{\mathrm{top}} \cap \mathscr{P}_{\mathrm{mor}}$ w.r.t. a given topology which is also first countable, then R satisfies $\mathcal{P}$.*

Proof. We only prove (ii); (i) is similar. By Theorem 1.7.3, $\mathrm{End}(R_S)$ satisfies $\mathcal{P}$. For all $r \in R$, define $\widehat{r} \in \mathrm{End}(R_S)$ by $\widehat{r}(x) = xr$ and observe that $\mathrm{Cent}_{\mathrm{End}(R_S)}(\{\widehat{r} \,|\, r \in R\}) \cong \mathrm{End}(R_R) = R$, hence R has $\mathcal{P}$ by Theorem 1.7.1. ☐

Let C be a commutative local ring. Azumaya proved in [**5**, Th. 22] that C is Henselian if and only if every *commutative* C-algebra R with R_C f.g. is semiperfect. This was improved by Vámos to non-commutative C-algebras in which all non-units are integral over C; see [**96**, Lm. 12]. Given the previous corollary, Azumaya and Vámos' results suggest that the notions of Henselian and quasi-π_∞-regular might sometimes coincide. This is verified in the following proposition.

Proposition 1.8.7. *Let R be a rank-1 valuation ring. Then R is Henselian if and only if R is quasi-π_∞-regular w.r.t. the topology induced by the valuation.*

Proof. Assume R is quasi-π_∞-regular. Observe that any free R-module is Hausdorff w.r.t. the standard topology, hence Corollary 1.8.6(i) implies that any R-algebra A such that A_R is free of finite rank is semiperfect. Thus, by [**5**, Th. 19], R is Henselian.

Conversely, assume R is Henselian. Denote by ν the (additive) valuation of R. Since ν is of rank 1, we may assume ν take values in $(\mathbb{R}, +)$. For every $\delta \in \mathbb{R}$, let $I_\delta = \{x \in R \,|\, \nu(x) > \delta\}$. Then $\{\mathrm{M}_n(I_\delta) \,|\, \delta \in [0,\infty)\}$ is a local basis for $\mathrm{M}_n(R)$. Let $a \in \mathrm{M}_n(R)$. By the Cayley-Hamilton theorem, a is integral over R, hence $R[a]$ is a f.g. R-module. Let $J = \mathrm{Jac}(R) \cdot R[a]$. Then $J \trianglelefteq R[a]$ and it is well known that $J \subseteq \mathrm{Jac}(R[a])$. The ring $R[a]/J$ is artinian, hence $a + J$ has an associated idemptent $\varepsilon \in \mathrm{E}(R[a]/J)$ (i.e. ε satisfies conditions (A)–(C) of Lemma 1.4.2). By [**5**, Th. 22], J is idempotent lifting, hence there is $e \in \mathrm{E}(R[a])$ such that $e + J = \varepsilon$. Let $f = 1 - e$. Then $a = eae + faf$ (since $R[a]$ is commutative). Furthermore, $eae + J$ is invertible in $\varepsilon(R[a]/J)\varepsilon$, hence eae is invertible in $eR[a]e$ and in particular in $e\mathrm{M}_n(R)e$. Next, $(faf)^k \in J \subseteq \mathrm{M}_n(\mathrm{Jac}(R)) = \mathrm{M}_n(I_0)$ for some $k \in \mathbb{N}$. This means $(faf)^k \in \mathrm{M}_n(I_\delta)$ for some $0 < \delta \in \mathbb{R}$, which implies $(faf)^m \xrightarrow{m\to\infty} 0$. Thus, e satisfies conditions (A),(B) and (C′) w.r.t. a and we may conclude that $\mathrm{M}_n(R)$ is quasi-π-regular for all $n \in \mathbb{N}$. ☐

Using the ideas in the proof of Theorem 1.8.3, we can also obtain:

Theorem 1.8.8. *Let R be an LT ring and let $E: 0 \to A \to B \to C \to 0$ be an exact sequence of right R-modules such that B is projective and A and B are Hasudorff. Endow* $\mathrm{End}(A)$ *and* $\mathrm{End}(B)$ *with the natural topology and* $\mathrm{End}(C)$ *with τ_E and let I_E denote the closure of the zero ideal in* $\mathrm{End}(C)$. *Then:*

(i) *If* $\mathrm{End}(A)$ *and* $\mathrm{End}(B)$ *are first countable and satisfy $\mathcal{P} \in \mathscr{P}_{\mathrm{top}}$, then so is* $\mathrm{End}(C)/I_E$.
(ii) *If* $\mathrm{End}(A)$ *and* $\mathrm{End}(B)$ *are quasi-π-regular, then so is* $\mathrm{End}(C)/I_E$.
(iii) *If* $\mathrm{End}(A)$ *and* $\mathrm{End}(B)$ *are quasi-π-regular and semiperfect, then* $\mathrm{End}(C)$ *is semiperfect.*

In light of the previous results, one might wonder under what conditions all right f.p. modules over an LT ring are Hausdorff. This is treated in the next section and holds, in particular, for right noetherian pro-semiprimary rings.

We finish this section by noting that we can take a different approach for complete Hausdorff modules. For the following discussion, a right R-module M will be called *complete* if the natural map $M \to \varprojlim \{M/MJ\}_{J\in\mathcal{I}_R}$ is an isomorphism.[12]

PROPOSITION 1.8.9. *(i) Let R be a complete first countable Hausdorff LT ring. Then any* Hausdorff *f.g. right R-module is complete.*

(ii) Let $\mathcal{P} \in \mathscr{P}_{\mathrm{disc}}$ and let R be an LT ring such that R/I has $\mathcal{P}$ for all $I \in \mathcal{I}_R$. Let M be a complete right R-module such that M/JM is f.p. as a right R/J-module for all $J \in \mathcal{I}_R$ (e.g. if M is f.p., or if M is f.g. and R is strictly pro-right-artinian). Then $\mathrm{End}(M)$ *is pro-$\mathcal{P}$ w.r.t. τ_1^M. If moreover R is semiperfect and M is f.g., then* $\mathrm{End}(M)$ *is semiperfect.*

PROOF. (i) This is a well-known argument: Let $\mathcal{B}$ be a countable local basis of R consisting of ideals. Without loss of generality, we may assume $\mathcal{B} = \{J_n\}_{n=1}^{\infty}$ with $J_1 \supseteq J_2 \supseteq \dots$. Let M be a f.g. Hausdorff R module and let $\{x_1, \dots, x_n\}$ be a set of generators of M. Since M is Hausdorff, it is enough to show that any sum $\sum_{i=1}^{\infty} m_i$ with $m_i \in MJ_i$ converges in M. Indeed, write $m_i = \sum_{j=1}^{n} x_j r_{ij}$ with $r_{i1}, \dots, r_{in} \in J_i$. Then $\sum_{i=1}^{\infty} r_{ij}$ converges in R for all j, hence $\sum_{i=1}^{\infty} m_i$ converges to $\sum_{j=1}^{n} x_j r_j$ where $r_j = \sum_{i=1}^{\infty} r_{ij}$.

(ii) Throughout, J denotes an open ideal of R. We first note that if M is f.p., then there is an exact sequence $R^n \to R^m \to M \to 0$ for some $n, m \in \mathbb{N}$. Tensoring it with R/J, we get $(R/J)^n \to (R/J)^m \to M/MJ \to 0$, implying M/J is a f.p. R/J-module. Next, if M is f.g. and R is strictly pro-right-artinian, then M/MJ is a f.g. module over R/J which is right artinian, hence M/J is f.p. over R/J.

Now, since M/MJ is f.p. over R/J, $\mathrm{End}(M/MJ)$ satisfies $\mathcal{P}$ by Theorem 1.7.3. There is a natural map $\mathrm{End}(M) \to \mathrm{End}(M/MJ)$ whose kernel is $\mathrm{Hom}(M, MJ)$. Assign $\mathrm{End}(M)$ the natural topology. Then since M is complete, $\mathrm{Hom}(M, M) \cong \varprojlim \{\mathrm{End}(M/MJ)\}_{J\in\mathcal{I}}$ as topological rings and therefore, $\mathrm{End}(M)$ is pro-$\mathcal{P}$.

Finally, assume M is f.g. and R is semiperfect. Then by Proposition 1.2.2, M admits a projective cover P which is easily seen to be finitely generated. Assume $M = M_1 \oplus \dots \oplus M_t$. Then each M_i is f.g. and thus has a projective cover P_i. Necessarily $P \cong P_1 \oplus \dots \oplus P_t$. By Proposition 1.2.3, $\mathrm{End}(P_R)$ is semiperfect, hence there is a finite upper bound on the cardinality of sets of orthogonal idempotents. This means t is bounded and hence, $\mathrm{End}(M)$ cannot contain an infinite set of orthogonal idempotents. By Lemma 1.5.9(ii), this implies $\mathrm{End}(M)$ is semiperfect. □

1.9. LT Rings with Hausdorff Finitely Presented Modules

In this section, we present sufficient conditions on an LT ring guaranteeing all right f.p. modules are Hausdorff (w.r.t. the natural topology). The discussion leads to an interesting consequence about noetherian pro-semiprimary rings.

We begin by noting a famous result that solves the problem for many noetherian rings with the Jacobson topology. For proof and details, see [**80**, Th. 3.5.28].

THEOREM 1.9.1 (Jategaonkar-Schelter-Cauchon). *Let R be an almost fully bounded noetherian ring*[13] *whose primitive images are artinian (e.g. a noetheorian PI ring). Assign R the* $\mathrm{Jac}(R)$*-topology or any stronger linear ring topology. Then any f.g. right R-module is Hausdorff.*

[12] Completeness can also be defined for non-Hausdorff topological abelian groups; see [**99**].

[13] A ring R is *almost bounded* if essential submodules of faithful f.g. right R-modules are also faithful. A ring R is *almost fully bounded* if any prime homeomorphic image of R is fully bounded. See [**80**, §3.5] for details.

EXAMPLE 1.9.2. The assumption that *all* powers of $\mathrm{Jac}(R)$ are open in Theorem 1.9.1 cannot be dropped: Let R be a Dedekind domain with exactly two prime ideals P and Q. Then R is noetherian, almost fully bounded, and any primite image of R is artinian. Let $n \in \mathbb{N} \cup \{0\}$ and let $\mathcal{B} = \{P^m Q^n \mid m \in \mathbb{N}\}$. Assign R the unique topology with local basis $\mathcal{B}$. Clearly $\mathrm{Jac}(R)^k = P^k Q^k$ is open for all $1 \le k \le n$. However, $\overline{\mathrm{Jac}(R)^{n+1}} = \overline{P^{n+1}Q^{n+1}} = \bigcap_{m=1}^{\infty}(P^{n+1}Q^{n+1} + P^m Q^n) = \bigcap_{m=1}^{\infty}(P^{\min\{n+1,m\}}Q^n) = P^{n+1}Q^n$, so $\mathrm{Jac}(R)^{n+1}$ is not closed. In particular, by $(*)$ below, $R/\mathrm{Jac}(R)^{n+1}$ is a f.g. non-Hausdorff R-module.

When considering quasi-π-regular rings, there is actually no point in taking a topology stronger than the Jacobson topology in Theorem 1.9.1, because for right noetherian rings the latter is the largest topology making the ring quasi-π-regular.

PROPOSITION 1.9.3. *Let R be an LT semilocal ring and let τ be the topology on R. Assume R is quasi-π-regular w.r.t. τ and R/I is semiprimary for all $I \in \mathcal{I}_R$ (e.g. if R is right noetherian or pro-semiprimary w.r.t. τ). Then R is quasi-π-regular w.r.t. the Jacobson topology and the latter contains τ.*

PROOF. We first note that if R is right noetherian, then for all $I \in \mathcal{I}_R$, R/I is right noetherian and π-regular, hence by Remark 1.2.9, R/I is semiprimary. If R is pro-semiprimary, then R/I is semiprimary for all $I \in \mathcal{I}_R$ by Lemma 1.8.2.

Let τ_{Jac} denote the Jacobson topology and let $a \in R$. Then a has an associated idempotent e w.r.t. τ. Let $f = 1 - e$ and observe that $\mathrm{Jac}(R)$ is open by Remark 1.5.12(ii). Then there is $n \in \mathbb{N}$ such that $(faf)^n \in \mathrm{Jac}(R)$ and it follows that $(faf)^n \xrightarrow{n\to\infty} 0$ w.r.t. τ_{Jac}. Therefore, e is the associated idempotent of a w.r.t. τ_{Jac}, hence R is quasi-π-regular provided we can verify τ_{Jac} is Hausdorff. This holds since $\tau_{\mathrm{Jac}} \supseteq \tau$, by the proof of Corollary 1.5.17 (which still works under our weaker assumptions). □

Stronger linear topologies are "better" since they have more Hausdorff modules. Note that the topology of an arbitrary qausi-π_∞-regular ring can be stronger than the Jacobson topology. For example, take any non-semiprimary perfect ring R with $\bigcap_{n\in\mathbb{N}} \mathrm{Jac}(R) = \{0\}$ (e.g. $R = \mathbb{Q}[x_1, x_2, x_3, \dots \mid x_m^2 = x_n x_m = 0\ \forall n > 2m]$) and give it the discrete topology.

The next result will rely on the following observation:

- $(*)$ Let R be an LT ring. If M is a right R-module and N is a submodule, then $\overline{N/N} = \overline{N}/N$. In particular, M/N is Hausdorff if and only if N is closed.

Indeed, $\overline{N/N} = \bigcap_{J\in\mathcal{I}_R}(M/N)J = \bigcap_{J\in\mathcal{I}_R}(MJ+M)/N = (\bigcap_{J\in\mathcal{I}_R}(MJ+M))/N = \overline{N}/N$. We will also need the following theorem. For proof, see [**21**, §7.4].

THEOREM 1.9.4. *Let $\{X_i, f_{ij}\}$ be an I-indexed inverse system of non-empty sets. Assume that for each $i \in I$ we are given a family of subsets $T_i \subseteq P(X_i)$ such that for all $i \le j$ in I we have:*

- (a) *$X_i \in T_i$ and T_i is closed under (arbitrary large) intersection.*
- (b) Finite Intersection Property: *If $L \subseteq T_i$ is such that the intersection of finitely many of the elements of L is non-empty, then $\bigcap_{A\in L} A \neq \phi$.*
- (c) *For all $A \in T_j$, $f_{ij}(A) \in T_i$.*
- (d) *For all $x \in X_i$, $f_{ij}^{-1}(x) \in T_j$.*

Then $\varprojlim \{X_i\}_{i\in I}$ is non-empty.[14]

[14] This can be compared to the following topological fact: An inverse limit of an inverse system of non-empty Hausdorff compact topological spaces is non-empty and compact.

LEMMA 1.9.5. *Let R be a ring and let M be a right R-module. Let $\{M_i\}_{i\in I}$ be a family of submodules of M and let $\{x_i\}_{i\in I}$ be elements of M. Then $\bigcap_{i\in I}(x_i+M_i)$ is either empty or a coset of $\bigcap_{i\in I} M_i$.*

PROOF. This is straightforward. □

THEOREM 1.9.6. *Let R be strictly pro-right-artinian. Then any f.g. submodule of a Hausdorff right R-module is closed.*

PROOF. Let $\mathcal{B}$ be a local basis of ideals such that R/J is right artinian for all $J \in \mathcal{B}$. Assume M is a Hausdorff right R-module, let $m_1, \dots, m_k \in M$ and $N = \sum_{i=1}^{k} m_i R$. We will show that $m \in \overline{N}$ implies $m \in N$.

Let $m \in \overline{N}$. For every $J \in \mathcal{B}$ define

$$X_J = \left\{ (a_1, \dots, a_k) \in (R/J)^k \, : \, \sum_i (m_i + MJ)a_i = m + MJ \right\} .$$

Observe that $m \in \overline{N} = \bigcap_{J\in\mathcal{B}}(N+MJ)$, hence for all $J \in \mathcal{B}$ there are $b_1, \dots, b_k \in R$ and $z \in MJ$ such that $\sum m_i b_i = m + z$, implying $X_J \neq \phi$. For all $J \subseteq I$ in $\mathcal{B}$, let f_{IJ} denote the map from $(R/J)^k$ to $(R/I)^k$ given by sending $(b_1 + J, \dots, b_k + J)$ to $(b_1 + I, \dots, b_k + I)$. Then $f_{IJ}(X_J) \subseteq X_I$. It easy to check that $\{X_I, f_{IJ}|_{X_J}\}$ is an inverse system of sets.

For all $J \in \mathcal{B}$, define T_J to be the set consisting of the empty set together with all cosets of (right) R-submodules of $(R/J)^k$ contained in X_J. We claim that conditions (a)-(d) of Theorem 1.9.4 hold. Indeed, X_J is easily seen to be a coset of a submodule of $(R/J)^k$, thus $X_J \in T_J$. In addition, by Lemma 1.9.5, T_J is closed under intersection, so (a) holds. Since R/J is right artinian, so is the $(R/J)^k$ (as a right R-module). Lemma 1.9.5 then implies that cosets of submodules of $(R/J)^k$ satisfy DCC, hence (b) holds. Conditions (c) and (d) are straightforward. Therefore, we may apply Theorem 1.9.4 to deduce that $\varprojlim X_J$ is non-empty.

Let $x \in \varprojlim \{X_J\}_{J\in\mathcal{B}}$. Then x consists of tuples $\{(a_1^{(J)}, \dots, a_k^{(J)}) \in (R/J)^k\}_{J\in\mathcal{B}}$ that are compatible with the maps $\{f_{IJ}\}$. As R is complete, there are $b_1, \dots, b_k \in R$ such that $a_i^{(J)} = b_i + J$ for all $1 \leq i \leq k$ and $J \in \mathcal{B}$. It follows that $m - \sum_i m_i b_i \in \bigcap_{J\in\mathcal{B}} MJ$. As M is Hausdorff, the right hand side is $\{0\}$, so $m = \sum_i m_i b_i \in N$. □

REMARK 1.9.7. Theorem 1.9.6 and its consequences actually hold for the larger class of strictly pro-*right-finitely-cogenerated* rings. A module M over a ring R is called *finitely cogenerated*[15] (abbrev.: f.cog.) if its submodules satisfy the Finite Intersection Property (condition (b) in Theorem 1.9.4). This is equivalent to soc(M) being f.g. and essential in M (see [**58**, Pr. 19.1]). A ring R called *right f.cog.* if R_R is finitely cogenerated. (For example, any right pseudo-Frobeniuos ring is right f.cog.) Among the examples of strictly pro-*right-finitely-cogenerated* rings are complete rank-1 valuation rings. Indeed, if $\nu : R \to \mathbb{R}$ is an (additive) valuation, and R is complete w.r.t. ν, then $R = \varprojlim \{R/\{x \in R \,|\, \nu(x) > n\}\}_{n\in\mathbb{N}}$. For a detailed discussion about f.cog. modules and rings, see [**95**] and [**58**, §19].

Notice that a complete semilocal ring is strictly pro-right-artinian if and only if its Jacobson radical is f.g. as a right module. The latter condition is commonly used when studying complete semilocal rings (e.g. [**79**]). In particular, Hinohara proved Theorem 1.9.6 for complete semilocal rings satisfying it ([**48**, Lm. 3]). (Other authors usually assume the ring is right noetherian.) By (∗) we now get:

[15] Other names used in the literature are “co-finitely generated”, “finitely embedded” or “essentially artinian”.

Corollary 1.9.8. *Let R be a strictly pro-right-artinian ring. Then any f.p. right R-module is Hausdorff.*

We can now prove that under mild assumptions, strictly pro-right-artinian rings are complete semilocal.

Corollary 1.9.9. *Let R be a strictly pro-right-artinian ring. If $J \subseteq \mathrm{Jac}(R)$ is an ideal that is f.g. as a right ideal, then R is complete in the J-adic topology (i.e. $R \cong \varprojlim \{R/J^n\}_{n\in\mathbb{N}}$). If moreover R/J is right artinian, then the topology on R is the Jacobson topology! In particular, if $\mathrm{Jac}(R)$ is f.g. as a right ideal, then the topology on R is the Jacobson topology and R is complete semilocal.*

Proof. Let $\mathcal{B}$ be a local basis of ideals of R such that R/I is right artinian for all $I \in \mathcal{B}$. We identify R with its natural copy in $\prod_{I\in\mathcal{B}} R/I$. Since J is f.g. as a right ideal, then so are its powers. Therefore, by Theorem 1.9.6, J^n is closed for all $n \in \mathbb{N}$.

Let φ denote the standard map from R to $\varprojlim \{R/J^n\}_{n\in\mathbb{N}}$. Define a map $\psi : \varprojlim \{R/J^n\}_{n\in\mathbb{N}} \to R$ as follows: Let $r \in \varprojlim \{R/J^n\}_{n\in\mathbb{N}}$ and let r_n denote the image of r in R/J^n. By Corollary 1.5.17, for all $I \in \mathcal{B}$, there is $n \in \mathbb{N}$ (depending on I) such that $J^n \subseteq I$. Let r_I denote the image of r_n in R/I. It is easy to check that r_I is independent of n and that $\widehat{r} := (r_I)_{I\in\mathcal{B}} \in R$. Define $\psi(r) = \widehat{r}$.

It is straightforward to check that $\psi \circ \varphi = \mathrm{id}$. Therefore, we are done if we show that ψ is injective. Let $y \in \ker\psi$ and let $y_n + J^n$ be the image of y in R/J^n. Then for all $I \in \mathcal{B}$, $J^n \subseteq I$ implies $y_n \in I$. This means $y_n \in \bigcap_{J^n\subseteq I\in\mathcal{B}} I = \overline{J^n} = J^n$, so $y_n + J^n = 0 + J^n$ for all $n \in \mathbb{N}$, hence $y = 0$.

Now assume R/J is right artinian. Then $\mathrm{Jac}(R)^k \subseteq J \subseteq \mathrm{Jac}(R)$ for some $k \in \mathbb{N}$, hence the Jacobson topology and the J-adic topology coincide. By Proposition 1.9.3, the topology on R is contained in the Jacobson topology, so we only need to show the converse. Let $n \in \mathbb{N}$. It is enough to show that J^n is open. Indeed, since J_R is f.g., then so is $(J^i/J^{i+1})_R$ $(i \geq 0)$. As $(R/J)_R$ has finite length, $(J^i/J^{i+1})_R$ has finite length. Thus, $(R/J^n)_R$ have finite length as well. Since J^n is closed, J^n is an intersection of open ideals. As $(R/J^n)_R$ is of finite length, J^n is the intersection of finitely many of those ideals, hence open. □

Corollary 1.9.10. *Let R be a right noetherian pro-π-regular ring. Then the topology on R is the Jacobson topology, R is strictly pro-right-artinian w.r.t. it and any right ideal of R is closed. In particular, R is semilocal complete.*

Proof. By Lemma 1.8.2, R/I is π-regular for all $I \in \mathcal{I}_R$, hence Remark 1.2.9 implies R/I is right artinian for all $I \in \mathcal{I}_R$ (since R/I is right noetherian). Therefore, R is pro-right-artinian, with $\mathrm{Jac}(R)_R$ finitely generated. Now apply the previous corollary. □

The next example demonstrates that Theorem 1.9.6 fails for pro-artinian rings (and in particular for pro-semiprimary rings). It also implies that there are pro-artinian rings that are not strictly pro-right-artinian.

Example 1.9.11. Let $S = \mathbb{Q}(x)[t \,|\, t^3 = 0]$. For all $n \in \mathbb{N}$ define $R_n = \mathbb{Q}(x^{2^n}) + \mathbb{Q}(x)t + \mathbb{Q}(x)t^2 \subseteq S$ and $I_n = \mathbb{Q}(x^{2^n})t^2 \subseteq S$. Then R_n is an artinian ring and $I_n \trianglelefteq R_n$. For $n \leq m$ define a map $f_{nm} : R_m/I_m \to R_n/I_n$ by $f_n(x + I_m) = x + I_n$. Then $\{R_n/I_n, f_{nm}\}$ is an inverse system of artinian rings. Let $R = \varprojlim \{R_n/I_n\}_{n\in\mathbb{N}}$. Then R can be identified with $\mathbb{Q} + \mathbb{Q}(x)t + Vt^2$ where V is the $\mathbb{Q}$-vector space $\varprojlim \{\mathbb{Q}(x)/\mathbb{Q}(x^{2^n})\}_{n\in\mathbb{N}}$ (R does not embed in S). Observe that $\mathbb{Q}(x)$ is dense in V, but $\mathbb{Q}(x) \neq V$ since V is not countable (it contains a copy of all power series $\sum a_n x^{2^n} \in \mathbb{Q}[[x]]$). Therefore, the ideal $tR = \mathbb{Q}t + \mathbb{Q}(x)t^2$ is not closed in R and by $(*)$, R/tR is a non-Hausdorff f.p. module. We also note that $\mathrm{Jac}(R)^2 = \mathbb{Q}(x)t^2$ is not closed (but $\mathrm{Jac}(R)$ must be closed by Proposition 1.5.11).

We conclude by specializing the results of the previous section to first countable strictly pro-right artinian rings. (We are guaranteed that all f.p. modules are Hausdorff in this case). By Corollary 1.9.10, this family include all noetherian pro-semiprimary rings. More general statements can be obtained by applying Remark 1.9.7.

Corollary 1.9.12. *(i) Let R be a first countable pro-right-artinian ring and let M be a f.p. right R-module. Then* $\mathrm{End}(M_R)$ *is pro-semiprimary and first countable (w.r.t. τ_2^M). If R is semiperfect (e.g. if R is right noetherian), then* $\mathrm{End}(M_R)$ *is semiperfect.*

(ii) Let S be commutative first countable pro-right-artinian ring and let R be an S-algebra s.t. R is f.p. as an S-module. Then R is pro-semiprimary (w.r.t. some topology). If S is semiperfect (e.g. if S is right noetherian), then R is semiperfect.

1.10. Further Remarks

It is likely that the theory of semi-invariant subrings developed in section 1.5 can be extended to *right* linearly topologized rings, i.e. topological rings having a local basis consisting of *right* ideals. This actually has the following remarkable implication (compare with Corollary 1.7.4 and Corollary 1.8.6):

Conjecture 1.10.1. *Let $S \in \mathscr{LTR}_2$ be a semiperfect quasi-π_∞-regular ring and let $\varphi : S \to R$ be a ring homomorphism. Assume that:*

(a) *When considered as a right S-module via φ, R is f.p. and Hausdorff.*

(b) *For all $r \in R$ and $I \in \mathcal{I}_S$, there is $J \in \mathcal{I}_S$ such that $R\varphi(J)r \subseteq R\varphi(I)$.*[16]

Then R is semiperfect and quasi-π_∞-regular (w.r.t. some topology).

The proof should be along the following lines: For any right S-module M, let W denote the ring of continuous $\mathbb{Z}$-homomorphisms from M to itself. Then W can be made into a *right LT* ring by taking $\{\mathrm{B}(J) \mid J \in \mathcal{I}_S\}$ as a local basis where $\mathrm{B}(J) = \{f \in W \,:\, \mathrm{im}\, f \subseteq MJ\}$ (this is the topology of uniform convergence).[17] Clearly W contains $\mathrm{End}(M_S)$ as a topological ring (endow $\mathrm{End}(M_S)$ with τ_1^M). Now take $M = R$ (where R is viewed as a right S-module via φ). Then condition (a) implies $\mathrm{End}(R_S)$ is semiperfect and quasi-π_∞-regular w.r.t. τ_1^R (Theorem 1.8.3). Condition (b) implies that for all $r \in R$, the map $\widehat{r} : x \mapsto xr$ from R to itself is continuous and hence lie in W. Since we assume the results of section 1.5 extend to right LT rings, $R \cong \mathrm{End}(R_R) = \mathrm{Cent}_{\mathrm{End}(R_S)}(\{\widehat{r} \mid r \in R\})$ is a T-semi-invariant subring of $\mathrm{End}(R_S)$, so R is semiperfect and quasi-π_∞-regular.

Examples of rings satisfying conditions (a) and (b) can be produced by taking R to be: (1) a twisted group algebra $S^\alpha G$ where G is a *finite* group and $\alpha : G \to \mathrm{Aut_c}\, G$ is a group homomorphism or (2) a "crossed product", i.e. $R = \mathrm{CrossProd}(S, \psi, G)$ where S is commutative, G is finite and acts on S via continuous automorphisms and $\psi \in H^2(G, S^\times)$. (Further examples can be produced by taking quotients.) However, we can show directly that the conjecture holds in these special cases. Indeed, that G is finite implies $\mathcal{B} = \{\bigcap_{g\in G} g(I) \mid I \in \mathcal{I}_R\}$ is a local basis of S and we have $RJ \subseteq JR$ for all $J \in \mathcal{B}$. For any right R-module M, let $W' = \{f \in W \,:\, f(MJ) \subseteq MJ\ \forall J \in \mathcal{B}\}$ (with W as in the previous paragraph).

[16] This is equivalent to saying that the topology on R spanned by cosets of the left ideals $\{R\varphi(I) \mid I \in \mathcal{I}_S\}$ is a ring topology; see [**99**, §3].

[17] Caution: Not every filter base of right ideals gives rise to a ring topology. By [**99**, §3], we need to check that for all $f \in W$ and $I \in \mathcal{I}_S$ there is $J \in \mathcal{I}_S$ such that $f\,\mathrm{B}(J) \subseteq \mathrm{B}(I)$. Indeed, we can take any J with $f(MJ) \subseteq MI$ and such J exists since f is continuous.

Then W' is a *linearly topologized* ring w.r.t. the topology induced from W (as seen by taking the local basis $\{W' \cap \mathrm{B}(J) \mid J \in \mathcal{B}\}$). In addition, when $M = R_S$, $\widehat{r}$ of the previous paragraph lies in W' (since $RJ \subseteq JR$ for all $J \in \mathcal{B}$). Therefore, repeating the argument of the last paragraph with W' instead of W, we get that R is semiperfect and quasi-π_∞-regular.

We could neither find examples nor contradict the existence of the following:

(1) a pro-semiprimary ring that is not complete semilocal (i.e. complete w.r.t. its Jacobson topology);
(2) a complete semilocal ring, endowed with the Jacobson topology, with a non-Hausdorff f.p. module.

1.11. Addendum: When Do $\tau_1^M, \tau_2^M, \ldots$ Coincide?

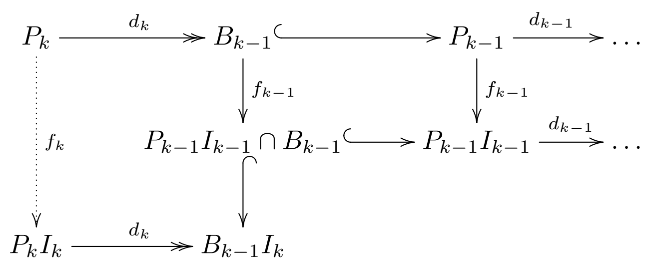
This addendum is dedicated to the question of when the topology obtained from a resolution is the natural topology. For that purpose, we briefly recall the Artin-Rees property for ideals. For details and proofs of the statements to follow, see [**80**, §3.5D].

Let R be a *right noetherian* ring. An ideal $I \trianglelefteq R$ is said to satisfy the *Artin-Rees property* (abbreviated: AR-property) if for any right ideal $A \leq R$ there is $n \in \mathbb{N}$ such that $I^n \cap A \subseteq AI$. This is well known to imply that for any f.g. right R-module M and a submodule N, there is $n \in \mathbb{N}$ such that $MI^n \cap N \subseteq NI$. For example, by [**80**, p. 462, Ex. 19], every *polycentral* ideal (e.g. an ideal generated by central elements) satisfies the AR-property. In addition, if R is *almost bounded* (e.g. a PI ring), then all ideals of R are satisfy the AR-property.

Now let R be any LT ring. A right R-module M is said to satisfy the *topological Artin-Rees property* (abbreviated: TAR-property) if for any submodule $N \subseteq M$ and any $I \in \mathcal{I}_R$ there is $J \in \mathcal{I}_R$ such that $MJ \cap N \subseteq NI$. (Equivalently, the induced topology and the natural topology coincide for any submodule of M). For example, if R is right noetherian, $J \trianglelefteq R$ and R is given the J-adic topology, then all f.g. right R-modules satisfy the TAR-property if and only if J satisfies the AR-property.

PROPOSITION 1.11.1. *Let R be an LT ring, let M be a right R-module and let $P : P_{n-1} \to \cdots \to P_0 \to P_{-1} = M \to 0$ be a projective resolution of M. Assume that $P_0, \ldots, P_{n-1}$ have the TAR-property. Then τ_P is the natural topology on* $\mathrm{End}(M)$*.*

PROOF. Denote by d_i the map $P_i \to P_{i-1}$ and let $B_i = \ker d_i$. We will prove that for all $I \in \mathcal{I}_R$ there is $J \in \mathcal{I}_R$ such that $\mathrm{Hom}(M, MJ) \subseteq \mathrm{B}(I, P)$. Given $I \in \mathcal{I}_R$, we define a sequence of open ideals $I_{n-1}, I_{n-2}, \ldots, I_{-1}$ as follows: Let $I_{n-1} = I$. Given I_i, take I_{i-1} to be an open ideal such that $P_{i-1}I_{i-1} \cap B_{i-1} \subseteq B_{i-1}I_i$ and $I_{i-1} \subseteq I_i$ (the existence of I_{i-1} follows from the TAR-property). We claim that $\mathrm{Hom}(M, MI_{-1}) \subseteq \mathrm{B}(I, P)$. To see this, let $f_{-1} \in \mathrm{Hom}(M, MI_{-1})$ and assume we have constructed maps $f_i \in \mathrm{Hom}(P_i, P_iI_i)$ for all $-1 \leq i < k$ such that $d_if_i = f_{i-1}d_i$. Then it is enough to show there is $f_k \in \mathrm{Hom}(P_k, P_kI_k)$ such that $d_kf_k = f_{k-1}d_k$. The argument to follow is illustrated in the following diagram:



That $d_{k-1}f_{k-1} = f_{k-2}d_{k-1}$ implies $f_{k-1}(B_{k-1}) \subseteq B_{k-1}$. As $\operatorname{im} f_{k-1} \subseteq P_{k-1}I_{k-1}$, we get that $f_{k-1}(B_{k-1}) \subseteq P_{k-1}I_{k-1} \cap B_{k-1} \subseteq B_{k-1}I_k$ (by the definition of I_{k-1}). Since the map $P_kI_k \to B_{k-1}I_k$ is onto (because $\operatorname{im}(d_k) = B_{k-1}$), we can lift $f_{k-1}d_k : P_k \to B_{k-1}I_k$ to a homomorphism $f_k : P_k \to P_kI_k$, as required. □

Remark 1.11.2. The proof still works if we replace the assumption that P_{n-1} is projective with d_{n-1} is injective.

Corollary 1.11.3. *Let R be an LT right noetherian ring admitting local basis of ideals $\mathcal{B}$ such that: (1) All ideals in $\mathcal{B}$ have the AR-property (e.g. if all ideals in $\mathcal{B}$ are generated by central elements or if R is PI) and (2) all powers of ideals in $\mathcal{B}$ are open. Then $\tau_1^M = \tau_2^M = \dots$ for any f.g. right R-module M.*

Proof. Let $n \in \mathbb{N}$. Since R is right noetherian, any f.g. R-module admits a resolution of length n consisting of f.g. projective modules. The assumptions (1) and (2) are easily seen to imply that any f.g. R-module satisfies the TAR-property. Therefore, by Proposition 1.11.1, τ_n^M is the natural topology on $\operatorname{End}(M)$. □

Example 1.11.4. Condition (2) in Corollary 1.11.3 is essential even when all ideals of R have the AR-property: Assign $\mathbb{Z}$ the unique topology with local basis $\mathcal{B} = \{2 \cdot 3^n\mathbb{Z} \,|\, n \geq 0\}$ and let $M = \mathbb{Z}/4 \times \mathbb{Z}/2$. Since $MI = 2M$ for all $I \in \mathcal{B}$, the natural topology on $\operatorname{End}(M)$ is obtained from the local basis $\{\operatorname{Hom}(M, 2M)\}$. ($M$ is not Hausdorff). Let $I = 2\mathbb{Z} \in \mathcal{B}$ and consider the projective resolution

$$P: \ 4\mathbb{Z} \times 2\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z} \to M \to 0 \ .$$

Define $f_{-1} : M \to MI = 2M$ by $f(x + 4\mathbb{Z}, y + 2\mathbb{Z}) = (2y + 4\mathbb{Z}, 0)$. Then any lifting $f_0 \in \operatorname{End}(\mathbb{Z} \times \mathbb{Z})$ of f_{-1} must satisfy $f_0(0, 1) = (4x + 2, 2y)$ for some $x, y \in \mathbb{Z}$. This means that any lifting $f_1 \in \operatorname{End}(4\mathbb{Z} \times 2\mathbb{Z})$ of f_0 (there is only one such lifting) satisfies $f_1(0, 2) = (8x + 4, 4y) \notin 8\mathbb{Z} \times 4\mathbb{Z} = (4\mathbb{Z} \times 2\mathbb{Z})I$. Therefore, $f_1 \notin \operatorname{Hom}(4\mathbb{Z} \times 2\mathbb{Z}, (4\mathbb{Z} \times 2\mathbb{Z})I)$, implying $f_{-1} \notin \mathrm{B}(P, I)$. But this means that $\mathrm{B}(P, I) \subsetneq \operatorname{Hom}(M, 2M)$, hence $\tau_2^M \neq \tau_1^M$.

CHAPTER 2

Bilinear Forms over Rings

Bilinear forms over (non-commutative) rings were considered by various authors (e.g. [**6**], [**55**], [**10**], [**56**]), but the base ring was always assumed to have an involution. In this chapter, we present a new notion of bilinear forms over arbitrary rings (no involution is needed) and show that it generalizes all the definition mentioned. (In particular, our definition includes sesquilinear forms over rings with involution.)

We then consider four basic properties of bilinear forms: the adjoint map is injective (i.e.: being nondegenerate), the adjoint map is surjective, having a corresponding anti-endomorphism and having a unique asymmetry map. All these properties have left and right versions. While all eight properties are equivalent for sesquilinear forms over division rings, this is not the case for our general setting. We therefore set to determine the logical implications between (subsets of) these conditions and demonstrate the non-implications. (Some parts of this project are still open.) In addition, we examine whether these properties are preserved under orthogonal sums.

Next, we present *categories with a double duality* which generalize *hermitian categories* (or *categories with duality*). The latter are the categorical analogues of bilinear or quadratic forms (see [**71**], [**7**] or [**86**, Ch. 7]) and likewise, categories with a double duality are a categorical analogue of our notion. We explain how our definition is connected to the classical one and show that our notion of bilinear forms cannot be naturally understood as a special case of a hermitian category.

We finish the chapter with applying our new definition to solve a problem suggested to the author by D. Saltman: For a ring R, what are the implications between the following three properties: (1) there is S, Morita equivalent to R, with an involution, (2) there is S, Morita equivalent to R, with an anti-automorphism and (3) R is Morita equivalent to R^{op}. Clearly (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and in [**82**], Saltman proved (2)$\Longrightarrow$(1) for Azumaya algebras. We show that (2)$\not\Longrightarrow$(1) in general, and for a large class of rings (e.g. semiperfect rings), (3)$\Longrightarrow$(2).

The results of this chapter will also be used in Chapters 3 and 4. Some of these results are described in [**39**] and [**40**].

Section 2.1 presents our new notion of bilinear forms, and in section 2.2 we study their basic properties. Sections 2.3, 2.4 and 2.5 are concerned with determining the implications and non-implications between the left and right versions of the four properties mentioned above; section 2.3 determines the implications, section 2.4 presents counterexamples, and section 2.5 demonstrates that in special cases one can strengthen the results of section 2.3. Section 2.6 defines and studies orthogonal sums. At the end of this section several constructions of Witt and Witt-Grothendick groups are considered. In section 2.7 we introduce categories with a double duality and relate them to hermitian categories. Section 2.9 presents the application briefly described before.

We note that sections 2.3-2.5 and section 2.7 are not mandatory and the reader can skip either of them without loss of continuity.

2.1. Definitions

In this section we present our new definition of bilinear forms over rings and all notions derived from it. However, before doing so, let us briefly recall the definition that is common in the literature nowadays. Throughout, bilinear forms are not assumed to be symmetric.

Let R be a ring with involution $*$. Recall that a *sesquilinear space* over R is a pair (M, b) such that M is a right R-module and $b : M \times M \to R$ is a biadditive map satisfying

$$b(xr, y) = r^* b(x, y), \qquad b(x, yr) = b(x, y)r, \qquad \forall x, y \in M,\ r \in R\ .$$

In this case, b is called a *sesquilinear form.* If there is $\lambda \in \mathrm{Cent}(R)$ such that $\lambda\lambda^* = 1$ and b satisfies the additional condition $b(x, y) = \lambda b(y, x)^*$, then b is called a λ-hermitian form. The ring R is usually taken to be a division ring or a commutative ring. If R is a field and $*$ is the identity, then sesquilinear (1-hermitian, (-1)-hermitian) forms become classical bilinear (symmetric bilinear, anti-symmetric bilinear) forms.

Hermitian categories or *categories with duality* generalize sesquilinear forms and they will be briefly described in section 2.7.

To present our new notion of bilinear forms, we will need the following definition:

DEFINITION 2.1.1. *Let R be a ring. A* (right) double R-module *is an additive group M together with two operations $\odot_0, \odot_1 : M \times R \to M$ such that M is a right R-module with respect to each of $\odot_0$, $\odot_1$ and*

$$(m \odot_0 a) \odot_1 b = (m \odot_1 b) \odot_0 a \qquad \forall\ m \in M,\ a, b \in R\ .$$

We let M_i denote the R-module obtained by letting R act on M via $\odot_i$.

The category of (right) double R-modules will be denoted by DMod-R. *For $M, N \in$ DMod-R, we define* $\mathrm{Hom}(M, N) = \mathrm{Hom}_R(M_0, N_0) \cap \mathrm{Hom}_R(M_1, N_1)$. *This makes* DMod-$R$ *into an abelian category. (The category* DMod-R *is isomorphic to* Mod-$(R \otimes_{\mathbb{Z}} R)$ *and also to the category of* (R^{op}, R)*-bimodules.)*[1]

Let R be any ring (R need not be commutative; no involution on R is required). A *bilinear space* over R is a triplet (M, b, K) such that $M \in$ Mod-R, $K \in$ DMod-R and $b : M \times M \to K$ is a biadditive map satisfying:

$$b(xr, y) = b(x, y) \odot_0 r, \qquad b(x, yr) = b(x, y) \odot_1 r \qquad \forall x, y \in M,\ r \in R\ .$$

In this case, b is called a *bilinear form.*

An *anti-isomorphism* of K is a bijective map $\kappa : K \to K$ satisfying:

$$(k \odot_i a)^\kappa = k^\kappa \odot_{1-i} a \qquad \forall a \in R,\ k \in K,\ i \in \{0, 1\}\ .$$

If additionally $\kappa \circ \kappa = \mathrm{id}_K$, then κ is called an *involution.* Given such an involution, b is called κ-symmetric if

$$b(x, y) = b(y, x)^\kappa \qquad \forall x, y \in M\ .$$

EXAMPLE 2.1.2. Let $(R, *)$ be a ring with involution. We can make any sesquilinear form $b : M \times M \to R$ fit into our definition; simply turn R into a double R-module by defining $r \odot_0 a = a^* r$ and $r \odot_1 a = ra$ for all $a, r \in R$. Moreover, if b is λ-hermitian, then b is κ_λ-symmetric where $\kappa_\lambda : R \to R$ is defined by $r^{\kappa_\lambda} = \lambda r^*$.

[1] The reader might think that it would be simpler if we were to use (R^{op}, R)-bimodules instead of double R-modules. However, the latter saves notation, prevents ambiguity and makes the proofs in the following sections more comprehensible.

After presenting our definitions, it remains to generalize common properties of bilinear forms to our general setting. Henceforth, R is a ring and K is some fixed double R-module.

We begin by introducing the adjoint maps. Given $M \in \text{Mod-}R$ and $i \in \{0,1\}$, the *i-K-dual* (or just i-dual) of M is defined to be $M^{[i]} := \mathrm{Hom}_R(M, K_{1-i})$.[2] Note $M^{[i]}$ is naturally a right R-module w.r.t. the operation $(fr)(m) = (fm) \odot_i r$ (for all $f \in M^{[i]}$, $r \in R$ and $m \in M$). Moreover, $M \mapsto M^{[i]}$ is a left-exact contravariant functor from Mod-R to itself, which we denote by $[i]$. In section 2.2, we will show that if $[0]$ is considered as a (covariant) functor from $(\text{Mod-}R)^{\text{op}}$ to Mod-R and $[1]$ is considered as a functor form Mod-R to $(\text{Mod-}R)^{\text{op}}$, then $[0]$ is left adjoint to $[1]$.

Let $b : M \times M \to K$ be a bilinear form. The *left adjoint* and *right adjoint* of b are defined as following:

$$\mathrm{Ad}_b^\ell : M \to M^{[0]}, \quad (\mathrm{Ad}_b^\ell m)(n) = b(m,n) \ ,$$
$$\mathrm{Ad}_b^r : M \to M^{[1]}, \quad (\mathrm{Ad}_b^r m)(n) = b(n,m) \ ,$$

for all $m, n \in M$. It can be easily checked that Ad_b^ℓ and Ad_b^r are right R-linear. We say that:

(R0) b is *right regular* if Ad_b^r is bijective;
(R1) b is *right injective* if Ad_b^r is injective;
(R2) b is *right surjective* if Ad_b^r is surjective.

Denote the left analogues of (R0),(R1),(R2) by (L0),(L1),(L2). Note that being right injective means that $b(M,m) = 0$ implies $m = 0$, namely b is *right nondegenerate*. Therefore, forms not satisfying (R1) will be called *right degenerate*. Being right surjective implies that any $f \in M^{[1]}$ is of the form $x \mapsto b(x,m)$ for some $m \in M$.

By addressing a bilinear form as satisfying a property without indicating whether it is the left or right version of that property, we mean that the form satisfies both versions. For example, "b is regular" means "b is left and right regular" and likewise for all properties defined in this section.

We now turn to define the corresponding anti-endomorphism of a bilinear form (see [**57**, Ch. 1] to compare with the classical definition). With notation as above:

(R3) b is called *right stable* if for every $\sigma \in \mathrm{End}(M_R)$ there exists a *unique* $\sigma' \in \mathrm{End}(M_R)$ satisfying $b(\sigma x, y) = b(x, \sigma' y)$ for all $x, y \in M$.

Denote the left analogue of (R3) by (L3). If b is right stable, then the map $*$ sending σ to σ' is an anti-endomorphism of $\mathrm{End}(M_R)$, called the *(right) corresponding anti-endomorphism* of b. Example 2.1.4 below shows that even when b is right regular, $*$ need not be injective nor surjective, hence we use anti-*endo*morphisms rather than anti-*auto*morphism. (Moreover, this example shows that *any* anti-endomorphism can be understood as a corresponding anti-endomorphism of some right regular bilinear form.) It is easy to verify that if b is κ-symmetric, where κ is an involution of K, then its corresponding anti-endomorphism is in an involution. The connection between bilinear forms and anti-endomorphisms will be discussed extensively in the next chapter.

Next, we define asymmetry maps, which are important tools in studying non-symmetric forms (see [**76**] and [**75**] for classical applications). Let κ be an anti-isomorphism of K. A *right κ-asymmetry* (resp. left κ-asymmetry) of b is a map $\lambda \in \mathrm{End}(M_R)$ such that $b(x,y)^\kappa = b(y, \lambda x)$ (resp. $b(x,y)^\kappa = b(\lambda y, x)$) for all $x, y \in M$. It is natural to consider the following property:

(R4) b has a unique right κ-asymmetry.

[2] The reason that we do not define $M^{[i]}$ to be $\mathrm{Hom}_R(M, K_i)$ is because we want $R^{[i]}$ to isomorphic to K_i via $f \leftrightarrow f(1)$.

Again, denote the left analogue by (L4). We will sometimes need to distinguish between two anti-isomorphisms and then we will write (R4)-κ, (L4)-κ instead of (R4), (L4). The inverse of an invertible κ-asymmetry is always a left κ^{-1}-asymmetry. (However, the asymmetry need not be invertible even when it is unique! See Example 2.4.12.)

REMARK 2.1.3. It might seem odd to consider κ-asymmetries for κ that is not an involution. However, we will see below that this is natural in some situations. Moreover, some double R-modules admit an anti-isomorphism but no involution (see Example 2.4.14).

Finally, the following property will also be useful:

(R5) b is called *right semi-stable* if for all $\sigma \in \mathrm{End}(M_R)$, $b(x, \sigma y) = 0$ for all $x, y \in M$ implies $\sigma = 0$.

Being semi-stable can be considered as a weaker version of nondegeneracy. It implies the uniqueness of σ' and λ from (R3) and (R4), provided they exist.

EXAMPLE 2.1.4. Let R be a ring and let $*$ be an anti-endomorphism of R. Let K be the double R-module obtained from R by defining

$$r \odot_0 s = s^* r, \qquad r \odot_1 s = rs \qquad \forall\, r, s \in R\ .$$

Define $b : R \times R \to K$ by $b(x, y) = x^* y$. Then b is a bilinear form. As $b(R, x) = 0$ implies $x = 0$ (since $x = b(1, x) = 0$), b is right injective. In addition, it is straightforward to check for all $f \in R^{[1]} = \mathrm{Hom}_R(R_R, K_0)$, $\mathrm{Ad}^r_b(f(1)) = f$, hence b is also surjective. Therefore, b is right regular and we will later show that this implies b is right stable.

Now observe that for all $r, x, y \in R$, $b(rx, y) = (rx)^* y = x^* r^* y = b(x, r^* y)$. Thus, identifying $\mathrm{End}(R_R)$ with R via $f \leftrightarrow f(1)$, the corresponding anti-endomorphism of b is $*$. It is also straightforward to check that $\ker(\mathrm{Ad}^\ell_b) = \ker(*)$ and $\mathrm{im}(\mathrm{Ad}^\ell_b) = \mathrm{im}(*)$ (once identifying $R^{[0]} = \mathrm{Hom}(R_R, K_1) = \mathrm{End}(R_R)$ with R as before). Hence, b is left injective (surjective) if and only if $*$ is. In particular, if $*$ is not injective nor surjective, then b is not left injective nor left surjective (and also not left semi-stable by Proposition 2.3.4 below), despite the fact b is right regular.

The following proposition is easy to prove:

PROPOSITION 2.1.5. *Let $(D, *)$ be a division ring with involution and let (M, b) be a sesquilinear space over $(D, *)$ with $\dim M_D < \infty$. Then all ten conditions (R1)-(R5),(L1)-(L5) are equivalent (where (R4), (L4) are considered w.r.t. the involution $\kappa = * : D \to D$).*

PROOF (SKETCH). We will see below that (R0) implies (R1)-(R4) and any of (R1)-(R4) imply (R5), so it is enough to verify (R5) $\implies$ (R0) and (L0). Indeed, identify M with D^n, where $n = \dim M_D$. Then b is necessarily given by $b(x, y) = (x^*)^T A y$ for some $A \in \mathrm{M}_n(D)$ (the elements of D^n are considered as column vectors and $*$ acts on D^n component-wise). It is now easy to see that (R5) is equivalent to $\mathrm{ann}^r A = 0$ and (R0) (resp. (L0)) is equivalent to A being invertible. The proposition follows immediately since A is invertible $\iff$ $\mathrm{ann}^r A = 0$. □

Moreover, we shall prove in section 2.5 that Proposition 2.1.5 remains true upon replacing D with a *quasi-Frobenius* ring, provided M is faithful. Despite this, without special assumptions on the base ring, no analogue of the last proposition holds. For example, it turns out that none of the conditions (R1)-(R4),(L1)-(L4) implies any of the others. (Part of this already follows from Example 2.1.4.)

In section 2.3 we prove a list of logical implications between subsets of the conditions (R1)-(R4),(L1)-(L4), and we conjecture the this list explains *all* implications between subsets of these conditions. What prevents us from declaring all implications as determined from our list is the absence of several counterexamples. The counterexamples that we do have and (hopefully all) the missing ones are described in section 2.4.

We note that when determining the implications, it is important to distinguish between three cases: (I) K is a not assumed to have an anti-isomorphism (so (R4) and (L4) are irrelevant); (II) K is assumed to have an anti-isomorphism; and (III) K is assumed to have an *augmentable* anti-isomorphism (e.g. an involution; see section 2.3 for the definition). Case I is completely solved in the sense that we are able to show that *all* implications are derived from (R0)$\Longrightarrow$(R3) and its left analogue. The other cases are more complicated and they admit different lists of implications. (For example, (R0)$\Longrightarrow$(L1) in cases II and III but not in Case I, and (L3) $\wedge$ (R4)-κ $\Longrightarrow$ (L4)-κ^{-1} in Case III, but not in Case II.)

2.2. Basic Properties

Let R be a ring and let K be a fixed double R-module. In this section, we prove some categorical results regarding bilinear forms and the functors [0] and [1]. These will serve as an infrastructure for the rest of the chapter and will also provide the intuition and justification for the categorical definition of bilinear forms given in section 2.7.

To avoid extra parentheses in the proofs, we adopt the following notation until the end of the section. The value of a function f at x will be denoted by fx, rather than $f(x)$. To distinguish application of a function from multiplication by a scalar, the latter will be written explicitly, i.e. we will write $m \cdot r$ rather than mr whenever $m \in M \in \text{Mod-}R$ and $r \in R$. Composition of functions will also be written explicitly.

Proposition 2.2.1. *Let $M \in \text{Mod-}R$. There are* natural *R-module homomorphisms $\Psi = \Psi_M : M \mapsto M^{[0][1]}$ and $\Phi = \Phi_M : M \mapsto M^{[1][0]}$ given by:*

$$(\Psi x)(f) = f(x) \qquad \forall x \in M,\ f \in M^{[0]},$$

$$(\Phi x)(f) = f(x) \qquad \forall x \in M,\ f \in M^{[1]}.$$

In addition, the maps Φ, Ψ satisfy:

$$\mathrm{id}_{M^{[0]}} = \Psi_M^{[0]} \circ \Phi_{M^{[0]}},$$

$$\mathrm{id}_{M^{[1]}} = \Phi_M^{[1]} \circ \Psi_{M^{[1]}},$$

i.e. the following diagrams commute

$$\begin{array}{ccc} M^{[0]} & \xrightarrow{\Phi_{M^{[0]}}} & M^{[0][1][0]} \\ & \overset{\mathrm{id}}{\searrow} & \downarrow \Psi_M^{[0]} \\ & & M^{[0]} \end{array} \qquad \begin{array}{ccc} M^{[1]} & \xrightarrow{\Psi_{M^{[1]}}} & M^{[1][0][1]} \\ & \overset{\mathrm{id}}{\searrow} & \downarrow \Phi_M^{[1]} \\ & & M^{[1]} \end{array}$$

Proof. That Φ and Ψ are R-module homomorphisms is straightforward. We will only check that Ψ is natural (in the categorical sense) and that $\mathrm{id}_{M^{[0]}} = \Psi_M^{[0]} \circ \Phi_{M^{[0]}}$. The rest follows by symmetry. Let $A, B \in \text{Mod-}R$, $\varphi \in \mathrm{Hom}_R(A, B)$, $x \in B$ and $f \in B^{[0]}$. Then:

$$((\varphi^{[0][1]} \circ \Psi_A)x)f = (\varphi^{[0][1]}(\Psi_A x))f = ((\Psi_A x) \circ \varphi^{[0]})f = (\Psi_A x)(\varphi^{[0]} f) =$$

$$= (\Psi_A x)(f \circ \varphi) = f(\varphi x) = (\Psi_B(\varphi x))f = ((\Psi_B \circ \varphi)x)f,$$

hence $\varphi^{[0][1]} \circ \Psi_A = \Psi_B \circ \varphi$, implying Ψ is natural. Next, for all $f \in M^{[0]}$ and $m \in M$:

$$(\Psi_M^{[0]}(\Phi_{M^{[0]}} f))m = (\Phi_{M^{[0]}} f)(\Psi_M m) = (\Psi_M m)(f) = f(m) ,$$

hence $\Psi_M^{[0]} \circ \Phi_{M^{[0]}} = \mathrm{id}_{M^{[0]}}$, as desired. □

Remark 2.2.2. At this point, the reader is advised to keep in mind that [0] corresponds to *left* and [1] corresponds to *right*, in the sense that the left (resp. right) adjoint always take values in the 0-dual (resp. 1-dual). The reader is also advised to remember that Ψ is a morphism of functors from $\mathrm{id}_{\text{Mod-}R}$ to [0][1] and Φ is a morphism of functors from $\mathrm{id}_{\text{Mod-}R}$ to [1][0].

Corollary 2.2.3. *Let $A, B \in$ Mod-R, then there is an additive natural isomorphism*

$$I = I_{A,B} : \mathrm{Hom}(B, A^{[1]}) \longrightarrow \mathrm{Hom}(A, B^{[0]})$$

given by $I_{A,B}(f) = f^{[0]} \circ \Phi_A$. The inverse of I is given by $I_{A,B}^{-1}(g) = g^{[1]} \circ \Psi_B$.

Proof. We leave it to the reader to check $I_{A,B}$ is indeed a natural additive map from $\mathrm{Hom}(B, A^{[1]})$ to $\mathrm{Hom}(A, B^{[0]})$, and only check that $g \mapsto g^{[1]} \circ \Psi_B$ is the inverse of $I_{A,B}$. Indeed, for $f \in \mathrm{Hom}(B, A^{[1]})$

$$(I_{A,B} f)^{[1]} \circ \Psi_B = (f^{[0]} \circ \Phi_A)^{[1]} \circ \Psi_B = \Phi_A^{[1]} \circ f^{[0][1]} \circ \Psi_B = \Phi_A^{[1]} \circ \Psi_{A^{[1]}} \circ f = f .$$

(In the third equality we used the naturality of Φ.) That $I_{A,B}(g^{[1]} \circ \Psi_B) = g$ follows by symmetry. □

Remark 2.2.4. Corollary 2.2.3 implies that if one considers [0] as a *covariant* functor from Mod-R to $(\text{Mod-}R)^{\mathrm{op}}$ and [1] as a *covariant* functor from $(\text{Mod-}R)^{\mathrm{op}}$ to Mod-R, then [0] is left adjoint to [1]. (See [**42**] for definition and details.)

Let $A, B \in$ Mod-R. Call a biadditive map $b : A \times B \to K$ a *bilinear pairing* if

$$b(xr, y) = b(x, y) \odot_0 r \quad \text{and} \quad b(x, yr) = b(x, y) \odot_1 r \qquad \forall\, x \in A, y \in B, r \in R .$$

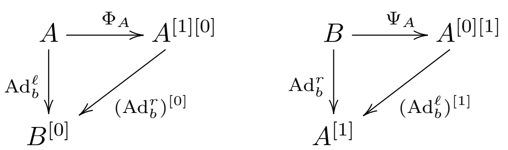
As with bilinear forms, we can define left and right adjoint maps $\mathrm{Ad}_b^\ell : A \to B^{[0]}$ and $\mathrm{Ad}_b^r : B \to A^{[1]}$. Since clearly any of Ad_b^ℓ, Ad_b^r determine b, it is expected that each of Ad_b^ℓ, Ad_b^r would determine the other. This is verified in the following corollary:

Corollary 2.2.5. *Let $b : A \times B \to K$ be a bilinear pairing. Then*

$$\mathrm{Ad}_b^r = I_{A,B}^{-1}(\mathrm{Ad}_b^\ell) = (\mathrm{Ad}_b^\ell)^{[1]} \circ \Psi_B ,$$

$$\mathrm{Ad}_b^\ell = I_{A,B}(\mathrm{Ad}_b^r) = (\mathrm{Ad}_b^r)^{[0]} \circ \Phi_A ,$$

i.e. the following diagrams commute:



Proof. We only check the first equality. The second follows by symmetry. Let $x \in A$ and $y \in B$. Then for all $x \in A$ and $y \in B$:

$$(((\mathrm{Ad}_b^\ell)^{[1]} \circ \Psi_B)y)x = ((\mathrm{Ad}_b^\ell)^{[1]}(\Psi_B y))x = (\Psi_B y)(\mathrm{Ad}_b^\ell x) =$$

$$= (\mathrm{Ad}_b^\ell x)y = b(x, y) = (\mathrm{Ad}_b^r y)x$$

hence $\mathrm{Ad}_b^r = (\mathrm{Ad}_b^\ell)^{[1]} \circ \Psi_B = I_{A,B}^{-1}(\mathrm{Ad}_b^\ell)$. □

PROPOSITION 2.2.6. *Let $b : A \times B \to K$ be a bilinear pairing and let $\sigma \in \mathrm{End}(A)$ and $\tau \in \mathrm{End}(B)$. Then $b(\sigma x, y) = b(x, \tau y)$ for all $x \in A$ and $y \in B$ $\iff$ the left diagram commutes $\iff$ the right diagram commutes*

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & A \\ {\scriptstyle \mathrm{Ad}_b^\ell}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_b^\ell} \\ B^{[0]} & \xrightarrow[\tau^{[0]}]{} & B^{[0]} \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\tau} & B \\ {\scriptstyle \mathrm{Ad}_b^r}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_b^r} \\ A^{[1]} & \xrightarrow[\sigma^{[1]}]{} & A^{[1]} \end{array}$$

PROOF. That each of the diagrams is equivalent to $b(\sigma x, y) = b(x, \tau y)$ for all $x \in A$ and $y \in B$ is straightforward. However, the diagrams' equivalence can also be shown directly (this is important for section 2.7). Indeed, if $\mathrm{Ad}_b^\ell \circ \sigma = \tau^{[0]} \circ \mathrm{Ad}_b^\ell$ (i.e. the left diagram commutes), then $\sigma^{[1]} \circ (\mathrm{Ad}_b^\ell)^{[1]} = (\mathrm{Ad}_b^\ell)^{[1]} \circ \tau^{[0][1]}$ and this implies

$$\sigma^{[1]} \circ \mathrm{Ad}_b^r \overset{(2.2.5)}{=} \sigma^{[1]} \circ (\mathrm{Ad}_b^\ell)^{[1]} \circ \Psi_B = (\mathrm{Ad}_b^\ell)^{[1]} \circ \tau^{[0][1]} \circ \Psi_B = (\mathrm{Ad}_b^\ell)^{[1]} \circ \Psi_B \circ \tau \overset{(2.2.5)}{=} \mathrm{Ad}_b^r \circ \tau,$$

so the second diagram commutes. □

We now turn to explain what is the categorical meaning of an anti-isomorphism (or an involution) $\kappa : K \to K$. Recall that we let K_i denote the R module obtained by letting R act on K via $\odot_i$.

PROPOSITION 2.2.7. *There is a one-to-one correspondence between isomorphisms of functors $u : [0] \to [1]$ and anti-isomorphisms of K.*

PROOF. Throughout, r, m, f stand for elements of R, M, $M^{[0]}$, respectively.

Let κ be an anti-isomorphism of K. For all $M \in$ Mod-R, define $u_{\kappa,M} : M^{[0]} \to M^{[1]}$ by

$$u_{\kappa,M}(f) = \kappa \circ f \ .$$

Note that $\kappa \circ f \in M^{[1]}$ since $(\kappa \circ f)(m \cdot r) = (f(m \cdot r))^\kappa = ((fm) \odot_1 r)^\kappa = (fm)^\kappa \odot_0 r = ((\kappa \circ f)m) \odot_0 r$. In addition, $u = u_{\kappa,M}$ is an R-module homomorphism because

$$(u(f \cdot r))m = ((f \cdot r)m)^\kappa = ((fm) \odot_0 r)^\kappa = (fm)^\kappa \odot_1 r = ((uf)m) \odot_1 r = ((uf) \cdot r)m\,.$$

We leave it to the reader to check that $u_{\kappa,M}$ is a natural isomorphism and, therefore, $u_\kappa : [0] \to [1]$ is a functor isomorphism.

Now assume we are given a functor isomorphism $u : [0] \to [1]$. Observe that $R^{[i]} \cong K_i$ via $f \mapsto f(1_R)$ where $i \in \{0, 1\}$. For all $k \in K$, denote by f_k the unique element of $R^{[0]}$ satisfying $f_k(1_R) = k$. Define $\kappa : K \to K$ by $k^\kappa = (uf_k)1_R$.[3] Let $r \in R$. Since the map $k \mapsto f_k$ is an R-module homomorphism (from K_0 to $R^{[0]}$),

$$(k \odot_0 r)^\kappa = (uf_{k \odot_0 r})1_R = (u(f_k \cdot r))1_R = ((uf_k) \cdot r)1_R = ((uf_k)1_R) \odot_1 r = k^\kappa \odot_1 r$$

for all $k \in K$. Now consider the homomorphism $\psi : R_R \to R_R$ given by $\psi(x) = r \cdot x$. Since u is natural, $u_R \circ \psi^{[0]} = \psi^{[1]} \circ u_R$. In addition, for all $k \in K$, $(\psi^{[0]} f_k)1_R = f_k(\psi 1_R) = f_k(1_R \cdot r) = k \odot_1 r$, hence $\psi^{[0]} f_k = f_{k \odot_1 r}$. Therefore:

$$\begin{aligned} (k \odot_1 r)^\kappa &= (uf_{k \odot_1 r})1_R = (u(\psi^{[0]} f_k))1_R = (\psi^{[1]}(uf_k))1_R \\ &= (uf_k)(\psi 1_R) = (uf_k)(1_R \cdot r) = ((uf_k)1_R) \odot_0 r = k^\kappa \odot_0 r \ . \end{aligned}$$

It follows that $(k \odot_i r)^\kappa = k^\kappa \odot_{1-i} r$ for all $r \in R$, $k \in K$ and $i \in \{0, 1\}$, hence $\kappa = \kappa(u)$ is an anti-isomorphism of K.

Clearly $\kappa(u_\kappa) = \kappa$. To see that $u_{\kappa(u)} = u$, first observe that for all $k \in K$, $(\kappa \circ f_k)1_R = (f_k 1_R)^\kappa = k^\kappa = (uf_k)1_R$, hence $uf = \kappa(u) \circ f$ for all $f \in R^{[0]}$. Now, let $M \in$ Mod-R, $f \in M^{[0]}$ and $m \in M$. Define $\varphi : R_R \to M$ by $\varphi(r) = m \cdot r$.

[3] It is easy to check that under the identification $R^{[i]} \cong K_i$, κ is in fact u_R.

Then φ gives rise to maps $\varphi^{[0]}: M^{[0]} \to R^{[0]}$ and $\varphi^{[1]}: M^{[1]} \to R^{[1]}$ satisfying $u_R \circ \varphi^{[0]} = \varphi^{[1]} \circ u_M$. Since $f \circ \varphi \in R^{[0]}$, $u_R(f \circ \varphi) = \kappa(u) \circ (f \circ \varphi)$. Therefore:

$$(fm)^{\kappa(u)} = (f(\varphi 1_R))^{\kappa(u)} = (\kappa(u) \circ f \circ \varphi)1_R = (u_R(f \circ \varphi))1_R =$$

$$= ((u_R \circ \varphi^{[0]})f)1_R = ((\varphi^{[1]} \circ u_M)f)1_R = ((u_M f) \circ \varphi)1_R = (u_M f)(\varphi 1_R) = (u_M f)m$$

and it follows that $u_M f = \kappa(u) \circ f$, as required. □

The map $u_{\kappa,M}$ defined in the last proof will be used throughout the chapter. Involutions of K correspond to natural isomorphisms $u : [0] \to [1]$ satisfying an additional condition that will be explained in Section 2.7.

Proposition 2.2.8. *Let $b : M \times M \to K$ be a bilinear form, let κ be an anti-isomorphism of K and let $\lambda \in \mathrm{End}(M)$. Then λ is a right κ-asymmetry of b if and only if $u_{\kappa,M} \circ \mathrm{Ad}^\ell_b = \mathrm{Ad}^r_b \circ \lambda$, i.e. the following diagram commute:*

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M \\ {\scriptstyle \mathrm{Ad}^\ell_b}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}^r_b} \\ M^{[0]} & \xrightarrow[u_{\kappa,M}]{\cong} & M^{[1]} \end{array}$$

In particular, b is κ-symmetric if and only if $u_{\kappa,M} \circ \mathrm{Ad}^\ell_b = \mathrm{Ad}^r_b$.

Proof. This is straightforward. □

Proposition 2.2.9. *Let $b : M \times M \to K$ be a bilinear form. The following are equivalent:*

(a) *b is right semi-stable.*
(b) *For all $\sigma, \tau \in \mathrm{End}(M_R)$, $b(x, \sigma y) = b(x, \tau y)$ for all $x, y \in M$ implies $\sigma = \tau$.*
(c) $\mathrm{Hom}(M_R, \ker \mathrm{Ad}^r_b) = 0$.
(d) *For all $\sigma, \tau \in \mathrm{End}(M_R)$, $\mathrm{Ad}^r_b \circ \sigma = \mathrm{Ad}^r_b \circ \tau$ implies $\sigma = \tau$.*

If b is right injective, right stable or has a unique right κ-asymmetry (for some anti-isomorphism κ of K), then b is right semi-stable.

Proof. The equivalence of (a), (b) and (d) is straightforward. As for (c), $\sigma \in \mathrm{Hom}_R(M, \ker \mathrm{Ad}^r_b) \iff \sigma \in \mathrm{End}(M_R)$ and $\sigma(M) \subseteq \ker \mathrm{Ad}^r_b \iff \sigma \in \mathrm{End}(M_R)$ and $b(x, \sigma M) = 0$ for all $x \in M \iff \sigma \in \mathrm{End}(M_R)$ and $b(x, \sigma y) = 0$ for all $x, y \in M$. Therefore (b) $\iff$ (c).

If b is right injective, then b is right semi-stable by (c). If b is right stable and $\sigma \in \mathrm{End}(M_R)$ is such that $b(x, \sigma y) = 0$ for all $x, y \in M$, then $b(0x, y) = 0 = b(x, \sigma y)$ for all $x, y \in M$. The stableness implies σ is the only endomorphism of M satisfying this, hence $\sigma = 0$, implying b is right semi-stable. If b has a *unique* right κ-asymmetry λ and σ is as before, then $\lambda + \sigma$ is also a right κ-asymmetry, so the uniqueness implies $\sigma = 0$. □

Let M be a right R-module such that $\mathrm{Hom}(M, N) \neq 0$ for any nonzero submodule $N \leq M$. Then condition (c) of the previous proposition implies that a bilinear form defined on M is right (resp. left) semi-reflexive if and only if it is right (resp. left) injective. An important family of modules satisfying the previous condition is the *generators* of Mod-R; a module $M \in$ Mod-R is called a generator (of Mod-R) if any right R-module is an epimorphic image of $\bigoplus_{i \in I} M$ for some set I. This turns out to be equivalent to R_R being a summand of M^n for some $n \in \mathbb{N}$. (See [**58**, Th. 18.8] for additional equivalent conditions). We now get the following corollary.

Corollary 2.2.10. *Let $b : M \times M \to K$ be a bilinear form such that M is a generator. Then b is right semi-stable $\iff$ b is right injective.*

2.3. Implications

In this section, we will determine what we conjecture to be all implications between subsets of the conditions (R1)-(R4),(L1)-(L4) as well as other useful results. The conditions (R5) and (L5) will also be treated but they are of less interest since they are implied by (R1),(R3),(R4) and (L1),(L3),(L4) respectively (Proposition 2.2.9).

Since the existence of an anti-isomorphism or an involution on K effects some of the implications, we will distinguish between the following three cases:

(I) K is not assumed to have an anti-isomorphism.
(II) K has an anti-isomorphism κ.
(III) K has an *augmentable* anti-isomorphism κ (e.g. if K has an involution).

In Case I, the conditions (R4) and (L4) are irrelevant where in the other cases we also have to treat (R4)-κ and (L4)-κ^{-1}. We will define augmentable anti-endomorphisms when we discuss Case III.

Remark 2.3.1. It is also reasonable to add (R4)-κ^{-1} and (L4)-κ to our list of properties, and these conditions are actually mentioned when discussing Case II. However, we suspect this will open a pandora box, as one could also add (R4)-κ^n and (L4)-κ^n for any odd integer n. When κ is augmentable, this issue is irrelevant since in this case having a (unique) right (left) κ-asymmetry is equivalent to having a (unique) right (left) κ^n-asymmetry, where n is any odd integer.

Remark 2.3.2. For any double R-module K, define K^{op} to be the set of formal symbols $\{k^{\mathrm{op}} \mid k \in K\}$ endowed with the double R-module structure given by

$$k^{\mathrm{op}} \odot_0 a = (k \odot_1 a)^{\mathrm{op}}, \qquad k^{\mathrm{op}} \odot_1 a = (k \odot_0 a)^{\mathrm{op}},$$

and $k^{\mathrm{op}} + k'^{\mathrm{op}} = (k + k')^{\mathrm{op}}$ ($k, k' \in K$, $a \in R$). Then any anti-isomorphism of K can be understood as a double R-module isomorphism from K to K^{op}. We can now describe Case I as $K \ncong K^{\mathrm{op}}$ and Case II as $K \cong K^{\mathrm{op}}$. Case III assumes a stronger kind of isomorphism from K to K^{op}.

Throughout, R is a ring, K is a fixed double R-module and (M, b, K) is a bilinear space. Unless specified explicitly, we do not assume $K \ncong K^{\mathrm{op}}$ or $K \cong K^{\mathrm{op}}$.

2.3.1. Case I.

Proposition 2.3.3. *If b is right regular, then b is right stable.*

Proof. Since b is injective, it is semi-stable. Therefore, it is enough to show that for all $\sigma \in \mathrm{End}(M_R)$ there is $\sigma' \in \mathrm{End}(M_R)$ such that $b(\sigma x, y) = b(x, \sigma' y)$ (the uniqueness of σ' is guaranteed). By Proposition 2.2.6, this is equivalent to $\sigma^{[1]} \circ \mathrm{Ad}_b^r = \mathrm{Ad}_b^r \circ \sigma'$, so take $\sigma' = (\mathrm{Ad}_b^r)^{-1} \circ \sigma^{[1]} \circ \mathrm{Ad}_b^r$. □

Proposition 2.3.4. *Assume b is right stable and let $*$ be its corresponding anti-endomorphism. Then b is left stable if and only if $*$ is invertible and left semi-stable if and only if $*$ is injective.*

Proof. The form b is left semi-stable $\iff$ for all $\tau \in \mathrm{End}(M_R)$, $b(\tau x, y) = 0$ implies $\tau = 0$ $\iff$ $b(x, \tau^* y) = 0$ implies $\tau = 0$ $\iff$ $\tau^* = 0$ implies $\tau = 0$ (since b is right semi-stable). Therefore, b is left semi-stable if and only if $*$ is injective.

If $*$ is bijective, then for all $\sigma \in \mathrm{End}(M_R)$ and $x, y \in M$, $b(x, \sigma y) = b(\sigma^{*^{-1}} x, y)$. By the previous argument, b is left semi-stable, hence $\sigma^{*^{-1}}$ is uniquely determined by σ, thus b is left stable. On the other hand, if b is left stable, then there is an anti-homomorphism $\sharp : \mathrm{End}(M_R) \to \mathrm{End}(M_R)$ satisfying $b(x, \sigma y) = b(\sigma^\sharp x, y)$ for all $x, y \in M$. This implies $b(x, \sigma y) = b(\sigma^\sharp x, y) = b(x, \sigma^{\sharp *} y)$ and since b is right semi-stable, $\sigma = \sigma^{\sharp *}$ for all $\sigma \in \mathrm{End}(M)$. Therefore, $* \circ \sharp = \mathrm{id}_{\mathrm{End}(M)}$ and by symmetry, $\sharp \circ * = \mathrm{id}_{\mathrm{End}(M)}$, hence $*$ is bijective. □

Surprisingly, all implications between subsets of (R1)-(R3),(L1)-(L3) in Case I can be explained by (R1) $\wedge$ (R2) $\Longrightarrow$ (R3) and its left analogue (the $\wedge$ sign denotes logical "and"). This will be verified (with counterexamples) in the next section.

2.3.2. Case II.

LEMMA 2.3.5. *Assume K has an anti-isomorphism κ. Then:*

(i) *If b has a unique right κ-asymmetry, then it is (left and right) semi-stable.*
(ii) *If b is right semi-stable and has a right κ-asymmetry, then it is unique.*

PROOF. (i) By Proposition 2.2.9, b is right semi-stable. Now assume $b(\sigma x, y) = 0$ for all $x, y \in M$. Then $b(x, \sigma y)^\kappa = b(\sigma y, \lambda x) = 0$. Since b is right semi-stable, this means $\sigma = 0$, implying b is left semi-stable.

(ii) This is clear from Proposition 2.2.9(b). □

PROPOSITION 2.3.6. *Assume K has an anti-isomorphism κ. Then:*

(i) *If b is right regular, then it has a unique right κ-asymmetry.*
(ii) *If b has a right κ-asymmetry and b is right injective, then b is injective.*
(iii) *If b has a right κ-asymmetry and b is left surjective, then b is surjective.*

PROOF. (i) Take $\lambda = (\mathrm{Ad}^r_b)^{-1} \circ u_{\kappa,M} \circ \mathrm{Ad}^\ell_b$. This is a right κ-asymmetry by Proposition 2.2.8. The uniqueness follows from Lemma 2.3.5(ii), since b is right semi-stable.

(ii) Let λ be a right κ-asymmetry and assume $b(x, M) = 0$. Then $b(M, x) = b(x, \lambda M)^{\kappa^{-1}} = 0$ and since b is right injective $x = 0$.

(iii) By Proposition 2.2.8, $\mathrm{Ad}^\ell_b = u^{-1}_{\kappa,M} \circ \mathrm{Ad}^r_b \circ \lambda$. Since $u_{\kappa,M}$ is an isomorphism, that Ad^ℓ_b is surjective implies Ad^r_b is surjective. □

COROLLARY 2.3.7. *Assume K has an anti-isomorphism and b is right regular. Then b is left injective.*

In contrast to the last corollary, in Case I, (R0) does not imply (L1) and not even (L5); see Example 2.4.3.

PROPOSITION 2.3.8. *Let κ be an anti-isomorphism of K and assume b has a unique right κ-asymmetry λ. Then the following are equivalent:*

(a) *b has a left κ^{-1}-asymmetry.*
(b) *b has a unique left κ^{-1}-asymmetry.*
(c) *λ is right invertible.*
(d) *λ is invertible.*

When these conditions hold, b is right regular (injective, surjective) if and only if b is left regular (injective, surjective).

PROOF. (a) $\Longleftrightarrow$ (b): By Lemma 2.3.5(i), b is right an left semi-stable, so by the left analogue of Lemma 2.3.5(ii), (a)$\Longrightarrow$(b). The opposite direction is obvious.

(b)$\Longrightarrow$(d): Let λ' be the left κ^{-1}-asymmetry of b. Then $b(x, y)^\kappa = b(y, \lambda x) = b(\lambda'\lambda x, y)^\kappa$. Since b is left semi-stable (Lemma 2.3.5(i)), $\lambda'\lambda = \mathrm{id}_M$ and by symmetry $\lambda\lambda' = \mathrm{id}_M$.

(d)$\Longrightarrow$(c): This is clear.

(c)$\Longrightarrow$(a): If λ' is a right inverse of λ, then $b(\lambda' x, y) = b(y, \lambda\lambda' x)^{\kappa^{-1}} = b(x, y)^{\kappa^{-1}}$ for all $x, y \in M$, hence λ' is a left κ^{-1}-asymmetry.

To finish, if conditions (a)-(d) hold, then λ is invertible. By Proposition 2.2.8, $\mathrm{Ad}_b^\ell = u_{\kappa,M}^{-1} \circ \mathrm{Ad}_b^r \circ \lambda$. Since $u_{\kappa,M}$ and λ are invertible, Ad_b^ℓ is bijective (injective, surjective) if and only if Ad_b^r is. □

PROPOSITION 2.3.9. *Assume b is right stable with corresponding anti-endomorphism $*$ and let κ be an anti-isomorphism of K. Then:*

(i) *If λ is a right κ-asymmetry of b, then $*$ is injective and $\sigma^{**}\lambda = \lambda\sigma$ for all $\sigma \in \mathrm{End}(M_R)$.*

(ii) *If λ' is a* left *κ-asymmetry of b, then $(\lambda')^*$ is a right κ-asymmetry of b. In this case b has unique right and left κ-asymmetries.*[4]

PROOF. (i) b is right semi-stable, hence λ is the unique κ-asymmetry of b (Lemma 2.3.5(ii)). By Lemma 2.3.5(i), b is left semi-stable, so $*$ is injective by Proposition 2.3.4. In addition, for all $x, y \in M$ and $\sigma \in \mathrm{End}(M_R)$, $b(x, \lambda\sigma y) = b(\sigma y, x)^\kappa = b(y, \sigma^* x)^\kappa = b(\sigma^* x, \lambda y) = b(x, \sigma^{**}\lambda y)$, implying $\lambda\sigma = \sigma^{**}\lambda$ (since b is right semi-stable).

(ii) For all $x, y \in M$, $b(x, (\lambda')^* y) = b(\lambda x, y) = b(y, x)^\kappa$ implying $(\lambda')^*$ is a right κ-asymmetry. The uniqueness follows from Lemma 2.3.5, since b is right semi-stable. □

COROLLARY 2.3.10. *Let κ be an anti-isomorphism of K and assume b has an invertible right κ-asymmetry λ. Then b is right stable if and only if b is left stable. In this case, if $*$ is the corresponding anti-isomorphism of b, then $\sigma^{**} = \lambda\sigma\lambda^{-1}$.*

PROOF. Assume b is right stable and let $*$ be its corresponding anti-isomorphism. Then by Proposition 2.3.9(i), $\sigma^{**} = \lambda\sigma\lambda^{-1}$ for all $\sigma \in \mathrm{End}_R(M)$, hence is $*^2 := * \circ *$ bijective. Therefore, $*$ is bijective and by Proposition 2.3.4, b is left stable. The opposite direction follows by symmetry. □

DEFINITION 2.3.11. *Let (M, b, K) be a bilinear space and let κ be an anti-isomorphism of K. An* augmentation map *for b (w.r.t. κ) is a map $\gamma \in \mathrm{End}_R(M)$ such that $b(x, \gamma y) = b(x, y)^{\kappa\kappa}$ for all $x, y \in M$.*

LEMMA 2.3.12. *Assume K has an anti-isomorphism κ. Then:*

(i) *If b is right semi-stable, then it has at most one augmentation map.*

(ii) *If b has a left or right κ-asymmetry λ and b is right stable with corresponding anti-endomorphism $*$, then $\gamma = \lambda^*\lambda$ is an augmentation map for b.*

(iii) *Assume b is right semi-stable. If γ is an augmentation map and b is left surjective or left stable, then $\gamma \in \mathrm{Cent}(\mathrm{End}_R(M))$.*

(iv) *If b has a left κ-asymmetry λ and γ is an augmentation map, then $b(\gamma x, y) = b(x, y)^{\kappa\kappa}$ for all $x, y \in M$. If moreover b is right stable with corresponding anti-endomorphism $*$, then $\gamma^* = \gamma$ (so by (ii), $\lambda^*\lambda = \lambda^*\lambda^{**}$).*

(v) *If b is right regular, then b has an augmentation map, and it is* invertible.

(vi) *If b is right injective and b has an augmentation map, then it is injective.*

PROOF. (i) This easily follows from Proposition 2.2.9(b).

(ii) Assume λ is a right κ-asymmetry. Then for all $x, y \in M$, $b(x, y)^{\kappa\kappa} = b(y, \lambda x)^\kappa = b(\lambda x, \lambda y) = b(x, \lambda^*\lambda y)$. If λ is a left κ-asymmetry, then $b(x, y)^{\kappa\kappa} = b(\lambda y, x)^\kappa = b(\lambda x, \lambda y) = b(x, \lambda^*\lambda y)$. In both cases $\lambda^*\lambda$ is an augmentation map.

(iii) Let $\sigma \in \mathrm{End}_R(M)$ and assume b is left surjective. Fix some $x \in M$. Then $y \mapsto b(x, \sigma y)$ lies in $M^{[0]}$, hence there is some $x' \in M$ such that $b(x, \sigma y) = b(x', y)$ for all $y \in M$. Therefore

$$b(x, \gamma\sigma y) = b(x, \sigma y)^{\kappa\kappa} = b(x', y)^{\kappa\kappa} = b(x', \gamma y) = b(x, \sigma\gamma y)\ .$$

[4] But κ^{-1}-asymmetries do not exist in general; see Example 2.4.13.

Since this holds for all $x, y \in M$ and since b is right semi-stable, $\gamma\sigma = \sigma\gamma$. Now assume b is left stable. Then there exists $\sigma' \in \mathrm{End}_R(M)$ such that $b(\sigma' x, y) = b(x, \sigma y)$ for all $x, y \in M$. Thus

$$b(x, \gamma\sigma y) = b(x, \sigma y)^{\kappa\kappa} = b(\sigma' x, y)^{\kappa\kappa} = b(\sigma' x, \gamma y) = b(x, \sigma\gamma y)$$

and as before we get $\gamma\sigma = \sigma\gamma$.

(iv) For all $x, y \in M$, $b(\gamma x, y) = b(\lambda y, \gamma x)^{\kappa^{-1}} = b(\lambda y, x)^{\kappa} = b(x, y)^{\kappa\kappa}$. The second assertion follows since $b(x, \gamma y) = b(x, y)^{\kappa\kappa} = b(\gamma x, y) = b(x, \gamma^* y)$ for all $x, y \in M$.

(v) By Proposition 2.3.6(i), b has a right asymmetry λ and by Proposition 2.3.3, b is right stable. Let $*$ be the corresponding anti-isomorphism of b. Then by (ii), $\gamma = \lambda^*\lambda$ is an augmentation map for b w.r.t. κ. We can now apply the same argument to κ^{-1} and get an augmentation map w.r.t. κ^{-1}, γ'. Then for all $x, y \in M$, $b(x, y) = b(x, \gamma' y)^{\kappa\kappa} = b(x, \gamma\gamma' y)$ and it follows that $\gamma\gamma' = \mathrm{id}_M$. By symmetry, $\gamma'\gamma = \mathrm{id}_M$, hence γ is invertible.

(vi) Let γ be an augmentation map of b and assume $\gamma y = 0$ for some $y \in M$. Then $b(x, y) = b(x, \gamma y)^{\kappa^{-2}} = 0$ and since b is right injective, $y = 0$. ☐

Proposition 2.3.13. *Assume K has an anti-isomorphism κ, and b is right regular with corresponding anti-isomorphism $*$. If $*$ is surjective, then b is left regular.*

Proof. By Proposition 2.3.6(i), b has a right κ-asymmetry λ and by Lemma 2.3.12, $\lambda^*\lambda$ is an augmentation map for κ and it is invertible, hence λ is left invertible. Assume $*$ is surjective. Then there exists $\mu \in \mathrm{End}_R(M)$ such that $\mu^* = \lambda$. Taking $\sigma = \mu$ in Proposition 2.3.9(i), we get $\lambda^*\lambda = \mu^{**}\lambda = \lambda\mu$, so λ is also right invertible (and necessarily $\mu = \lambda^*$, implying $\lambda^{**} = \lambda$). We are now through by Proposition 2.3.8. ☐

Before proceeding to Case III, let us summarize the implications proved so far. We conjecture that in Case II, all implications between subsets of (R1)-(R3),(R4)-κ and their left analogues can be explained by the following list and its left analogue:

(1) (R0) $\Longrightarrow$ (R3) $\wedge$ (R4) (Prp. 2.3.3, Prp. 2.3.6(i));
(2) (R1) $\wedge$ (R4) $\Longrightarrow$ (L1) (Prp. 2.3.6(ii));
(3) (L2) $\wedge$ (R4) $\Longrightarrow$ (R2) (Prp. 2.3.6(iii));
(4) (R4)-κ $\wedge$ (L4)-κ^{-1} $\wedge$ (R1) $\Longrightarrow$ (L1) (Prp. 2.3.8);
(5) (R4)-κ $\wedge$ (L4)-κ^{-1} $\wedge$ (R2) $\Longrightarrow$ (L2) (Prp. 2.3.8);
(6) (R4)-κ $\wedge$ (L4)-κ^{-1} $\wedge$ (R3) $\Longrightarrow$ (L3) (Prp. 2.3.8);
(7) (R0) $\wedge$ (L3) $\Longrightarrow$ (L0) (Prp. 2.3.13).

Note that (R0) is equivalent to (R1) $\wedge$ (R2) and the same holds for (L0). We also have shown that:

(8) (L4)-κ $\wedge$ (R3) $\Longrightarrow$ (R4)-κ (Prp. 2.3.9(ii));
(9) (R4) $\Longrightarrow$ (R5) $\wedge$ (L5) (Lm. 2.3.5(i)).

2.3.3. Case III. The definition of augmentable anti-endomorphisms of K is inspired by the map γ of Lemma 2.3.12.

Definition 2.3.14. *Call an anti-endomorphism κ of K* weakly augmentable *if there exist a natural transformation $\gamma : \mathrm{id}_{\text{Mod-}R} \to \mathrm{id}_{\text{Mod-}R}$ such that for all $A, B \in \text{Mod-}R$ and any bilinear pairing $b : A \times B \to K$*

$$b(x, \gamma_B y) = b(x, y)^{\kappa\kappa} \qquad \forall x \in A,\ y \in B\ . \tag{2}$$

If γ can be chosen to be a natural isomorphism (i.e. $\gamma_B \in \mathrm{End}(B)$ is an isomorphism for all $B \in \text{Mod-}R$), call κ an augmentable *anti-isomorphism. The transformation*

γ is called an augmentation transformation of κ *(or just an augmentation, for brevity).*

Observe that since γ is natural, γ_A is central in $\mathrm{End}_R(A)$ for all $A \in \text{Mod-}R$. The augmentation γ is not uniquely determined by κ, as demonstrated in Example 2.3.15(ii) below, but if K_1 is faithful (which is equivalent to K_0 being faithful, since K has an anti-isomorphism), then γ is uniquely determined. This will be verified later in Proposition 2.7.6 below (take $X = R_R$). Moreover, we shall prove below that for any anti-isomorphism κ there is a unique "augmentation transformation" γ which is defined on the subcategory of *right reflexive* R-modules (Proposition 2.7.7).

The reason that in the definition γ depends only on B and not on A and b is that if γ is an augmentation of $b : A \times B \to K$, then it is also an augmentation of $(x, y) \mapsto b(\sigma x, y) : A' \times B \to K$ for any $\sigma \in \mathrm{Hom}_R(A', A)$. A deeper justification for this will appear later in section 2.7.3, where we shall see that there is a unique natural transformation $\widehat{\gamma} : [0] \to [0]$ such that $\widehat{\gamma}_B \circ \mathrm{Ad}^\ell_b = \mathrm{Ad}^\ell_{\kappa^2 \circ b}$ for any bilinear pairing $b : A \times B \to K$.

EXAMPLE 2.3.15. (i) Any involution of K is augmentable (take $\gamma = \mathrm{id}_{\text{Mod-}R}$).

(ii) Consider $\mathbb{Q}$ as a double $\mathbb{Z}$-module by letting $\odot_0$ and $\odot_1$ be the standard action on $\mathbb{Q}$. Then the map $x \mapsto 2x$ is an anti-isomorphism of $\mathbb{Q}$ (as a double $\mathbb{Z}$-module) and it is weakly augmentable but not augmentable. Indeed, $\gamma = 4\,\mathrm{id}_{\text{Mod-}\mathbb{Z}}$ in this case, and $4\,\mathrm{id}_{\text{Mod-}\mathbb{Z}}$ is not invertible for all $\mathbb{Z}$-modules. (The augmentation γ is uniquely determined since $\mathbb{Q}$ is faithful.)

(iii) Let K be a 2-torsion $\mathbb{Z}$-module. Then K can be considered as a double $\mathbb{Z}$-module by letting $\odot_0$ and $\odot_1$ be the standard action of $\mathbb{Z}$ on K. Then $\kappa := \mathrm{id}_K$ is an involution of K. However, $n\,\mathrm{id}_{\text{Mod-}R}$ is an augmentation of κ for any odd $n \in \mathbb{Z}$. In particular, $\mathrm{id}_{\text{Mod-}R}$ and $(-1)\,\mathrm{id}_{\text{Mod-}R}$ are two different invertible augmentations of κ.

(iv) Let R be a ring and let $*$ be an anti-endomorphism of R. Assume that there is $\lambda \in R$ such that $\lambda^*\lambda \in \mathrm{Cent}(R) \cap R^\times$ and $x^{**}\lambda = \lambda x$ for all $x \in R$. Let K be the double R-module obtained from R and $*$ in Example 2.1.4. Then the map $\kappa : K \to K$ defined by $k^\kappa = k^*\lambda$ satisfies

$$(k \odot_1 r)^\kappa = (kr)^*\lambda = r^*k^*\lambda = k^\kappa \odot_0 r \ ,$$

$$(k \odot_0 r)^\kappa = (r^*k)^*\lambda = k^*r^{**}\lambda = k^*\lambda r = k^\kappa \odot_1 r \ ,$$

for all $k \in K$ and $r \in R$. In addition, for all $k \in K$, we have

$$k^{\kappa\kappa} = (k^*\lambda)^*\lambda = \lambda^*k^{**}\lambda = \lambda^*\lambda k \ ,$$

hence κ^2 is invertible (since $\lambda^*\lambda \in R^\times$) and it follows that κ is an anti-isomorphism of K. Since $\lambda^*\lambda$ is also central, κ is augmentable because we can take $\gamma = \lambda^*\lambda\,\mathrm{id}_{\text{Mod-}R}$. In Proposition 2.4.1, we show that all augmentable anti-isomorphisms of K are obtained in this manner.

Part (i) of the following proposition shows that there is no need for a "left analogue" for the augmentation transformation.

PROPOSITION 2.3.16. *Let κ be a weakly augmentable anti-isomorphism of K with augmentation γ. Then:*

(i) *For any bilinear pairing $b : A \times B \to K$ we have*

$$b(\gamma_A x, y) = b(x, y)^{\kappa\kappa} \qquad \forall x \in A,\ y \in B \ .$$

(ii) *Assume κ is augmentable. If (M, b, K) is a bilinear space and $n \in \mathbb{Z}$ is odd, then b has a (unique) right (left) κ-asymmetry if and only if b has a (unique) right (left) κ^n-asymmetry.*

(iii) *If (M, b, K) is a bilinear space and b is right stable with corresponding anti-endomorphism $*$, then $\gamma_M^* = \gamma_M$.*

Proof. (i) Let $b : A \times B \to K$ be a bilinear pairing. Then $b' : B \times A \to K$ defined by $b'(y, x) = b(x, y)^\kappa$ is also a bilinear pairing. Therefore, $b'(y, \gamma_A x) = b'(y, x)^{\kappa\kappa}$. But this implies

$$b(\gamma_A x, y) = (b'(y, \gamma_A x))^{\kappa^{-1}} = (b'(y, x)^{\kappa\kappa})^{\kappa^{-1}} = b(x, y)^{\kappa\kappa}$$

for all $x \in A$ and $y \in B$, as required.

(ii) It is straightforward to check that $\lambda \in \mathrm{End}_R(M)$ is a right (left) κ-asymmetry of b if and only if $\gamma_M^m \lambda$ is a right (left) κ^{2m+1}-asymmetry of b, where m can be any integer. (The left version follows from (i).) The claim then follows immediately.

(iii) For all $x, y \in M$ we have $b(x, \gamma_M y) = b(x, y)^{\kappa\kappa} = b(\gamma_M x, y) = b(x, \gamma_M^* y)$ (the second equality follows from (i)), hence $\gamma_M^* = \gamma_M$. □

We now have the following improvement of Proposition 2.3.9.

Proposition 2.3.17. *Let (M, b, K) be a bilinear space where κ is augmentable with augmentation γ. Assume b is right stable with corresponding anti-isomorphism $*$, and b has a left κ^{-1}-asymmetry λ'. Then b has a (unique) right κ-asymmetry given by $\lambda = (\lambda')^* \gamma_M$ and $\sigma^{**} = \lambda\sigma\lambda^{-1}$ for all $\sigma \in \mathrm{End}(M)$. In particular, $*$ is bijective, $*^2$ is an inner automorphism of $\mathrm{End}_R(M)$ and b is left stable.*

Proof. By Proposition 2.3.9(ii), $(\lambda')^*$ is a right κ^{-1}-asymmetry and by Proposition 2.3.16(ii), $\lambda := \gamma_M(\lambda')^*$ a right κ-asymmetry. Let $\mu = \gamma_M \lambda'$ and observe that $\mu^* = \lambda$. Then by arguing as in the proof of Proposition 2.3.13, we get that λ is invertible and $\sigma^{**} = \lambda\sigma\lambda^{-1}$ for all $\sigma \in \mathrm{End}_R(M)$. All other assertions follow immediately (b is left stable by Proposition 2.3.4). □

Corollary 2.3.18. *Let (M, b, K) be a right stable bilinear space with κ augmentable. Then the following are equivalent:*

(a) *b is left stable and has unique left and right κ- and κ^{-1}-asymmetries.*
(b) *b has a left κ^{-1}-asymmetry.*
(c) *b has a right κ-asymmetry which is right invertible.*
(d) *b has a right κ-asymmetry λ satisfying $\lambda^{**} = \lambda$.*

If moreover M does not contain an infinite direct sum of its non-zero summands (e.g. if M is noetherian or has finite uniform dimension), then these conditions are also equivalent to:

(e) *b has a right κ-asymmetry.*

Proof. (b)$\Longrightarrow$(a) and (a)$\Longrightarrow$(d) easily follow from Propositions 2.3.17 (note $\lambda^{**} = \lambda(\lambda)\lambda^{-1} = \lambda$ in this case) and 2.3.16(ii).

(d)$\Longrightarrow$(c): By taking $\sigma = \lambda^*$ in Proposition 2.3.9(i) we get $\lambda^*\lambda = (\lambda^*)^{**}\lambda = \lambda\lambda^*$. By Lemma 2.3.12(ii), $\lambda^*\lambda = \gamma_M$, which is invertible. Thus, λ is invertible.

(c)$\Longrightarrow$(b): Let λ be a right asymmetry of b which is right-invertible. By Lemma 2.3.12(ii), $\lambda^*\lambda = \gamma_M$, so λ is also left-invertible, hence invertible. Therefore, λ^{-1} is a left κ^{-1}-asymmetry.

We now assume M does not contain an infinite sum of its non-zero summands and prove (c) $\Longleftrightarrow$ (e). Indeed, (c)$\Longrightarrow$(e) is a tautology. To see the converse, note that λ is left invertible by the argument in (c)$\Longrightarrow$(b). The assumption on M implies that $\mathrm{End}_R(M)$ does not contain an infinite set of non-zero orthogonal idempotents. By [**58**, p. 231], this means M is Dedekind finite[5], thus λ is right invertible. □

[5] A ring R is called Dedekind finite if $xy = 1$ implies $yx = 1$.

The last corollary shows that (R3) and (L4)-κ^{-1} imply (R4)-κ, which is false in Case II. We believe that in Case III, all implications between subsets of (R1)-(R3),(R4)-κ and their left analogues can be obtained from the following list and its left analogue:

(1) (R0) $\Longrightarrow$ (R4) (Prp. 2.3.6(i));
(2) (R1) $\wedge$ (R4) $\Longrightarrow$ (L1) (Prp. 2.3.6(ii));
(3) (L2) $\wedge$ (R4) $\Longrightarrow$ (R2) (Prp. 2.3.6(iii));
(4) (R4)-κ $\wedge$ (L4)-κ^{-1} $\wedge$ (R1) $\Longrightarrow$ (L1) (Prp. 2.3.8);
(5) (R4)-κ $\wedge$ (L4)-κ^{-1} $\wedge$ (R2) $\Longrightarrow$ (L2) (Prp. 2.3.8);
(6) (L4)-κ^{-1} $\wedge$ (R3) $\Longrightarrow$ (R4)-κ $\wedge$ (L3) (Prp. 2.3.17).

In addition, if we assume that the base module M does not contain an infinite direct sum of its non-zero summands, then:

(7) (R4)-κ $\wedge$ (R3) $\Longrightarrow$ (L4)-κ^{-1} (Cr. 2.3.18).

2.4. Counterexamples

In this section, we demonstrate the non-implications between the properties (R1)-(R5) and (L1)-(L5). The examples will be divided according to their relevance to Cases I, II and III of the previous section. Since examples for Case III are also relevant to Case II, but not necessarily vice versa, the examples of Case II appear *after* the examples of Case III.

We begin with a propositions that will help us to generate examples.

Proposition 2.4.1. *Let R be a ring and let $*$ be an anti-endomorphism of R. Make R into a double R-module by defining:*

$$r \odot_0 s = s^* r, \qquad r \odot_1 s = rs \qquad \forall\, r, s \in R\ ,$$

and let K denote the R-module thus obtained. Define $b : R \times R \to K$ by $b(x, y) = x^* y$. Then:

(i) *b is a right regular bilinear form and under the natural identification $\mathrm{End}(R_R) \cong R$ (via $f \leftrightarrow f(1)$), the corresponding anti-endomorphism of b is $*$. In addition, $\ker(\mathrm{Ad}^\ell_b) = \ker(*)$ and $\mathrm{im}(\mathrm{Ad}^\ell_b) = \mathrm{im}(*)$ under the identification $R^{[0]} = \mathrm{Hom}(R_R, K_1) = \mathrm{End}(R_R) \cong R$.*
(ii) *Assume there is $\lambda \in R$ such that $\lambda^*\lambda \in R^\times$ and $x^{**}\lambda = \lambda x$ for all $x \in R$. Define $\kappa : K \to K$ by $r^\kappa = r^*\lambda$. Then κ is an anti-isomorphism and under the identification $\mathrm{End}(R_R) \cong R$, λ is a right κ-asymmetry of b.*
(iii) *In the assumptions of (ii), if $\lambda^*\lambda \in \mathrm{Cent}(R)$, then κ is augmentable and its augmentation transformation is $\gamma = \lambda^*\lambda\, \mathrm{id}_{\mathrm{Mod}\text{-}R}$.*
(iv) *In the assumptions of (ii), if $\lambda^*\lambda = 1$, then κ is an involution of K.*
(v) *Any anti-isomorphism (resp.: augmentable anti-isomorphism, involution) κ of K is obtained from some $\lambda \in R$ as in (ii) (resp.: (iii), (iv)).*

Proof. Recall that under the natural identification $\mathrm{End}(R_R) \cong R$, an element $r \in R$ corresponds to the homomorphism $x \mapsto rx$.

(i) This was verified in Example 2.1.4.

(ii) That κ is an anti-isomorphism follows from Example 2.3.15(iv). (The assumption $\lambda^*\lambda \in \mathrm{Cent}(R)$ was not used to show this.) Next, $b(x, y)^\kappa = (x^*y)^*\lambda = y^*x^{**}\lambda = y^*\lambda x = b(y, \lambda x)$, hence λ is the right κ-asymmetry of b.

(iii) This was shown in Example 2.3.15(iv).

(iv) For all $k \in K$, $k^{\kappa\kappa} = (k^*\lambda)^*\lambda = \lambda^* k^{**}\lambda = \lambda^*\lambda k = k$, hence κ is an involution.

(v) Let $* : R \to R$ denote the corresponding anti-endomorphism of b and assume κ is an anti-isomorphism of K. Then by Proposition 2.3.6(i), b has a right κ-asymmetry $\lambda \in R$. Then for all $k \in K$, $k^\kappa = b(1,k)^\kappa = b(k,\lambda) = k^*\lambda$. In addition, by Proposition 2.3.9(i), $x^{**}\lambda = \lambda x$ for all $x \in R$. Therefore, for all $k \in K$, $k^{\kappa\kappa} = \lambda^*\lambda k$ (see the computation in (iv)) and since κ^2 is invertible, $\lambda^*\lambda \in R^\times$. Thus, κ is obtained from λ as in (ii). Now assume κ is augmentable with augmentation transformation γ. Then by Lemma 2.3.12, $\gamma_R = \lambda^*\lambda$ and the naturalness of γ implies $\lambda^*\lambda = \gamma_R \in \mathrm{Cent}(R)$, as required. If moreover κ is an involution, then $\gamma = \mathrm{id}$, hence $\lambda^*\lambda = 1$, as in (iv). $\square$

Remark 2.4.2. Assume the notation of part (ii) of the last proposition. That $\lambda^*\lambda \in R^\times$ does not imply λ is invertible, even when $\lambda^*\lambda = 1$; see Example 2.4.12. In addition, $\lambda^*\lambda$ commutes with $\mathrm{im}(*)$. Indeed, for all $x \in R$:

$$\lambda^*\lambda x^* = \lambda^* x^{***}\lambda = (x^* * \lambda)^*\lambda = (\lambda x)^*\lambda = x^*\lambda^*\lambda\ .$$

Therefore, if $*$ is surjective, then κ is augmentable, and further arguing would show b is regular. This agrees with Proposition 2.3.13.

2.4.1. Case I.

Example 2.4.3. Let R be a ring, let $*$ be an anti-endomorphism of R and define b as in Proposition 2.4.1. Then b is right regular (and hence right injective, surjective and stable). It is left injective (resp. surjective) if and only if $*$ is, and by Proposition 2.3.4, it is left stable (semi-stable) if and only $*$ is bijective (injective). Therefore, we see that:

(1) (R0) $\not\Longrightarrow$ (L5) $\vee$ (L2)[6] by taking $R = F[x \,|\, x^2 = 0]$ with F a field and defining $* : R \to R$ by $f(x)^* = f(0)$.
(2) (R0) $\wedge$ (L2) $\not\Longrightarrow$ (L5) (and in particular (R0) $\wedge$ (L2) $\not\Longrightarrow$ (R1) $\vee$ (R3)) by taking $R = F^{\mathbb{N}} = F \times F \times \cdots$ with F a field and defining $* : R \to R$ by $(x_1, x_2, \dots)^* = (x_2, x_3, \dots)$.
(3) (R0) $\wedge$ (L1) $\not\Longrightarrow$ (L2) $\vee$ (L3) by taking $R = \mathbb{Q}(x)$ and defining $* : R \to R$ by $f(x)^* = f(x^2)$.

We set some general notation for the next examples: Let F be a field and let T denote the ring of 2×2 upper triangular matrices over F. Define M and J to be the ideals of T consisting of matrices of the forms $\left[\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right]$, respectively. Then it is easy to verify that:

$$\mathrm{End}_T(M) \cong \mathrm{End}_T(J) \cong \mathrm{End}_T(M/J) \cong F,$$
$$\mathrm{Hom}_T(M,J) \cong \mathrm{Hom}(M/J,M) \cong \mathrm{Hom}_T(M/J,J) \cong \mathrm{Hom}(J,M/J) \cong 0,$$
$$\mathrm{Hom}_T(M,M/J) \cong \mathrm{Hom}_T(J,M) \cong F,$$

where all isomorphisms are F-vector spaces isomorphisms or F-algebras isomorphisms (whichever appropriate). In particular, M, J, M/J are LE-modules (i.e. modules with local endomorphism ring), hence $M^k \oplus J^m \oplus (M/J)^n \cong_T M^{k'} \oplus J^{m'} \oplus (M/J)^{n'}$ implies $k = k'$, $m = m'$ and $n = n'$. (This follows from the Krull-Schmidt Theorem; see [**80**, §2.9]. Moreover, any f.g. right T-module is a sum of copies of M, J and M/J.)

Example 2.4.4. (R2)$\wedge$(R3)$\wedge$(R4)$\wedge$(L2)$\wedge$(L3)$\wedge$(L4) $\not\Longrightarrow$ (R1)$\vee$(L1): Make $K = M/J$ into a double R-module by taking both $\odot_0$ and $\odot_1$ to be the standard right action of T on K (this works since $T/\operatorname{ann}^r K$ is commutative). Note that $\kappa := \mathrm{id}_K$ is an involution of K. Define $b : M \times M \to M/J$ by $b(m_1, m_2) = m_1 m_2 + J$. Then b is a κ-symmetric bilinear form and it is easy to check that b is degenerate (i.e. neither right nor left injective) with $\ker \mathrm{Ad}_b^r = \ker \mathrm{Ad}_b^\ell = J$. Since $\mathrm{Hom}_T(M,J) = 0$, b is semi-stable. It is now routine to verify that id_M is a left

[6] The sign "$\vee$" denotes logical "or".

and right κ-asymmetry and that b is stable (use the fact $\mathrm{End}(M_T) \cong F$; the corresponding anti-endomorphism is id_F). The form b is surjective since for all $i \in \{0,1\}$, $\dim_F M^{[i]} = \dim_F \mathrm{Hom}_T(M, M/J) = 1$ and Ad_b^ℓ, Ad_b^r are non-zero F-linear maps.

Example 2.4.5. (R0) $\wedge$ (L2) $\wedge$ (L3) $\not\Longrightarrow$ (L1): Make $K := M$ into a double T-module by defining

$$m \odot_0 t = tm \qquad \text{and} \qquad m \odot_1 t = mt \qquad \forall m \in M, x \in T \ ,$$

(this works since $T/\mathrm{ann}^\ell M \cong F$), and let $b : M \times M \to K$ be given by $b(x,y) = xy$. It is easy to check that b is right but not left injective. In addition, $K_0 \cong M/J \oplus M/J$ and therefore, $\dim_F M^{[1]} = \dim_F \mathrm{Hom}(M_T, K_0) = 2 \dim_F \mathrm{Hom}_T(M, M/J) = 2$. Dimension considerations now imply Ad_b^r is bijective, hence b is right regular. Therefore, b is right stable and since $\mathrm{End}_T(M) \cong F$ as F-algebras and the corresponding anti-isomorphism $*$ is F-linear, $*$ must be id_F. Thus, $*$ is bijective, implying b is left stable (by Proposition 2.3.4). To finish, $\dim_F M^{[0]} = \dim_F \mathrm{Hom}_T(M, M) = \dim F_F = 1$, so since $\mathrm{Ad}_b^\ell \neq 0$, b is left surjective. (Note that K cannot have an anti-isomorphism for otherwise we would get a contradiction to Corollary 2.3.7.)

Example 2.4.6. (R0) $\wedge$ (L1) $\wedge$ (L3) $\not\Longrightarrow$ (L2): Make $K = \mathrm{M}_2(F) \times F$ into a double T-module by defining:

$$(U, x) \odot_0 A = (A^T U, cx) \qquad \text{and} \qquad (U, x) \odot_1 A = (UA, ax)$$

for all $U \in \mathrm{M}_2(F)$, $x \in F$ and $A = \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right] \in T$. Define $b : M \times M \to K$ by $b(x,y) = (x^T y, 0)$. It is straightforward to check that b is injective. Note that $K_0 \cong M \oplus M \oplus J$, hence $\dim_F M^{[1]} = \dim_F \mathrm{Hom}_T(M, M \oplus M \oplus J) = 2$, so dimension constraints imply Ad_b^r is bijective. The argument of the last example shows that in this case b is also left stable. On the other hand, $K_1 \cong M \oplus M \oplus M/J$, hence $\dim_F M^{[0]} = \dim_F \mathrm{Hom}_T(M, M \oplus M \oplus M/J) = 3$, so Ad_b^ℓ cannot be surjective, implying b is not left surjective. (In this case K does not have an anti-isomorphism κ since $M^{[0]} \not\cong M^{[1]}$; see Proposition 2.2.7. Alternatively, K cannot have an anti-isomorphism since this would contradict Proposition 2.3.13.)

Example 2.4.7. (R1)$\wedge$(R3)$\wedge$(L2)$\wedge$(L3) $\not\Longrightarrow$ (R2)$\vee$(L1): Make $K = M \times F$ into a double T-module by defining:

$$(m, x) \odot_0 t = (tm, ax) \quad \text{and} \quad (m, x) \odot_1 t = (mt, cx)$$

for all $m \in M$, $x \in F$ and $t = \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right] \in T$. Define $b : M \times M \to K$ by $b(x,y) = (xy, 0)$. By restricting the range of b to be $M \times \{0\} \subseteq K$, we get the bilinear form of Example 2.4.5. Therefore, b is right injective, stable and left degenerate. However, $K_0 \cong M/J \oplus M/J \oplus M/J$, hence $\dim_F M^{[0]} = \dim_F \mathrm{Hom}_T(M, (M/J)^3) = 3$, implying b is not right surjective. On the other hand, $K_1 \cong M \oplus J$, hence $\dim_F M^{[1]} = \dim_F \mathrm{Hom}_T(M, M \oplus J) = 1$, so b is left surjective (since $\mathrm{Ad}_b^\ell \neq 0$).

It is possible (but tedious) to check that the previous examples, together with Example 2.4.9 below, imply that in case K is not assumed to have an anti-isomorphism (i.e. Case I), all the implications between subsets of the properties (R1)-(R3) and (L1)-(L3) are explained by "(R1) and (R2) $\Longrightarrow$ (R3)" and its left analogue.

2.4.2. Case III. (We preceded Case III to Case II since examples relevant to the former are relevant to the latter but not vice versa.)

Example 2.4.8. (R1) $\wedge$ (R3) $\wedge$ (R4)-κ $\wedge$ (L1) $\wedge$ (L3) $\wedge$ (L4)-κ^{-1} $\not\Longrightarrow$ (R2) $\vee$ (L2): Make $\mathbb{Z}$ into a double $\mathbb{Z}$-module by letting both $\odot_0$ and $\odot_1$ be the standard action of $\mathbb{Z}$ on itself. Define $b : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $b(x,y) = 2xy$ and let $\kappa = \mathrm{id}_{\mathbb{Z}}$. Then b is a κ-symmetric bilinear form over $\mathbb{Z}$. The rest of the details are left to the reader.

EXAMPLE 2.4.9. (R1) $\wedge$ (R4)-κ $\wedge$ (L1) $\wedge$ (L4)-κ^{-1} $\not\Longrightarrow$ (R2) $\vee$ (R3) $\vee$ (L2) $\vee$ (L3): Consider $\mathbb{Z}$ as a double $\mathbb{Z}$-module like in the previous example and define $b : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ by $b(x, y) = x^T \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right] y$ (the elements of $\mathbb{Z}^2$ are considered as column vectors). Then b is injective and it has a unique right and left $\mathrm{id}_{\mathbb{Z}}$-asymmetries (given by $\mathrm{id}_{\mathbb{Z}^2}$). It is also easy to see b is not right nor left surjective.

We claim b is not right stable. To see this, identify $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^2)$ with $\mathrm{M}_2(\mathbb{Z})$ and let $\sigma, \sigma' \in \mathrm{M}_2(\mathbb{Z})$. Then $b(\sigma x, y) = b(x, \sigma' y)$ for all $x, y \in \mathbb{Z}^2$ if and only if $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right] \sigma' = \sigma^T \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$. By letting $\sigma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and working in $\mathrm{M}_2(\mathbb{Q})$, we see that necessarily

$$\sigma' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a & 2c \\ b/2 & d \end{bmatrix} .$$

Therefore, for $\sigma = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ there is no $\sigma' \in \mathrm{M}_2(\mathbb{Z})$ as above and thus b is not right stable. The form b is also not left stable by Corollary 2.3.10.

EXAMPLE 2.4.10. (R1) $\wedge$ (R3) $\wedge$ (L1) $\wedge$ (L3) $\not\Longrightarrow$ (R2) $\vee$ (R4) $\vee$ (L2) $\vee$ (L4): Let F be a field and let $\pm 1 \neq \alpha \in F^\times$. Make F^2 into a double F-module by letting both $\odot_0$ and $\odot_1$ be the standard action of F on F^2. Then $b : F \times F \to F^2$ defined by $b(x, y) = (xy, \alpha xy)$ is a bilinear form. It is easy to check that b is injective and stable (the corresponding anti-isomorphism is $\mathrm{id}_{\mathrm{End}(F_F)}$). Dimension constraints also imply b neither left nor right surjective. Define $\kappa : F^2 \to F^2$ by $(a, b)^\kappa = (b, a)$. Then κ is an involution of F^2 and we claim b has neither left nor right κ-involution. Indeed, assume λ is a right κ-asymmetry and identify $\mathrm{End}(F_F)$ with F. Then $(\alpha, 1) = b(1, 1)^\kappa = b(1, \lambda) = (\lambda, \lambda\alpha)$ which is impossible if $\alpha \neq \pm 1$. Similarly, b has no left κ-asymmetry. Despite the former, b has right and left id_{F^2}-asymmetries (given by id_F).

EXAMPLE 2.4.11. (R3) $\wedge$ (R4)-κ $\wedge$ (L3) $\wedge$ (L4)-κ^{-1} $\not\Longrightarrow$ (R1) $\vee$ (R2) $\vee$ (L1) $\vee$ (L2): With the notation of Example 2.4.4, $K \times K$ is a double T-module and $\kappa = \mathrm{id}_{K \times K}$ is an an involution of K. Define $b' : M \times M \to K \times K$ by $b'(x, y) = (b(x, y), 0)$. Then b' satisfies (R3), (R4)-κ, (L3) and (L4) but not (R1), (R2), (L1) or (L2). The details are left to the reader.

EXAMPLE 2.4.12. (R0) $\not\Longrightarrow$ (L2) $\vee$ (L3) $\vee$ (L4): Let M be the free monoid over $\{x_0, x_1, x_2, \dots\}$ subject to the relations:

$$x_{2k+1} x_{2k} = 1 = x_{2k+1} x_{2k+2},$$

$$x_{n+2+2k} x_{2k} = x_{2k} x_{n+2k}, \qquad x_{2k+1} x_{n+2k+3} = x_{n+2k+1} x_{2k+1},$$

for all $n, k \geq 0$. The map $* : M \to M$ defined by sending the word $x_{i_1} x_{i_2} \dots x_{i_r}$ to $x_{i_r+1} \dots x_{i_2+1} x_{i_1+1}$ is a well defined anti-endomorphism of M satisfying:

$$w^{**} x_0 = x_0 w \tag{3}$$

for all $w \in M$. In addition, $x_0^* x_0 = x_1 x_0 = 1$. Let F be a field and let $R = FM$ be the monoid algebra of M over F. Then $*$ extends to an anti-endomorphism of M, satisfying (3) for all $w \in R$. Define b and K as in Proposition 2.4.1. Then b is right regular. By taking $\lambda = x_0$ in part (iv) of that proposition, we see that K admits an involution κ, and x_0 is a right κ-asymmetry of b. We claim b does not have a left κ-asymmetry. Indeed, by Proposition 2.3.18, it is enough to verify $x_2 = x_0^{**} \neq x_0$. Since showing this involves a long and technical argument, we leave it for the addendum. b cannot be left stable since this would contradict Proposition 2.3.13, and b cannot be left surjective since this would imply b is left regular (because b is already injective by Proposition 2.3.6(ii)), and hence has a left κ-asymmetry.

In order to show that the list of implications for Case III given in subsection 2.3.3 is complete, one also has to demonstrate the following non-implications, and possibly more:

- (R2) $\wedge$ (R3) $\wedge$ (L2) $\wedge$ (L3) $\not\Longrightarrow$ (R4)-κ (even when K has an involution),
- (R2) $\wedge$ (R4)-κ $\wedge$ (L2) $\wedge$ (L4)-κ^{-1} $\not\Longrightarrow$ (R3),
- (R3) $\wedge$ (R4)-κ $\wedge$ (L2) $\not\Longrightarrow$ (L4)-κ^{-1}.

The problem with all these non-implications is that b has to be right or left surjective. In addition, we could not produce an example of a bilinear form having a unique right asymmetry with non-trivial kernel.

2.4.3. Case II.

EXAMPLE 2.4.13. (R1), (R3), (R4)-κ, (L1), (L3) $\not\Longrightarrow$ (R2), (L2) or (L4)-κ^{-1}: Make $\mathbb{Q}$ into a double $\mathbb{Z}$-module by letting $\odot_0$ and $\odot_1$ be the standard action of $\mathbb{Z}$ on $\mathbb{Q}$. Define $\kappa : \mathbb{Q} \to \mathbb{Q}$ by $\kappa(q) = 2q$ and $b : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ by $b(x, y) = xy$. The details are left to the reader. (Note that κ is not augmentable, for otherwise this would contradict Corollary 2.3.18; see Example 2.3.15(ii) for a direct verification of this fact.)

Provided that all the missing non-implications for Case III are shown, the following non-implications (and possibly more) are needed in order to show that the list of implications for Case II given in subsection 2.3.3 is complete.

- (R1), (R3), (L2) and (L4)-κ^{-1} $\not\Longrightarrow$ (R2), (R4)-κ, (L1) or (L3),
- (R1), (R4)-κ, (L1), (L3) $\not\Longrightarrow$ (R3),
- (R2), (R3), (R4)-κ, (L2) and (L3) $\not\Longrightarrow$ (R1), (L1) and (L4)-κ^{-1}.

Note that κ cannot be augmentable in such examples.

2.4.4. Further Examples. The following examples demonstrate that there exist:

(1) regular bilinear spaces (M, b, K) s.t. K has no anti-isomorphism,
(2) regular bilinear spaces (M, b, K) s.t. K has an augmentable anti-isomorphism, but no involution.

We also construct an example of a right regular bilinear form taking values in a double R-module with an an anti-isomorphism which we believe (but still cannot prove) to have no augmentable anti-isomorphisms.

EXAMPLE 2.4.14. Let R be a ring and let $*$ be an anti-*auto*morphism such that $*^2$ is not an inner automorphism (e.g. take R to be commutative and let $*$ be an automorphism of R such that $*^2 \neq \mathrm{id}$). Define b and K as in Proposition 2.4.1. Then b is regular, but K does not have an anti-isomorphism. Indeed, if K had an anti-isomorphism κ, then the proof of Proposition 2.3.13 would imply $*^2$ is inner, which contradicts our assumptions. (This implies that having an anti-isomorphism is quite rare in general.)

EXAMPLE 2.4.15. Keeping the notation of the previous example, if we choose $*$ such that $r^{**}\lambda = \lambda r$ for some $\lambda \in R$ with $\lambda^*\lambda \in R^\times$ but λ cannot be taken to satisfy $\lambda^*\lambda = 1$, then K admits an anti-isomorphism but no involution (Proposition 2.4.1). If we take λ to be invertible, then $*$ is bijective, hence by Remark 2.4.2(ii), $\lambda^*\lambda \in \mathrm{Cent}(R)$ and it follows that κ is augmentable (Proposition 2.4.1(iii)).

Such an example can be constructed as follows: Let M denote the free monoid over $\{x_n\}_{n\in\mathbb{Z}}$. The group $F = \mathrm{FreeAb}\langle y, z\rangle$ acts on M by $y(x_n) = x_{n+2}$ and $z(x_n) = x_{n-2}$ for all $n \in \mathbb{Z}$. We can thus form the semi-direct product $S = M \rtimes F$. Namely, S consists of pairs $(m, f) \in M \times F$ and $(m, f) \cdot (m', f') = (mf(m'), ff')$. Identify x_n with $(x_n, 1_F) \in S$ and $y, z \in F$ with $(1_M, y), (1_M, z) \in S$. Then

$yx_n = x_{n+2}y$ and $zx_n = x_{n-2}z$ for all $n \in \mathbb{N}$. Define an anti-isomorphism $* : S \to S$ by $x_n^* = x_{n+1}$, $y^* = z$ and $z^* = y$. Then $*$ is easily seen to satisfy

$$x_n^{**}y = yx_n, \qquad y^{**}y = yy, \qquad z^{**}y = yz,$$

hence $*^2$ is an inner isomorphism of S. Let F be a field and let $R = FS$ be the monoid algebra of S over F. Then $*$ extends to R and $x^{**}y = yx$ for all $x \in R$.

We claim there is no $y' \in R^\times$ with $(y')^*y' = 1$ such that $x^{**}y' = y'x$ for all $x \in R$. Assume by contradiction such y' exists. Then leading-term considerations easily imply $y' = \alpha y^t z^s$ for some $t, s \in \mathbb{Z}$ and $\alpha \in F$. Since $x_2 y' = y' x_0$, $t - s$ must be 1. But then $(y')^*y' = \alpha^2 y^{t+s} z^{t+s} \neq 1$, since $t + s$ is odd. Therefore, $*^2$ is inner, but there is no $\lambda \in R$ with $\lambda^*\lambda = 1$ such that $x^{**}\lambda = \lambda x$ for all $x \in R$.

Remark 2.4.16. The reader might wonder why we had to construct such a complicated ring in the last example. The reason lies in the fact that although $*^2$ must be inner, $*$ cannot be a composition of an inner automorphism with an involution. (Indeed, a direct computation would show that if $r^* = xr^\sharp x^{-1}$ for some involution $\sharp$, then $*^2$ is inner, but we can take $\lambda = x(x^{-1})^*$ which clearly satisfies $\lambda^*\lambda = 1$.)

Example 2.4.17. Keeping the notation of the previous examples, if we can we choose $*$ such that $r^{**}\lambda = \lambda r$ for some $\lambda \in R$ with $\lambda^*\lambda \in R^\times \setminus \mathrm{Cent}(R)$ and λ cannot be chosen to satisfy $\lambda^*\lambda \in R^\times \cap \mathrm{Cent}(R)$, then K has an anti-isomorphism, but no augmentable anti-isomorphism (Proposition 2.4.1). Note that $*$ cannot be surjective (since $\lambda^*\lambda$ commutes with $\mathrm{im}(*)$), hence λ cannot be invertible. We believe the following construction satisfies the previous requirements, we were unable to prove that λ cannot be chosen to satisfy $\lambda^*\lambda \in R^\times \cap \mathrm{Cent}(R)$.

The construction is similar to Example 2.4.12, with a major difference — the relation $x_1x_0 = 1$ and all relations following from it by applying $*$ are dropped and replaced with relations making x_1x_0 into an invertible element. Define M to be the free monoid on $\{x_0, x_1, x_2, \dots\} \cup \{y_0, y_1, y_2, \dots\}$ subject to the relations:

$$x_{2n+1}x_{2n}y_{2n} = y_{2n}x_{2n+1}x_{2n} = x_{2n+1}x_{2n+2}y_{2n+1} = y_{2n+1}x_{2n+1}x_{2n+2} = 1,$$

$$x_{n+2+2k}x_{2k} = x_{2k}x_{n+2k}, \qquad x_{2k+1}x_{n+2k+3} = x_{n+2k+1}x_{2k+1},$$

$$y_{n+2+2k}x_{2k} = x_{2k}y_{n+2k}, \qquad x_{2k+1}y_{n+2k+3} = y_{n+2k+1}x_{2k+1},$$

for all $n, k \geq 0$. Let $*$ be the unique anti-endomorphism sending x_n to x_{n+1} and y_n to y_{n+1}. Then $*$ extends to the monoid algebra $R = FM$ (F is a field) and $r^{**}x_0 = x_0 r$ for all $r \in R$. In addition, $x_0^* x_0 = x_1 x_0 \in R^\times$ (its inverse is y_0). (Note that R maps onto the ring constructed in Example 2.4.12 by sending all the y-s to 1.) It is left to check that $x_0^* x_0 \notin \mathrm{Cent}(R)$ (which should be technical by not impossible) and that there is no $\lambda \in R$ such that $r^{**}\lambda = \lambda r$ for all $r \in R$ and $\lambda^*\lambda \in R^\times \cap \mathrm{Cent}(R)$. (We have no clue how to show the latter.)

2.5. Special Cases

In this section, we demonstrate how the results of section 2.3 can be improved by adding extra assumptions on $[0], [1], \Psi$ and Φ, e.g.:

(1) One or both of $[0], [1]$ is exact. (The functors $[0], [1]$ are only left-exact.)
(2) One or both of Ψ, Φ is injective.
(3) One or both of Ψ, Φ is bijective.

We will also explain what K and M should satisfy for these assumptions to hold. Explicit examples are also presented. (Recall that Ψ (resp. Φ) was defined to be a the natural transformation from $\mathrm{id}_{\text{Mod-}R}$ to $[0][1]$ (resp. $[1][0]$) given by $(\Psi_M f)x = fx$ for all $M \in \text{Mod-}R$, $x \in M$, $f \in M^{[0]}$ (resp. $(\Phi_M f)x = fx$ for all $M \in \text{Mod-}R$, $x \in M$, $f \in M^{[1]}$).)

For the discussion to follow, K is a fixed double R-module. Recall that K_i denotes K considered as an R-module via $\odot_i$. We will make repeated use of Corollary 2.2.5, which states that for any bilinear space (M, b, K), the following diagrams commute:

$$(4)\qquad \begin{array}{ccc} M & \xrightarrow{\Phi_M} & M^{[1][0]} \\ {\scriptstyle \mathrm{Ad}_b^\ell}\downarrow & \swarrow {\scriptstyle (\mathrm{Ad}_b^r)^{[0]}} & \\ M^{[0]} & & \end{array} \qquad\qquad \begin{array}{ccc} M & \xrightarrow{\Psi_M} & M^{[0][1]} \\ {\scriptstyle \mathrm{Ad}_b^r}\downarrow & \swarrow {\scriptstyle (\mathrm{Ad}_b^\ell)^{[1]}} & \\ M^{[1]} & & \end{array}$$

In addition, we will freely use the fact that if $A, B \in \text{Mod-}R$, $f : A \to B$ and $i \in \{0, 1\}$, then f is surjective (bijective) implies $f^{[i]} : B^{[i]} \to A^{[i]}$ is injective (bijective). (This holds since by definition $[i]$ is the functor $\mathrm{Hom}(\_\,, K_{1-i})$, which is well-known to be left-exact.)

We begin with the following definition regarding the natural transformations Ψ and Φ.

Definition 2.5.1. *A module $M \in \text{Mod-}R$ is called right (left)* semi-reflexive *if Ψ_M (Φ_M) is injective and right (left)* reflexive *if Ψ_M (Φ_M) is bijective.*[7] *We will say M is semi-reflexive (reflexive) if it is both left and right semi-reflexive (reflexive).*

Reflexive and semi-reflexive modules appear when considering injective and regular bilinear forms.

Proposition 2.5.2. *Let (M, b, K) be a bilinear space. Then:*

(i) *If b is right injective, then M is right semi-reflexive (i.e. Ψ_M is injective).*
(ii) *In b is right regular and left surjective, then M is right reflexive. (In particular, if b is regular, then M is reflexive.)*

Proof. This follows from (4). (In (ii), $(\mathrm{Ad}_b^\ell)^{[1]}$ is injective since Ad_b^ℓ is surjective.) □

In light of this proposition, if one is interested only in regular bilinear forms, then it makes sense to assume Φ and Ψ are isomorphisms. However, if one is interested in right (but not necessarily left) regular forms, then one can only assume Ψ is injective; requiring Ψ to be bijective will result in the exclusion of some examples, as demonstrated in the following example.

Example 2.5.3. Let R be a ring and let $*$ be an anti-endomorphism. Define K and $b : R \times R \to K$ as in Proposition 2.4.1. Then b is right regular. Let us compute Ψ_R and Φ_R explicitly.

First, $R^{[i]} \cong K_i$ and the isomorphism is given by $[r \mapsto k \odot_{1-i} r] \mapsto k$. In particular, $R^{[1]} \cong K_1 = R_R$ and hence $R^{[1][0]} \cong R^{[0]} \cong K_0$ where the isomorphism is given by $[[r \mapsto r^* k] \mapsto sk] \mapsto s \in K_0$ $(r, s, k \in R = K)$. Since $\Phi_R(x) = [[r \mapsto r^* k] \mapsto x^* k]$, Φ_R is just $*$ once identifying $R^{[1][0]}$ with K_0. On the other hand, $R^{[0][1]} \cong \mathrm{Hom}_R(K_0, K_0) \cong \mathrm{End}_R(K_0)$ and, identifying both modules, Ψ_R is easily seen to be the map sending x to $[k \mapsto kx] \in \mathrm{End}_R(K_0)$.

It is now easy to see that Ψ_R is always injective (as it must be, since b is right regular), but Φ_R is injective (surjective) if and only if $*$ is. If we take $(R, *)$ to be as in Example 2.4.3(i), then Ψ_R is not surjective and Φ_R is not injective nor surjective. In particular, R is right semi-reflexive and not left semi-reflexive.

Proposition 2.5.4. *Let (M, b, K) be a bilinear space. Then:*

---

[7] Compare this with *torsionless* and *reflexive* modules in [**58**]: Let $M \in \text{Mod-}R$ and consider $M^* := \mathrm{Hom}(M_R, R_R)$ as a *left* R-module by $(r \cdot f)m := r \cdot (fm)$. Then there is a standard map $i : M \to M^{**} := \mathrm{Hom}({}_R M^*, {}_R R)$ given by $(ix)f = fx$ (note M^{**} is a *right* R-module). The module M is called *reflexive* (resp. *torsionless*) if i is a bijection (resp. injection).

(i) *If M is right semi-reflexive and b is left surjective then b is right injective.*
(ii) *If M is right reflexive and b is left regular, then b is right regular.*
(iii) *If* $[1]$ *is exact, M is right reflexive and b is left injective, then b is right surjective.*
(iv) *If K has an anti-isomorphism, then M is right semi-reflexive (reflexive) $\iff$ M is left semi-reflexive (reflexive), and* $[1]$ *is exact* $\iff$ $[0]$ *is exact.*

Proof. (i), (ii) and (iii) easily follow (4) where in (iii), the fact $[1]$ is exact implies that if Ad_b^ℓ is injective, then $(\mathrm{Ad}_b^\ell)^{[1]}$ is surjective.

(iv) If κ is an involution, then by Proposition 2.2.7, $[0] \cong [1]$, hence $[0]$ is exact if and only if $[1]$ is exact. The first equivalence follows from Proposition 2.7.5(iii) below, which says that there is a natural isomorphism $\delta_M : M^{[1][0]} \to M^{[0][1]}$ such that $\delta_M \circ \Phi_M = \Psi_M$. □

Corollary 2.5.5. *Let (M, b, K) be a bilinear space. If* $[0]$ *and* $[1]$ *are exact, M is reflexive and b is injective, then b is regular.*

Reflexive and semi-reflexive modules also have the following nice properties:

Proposition 2.5.6. *Let $A, B \in$ Mod-R.*
(i) *The map $f \mapsto f^{[0]}$ from* $\mathrm{Hom}(A^{[1]}, B^{[1]})$ *to* $\mathrm{Hom}(B^{[1][0]}, A^{[1][0]})$ *is injective.*
(ii) *Assume B is left semi-reflexive, then the map $f \mapsto f^{[1]}$ from* $\mathrm{Hom}(A, B)$ *to* $\mathrm{Hom}(B^{[1]}, A^{[1]})$ *is injective.*
(iii) *Assume A and B are left reflexive, then the map $f \mapsto f^{[1]}$ from* $\mathrm{Hom}(A, B)$ *to* $\mathrm{Hom}(B^{[1]}, A^{[1]})$ *is bijective.*

Proof. (i) Let $f \in \mathrm{Hom}(A^{[1]}, B^{[1]})$ and assume $f^{[0]} = 0$. Then $f^{[0][1]} = 0 \Longrightarrow 0 = \Phi_B^{[1]} \circ f^{[0][1]} \circ \Psi_{A^{[1]}} = \Phi_B^{[1]} \circ \Psi_{B^{[1]}} \circ f = \mathrm{id}_{B^{[1]}} \circ f = f$.

(ii) Let $f \in \mathrm{Hom}(A, B)$ be such that $f^{[1]} = 0$. Then $f^{[1][0]} = 0 \Longrightarrow 0 = f^{[1][0]} \circ \Phi_A = \Phi_B \circ f$. Since Φ_B is injective, it follows that $f = 0$.

(iii) In view of (ii), we only need to prove that for any $g \in \mathrm{Hom}(B^{[1]}, A^{[1]})$ there is $f \in \mathrm{Hom}(A, B)$ such that $f^{[1]} = g$. Indeed, let $f = \Phi_B^{-1} \circ g^{[0]} \circ \Phi_A$. Then $(f^{[1]} - g)^{[0]} \circ \Phi_A = f^{[1][0]} \circ \Phi_A - g^{[0]} \circ \Phi_A = \Psi_B \circ f - \Psi_B \circ f = 0$. Since A is left reflexive, this implies $(f^{[1]} - g)^{[0]} = 0$, so by (i), $f^{[1]} = g$. □

Note that $[i] = \mathrm{Hom}(\_, K_{1-i})$ is exact if and only if K_{1-i} is injective. A sufficient and necessary condition for Ψ (resp. Φ) to be injective (for all modules) is that K_1 (resp. K_0) would be is a *cogenerator*. Recall that module $U \in$ Mod-R is called a cogenerator if it satisfies any of the following equivalent conditions (see [**58**, Prp. 19.6]):

(a) For all $A, B \in$ Mod-R and $0 \neq f \in \mathrm{Hom}(A, B)$, there exists $g \in \mathrm{Hom}(B, U)$ such that $g \circ f \neq 0$.
(b) Any right R-module embeds in U^α for some cardinal α.
(c) For every $M \in$ Mod-R and $x \in M$, there exists $f \in \mathrm{Hom}(M, U)$ such that $f(x) \neq 0$.

Indeed, Ψ_M (resp. Φ_M) is injective if and only if for all $x \in M$, there is $f \in \mathrm{Hom}(M, K_1)$ (resp. $f \in \mathrm{Hom}(M, K_0)$) s.t. $f(x) \neq 0$ and by condition (c), this holds for all $M \in$ Mod-R if and only if K_1 (resp. K_0) is a cogenerator. It is unreasonable to ask for Ψ_M or Φ_M to be bijective for all $M \in$ Mod-R since $\mathrm{Hom}(\_, K_i)$ takes direct sums to products. However, this might hold for all modules satisfying some finiteness condition, as will be demonstrated below.

We shall now apply our previous observations to *quasi-Frobenius* and *pseudo-Frobenius* rings (abbrev. QF and PF respectively); such rings admit a very rich supply of cogenerators and generators. A ring R is called *right PF* if it satisfies one of the following equivalent conditions (see [**58**, Th. 19.25] or [**54**, Ch. 12]):

(a) All faithful right R-modules are generators.
(b) R_R is an injective cogenerator.
(c) R is semilocal (or semiperfect), right self injective and $\mathrm{soc}(R_R) \subseteq_e R_R$.

A QF ring is an artinian right PF ring. This turns out to be equivalent to being one-sided noetherian and one-sided self-injective (see [**58**, §15]). Examples of QF rings include local artinian rings with simple socle and finite group rings over other QF rings.

Example 2.5.7. Assume R is right PF and K_1 is faithful. Then K_1 is a generator and hence a cogenerator (since R_R, which is a summand of K_1^n, is a cogenerator). Thus, all right R-modules are right semi-reflexive and in particular, (L2)$\Longrightarrow$(R1) (Proposition 2.5.4(i)).

Specializing the previous example further yields even sharper results.

Proposition 2.5.8. *Let R be a PF ring with an anti-*automorphism *$*$ and let K be the double R-module defined in Proposition 2.4.1. Assume M is a f.g. right R-module. Then:*

(i) *M is reflexive.*
(ii) *The conditions (R1) and (L2) (resp. (L1) and (R2)) are equivalent for any bilinear form $b : M \times M \to K$. If M is faithful, then they are also equivalent to (R5) (resp. (L5)).*

If moreover R is QF, then:

(iii) *The conditions (R0)-(R2),(L0)-(L2) are equivalent for any bilinear form $b : M \times M \to K$. If M is faithful, then they are also equivalent to (R3)-(R5),(R3)-(L5) (where (R4) and (L4) can be taken w.r.t. any anti-isomorphism of K). (Compare with Proposition 2.1.5.)*

Proof. (i) Note that $K_1 \cong K_0 \cong R_R$, hence K_1 and K_0 are injective, and thus [0] and [1] are exact. Moreover, the previous example implies all right R-modules are semi-reflexive, so we only have to show Ψ_M and Φ_M are onto.

It easy to see from Example 2.5.3 that R_R is reflexive. Since M is f.g. there is a surjection $R^n \to M$ for some n, and since [0] and [1] are exact, we get the following exact commutative diagram:

$$\begin{array}{ccccc} R^n & \twoheadrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Psi_{R^n}} & & \downarrow{\scriptstyle \Psi_M} & & \\ (R^n)^{[0][1]} & \overset{f}{\twoheadrightarrow} & M^{[0][1]} & \longrightarrow & 0 \end{array}$$

Now, Ψ_{R^n} is bijective (since R^n is right reflexive), hence $f \circ \Psi_{R^n}$ is onto which implies Ψ_M is onto. By symmetry, Φ_M is also onto, so we are through.

(ii) (L2)$\Longrightarrow$(R1) was shown in the previous example. The converse follows from (i) and Proposition 2.5.4(iii). If M is also faithful, then M is a generator, hence Corollary 2.2.10 implies (R1) $\Longleftrightarrow$ (R5).

(iii) The first assertion follows from (ii) if we show that (L1) $\Longleftrightarrow$ (L2). As M is artinian (since R is), it enough to prove $\mathrm{length}(M) = \mathrm{length}(M^{[0]})$ (since then $\mathrm{Ad}_b^\ell : M \to M^{[0]}$ is injective if and only if it is surjective). Indeed, by [**58**, Cr. 15.13], $X \in \text{Mod-}R$ is simple if and only if $X^* := \mathrm{Hom}(X_R, {}_RR_R) \in R\text{-Mod}$ is simple. Since R_R is injective, this is easily seen to imply that $\mathrm{length}(X) = \mathrm{length}(X^*)$ for any $X \in \text{Mod-}R$ of finite length (i.e. a f.g. X). Using $K_0 \cong K_1 \cong R_R$, it is easy to see that $\mathrm{length}({}_RX^*) = \mathrm{length}(X_R^{[0]})$ for all $X \in \text{Mod-}R$, so our claim follows immediately.

If M is also faithful, then by (ii), (R5)$\Longrightarrow$(R1). As (R0) $\Longrightarrow$ (R3)$\wedge$(R4) $\Longrightarrow$ (R3)$\vee$(R4) $\Longrightarrow$ (R5) (see section 2.3), we are through. □

2.6. Orthogonal Sums

In this section we define orthogonal sums of bilinear spaces and explore their relationship with the properties (R0)-(R5) and (L0)-(L5). At the end of the section, we discuss several possible constructions of Witt and Witt-Grothendick Groups. Throughout, K is a fixed double R-module.

Let (M_1, b_1, K) and (M_2, b_2, K) be two bilinear spaces. The *orthogonal sum* $(M_1, b_1, K) \perp (M_2, b_2, K)$ is defined to be $(M_1 \oplus M_2, b_1 \perp b_2, K)$ where:

$$(b_1 \perp b_2)((x_1, x_2), (y_1, y_2)) = b_1(x_1, y_1) \perp b_2(x_2, y_2) \ .$$

It is straightforward to check that $\mathrm{Ad}^r_{b_1 \perp b_2} = \mathrm{Ad}^r_{b_1} \oplus \mathrm{Ad}^r_{b_2}$ and $\mathrm{Ad}^\ell_{b_1 \perp b_2} = \mathrm{Ad}^\ell_{b_1} \oplus \mathrm{Ad}^\ell_{b_2}$ (once identifying $(M_1 \oplus M_2)^{[i]}$ with $M_1^{[i]} \oplus M_2^{[i]}$).

Proposition 2.6.1. *Let M_1, M_2 be right R-modules and let $(b, M_1 \oplus M_2, K)$ be a bilinear space. Let $b_i = b|_{M_i \times M_i}$ $(i = 1, 2)$. Then (b_i, M_i, K) is a bilinear space and $b = b_1 \perp b_2 \iff b(M_1, M_2) = b(M_2, M_1) = 0$.*

Proof. This is easy and left to the reader. □

Proposition 2.6.2. *Let (M_1, b_2, K) and (M_2, b_2, K) be two bilinear spaces. Then:*

(i) *$b_1 \perp b_2$ is right regular (injective, surjective) $\iff$ b_1 and b_2 are right regular (injective, surjective).*
(ii) *$b_1 \perp b_2$ is right stable (semi-stable) $\implies$ b_1 and b_2 is right stable (semi-stable).*

If in addition K has an anti-isomorphism κ, then:

(iii) *If $b_1 \perp b_2$ has a* unique *right κ-asymmetry λ, then b_1 and b_2 has unique right κ-asymmetries, namely $\lambda|_{M_1}$, $\lambda|_{M_2}$ and $\lambda = \lambda|_{M_1} \oplus \lambda|_{M_2}$. Conversely, if b_1, b_2 has right κ-asymmetries λ_1, λ_2, then $\lambda_1 \oplus \lambda_2$ is a right κ-asymmetry of $b_1 \perp b_2$ (but it need not be unique).*
(iv) *$b_1 \perp b_2$ is κ-symmetric $\iff$ b_1 and b_2 are κ-symmetric.*

Proof. Throughout, let $b = b_1 \perp b_2$.

(i) This is follows from $\mathrm{Ad}^r_{b_1 \perp b_2} = \mathrm{Ad}^r_{b_1} \oplus \mathrm{Ad}^r_{b_2}$.

(ii) Assume b is right semi-stable and let $\sigma \in \mathrm{End}(M_1)$ be such that $b_1(x, \sigma y) = 0$. Define $\tau = \sigma \oplus 0 \in \mathrm{End}(M_1 \oplus M_2)$. Then $b((x_1, x_2), \tau(y_1, y_2)) = b_1(x_1, \sigma y_1) + b_2(x_2, 0) = 0$, hence $\tau = 0$ (because b is right semi-stable). Thus, $\sigma = 0$ and b_1 is right semi-stable.

Now assume b is right stable with corresponding anti-endomorphism $*$ and let $e = \mathrm{id}_{M_1} \oplus 0 \in \mathrm{End}(M_1 \oplus M_2)$. Then $b(e(x_1, x_2), (y_1, y_2)) = b_1(x_1, y_1) + b_2(0, y_2) = b_1(x_1, y_1) + b_2(x_2, 0) = b((x_1, x_2), e(y_1, y_2))$, hence $e^* = e$. Let $\sigma \in \mathrm{End}(M_1)$ and define τ as before. Observe that $\tau^* = (e\tau e)^* = e^*\tau^*e^* = e\tau^*e$, thus there is $\sigma' \in \mathrm{End}(M_1)$ such that $\tau^* = \sigma' \oplus 0$. We now get $b_1(\sigma x_1, y_1) = b(\tau(x_1, x_2), (y_1, y_2)) = b((x_1, x_2), \tau^*(y_1, y_2)) = b_1(x_1, \sigma' y_1)$, so b_1 is right stable.

(iii) The second assertion is straightforward. To see the first assertion, note that b_1 and b_2 are semi-stable by (ii) and Lemma 2.3.5. Write $\lambda(x_1, x_2) = (\lambda_{11}x_1 + \lambda_{12}x_2, \lambda_{21}x_1 + \lambda_{22}x_2)$ with $\lambda_{ij} \in \mathrm{Hom}(M_j, M_i)$. Then it is straightforward to see

$$(5) \qquad b_1(x_1, y_1)^\kappa + b_2(x_2, y_2)^\kappa = b_1(y_1, \lambda_{11}x_1 + \lambda_{12}x_2) + b_2(y_2, \lambda_{21}x_1 + \lambda_{22}x_2) \ .$$

By taking x_1 and y_2 to be zero, one gets $0 = b_1(y_1, \lambda_{12}x_2) = b((y_1, y_2), (\lambda_{12}x_2, 0))$, hence $\lambda_{12} = 0$ (because b is right semi-stable). Similarly, $\lambda_{21} = 0$, so $\lambda = \lambda_{11} \oplus \lambda_{22}$. Taking $x_2 = y_2 = 0$ in (5), we get $b_1(x_1, y_1)^\kappa = b_1(y_1, \lambda_{11}x_1)$, hence λ_{11} is a right κ-asymmetry of b_1 and it unique since b_1 is right semi-stable. As the same argument applies to λ_{22} and b_2, we are through.

(iv) This is straightforward. □

In general, the orthogonal sum of two stable forms need neither be right nor left semi-stable, even when one form is regular and even when both forms have right and left κ-asymmetries. (However, the orthogonal sum of two injective forms is always semi-stable because it is injective.) This is demonstrated in the following examples.

EXAMPLE 2.6.3. Consider $\mathbb{Z}$ as a double $\mathbb{Z}$-module by letting $\odot_0$ and $\odot_1$ be the standard right action of $\mathbb{Z}$ on itself. Define $b_1, b_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $b_1(x, y) = xy$ and $b_2(x, y) = 2xy$. Then b_1 is regular, b_2 is injective and stable, and both forms have right and left $\mathrm{id}_{\mathbb{Z}}$-asymmetries. However, $b_1 \perp b_2$ is the bilinear form b of Example 2.4.9, which is not stable (but it is semi-stable because it is injective).

EXAMPLE 2.6.4. We use the general notation presented before Example 2.4.4. Make $K = T/J$ into a double T-module by letting $\odot_0$ and $\odot_1$ be the standard right action of T on K (this works because T/J is a commutative ring) and define $b : T \times T \to K$ by $b(x, y) = xy + J$. Observe that id_K is an involution and b is id_K-symmetric. Now, $T = M \oplus N$ where N is the right T-ideal consisting of matrices of the form $\left[\begin{smallmatrix} 0 & 0 \\ 0 & * \end{smallmatrix}\right]$. It is easy to check that $b(M, N) = b(N, M) = 0$, hence $b = b_1 \perp b_2$ where $b_1 = b|_{M \times M}$ and $b_2 = b|_{N \times N}$ (by Proposition 2.6.1). We claim b is not right nor-left semi-stable but b_1 is stable and b_2 is regular. Since b_1 and b_2 have (necessarily unique) left and right id_K-asymmetries, namely id_M and id_N, this implies b has left and right id_K-asymmetries, but they are not unique.

That b not left nor right semi-stable follows from $b(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x, y) = b(x, \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] y) = 0$ (for all $x, y \in T$). That b_1 is right stable is shown in the same manner as in Example 2.4.4. (Observe that $K_1 = K_0 \cong M/J \oplus J$ and $\mathrm{Hom}(M, J) = 0$, hence $\mathrm{Hom}(M, K_i) = \mathrm{Hom}(M, M/J)$.) To see b_2 is regular, note that $\dim_F N^{[i]} = \dim_F \mathrm{Hom}_T(N, K_i) = \dim_F(J, M/J \oplus J) = 1 = \dim_F N$, hence $\mathrm{Ad}^r_{b_2}$ and $\mathrm{Ad}^\ell_{b_2}$ must be bijective (for they are non-zero).

A sequence of bilinear spaces $\{(b_i, M_i, K)\}_{i=1}^t$ will be called right *joinable* (*semi-joinable*) if $b_1 \perp \cdots \perp b_t$ is right stable (semi-stable). (Note that this implies $b_1, \dots, b_t$ are right stable (semi-stable) by Proposition 2.6.2.) For example, the forms b_1, b_2 of Example 2.6.3 are semi-joinable, but neither left nor right joinable. The following proposition presents necessary and sufficient conditions for a set of forms to be joinable or semi-joinable.

PROPOSITION 2.6.5. *Let $\{(b_i, M_i, K)\}_{i=1}^t$ be bilinear spaces. Then*

(i) *$b_1 \perp \cdots \perp b_t$ is right semi-stable $\iff$ $\mathrm{Hom}(M_i, \ker \mathrm{Ad}^r_{b_j}) = 0$ for all $1 \le i, j \le t$ $\iff$ for all $\sigma \in \mathrm{Hom}(M_i, M_j)$, there is at most one $\sigma' \in \mathrm{Hom}(M_j, M_i)$ such that $\sigma^{[1]} \circ \mathrm{Ad}^r_{b_j} = \mathrm{Ad}^r_{b_i} \circ \sigma'$.*

(ii) *$b_1 \perp \cdots \perp b_t$ is right stable $\iff$ for all $\sigma \in \mathrm{Hom}(M_i, M_j)$, there exists unique $\sigma' \in \mathrm{Hom}(M_j, M_i)$ such that $\sigma^{[1]} \circ \mathrm{Ad}^r_{b_j} = \mathrm{Ad}^r_{b_i} \circ \sigma'$.*

PROOF. For brevity, let $M = \bigoplus_{i=1}^t M_i$, $b = b_1 \perp \cdots \perp b_t$ and $h_i = \mathrm{Ad}^r_{b_i}$. We will write elements of $\mathrm{End}(M)$ as $t \times t$ matrices where the (i, j) coordinate lies in $\mathrm{Hom}(M_j, M_i)$. Similar notation will be used for $\mathrm{Hom}(M, M^{[1]})$ and $\mathrm{End}(M^{[1]})$. Note that the equation $\sigma^{[1]} \circ \mathrm{Ad}^r_b = \mathrm{Ad}^r_b \circ \sigma'$ (where $\sigma, \sigma' \in \mathrm{End}(M)$) now becomes:

$$(6)\quad \begin{bmatrix} \sigma^{[1]}_{11} & \dots & \sigma^{[1]}_{t1} \\ \vdots & \ddots & \vdots \\ \sigma^{[1]}_{1t} & \dots & \sigma^{[1]}_{tt} \end{bmatrix} \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_t \end{bmatrix} = \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_t \end{bmatrix} \begin{bmatrix} \sigma'_{11} & \dots & \sigma'_{1t} \\ \vdots & \ddots & \vdots \\ \sigma'_{t1} & \dots & \sigma'_{tt} \end{bmatrix}$$

where $\sigma = (\sigma_{ij})$ and $\sigma' = (\sigma'_{ij})$.

(i) The equivalence of the last two conditions is straightforward (compare with the proof of Proposition 2.2.9). The equivalence of the first two conditions follows from the fact that $\mathrm{Hom}(M, \ker \mathrm{Ad}_b^r)$ can be understood as:

$$\mathrm{Hom}\left(\bigoplus_{i=1}^t M_i, \bigoplus_{i=1}^t \ker h_i\right) = \begin{bmatrix} \mathrm{Hom}(M_1, \ker h_1) & \dots & \mathrm{Hom}(M_t, \ker h_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}(M_1, \ker h_t) & \dots & \mathrm{Hom}(M_t, \ker h_t) \end{bmatrix}.$$

(ii) Assume b is right stable and let $\tau \in \mathrm{Hom}(M_{i_0}, \ker M_{j_0})$. Define $\sigma = (\sigma_{ij})$ by $\sigma_{ij} = 0$ for all $(i,j) \neq (j_0, i_0)$ and $\sigma_{j_0 i_0} = \tau$. Then there exists $\sigma' = (\sigma'_{ij}) \in \mathrm{End}(M)$ satisfying (6). This implies $\tau^{[1]} \circ h_{j_0} = (\sigma^{[1]})_{i_0 j_0} \circ h_{j_0} = h_{i_0} \circ \sigma'_{i_0 j_0}$, so for all $\tau \in \mathrm{Hom}(M_i, M_j)$ there exists $\tau' \in \mathrm{Hom}(M_j, M_i)$ such that $\tau^{[1]} \circ h_j = h_i \circ \tau'$ and τ' is unique by (i).

To see the converse, let $\sigma = (\sigma_{ij}) \in \mathrm{End}(M)$. Then for all i, j there is $\sigma'_{ij} \in \mathrm{Hom}(M_j, M_i)$ such that $\sigma_{ji}^{[1]} \circ h_j = h_i \circ \sigma'_{ij}$. Let $\sigma' = (\sigma'_{ij}) \in \mathrm{End}(M)$. Then (6) implies $\sigma^{[1]} \circ \mathrm{Ad}_b^r = \mathrm{Ad}_b^r \circ \sigma'$. Since b is right semi-stable (by (i)), σ' is the only element of $\mathrm{End}(M)$ satisfying $\sigma^{[1]} \circ \mathrm{Ad}_b^r = \mathrm{Ad}_b^r \circ \sigma'$. Hence b is right stable. □

Corollary 2.6.6. *Let $\{(b_i, M_i, K)\}_{i=1}^t$ be a sequence of bilinear spaces. Write $\{b_i \,|\, 1 \leq i \leq t\} = \{b'_1, \dots, b'_s\}$ (so $\{b'_j\}_{i=1}^s$ does not have multiplicities). Then $\{b_i\}_{i=1}^t$ are right joinable (semi-joinable) $\iff$ $\{b'_j\}_{i=1}^s$ are right joinable (semi-joinable). In particular, a bilinear form b is right stable (semi-stable) $\iff$ the form $b \perp \dots \perp b$ is right stable (semi-stable).*

Corollary 2.6.7. *Let $\{(b_i, M_i, K)\}_{i=1}^t$ be right stable (semi-table) bilinear spaces. Then $\{b_i\}_{i=1}^t$ are right joinable (semi-joinable) $\iff$ $\{b_1, \dots, b_t\}$ are pairwise right joinable (semi-joinable).*

Let (M_1, b_1, K) and (M_2, b_2, K) be two bilinear spaces. An *isometry* from b_1 to b_2 is an R-module isomorphism $\sigma : M_1 \to M_2$ such that $b_2(\sigma x, \sigma y) = b_1(x, y)$. If there exists such an isometry, then b_1 and b_2 are called *isometric* and we write $b_1 \cong b_2$. This an equivalence relation and its equivalence classes are called *isometry classes*. It is easy to see that each of the properties (R0)-(R5) and (L0)-(L5) is preserved under isometry of forms.

We finish this section with a short discussion about possible constructions of Witt and Witt-Grothendick groups using our notion of bilinear forms. As Witt and Witt-Grothendick groups are out of the scope of this paper, we are satisfied with presenting them over rings with involutions in which 2 is unit, referring the reader to [**86**], [**71**] and also [**6**], [**7**] for an extensive discussion.

Let $(R, *)$ be a ring with involution, let $\lambda \in \mathrm{Cent}(R)$ be such that $\lambda^* \lambda = 1$ and let $\mathscr{M}$ be an additive full subcategory of Mod-R such that the isomorphism classes of $\mathscr{M}$ form a set (e.g. finite projective modules). The *Witt-Grothendick group* of R, $*$, λ and $\mathscr{M}$, denoted $\widehat{W}(\lambda, \mathscr{M})$ consists of formal differences of isometry classes of regular λ-hermitian forms $h : M \times M \to R$ with $M \in \mathscr{M}$. The addition in $\widehat{W}(\lambda, \mathscr{M})$ is given by orthogonal sum (which is easily checked to be well-defined on isometry classes), i.e.:

$$([h_1] - [h_2]) \perp ([h_3] - [h_4]) = [h_1 \perp h_3] - [h_2 \perp h_4]$$

(here $h_1, \dots, h_4$ are λ-hermitian forms, $[h_i]$ denotes the isometry class of h_i and the negative signs are formal differences). The *Witt Group* of R, $*$, λ and $\mathscr{M}$, denoted $W(\lambda, \mathscr{M})$ is obtained modding out *metabolic* hermitian forms from $\widehat{W}(\lambda, \mathscr{M})$. These are the hermitian forms $h : M \times M \to R$ such that there is a summand of M, N, satisfying $N = N^\perp := \{x \in M \,|\, h(x, N) = 0\}$. (When R is a field, the metabolic forms are *hyperbolic forms*. For general rings, metabolic forms are *stably hyperbolic*; see [**86**, §7, Lm. 3.7].) Several texts have considered "non-symmetric"

Witt-Grothendick/Witt groups which are obtained by replacing hermitian forms with arbitrary regular (not-necessarily-symmetric) bilinear forms, e.g. [**38**].

The constructions of $\widehat{W}(\lambda,\mathscr{M})$ and $W(\lambda,\mathscr{M})$ can be carried out as is into our situation by replacing "λ-hermitian" with "κ-symmetric", where κ is an involution of K. (This turns out to result in a Witt-Grothendick/Witt group of a hermitian category, as the next section would imply.) However, several variations can be obtained by weakening the regularity assumption the forms, namely, one can construct $\widehat{W}$ and W using isometry classes of injective or surjective forms. Furthermore, in case of non-symmetric Witt-Grothendick/Witt groups, one can construct the group from isometry classes of right regular or left regular forms (rather than two-sided regular forms), thus obtaining left and right versions of the non-symmetric Witt-Grothendick/Witt group. However, Examples 2.6.3 and 2.6.4 imply that one cannot construct a Witt-Grothendick group from the isometry classes of stable or semi-stable forms.

2.7. Categories with a Double Duality

In this section, we present categories with double a duality which are a categorical generalization of our previous bilinear forms. We explain how our definition is connected to hermitian categories (or categories with duality; see [**71**], [**86**, Ch. 7], [**7**]), which generalize classical bilinear forms, and show that our new notion of bilinear forms cannot be understood as a special case of a hermitian category.

As in section 2.1, let us first recall what are hermitian categories. Our description follows [**7**], which calls hermitian categories *categories with duality.* We shall stick to that name henceforth. A category with duality is a triplet $(\mathscr{A},*,\omega)$ such that $\mathscr{A}$ is a category, $*:\mathscr{A}\to\mathscr{A}$ is a contravariant functor and $\omega:\mathrm{id}_{\mathscr{A}}\to *^2$ is a natural transformation (which is usually assumed to be an isomorphism) satisfying $\omega_M^*\circ\omega_{M^*}=\mathrm{id}_{M^*}$ for all $M\in\mathscr{A}$. If $\mathscr{A}$ is additive or exact, then $*$ is assumed to be additive or exact respectively. A *bilinear form* in $(\mathscr{A},*,\omega)$ is a pair (M,b) such that $M\in\mathscr{A}$ and $b\in\mathrm{Hom}_{\mathscr{A}}(M,M^*)$. Define $\widetilde{b}=b^*\circ\omega_M\in\mathrm{Hom}_{\mathscr{A}}(M,M^*)$. Then b and $\widetilde{b}$ play the role of the right and left adjoint maps. The map $b\mapsto\widetilde{b}$ is easily seen to be of order 2, hence b can be recovered from $\widetilde{b}$. A bilinear form (M,b) will be called symmetric if $b=\widetilde{b}$ and (right) regular if b is an isomorphism.

When ω is assumed to be an isomorphism (which is the case in all texts seen by the author), it is common to identify M with M^{**} via ω. In particular, $\widetilde{b}$ is identified with $b^*\in\mathrm{Hom}_{\mathscr{A}}(M^{**},M^*)$ and only the latter is used.

2.7.1. Definitions. Inspired by section 2.2, we define a *category with a double duality* to be a quintet $(\mathscr{A},[0],[1],\Phi,\Psi)$ consisting of a category $\mathscr{A}$ equipped with two contravariant functors $[0],[1]:\mathscr{A}\to\mathscr{A}$ and natural transformations $\Phi:\mathrm{id}_{\mathscr{A}}\to[1][0]$ and $\Psi:\mathrm{id}_{\mathscr{A}}\to[0][1]$ satisfying

$$\mathrm{id}_{M^{[0]}}=\Psi_M^{[0]}\circ\Phi_{M^{[0]}}\qquad\text{and}\qquad\mathrm{id}_{M^{[1]}}=\Phi_M^{[1]}\circ\Psi_{M^{[1]}}$$

for all $M\in\mathscr{A}$. If $\mathscr{A}$ is additive, then we will require $[0]$ and $[1]$ to be additive. (This implies Φ and Ψ respect the additivity of $\mathscr{A}$. It is a general fact about natural transformations between additive functors.) We do not require Φ and Ψ to be isomorphisms. The reason for this will be explained below.

By Corollary 2.2.5, there is a natural isomorphism $I_{A,B}:\mathrm{Hom}(B,A^{[1]})\to\mathrm{Hom}(A,B^{[0]})$ given by $I_{A,B}(f)=f^{[0]}\circ\Phi_A$. Note that $I_{A,B}$ also determines Φ and Ψ by $\Phi_A=I_{A,A^{[1]}}(\mathrm{id}_{A^{[1]}})$ and $\Psi_A=I^{-1}_{A^{[0]},A}(\mathrm{id}_{A^{[0]}})$, so a category with a double duality can also be defined as a quadruple $(\mathscr{A},[0],[1],I)$ with $\mathscr{A}$, $[0]$, $[1]$ as before and $I_{A,B}:\mathrm{Hom}(B,A^{[1]})\to\mathrm{Hom}(A,B^{[0]})$ being a natural isomorphism.

EXAMPLE 2.7.1. By Proposition 2.2.1, any ring R and a double R-module K give rise to a category with a double duality structure on Mod-R. However, as shown in Example 2.5.3, Ψ and Φ need not be isomorphisms, which explains why we did not assume that in the definition.

A *bilinear pairing* in $(\mathscr{A}, [0], [1], \Psi, \Phi)$ (or $\mathscr{A}$, for brevity) is a triplet (A, B, b) where $A, B \in \mathscr{A}$ and $b \in \mathrm{Hom}_{\mathscr{A}}(B, A^{[1]})$. In this case, let $\widetilde{b}$ denote $I_{A,B}(b) \in \mathrm{Hom}_{\mathscr{A}}(A, B^{[0]})$. (The maps b and $\widetilde{b}$ play the role of the right and left adjoint maps respectively.) A *bilinear form* in $\mathscr{A}$ is a pair (M, b) such that (M, M, b) is a bilinear pairing.

If (M, b) and (M', b') are two bilinear forms, then an *isometry* from (M, b) to (M', b') is an isomorphism $\sigma \in \mathrm{Hom}(M, M')$ satisfying $\sigma^{[1]} \circ b' \circ \sigma = b$. In this case (M, b) and (M', b') are called *isometric*.

A bilinear form (M, b) will be called:

(R0) *right regular* if b is an isomorphism;
(R1) *right monic* if b is monic;
(R2) *right epic* if b is epic;
(R3) *right stable* if for all $\sigma \in \mathrm{End}_{\mathscr{A}}(M)$ there exists *unique* $\tau \in \mathrm{End}_{\mathscr{A}}(M)$ such that $\sigma^{[1]} \circ b = b \circ \tau$ (or equivalently, $\widetilde{b} \circ \sigma = \tau^{[0]} \circ \widetilde{b}$; see the diagrams in Proposition 2.2.6);
(R5) *right semi-stable* if for all $\sigma, \tau \in \mathrm{End}_{\mathscr{A}}(M)$, $b \circ \sigma = b \circ \tau$ implies $\sigma = \tau$.

The left analogues of (R0)-(R2) and (R5) are defined by replacing b with $\widetilde{b}$, and the left analogue of (R3) is defined by replacing σ and τ.

Now let $u : [0] \to [1]$ be an isomorphism of functors. A *right u-asymmetry* of a bilinear form (M, b) is a map $\lambda \in \mathrm{End}_{\mathscr{A}}(M)$ such that $u_M \circ \widetilde{b} = b \circ \lambda$ (see Proposition 2.2.8 for explanation). We can now consider the following property:

(R4) The bilinear form (M, b) has a unique right u-asymmetry.

Left asymmetries are defined with respect to an isomorphism $u' : [1] \to [0]$; a left u'-asymmetry is a map $\lambda \in \mathrm{End}_{\mathscr{A}}(M)$ such that $u'_M \circ b = \widetilde{b} \circ \lambda$. Again, the inverse of an invertible right u-asymmetry is a left u^{-1}-asymmetry. A bilinear form (M, b) will be called *u-symmetric* if $b = u \circ \widetilde{b}$.

So far, it is clear from section 2.2 that our new definition of bilinear forms agrees with that of section 2.1 (if $\mathscr{A}$ is induced by a ring R and a double R-module K). However, not all results of sections 2.3 and 2.5 hold for bilinear forms in categories with a double duality, the reason being that maps that are monic and epic in $\mathscr{A}$ need not be invertible.

It is now left to explain what are the categorical analogues of an involutions and augmentable anti-isomorphisms. Rather than spelling out the definitions, which are simple yet not intuitive at all, we first explain what stands behind them, collecting some useful facts along the way. This will be done in the following two subsections.

2.7.2. Involutions. Let us restrict for a moment to the case where $\mathscr{A}$ arises from a ring R and a double R-module K. By Proposition 2.2.7, any isomorphism $u : [0] \to [1]$ corresponds to an anti-isomorphism κ of K. Since κ^{-1} is also an anti-isomorphism of K, we can define $\widetilde{u} = u_{\kappa^{-1}} : [0] \to [1]$, which functions a an alternative "inverse" of u. However, Proposition 2.2.7 does not tell us how to get $\widetilde{u}$ from u by purely categorical means and this is what we shall now tend to.

With κ as before, let $b : A \times B \to K$ be a bilinear paring. Then the map $b^{\kappa} : B \times A \to K$ defined by $b^{\kappa}(y, x) = b(x, y)$ is also a bilinear pairing. Let

$\mathrm{Bil}_K(A, B)$ denote the set of bilinear pairings $b : A \times B \to K$. Then by Corollary 2.2.5, $\mathrm{Bil}_K(A, B)$ is in one-to-one correspondence with $\mathrm{Hom}_R(A, B^{[0]})$. Therefore, the map $b \mapsto b^\kappa : \mathrm{Bil}_K(A, B) \to \mathrm{Bil}_K(B, A)$ gives rise to a map $v_{\kappa,A,B} : \mathrm{Hom}(A, B^{[0]}) \to \mathrm{Hom}(B, A^{[0]})$ and a direct computation shows that $v_{\kappa,A,B} = I_{B,A} \circ (u_{\kappa,B})_*$ where $(u_{\kappa,B})_* f = u_{\kappa,B} \circ f$.[8] Namely, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(A, B^{[0]}) & \xrightarrow{(u_{\kappa,B})_*} & \mathrm{Hom}(A, B^{[1]}) \\ & \searrow^{v_{\kappa,A,B}} & \downarrow I_{B,A} \\ & & \mathrm{Hom}(B, A^{[0]}) \end{array}$$

In the same manner, $v^{-1}_{\kappa,A,B} = v_{\kappa^{-1},B,A} = I_{A,B} \circ u_{\kappa^{-1},A}$, so we can extend the previous diagram as following:

$$(7) \qquad \begin{array}{ccc} \mathrm{Hom}(A, B^{[0]}) & \xrightarrow{(u_{\kappa,B})_*} & \mathrm{Hom}(A, B^{[1]}) \\ I_{A,B} \uparrow & \searrow^{v_{\kappa,A,B}} & \downarrow I_{B,A} \\ \mathrm{Hom}(B, A^{[1]}) & \xleftarrow[(u_{\kappa^{-1},A})_*]{} & \mathrm{Hom}(B, A^{[0]}) \end{array}$$

Let us now move back to arbitrary categories. Diagram (7) suggests that if an isomorphism $u : [0] \to [1]$ admits an "inverse" $\widetilde{u}$, then the following diagram should commute:

$$(8) \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathscr{A}}(A, B^{[0]}) & \xrightarrow{(u_B)_*} & \mathrm{Hom}_{\mathscr{A}}(A, B^{[1]}) \\ I_{A,B} \uparrow & & \downarrow I_{B,A} \\ \mathrm{Hom}_{\mathscr{A}}(B, A^{[1]}) & \xleftarrow[(\widetilde{u}_A)_*]{} & \mathrm{Hom}_{\mathscr{A}}(B, A^{[0]}) \end{array}$$

Therefore, we need to find an isomorphism $\widetilde{u} : [0] \to [1]$ such that

$$(9) \qquad (\widetilde{u}_A)_* = I^{-1}_{A,B} \circ (u_B)^{-1}_* \circ I^{-1}_{B,A} : \mathrm{Hom}(B, A^{[0]}) \to \mathrm{Hom}(B, A^{[1]})$$

(or less formally, $I \circ \widetilde{u}_* = (I \circ u_*)^{-1}$). It is therefore natural to ask whether $\widetilde{u}$ can be determined from $\widetilde{u}_*$. It turns out that the answer is yes and moreover, any natural transformation $f_{A,B} : \mathrm{Hom}(A, B^{[0]}) \to \mathrm{Hom}(A, B^{[1]})$ is of the form $(u_0)_*$ for some $u_0 : [0] \to [1]$. This is verified in the following proposition.

PROPOSITION 2.7.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be categories and let $F, G : \mathscr{B} \to \mathscr{A}$ be two contravariant functors. Then there is a one-to-one correspondence between natural transformations $u : F \to G$ and natural transformations $f_{A,B} : \mathrm{Hom}_{\mathscr{A}}(A, FB) \to \mathrm{Hom}_{\mathscr{A}}(A, GB)$ given by $u \mapsto u_*$ and $f \mapsto u_f$ where $(u_f)_A = f_{FA,A}(\mathrm{id}_{FA})$. In addition, u is an isomorphism if and only if u_* is.*[9]

[8] Other texts use $\mathrm{Hom}(A, u_{\kappa,B})$ to denote $(u_{\kappa,B})_*$. We chose the latter for brevity.

[9] By saying $f_{A,B}$ is natural, we mean that for all $A, A' \in \mathscr{A}$, $B, B' \in \mathscr{B}$, $\alpha \in \mathrm{Hom}_{\mathscr{A}}(A, A')$ and $\beta \in \mathrm{Hom}_{\mathscr{B}}(B, B')$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(A, FB) & \xrightarrow{f_{A,B}} & \mathrm{Hom}(A, GB) \\ F\beta \circ \_ \circ \alpha \uparrow & & \uparrow G\beta \circ \_ \circ \alpha \\ \mathrm{Hom}(A', FB') & \xrightarrow{f_{A',B'}} & \mathrm{Hom}(A', GB') \end{array}$$

Proof. It is straightforward to check that u_* is a natural transformation (the fact that u is natural is needed) and that $u_{u_*} = u$.

Let f be a natural transformation as above and let $u = u_f$. We first claim that $f = u_*$. Let $A \in \mathscr{A}$, $B \in \mathscr{B}$ and let $\psi \in \mathrm{Hom}_{\mathscr{A}}(A, FB)$. Then the following diagram commutes since f is natural:

$$\begin{array}{ccc} \mathrm{Hom}(A,FB) & \xrightarrow{f_{A,B}} & \mathrm{Hom}(A,GB) \\ {\scriptstyle \_\circ\psi}\uparrow & & \uparrow{\scriptstyle \_\circ\psi} \\ \mathrm{Hom}(FB,FB) & \xrightarrow{f_{FB,B}} & \mathrm{Hom}(FB,GB) \end{array}$$

Therefore, $f_{A,B}\psi = f_{A,B}(\mathrm{id}_{FB} \circ\psi) = f_{FB,B}(\mathrm{id}_{FB}) \circ \psi = u_B \circ \psi$, which implies $f_{A,B} = (u_B)_*$. It is left to verify that u is natural. Let $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $\psi \in \mathrm{Hom}_{\mathscr{A}}(A, B)$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(FB,FA) & \xrightarrow{f_{FB,A}} & \mathrm{Hom}(FB,GA) \\ {\scriptstyle F\psi\circ\_}\uparrow & & \uparrow{\scriptstyle G\psi\circ\_} \\ \mathrm{Hom}(FB,FB) & \xrightarrow{f_{FB,B}} & \mathrm{Hom}(FB,GB) \end{array}$$

Since $f = u_*$, we get $u_A \circ F\psi = f_{FB,A}(F\psi \circ \mathrm{id}_{FB}) = G\psi \circ (f_{FB,B}\,\mathrm{id}_{FB}) = G\psi \circ u_B$, as required.

If u is an isomorphism, then it is easy to see that so is u_*. If u_* is an isomorphism, then define $f_{A,B} : \mathrm{Hom}_{\mathscr{A}}(A, GB) \to \mathrm{Hom}_{\mathscr{A}}(A, FB)$ to be the inverse of u_*. By what we have just shown, $u_f : G \to F$ is a functor morphism. Observe that $(u \circ u_f)_* = (\mathrm{id}_G)_*$, hence the correspondence implies $u \circ u_f = \mathrm{id}_G$. Similarly, $u_f \circ u = \mathrm{id}_F$, hence u is an isomorphism. □

Remark 2.7.3. The last proposition slightly resembles Yoneda's Lemma (see [**42**, Th. 5.34]) and also Theorems 3.2 and 3.2* in [**53**] (described below). However, it seems that it cannot be rendered to either of them and moreover, Theorems 3.2 and 3.2* in [**53**] can be easily proved using it.

With Proposition 2.7.2 and (9) in mind, for any isomorphism $u : [0] \to [1]$ and $A \in \mathscr{A}$, we define:

$$\widetilde{u}_A = (I^{-1}_{A,A^{[0]}} \circ (u_{A^{[0]}})^{-1}_* \circ I^{-1}_{A^{[0]},A})(\mathrm{id}_{A^{[0]}}) \in \mathrm{Hom}(A^{[0]}, A^{[1]}) \ . \tag{10}$$

Then $\widetilde{u} : [0] \to [1]$ is a functor isomorphism and by the definition of $I_{A,B}$:

$$\begin{aligned} \widetilde{u}_A &= (I^{-1}_{A,A^{[0]}} \circ (u_{A^{[0]}})^{-1}_* \circ I^{-1}_{A^{[0]},A})(\mathrm{id}_{A^{[0]}}) \\ &= (I^{-1}_{A,A^{[0]}} \circ (u_{A^{[0]}})^{-1}_*)(\Psi_A) = I^{-1}_{A,A^{[0]}}(u^{-1}_{A^{[0]}} \circ \Psi_A) \\ &= (u^{-1}_{A^{[0]}} \circ \Psi_A)^{[1]} \circ \Psi_{A^{[0]}} = \Psi^{[1]}_A \circ (u^{[1]}_{A^{[0]}})^{-1} \circ \Psi_{A^{[0]}} \end{aligned} \tag{11}$$

This leads to the following definition:

Definition 2.7.4. *For any natural isomorphism $u : [0] \to [1]$, let $\widetilde{u} : [0] \to [1]$ be the natural isomorphism defined by $\widetilde{u}_A = \Psi^{[1]}_A \circ (u^{[1]}_{A^{[0]}})^{-1} \circ \Psi_{A^{[0]}}$. The map u will be called an* involution *(of $(\mathscr{A}, [0], [1], \Phi, \Psi)$) if $u = \widetilde{u}$.*

Note that we know from (9) and Proposition 2.7.2 that $\widetilde{u}$ is an isomorphism, but this is not obvious from our definition (since Ψ is not an isomorphism).

Proposition 2.7.5. *In the previous notation, $\widetilde{u}$ has the following properties:*

(i) $\widetilde{\widetilde{u}} = u$ *and* $\widetilde{u}^{-1}_A = \Phi^{[0]}_A \circ u^{[0]}_{A^{[1]}} \circ \Phi_{A^{[1]}}$ *for all* $A \in \mathscr{A}$.

(ii) *For any bilinear pairing* (A, B, b), $\widetilde{u}_A \circ \widetilde{u_B \circ \widetilde{b}} = b$. *Moreover,* u *is an involution* $\iff$ $u_A \circ \widetilde{u_B \circ \widetilde{b}} = b$ *for any bilinear pairing* $\iff$ *for all* $A, B \in \mathscr{A}$, $(u_A)_* \circ I_{B,A} \circ (u_B)_* \circ I_{A,B} = \mathrm{id}_{\mathrm{Hom}(B,A^{[0]})}$ *(or less formally,* $(u_* \circ I)^2 = \mathrm{id}$*).*

(iii) *Define* $\delta : [1][0] \to [0][1]$ *by* $\delta_M = \widetilde{u}_{M^{[0]}} \circ u_M^{[0]}$. *Then* δ_M *is a natural isomorphism satisfying* $\Psi = \delta \circ \Phi$ *and* $\delta_M = u_M^{[1]} \circ \widetilde{u}_{M^{[1]}}$. *(By symmetry, the natural transformation* $\widetilde{\delta}_M = u_{M^{[0]}} \circ \widetilde{u}_M^{[0]}$ *also satisfies these identities, but* $\delta \neq \widetilde{\delta}$ *in general.)*

(iv) *If* $\mathscr{A}$ *arises from a ring* R *and a double* R*-module* K *and* $u = u_\kappa$ *for some anti-isomorphism* κ *of* K*, then* $\widetilde{u} = u_{\kappa^{-1}}$. *In particular,* κ *is an involution if and only if* $u_\kappa = \widetilde{u}_\kappa$.

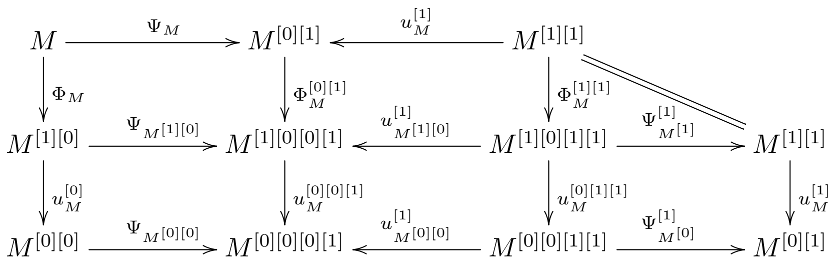
Proof. (i) That $\widetilde{\widetilde{u}}_* = u_*$ is straightforward from (9), hence by Proposition 2.7.2, $\widetilde{\widetilde{u}} = u$. To see the second equality, note that by diagram (8), $(\widetilde{u}_A^{-1})_* = (\widetilde{u}_A)_*^{-1} = I_{B,A} \circ (u_B)_* \circ I_{A,B}$. By Proposition 2.7.2,

$$u_A^{-1} = (I_{A^{[1]},A} \circ (u_{A^{[1]}})_* \circ I_{A,A^{[1]}})(\mathrm{id}_{A^{[1]}}) \ ,$$

and a computation similar to (11) would show $\widetilde{u}_A^{-1} = \Phi_A^{[0]} \circ u_{A^{[1]}}^{[0]} \circ \Phi_{A^{[1]}}$.

(ii) Observe that $\widetilde{u}_A \circ \widetilde{u_B \circ \widetilde{b}} = ((\widetilde{u}_A)_* \circ I_{B,A} \circ (u_B)_* \circ I_{A,B})(b)$ and the right hand side is b by (8). To see the second assertion, observe that the computation we just carried out implies $u_A \circ \widetilde{u_B \circ \widetilde{b}} = b$ for any $b \in \mathrm{Hom}_{\mathscr{A}}(B, A^{[0]})$ if and only if $(u_A)_* \circ I_{B,A} \circ (u_B)_* \circ I_{A,B} = \mathrm{id}_{\mathrm{Hom}(B,A^{[0]})}$. The latter is equivalent to the commutativity of (8) when replacing $\widetilde{u}_A$ with u_A. Since $(\widetilde{u}_A)_*$ is the only map making the diagram commutative, it follows that $(\widetilde{u}_A)_* = (u_A)_*$ for all $A \in \mathscr{A}$, so by Proposition 2.7.2, $\widetilde{u} = u$ and u is an involution. The oppositive implications are obvious. (The courageous reader is welcome to try verifying (i) and (ii) with direct computation. Beware: this is trickier than (iii) below).

(iii) Let $M \in \mathscr{A}$. Observe that the following diagram commutes:



(The squares commute because $\Psi : \mathrm{id} \to [0][1]$, $u^{[1]} : [1][1] \to [0][1]$ and $\Psi^{[1]} : [0][1][1] \to [1]$ are natural transformations. The top right triangle follows from Proposition 2.2.1). By moving along the border of the diagram from the top left object to the bottom right object, we see that

$$\Psi_{M^{[0]}}^{[1]} \circ (u_{M^{[0][0]}}^{[1]})^{-1} \circ \Psi_{M^{[0][0]}} \circ u_M^{[0]} \circ \Phi_M = u_M^{[1]} \circ \mathrm{id}_{M^{[1][1]}} \circ (u_M^{[1]})^{-1} \circ \Psi_M = \Psi_M \ .$$

However, by definition, the left hand side is $\delta_M \circ \Phi_M$, hence $\Psi = \delta \circ \Phi$. That $\widetilde{u}_{M^{[0]}} \circ u_M^{[0]} = u_M^{[1]} \circ \widetilde{u}_{M^{[1]}}$ follows by moving along the diagram from $M^{[1][0]}$ to the bottom left object along the second and third rows.

(iv) By (7), $(u_{\kappa^{-1}})_* = (\widetilde{u})_*$, hence $u_{\kappa^{-1}} = \widetilde{u}$ by Proposition 2.7.2. Therefore, u_κ is an involution $\iff$ $u_\kappa = u_{\kappa^{-1}}$ $\iff$ $\kappa = \kappa^{-1}$ (by Proposition 2.2.7). □

2.7.3. Augmentation. We will now define augmentable natural isomorphisms $u : [0] \to [1]$ using the ideas of the previous subsection. (The categorical definition of an augmentation map for a *given* bilinear form will be given a the end of this subsection). As with involutions, we first present the intuition behind the definition, so assume R is a ring, K is a double R-module, κ is an anti-isomorphism and $\gamma : \mathrm{id}_{\text{Mod-}R} \to \mathrm{id}_{\text{Mod-}R}$ is natural transformation. Set $u = u_\kappa$ and observe that for any bilinear pairing $b : A \times B \to K$ and $\gamma \in \mathrm{End}_R(B)$, the identity $b(x,y)^{\kappa\kappa} = b(x,\gamma_B y)$ is equivalent to $(I_{A,B} \circ (u_A)_* \circ I_{B,A} \circ (u_B)_*)(\mathrm{Ad}_b^\ell) = \gamma_B^{[0]} \circ \mathrm{Ad}_b^\ell$ (this is a straightforward computation). Therefore, that γ is an augmentation for κ is equivalent to $I_{A,B} \circ (u_A)_* \circ I_{B,A} \circ (u_B)_* = (\gamma_B^{[0]})_*$. The latter is illustrated in the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}(A,B^{[0]}) & \xrightarrow{(u_B)_*} & \mathrm{Hom}(A,B^{[1]}) & \xrightarrow{I_{B,A}} & \mathrm{Hom}(B,A^{[0]}) \\
\Big\downarrow{\scriptstyle (\gamma_B^{[0]})_*} & & & & \Big\downarrow{\scriptstyle (u_A)_*} \\
\mathrm{Hom}(A,B^{[0]}) & & \xleftarrow[\quad I_{A,B} \quad]{} & & \mathrm{Hom}(B,A^{[1]})
\end{array}$$

Now let $(\mathscr{A},[0],[1],\Phi,\Psi)$ be any category with a double duality and let $u : [0] \to [1]$ be an isomorphism of functors. Then by Proposition 2.7.2, there exists a unique natural transformation $\widehat{\gamma} = \widehat{\gamma}(u) : [0] \to [0]$ such that $(\widehat{\gamma}_B)_* = I_{A,B} \circ (u_A)_* \circ I_{B,A} \circ (u_B)_*$ for all $A, B \in \mathscr{A}$. In fact, by (9), $I_{A,B} \circ (u_*)_{B,A} \circ I_{B,A} = (\widetilde{u}_B^{-1})_*$. Hence

$$\widehat{\gamma}(u) = \widetilde{u}^{-1} \circ u$$

(since $(\widehat{\gamma}_B)_* = (\widetilde{u}_B^{-1})_* \circ (u_B)_* = (\widetilde{u}_B^{-1} \circ u_B)_*$). More explicitly, we have:

$$\begin{aligned}
\widehat{\gamma}_A &= (I_{A^{[0]},A} \circ (u_{A^{[0]}})_* \circ I_{A,A^{[0]}} \circ (u_A)_*)(\mathrm{id}_{A^{[0]}}) \\
&= (u_{A^{[0]}} \circ ((u_A \circ \mathrm{id}_{A^{[0]}})^{[0]} \circ \Phi_A))^{[0]} \circ \Phi_{A^{[0]}} \\
&= (u_{A^{[0]}} \circ u_A^{[0]} \circ \Phi_A)^{[0]} \circ \Phi_{A^{[0]}} \\
&= \Phi_A^{[0]} \circ u_A^{[0][0]} \circ u_{A^{[0]}}^{[0]} \circ \Phi_{A^{[0]}} \ .
\end{aligned}$$

We say that u is *augmentable* if there exists a natural transformation $\gamma : \mathrm{id}_{\mathscr{A}} \to \mathrm{id}_{\mathscr{A}}$ such that $\widetilde{u}^{-1} \circ u = \widehat{\gamma} = \gamma^{[0]}$. Such a γ will be called an *augmentation transformation* for u.

Given a bilinear form (M,b), a map $\gamma_0 \in \mathrm{End}_{\mathscr{A}}(M)$ will be called an *augmentation map for b* (w.r.t. u) if $\gamma_0^{[0]} = \widehat{\gamma}(u)_M$.

Proposition 2.7.6. *In the previous assumptions, if $\mathscr{A}$ contains a generator X such that the map $[0] : \mathrm{End}_{\mathscr{A}}(X) \to \mathrm{End}_{\mathscr{A}}(X^{[0]})$ is injective (we can replace $[0]$ with $[1]$ here since $[0] \cong [1]$), then u has at most one augmentation transformation.*

Proof. Assume $\beta, \gamma : \mathrm{id}_{\mathscr{A}} \to \mathrm{id}_{\mathscr{A}}$ are augmentations. Then $\beta^{[0]} = \gamma^{[0]} = \widehat{\gamma}$. In particular, $\beta_X^{[0]} = \gamma_X^{[0]}$, hence our assumptions imply $\beta_X = \gamma_X$. Now, that X is a generator is well known to imply $\beta = \gamma$. Indeed, if $A \in \mathscr{A}$ is any other object and $\beta_A \neq \gamma_A$, then there exists $\alpha : X \to A$ such that $\beta_A \circ \alpha \neq \gamma_A \circ \alpha$. But $\beta_A \circ \alpha = \alpha \circ \beta_X = \alpha \circ \gamma_X = \gamma_A \circ \alpha$, a contradiction. □

We finish this subsection by showing that under certain assumptions, all functor isomorphisms $u : [0] \to [1]$ are augmentable. Following section 2.5, call an object $A \in \mathscr{A}$, right (left) *semi-reflexive* if Ψ_A (Φ_A) is monic and right (left) *reflexive* if Ψ_A (Φ_A) is bijective.

PROPOSITION 2.7.7. *Assume all objects in $\mathscr{A}$ are right reflexive. Then any natural isomorphism $u:[0]\to[1]$ is augmentable and admits a unique augmentation transformation.*[10]

PROOF. We shall make use of Proposition 2.5.6 whose proof can be easily generalized to arbitrary categories with duality. Let $u:[0]\to[1]$ be a natural isomorphism and let $\widehat{\gamma}=\widehat{\gamma}(u):[0]\to[0]$ be as above. Since all objects in $\mathscr{A}$ are right reflexive, the map $f\mapsto f^{[0]}$ from $\operatorname{End}_{\mathscr{A}}(A)$ to $\operatorname{End}_{\mathscr{A}}(A^{[0]})$ is invertible for all $A\in\mathscr{A}$. Thus, for all $A\in\mathscr{A}$, there is *unique* $\gamma_A\in\operatorname{End}_{\mathscr{A}}(A)$ such that $\gamma_A^{[0]}=\widehat{\gamma}_A$. We are done if we prove that γ is natural. Indeed, let $A,B\in\mathscr{A}$ and $f\in\operatorname{Hom}_{\mathscr{A}}$. Then $(\gamma_B\circ f)^{[0]}=f^{[0]}\circ\widehat{\gamma}_B=\widehat{\gamma}_A\circ f^{[0]}=(f\circ\gamma_A)^{[0]}$, so Proposition 2.5.6 implies $\gamma_B\circ f=f\circ\gamma_A$. □

2.7.4. The Connection to Categories with Duality. It is clear that a triplet $(\mathscr{A},*,\omega)$ is a category with duality if and only if $(\mathscr{A},*,*,\omega,\omega)$ is a category with a double duality. Given a category with a double duality $(\mathscr{A},[0],[1],\Phi,\Psi)$ and an involution $u:[0]\to[1]$, it turns out that there is a natural transformation $\omega_u:\mathrm{id}_{\mathscr{A}}\to[0][0]$ such that $(\mathscr{A},[0],\omega_u)$ is a category with duality. This and much more is verified in the following theorem:

THEOREM 2.7.8. *Let $(\mathscr{A},[0],[1],\Phi,\Psi)$ be a category with a double duality. Then there is a one-to-one correspondence between involutions $u:[0]\to[1]$ and natural transformations $\omega:\mathrm{id}_{\mathscr{A}}\to[0][0]$ for which $(\mathscr{A},[0],\omega)$ is a category with duality. Moreover, if u is such an involution corresponding to $\omega:\mathrm{id}_{\mathscr{A}}\to[0][0]$, then there is a natural one-to-one correspondence between bilinear forms in $(\mathscr{A},[0],[1],\Phi,\Psi)$ and bilinear forms in $(\mathscr{A},[0],\omega)$ that sends u-symmetric forms to symmetric forms.*[11]

Before proving the theorem, we give an explicit example of this correspondence:

EXAMPLE 2.7.9. Let R be a ring, let K be a double R-module and define $[0]$ and $[1]$ as in section 2.1. By Propositions 2.2.7 and 2.7.5(iv), there is a one to one correspondence between involutions of K and involutions $u:[0]\to[1]$.

Let κ be an involution of K. For all $M\in\text{Mod-}R$, define $\omega_M:M\to M^{[0][0]}$ by $(\omega_M x)f=(fx)^\kappa$ where $x\in M$, $f\in M^{[0]}$. Keeping this convention for x and f, observe that $\omega_M x\in M^{[0][0]}$ since

$$(\omega x)(f\cdot r)=((f\cdot r)x)^\kappa=((fx)\odot_0 r)^\kappa=(fx)^\kappa\odot_1 r=((\omega x)f)\odot_1 r$$

and ω_M is R-linear since

$$(\omega(x\cdot r))f=(f(x\cdot r))^\kappa=((fx)\odot_1 r)^\kappa=(fx)^\kappa\odot_0 r=((\omega x)f)\odot_0 r=((\omega x)\cdot r)f\ .$$

In addition:

$$((\omega_M^{[0]}\circ\omega_{M^{[0]}})f)x=(\omega_M^{[0]}(\omega_{M^{[0]}}f))x=(\omega_{M^{[0]}}f)(\omega_M x)=((\omega_M x)f)^\kappa=(fx)^{\kappa\kappa}=fx,$$

hence $\omega_M^{[0]}\circ\omega_{M^{[0]}}=\mathrm{id}_{M^{[0]}}$. Therefore, $(\text{Mod-}R,[0],\omega)$ is a category with duality and it can be checked that correspondence in Theorem 2.7.8 sends u_κ to ω just defined.

We first prove the following lemma.

---

[10] Note that if there exists a functor isomorphism $u:[0]\to[1]$, then being right reflexive is equivalent to being left reflexive by Proposition 2.7.5(iii). Thus, the assumptions on $\mathscr{A}$ are effectively left-right symmetric.

[11] It is possible to define isomorphisms of categories with a double duality (see the next section) and then show that $(\mathscr{A},[0],[1],\Phi,\Psi)$ is isomorphic to $(\mathscr{A},[0],[0],\omega,\omega)$. As this is not needed for the chapter, we leave it to the reader to verify.

LEMMA 2.7.10. *Let $\mathscr{A}$ be a category and let $* : \mathscr{A} \to \mathscr{A}$ be a contravariant functor. The there is a one-to-one correspondence between natural transformations $\omega : \mathrm{id}_{\mathscr{A}} \to **$ and natural transformations $v_{A,B} : \mathrm{Hom}(A, B^*) \to \mathrm{Hom}(B, A^*)$. The correspondence is given by $\omega \mapsto v_\omega$ where v_ω is defined by $v_{\omega,A,B}(f) = f^* \circ \omega_B$ for $f \in \mathrm{Hom}(A, B^*)$ and $v \mapsto \omega_v$ where $\omega_{v,A} := v_{A^*,A}(\mathrm{id}_{A^*})$. In addition, if ω corresponds to v, then $\omega_A^* \circ \omega_{A^*} = \mathrm{id}_{A^*}$ for all $A \in \mathscr{A}$ if and only if $v_{A,B} \circ v_{B,A} = \mathrm{id}$ for all $A, B \in \mathscr{A}$.*

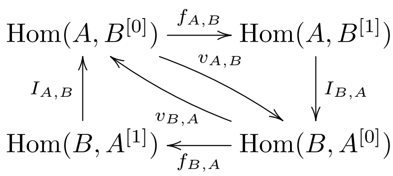
PROOF. We leave it to the reader to check that v_ω and ω_v are indeed natural. Given ω as above, $\omega_{v_\omega,A} = v_{\omega,A^*,A}(\mathrm{id}_{A^*}) = \mathrm{id}_{A^{**}} \circ \omega_A = \omega_A$ for all $A \in \mathscr{A}$, hence $\omega_{v_\omega} = \omega$. In addition, for any v as above and $f \in \mathrm{Hom}(A, B^*)$, the following diagram commutes (since v is natural):

$$\begin{array}{ccc} \mathrm{Hom}(B^*, B^*) & \xrightarrow{v_{B^*,B}} & \mathrm{Hom}(B, B^{**}) \\ \big\downarrow{\scriptstyle \_\circ f} & & \big\downarrow{\scriptstyle f^*\circ \_} \\ \mathrm{Hom}(A, B^*) & \xrightarrow{v_{A,B}} & \mathrm{Hom}(B, A^*) \end{array}$$

Therefore, $v_{\omega_v,A,B}(f) = f^* \circ \omega_{v,B} = f^* \circ v_{B^*,B}(\mathrm{id}_{B^*}) = v_{A,B}(\mathrm{id}_{B^*} \circ f) = v_{A,B}(f)$, hence $v_{\omega_v} = v$.

To finish, if ω corresponds to v, then for all $A, B \in \mathscr{A}$ and $f \in \mathrm{Hom}(B, A^*)$, $v_{A,B}(v_{B,A} f) = v_{A,B}(f^* \circ \omega_A) = (f^* \circ \omega_A)^* \circ \omega_B = \omega_A^* \circ f^{**} \circ \omega_B = \omega_A^* \circ \omega_{A^*} \circ f$ (in the last equality we used the fact ω is natural). Noting that we can take $B = A^*$ and $f = \mathrm{id}_{A^*}$, it follows that $\omega_A^* \circ \omega_{A^*} = \mathrm{id}_{A^*}$ for all $A \in \mathscr{A}$ if and only if $v_{A,B} \circ v_{B,A} = \mathrm{id}$ for all $A, B \in \mathscr{A}$. □

PROOF OF THEOREM 2.7.8. The natural transformations $v : \mathrm{Hom}(A, B^{[0]}) \to \mathrm{Hom}(B, A^{[0]})$ satisfying $v_{A,B} \circ v_{B,A} = \mathrm{id}$ are in one-to-one correspondence with natural transformations $f : \mathrm{Hom}(A, B^{[0]}) \to \mathrm{Hom}(A, B^{[1]})$ for which $I_{A,B} \circ f_{B,A} \circ I_{B,A} \circ f_{A,B} = \mathrm{id}$. Indeed, let v correspond to f if and only $v_{A,B} = I_{B,A} \circ f_{A,B}$. The correspondence is then obvious from the following diagram:



By Lemma 2.7.10, the v-s are in one-to-one correspondence with natural transformations $\omega : \mathrm{id}_{\mathscr{A}} \to [0][0]$ for which $(\mathscr{A}, [0], \omega)$ is a category with duality, and by Propositions 2.7.2 and 2.7.5(ii), the f-s are in one-to-one correspondence with involutions $u : [0] \to [1]$. Therefore, there is a one-to-one correspondence between natural transformations $\omega : \mathrm{id}_{\mathscr{A}} \to [0][0]$ for which $(\mathscr{A}, [0], \omega)$ is a category with duality and involutions $u : [0] \to [1]$.

Now let u correspond to ω. We will represent bilinear forms in $(\mathscr{A}, [0], [1], \Phi, \Psi)$ and $(\mathscr{A}, [0], \omega)$ by their *left adjoint* (rather than their *right adjoint*), i.e. as a pair $(M, \widetilde{b})$ with $M \in \mathscr{A}$ and $\widetilde{b} \in \mathrm{Hom}_{\mathscr{A}}(M, M^{[0]})$. By mapping each form to itself, we get a natural one-to-one correspondence between bilinear forms in $(\mathscr{A}, [0], [1], \Phi, \Psi)$ and in $(\mathscr{A}, [0], \omega)$. In addition, a form $(M, \widetilde{b})$ is u-symmetric in $(\mathscr{A}, [0], [1], \Phi, \Psi)$ $\iff$ $b = u_M \circ \widetilde{b}$ where $b = I_{M,M}^{-1}(\widetilde{b}) \iff \widetilde{b} = (I_{M,M} \circ (u_M)_*)(\widetilde{b}) \iff \widetilde{b} = v_{M,M}(\widetilde{b})$ where $v_{A,B} = I_{B,A} \circ (u_B)_* \iff \widetilde{b} = (\widetilde{b})^{[0]} \circ \omega_M$ (since $v = v_\omega$, by the construction of the correspondence) $\iff$ $\widetilde{b}$ is symmetric in $(\mathscr{A}, [0], \omega)$. □

Tracking along the proof, one can see that if $u : [0] \to [1]$ is an involution, then its corresponding $\omega : \mathrm{id}_{\mathscr{A}} \to [0][0]$ is given by simple formula:

$$\omega_A = (I_{A,A^{[0]}} \circ (u_A)_*)(\mathrm{id}_{A^{[0]}}) = I_{A,A^{[0]}}(u_A) = u_A^{[0]} \circ \Phi_A$$

and u can be recovered from ω by:

$$u_A = (I_{A,A^{[0]}}^{-1} \circ v_{\omega,A,A^{[0]}})(\mathrm{id}_{A^{[0]}}) = I_{A,A^{[0]}}^{-1}(\mathrm{id}_{A^{[0]}}^{[0]} \circ \omega_A) = \omega_A^{[1]} \circ \Psi_{A^{[0]}} \ .$$

(However, it is not clear from the formulas that u is an involution if and only if $\omega_A^{[0]} \circ \omega_{A^{[0]}} = \mathrm{id}_{A^{[0]}}$ for all $A \in \mathscr{A}$.)

Remark 2.7.11. We can now see that in a certain sense our definition of bilinear forms from section 2.1 cannot be explained as a special case of a category with duality. Indeed, there are rings R with a double R-module K admitting no involution (e.g. Example 2.4.14 and the examples following it), hence by Propositions 2.2.7 and 2.7.5(iv), there is no involution $u : [0] \to [1]$. But then Theorem 2.7.8 implies that there is no $\omega : \mathrm{id}_{\text{Mod-}R} \to [0][0]$ for which $(\text{Mod-}R, [0], \omega)$ is a category with duality, and hence $(\text{Mod-}R, [0], [1], \Phi, \Psi)$ cannot be equivalent to a category with a double duality coming from a category with duality (i.e. a c.w.d.d. of the form $(\mathscr{A}, *, *, \omega, \omega)$).

2.7.5. Further Remarks. Categories with a double duality can be generalized even more, if one is only interested in bilinear pairings and not in bilinear forms. Define a *pairing context* as a sextet $C = (\mathscr{A}, \mathscr{B}, [0], [1], \Phi, \Psi)$ such that $\mathscr{A}$ and $\mathscr{B}$ are categories, $[0] : \mathscr{B} \to \mathscr{A}$ and $[1] : \mathscr{A} \to \mathscr{B}$ are contravariant functors, and $\Phi : \mathrm{id}_{\mathscr{A}} \to [1][0]$, $\Psi : \mathrm{id}_{\mathscr{B}} \to [0][1]$ are natural transformations satisfying $\Psi_B^{[0]} \circ \Phi_{B^{[0]}} = \mathrm{id}_{B^{[0]}}$ and $\Phi_A^{[1]} \circ \Psi_{A^{[1]}} = \mathrm{id}_{A^{[1]}}$ for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$. As before, this induces a natural isomorphism $I_{A,B} : \mathrm{Hom}_{\mathscr{B}}(B, A^{[1]}) \to \mathrm{Hom}_{\mathscr{A}}(A, B^{[0]})$ ($A \in \mathscr{A}$, $B \in \mathscr{B}$) given by $I_{A,B}(b) = b^{[0]} \circ \Phi_A$, and Φ, Ψ can be recovered from I as described above. A bilinear pairing in C is a triplet (A, B, b) with $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $b \in \mathrm{Hom}_{\mathscr{B}}(B, A^{[1]})$. However, one cannot define bilinear forms without some identification of objects in $\mathscr{A}$ with objects in $\mathscr{B}$.

2.8. The Transfer Principle

The transfer principle of categories with duality says that, roughly speaking, every category with duality is "locally" the category of λ-hermitian forms over some ring with involution. This allows to *transfer* the theory of arbitrary categories with duality to the theory of hermitian forms; see [**86**, Ch. 7, §4] or [**71**] for details. In this section we extend this result to categories with a double duality. That is, we prove that every object in an additive category with a double duality is contained in an additive full subcategory that is isomorphic (as categories with a double duality) to a full subcategory of a category with duality obtained from some ring R and a double R-module K. Our new transfer principle also benefits the theory categories with duality as it allows transfer in situations that were not applicable before.

We begin with defining morphisms of categories with a double duality.

Definition 2.8.1. *Let* $(\mathscr{A}, [0], [1], \Phi, \Psi)$ *and* $(\mathscr{A}', [0]', [1]', \Phi', \Psi')$ *be categories with a double duality. A* morphism of categories with a double duality *from* $\mathscr{A}$ *to* $\mathscr{A}'$ *consists of a triplet* (F, θ_0, θ_1) *such that* $F : \mathscr{A} \to \mathscr{A}'$ *is a functor and* $\theta_i : F[i] \to [i]'F$ *is a natural isomorphism* ($i = 0, 1$) *satisfying*

$$\begin{aligned} \theta_{0,M^{[1]}} \circ F\Phi_M &= \theta_{1,M}^{[0]'} \circ \Phi'_{FM}, \\ \theta_{1,M^{[0]}} \circ F\Psi_M &= \theta_{0,M}^{[1]'} \circ \Psi'_{FM}. \end{aligned}$$

If $\mathscr{A}$ and $\mathscr{A}'$ are additive then F is required to be additive as well. The morphism (F, θ_0, θ_1) is called an equivalence of categories with a double duality *if F is an equivalence of categories.*

Let $\mathscr{A}_0$ be a full subcategory of $\mathscr{A}$. If F is only defined over $\mathscr{A}_0$, then (F, θ_0, θ_1) is called a sub-morphism *of categories with a double duality.*[12]

If $u : [0] \to [1]$ and $u' : [0]' \to [1]'$ are natural isomorphisms, then (F, θ_0, θ_1) is said to pass u to u' if $u'_{FM} \circ \theta_{0,M} = \theta_{1,M} \circ Fu_M$.

The following proposition shows that the theory of bilinear forms over a category with a double duality can be transferred (in a certain sense) along morphisms and sub-morphisms.

PROPOSITION 2.8.2. *Let (F, θ_0, θ_1) be a sub-morphism of categories with a double duality from $(\mathscr{A}, [0], [1], \Phi, \Psi)$ to $(\mathscr{A}', [0]', [1]', \Phi', \Psi')$, and let $\mathscr{A}_0$ denote the domain of F. Let $\mathscr{B}_0$ (resp. $\mathscr{B}'$) be the category of bilinear forms (M, b) over $\mathscr{A}_0$ (resp. $\mathscr{A}'$) with $M \in \mathscr{A}_0$ (resp. without restriction), with isometries as morphisms. Then:*

(i) *(F, θ_0, θ_1) induces a functor $G : \mathscr{B}_0 \to \mathscr{B}'$. The functor G is faithful (resp. faithful and full), provided F is.*
(ii) *If $u : [0] \to [1]$ and $u' : [0]' \to [1]'$ are natural isomorphisms such that (F, θ_0, θ_1) passes u to u', then G sends u-symmetric forms to u'-symmetric forms.*

PROOF. (i) Define $G : \mathscr{B}_0 \to \mathscr{B}$ by $G(M, b) = (FM, \theta_{1,M} \circ Fb)$ and $G\sigma = F\sigma$ for every bilinear form $(M, b) \in \mathscr{B}_0$ and isometry $\sigma : (M, b) \to (M', b')$ in $\mathscr{B}_0$. Observe that $G\sigma$ is indeed an isometry since $(G\sigma)^{[1]'} \circ Gb' \circ G\sigma = (F\sigma)^{[1]'} \circ \theta_{1,M'} \circ Fb' \circ F\sigma = \theta_{1,M} \circ F(\sigma^{[1]}) \circ Fb' \circ F\sigma = \theta_{1,M} \circ F(\sigma^{[1]} \circ b' \circ \sigma) = \theta_{1,M} \circ Fb = Gb$. That G preserves composition is straightforward. Now let $\sigma' : (M, b) \to (M', b')$ be another isometry. If F is faithful, then $G\sigma = G\sigma'$ implies $F\sigma = F\sigma'$, hence $\sigma = \sigma'$. This means G is faithful. Now assume F is also full and let τ be an isometry from $G(M, b)$ to $G(M', b')$. We need to find and isometry $\sigma : (M, b) \to (M', b')$ such that $G\sigma = \tau$. Since F is faithful and full, there is an *isomorphism* $\sigma \in \mathrm{Hom}_{\mathscr{A}}(M, M')$ such that $F\sigma = \tau$. We claim that σ is an isometry from (M, b) to (M', b'). Indeed, $(G\sigma)^{[1]'} \circ Gb' \circ G\sigma = Gb$, so the previous computation implies that $\theta_{1,M} \circ F(\sigma^{[1]} \circ b' \circ \sigma) = \theta_{1,M} \circ Fb$. Multiplying by $\theta_{1,M}^{-1}$ on the right yields $F(\sigma^{[1]} \circ b' \circ \sigma) = Fb$ and since F is faithful, we get $\sigma^{[1]} \circ b' \circ \sigma = b$, as required.

(ii) Let u, u' be as above and let (M, b) be a u-symmetric bilinear form. Recall that this implies that $b = u_M \circ \widetilde{b} = u_M \circ b^{[0]} \circ \Phi_M$. We need to prove that $(\theta_{1,M}) = u'_{FM} \circ (\theta_{1,M} \circ Fb)^{[0]'} \circ \Phi'_{FM}$. Indeed, $u'_{FM} \circ (\theta_{1,M} \circ Fb)^{[0]'} \circ \Phi'_{FM} = u'_{FM} \circ (Fb)^{[0]'} \circ \theta_{1,M}^{[0]'} \circ \Phi'_{FM} = u'_{FM} \circ (Fb)^{[0]'} \circ \theta_{0,M^{[0]}} \circ F\Phi_M = u'_{FM} \circ \theta_{0,M} \circ F(b^{[0]}) \circ F\Phi_M = \theta_{1,M} \circ Fu_M \circ F(b^{[0]}) \circ F\Phi_M = \theta_{1,M} \circ F(u_M \circ b^{[0]} \circ \Phi_M) = \theta_{1,M} \circ Fb$ (we used the fact θ_0 is natural and that (F, θ_0, θ_1) passes u to u'). □

Henceforth, $(\mathscr{A}, [0], [1], \Phi, \Psi)$ is an *additive* category with a double duality. Fix an object $A \in \mathscr{A}$ and let $R_A = \mathrm{End}_{\mathscr{A}}(A)$. We let $\mathscr{A}|_A$ denote the full subcategory of $\mathscr{A}$ whose objects are (isomorphic to) summands of A^n for some $n \in \mathbb{N}$. Let F_A be the functor $\mathrm{Hom}(A, \_)$. Observe that for every $M \in \mathscr{A}$, $\mathrm{Hom}(A, M)$ can be made into a right R_A-module by defining

$$f \cdot r = f \circ r \qquad \forall\ f \in \mathrm{Hom}(A, M),\ r \in R_A = \mathrm{End}(A)\ .$$

This makes F_A into an additive functor from $\mathscr{A}$ to Mod-R_A. The following proposition is well known.

---

[12] The reason we do not call F a morphism of c.w.d.d. from $\mathscr{A}_0$ to $\mathscr{A}'$ is that $\mathscr{A}_0$ might not be a c.w.d.d. Indeed, $[0]$ and $[1]$ are not assumed to send $\mathscr{A}_0$ into itself.

PROPOSITION 2.8.3. *Once restricted to $\mathscr{A}|_A$, the functor F_A is full and faithful. That is, for every $M, N \in \mathscr{A}|_A$, the following map is bijective:*

$$F_A : \mathrm{Hom}_{\mathscr{A}}(M, N) \to \mathrm{Hom}_{\text{Mod-}R_A}(F_A M, F_A N) \ .$$

PROOF. Since Hom is biadditive and M, N are summands of A^n for sufficiently large n, it is enough to check this for $M = N = A$, which is routine. ☐

REMARK 2.8.4. If $\mathscr{A}$ is a *Grothendieck category* (e.g. Mod-R for some ring R) and A is a generator of $\mathscr{A}$, then the GabrielŰ-Popesco Theorem ([**70**]) asserts that F_A is faithful and full on all of $\mathscr{A}$ (rather than just $\mathscr{A}|_A$).

REMARK 2.8.5. If all idempotents in $\mathscr{A}$ split, then $F_A(\mathscr{A}|_A) = \text{proj-}R_A$, the category of right finite projective R_A-modules.

Let $K_A := \mathrm{Hom}(A, A^{[1]})$. Then K_A can be made into a double R_A-module by defining

$$f \odot_0 r = r^{[1]} \circ f \qquad \text{and} \qquad f \odot_1 r = f \circ r$$

for all $f \in K_A$ and $r \in R_A$. Thus K_A induces an double duality structure on Mod-R_A, which, abusing the notation, we denote by $(\text{Mod-}R_A, [0], [1], \Phi, \Psi)$. The transfer principle is phrased in the following theorem.

THEOREM 2.8.6 (Transfer Principle). *There is a faithful full sub-morphism of categories with a double duality from $\mathscr{A}|_A$ to* Mod-R_A.

PROOF. By the previous proposition, F_A is an equivalence of categories from $\mathscr{A}|_A$ to its image. Hence it is enough to define natural transformations $\theta_i : F_A[i] \to [i]F_A$ such that $(F_A, \theta_0, \theta_1)$ is a sub-morphism of categories with a double duality.

Let $M \in \mathscr{A}$. Define $\theta_{i,M} : F_A(M^{[i]}) \to (F_A M)^{[i]}$ $(i = 0, 1)$ by

$$\begin{aligned} \theta_{0,M}(t) &= [f \mapsto t^{[1]} \circ \Psi_M \circ f] \ , \\ \theta_{1,M}(s) &= [f \mapsto f^{[1]} \circ s] \end{aligned}$$

where

$$\begin{aligned} t &\in F_A(M^{[0]}) = \mathrm{Hom}_{\mathscr{A}}(A, M^{[0]}) \ , \\ s &\in F_A(M^{[1]}) = \mathrm{Hom}_{\mathscr{A}}(A, M^{[1]}) \ , \\ f &\in F_A(M) = \mathrm{Hom}_{\mathscr{A}}(A, M) \ . \end{aligned}$$

(Observe that $(F_A M)^{[i]} = \mathrm{Hom}_{R_A}(\mathrm{Hom}_{\mathscr{A}}(A, M), (K_A)_i)$ and $(K_A)_i = \mathrm{Hom}(A, A^{[1]})$ considered as a right R_A-module w.r.t. $\odot_i$ defined above.) Throughout the proof, s, t, f would continue to denote arbitrary elements of the sets specified above and r is always an element of R_A.

We now have several technical checks to do. (The reader can skip to the end of the proof without loss of continuity.) The maps $\theta_{0,M}$ and $\theta_{1,M}$ are R_A-modules homomorphisms since

$$\begin{aligned} (\theta_{0,M}(t \cdot r))f &= ((t \circ r)^{[1]} \circ \Psi_M \circ f) = r^{[1]} \circ (t^{[1]} \circ \Psi_M \circ f) = (\theta_{0,M} t) f \odot_0 r \\ &= ((\theta_{0,M} t) \cdot r) f \\ (\theta_{1,M}(s \cdot r))g &= g^{[1]} \circ (s \circ r) = (g^{[1]} \circ s) \circ r = (\theta_{1,M} s) g \odot_1 r = ((\theta_{1,M} s) \cdot r) g \ . \end{aligned}$$

(The additivity of $\theta_{0,M}$ and $\theta_{1,M}$ is clear.) In addition, for every $M, M' \in \mathscr{A}$, $\sigma \in \mathrm{Hom}_{\mathscr{A}}(M', M)$ and $f' \in FM' = \mathrm{Hom}_{\mathscr{A}}(A, M')$, one has

$$\begin{aligned}
((F_A\sigma)^{[0]}(\theta_{0,M}t))f' &= (\theta_{0,M}t)((F_A\sigma)f') = (\theta_{0,M}t)(\sigma\circ f') = t^{[1]}\circ\Psi_M\circ(\sigma\circ f')\\
&= t^{[1]}\circ\sigma^{[0][1]}\circ\Psi_{M'}\circ f' = (\sigma^{[0]}\circ t)^{[1]}\circ\Psi_{M'}\circ f'\\
&= (F_A(\sigma^{[0]})t)^{[1]}\circ\Psi_{M'}\circ f' = (\theta_{0,M'}(F_A(\sigma^{[0]})t))f'\\
((F_A\sigma)^{[1]}(\theta_{1,M}s))f' &= (\theta_{1,M}s)((F_A\sigma)f') = (\theta_{1,M}s)(\sigma\circ f') = (\sigma\circ f')^{[1]}\circ s\\
&= f'^{[1]}\circ\sigma^{[1]}\circ s = f'^{[1]}\circ(F_A(\sigma^{[1]}))s = (\theta_{1,M}((F_A(\sigma^{[1]}))s))f'.
\end{aligned}$$

Thus, $(F_A\sigma)^{[0]}\circ\theta_{0,M} = \theta_{0,M'}\circ F_A(\sigma^{[0]})$ and $(F_A\sigma)^{[1]}\circ\theta_{1,M} = \theta_{1,M'}\circ F_A(\sigma^{[1]})$, implying θ_0 and θ_1 are natural. Next, we have

$$\begin{aligned}
((\theta_{0,M^{[1]}}\circ F_A\Phi_M)f)s &= (\theta_{0,M^{[1]}}((F_A\Phi_M)f))s = (\theta_{0,M^{[1]}}(\Phi_M\circ f))s\\
&= (\Phi_M\circ f)^{[1]}\circ\Psi_{M^{[1]}}\circ s = f^{[1]}\circ\Phi_M^{[1]}\circ\Psi_{M^{[1]}}\circ s = f^{[1]}\circ s\\
&= (\theta_{1,M}s)f = (\Phi_{F_AM}f)(\theta_{1,M}s) = (\theta_{1,M}^{[0]}(\Phi_{F_AM}f))s\\
((\theta_{1,M^{[0]}}\circ F_A\Psi_M)f)t &= (\theta_{1,M^{[0]}}((F_A\Psi_M)f))t = (\theta_{1,M^{[0]}}(\Psi_M\circ f))t\\
&= t^{[1]}\circ(\Psi_M\circ f) = (\theta_{0,M}t)f = (\Psi_{F_AM}f)(\theta_{0,M}t)\\
&= (\theta_{0,M}^{[1]}(\Psi_{F_AM}f))t\ ,
\end{aligned}$$

so $\theta_{0,M^{[1]}}\circ F_A\Phi_M = \theta_{1,M}^{[0]}\circ\Phi_{F_AM}$ and $\theta_{1,M^{[0]}}\circ F_A\Psi_M = \theta_{0,M}^{[1]}\circ\Phi_{F_AM}$. To finish, we need to show that $\theta_{0,M}$ and $\theta_{1,M}$ are bijective. As everything is additive and M is a summand of A^n for some $n\in\mathbb{N}$, it is enough to show that $\theta_{0,A}$ and $\theta_{1,A}$ are bijective. Indeed, the maps $\eta_i : (F_AA)^{[i]} \to F_A(A^{[i]}) = \mathrm{Hom}_{\mathscr{A}}(A, A^{[i]})$ defined by

$$\begin{aligned}
\eta_0(t') &= t'(\mathrm{id}_A)^{[0]}\circ\Phi_A\ ,\\
\eta_1(s') &= s'(\mathrm{id}_A)
\end{aligned}$$

for all $t' \in (F_AA)^{[0]} = \mathrm{Hom}_{R_A}(R_A, (K_A)_1)$ and $s' \in (F_AA)^{[1]} = \mathrm{Hom}_{R_A}(R_A, (K_A)_0)$ are inverses of $\theta_{0,A}$, $\theta_{1,A}$ since

$$\begin{aligned}
(\theta_{0,A}(\eta_0t'))r &= (\theta_{0,A}(t'(\mathrm{id}_A)^{[0]}\circ\Phi_A))r = (t'(\mathrm{id}_A)^{[0]}\circ\Phi_A)^{[1]}\circ\Psi_A\circ r\\
&= \Phi_A^{[1]}\circ t'(\mathrm{id}_A)^{[0][1]}\circ\Psi_A\circ r = \Phi_A^{[1]}\circ\Psi_{A^{[1]}}\circ t'(\mathrm{id}_A)\circ r\\
&= t'(\mathrm{id}_A)\odot_1 r = t'(\mathrm{id}_A\cdot r) = t'(r)\\
(\theta_{1,A}(\eta_1s')r &= ((\theta_{1,A})(s'(\mathrm{id}_A)))r = r^{[1]}\circ s'(\mathrm{id}_A) = s'(\mathrm{id}_A)\odot_0 r\\
&= s'(\mathrm{id}_A\cdot r) = s'(r)\ .
\end{aligned}$$

That $\eta_0\circ\theta_{0,A} = \mathrm{id}$ and $\eta_1\circ\theta_{1,A} = \mathrm{id}$ is easy and thus left to the reader. □

REMARK 2.8.7. We can also endow $\mathrm{Hom}_{\mathscr{A}}(A, A^{[0]})$ with a double R_A-module structure by letting

$$f\odot_0 r = f\circ r \qquad\text{and}\qquad f\odot_1 r = r^{[0]}\circ f\ .$$

The resulting double R-module is isomorphic to K_A and the isomorphism is the map $I_{A,A}^{-1} : \mathrm{Hom}(A, A^{[0]}) \to \mathrm{Hom}(A, A^{[1]})$. Identifying $\mathrm{Hom}(A, A^{[1]})$ with $\mathrm{Hom}(A, A^{[0]})$ in this way, the map θ_0 of the last proof can be described by the formula $\theta_{1,M}(t) = [f \mapsto f^{[0]}\circ t]$ (here $f^{[0]}\circ t$ lies in $\mathrm{Hom}(A, A^{[0]})$ rather than $\mathrm{Hom}(A, A^{[1]})$). Thus, although it is not clear at first sight, the definitions of θ_0 and θ_1 are basically the same up to 0-1 exchange.

The next proposition shows how F_A interacts with natural isomorphisms from [0] to [1].

PROPOSITION 2.8.8. *Assume that* $u : [0] \to [1]$ *is a natural isomorphism (resp. involution). Then* K_A *has anti-isomorphism (resp. involution)* κ *and* $(F_A, \theta_0, \theta_1)$ *passes* u *to* u_κ *(that is,* $u_{\kappa,F_AM} \circ \theta_{0,M} = \theta_{1,M} \circ F_A u_M$ *for all* $M \in \mathscr{A}|_A$*).*

PROOF. Define $\kappa : K_A \to K_A = \mathrm{Hom}(A, A^{[1]})$ by $\kappa(f) = ((u_A)_* \circ I_{A,A})f = u_A \circ f^{[0]} \circ \Phi_A$. It is straightforward to check that κ is an anti-isomorphism and by Proposition 2.7.5(ii), $\kappa^2 = \mathrm{id}$ when u is an involution. The equality $u_{\kappa,F_AM} \circ \theta_{0,M} = \theta_{1,M} \circ F_A u_M$ holds since for all $t \in F_A(M^{[0]}) = \mathrm{Hom}_{\mathscr{A}}(A, M^{[0]})$ and $f \in F_A M = \mathrm{Hom}_{\mathscr{A}}(A, M)$, we have

$$\begin{aligned}(u_{\kappa,F_AM}(\theta_{0,M}t))f &= (\kappa \circ (\theta_{0,M}t))f = \kappa((\theta_{0,M}t)f) = \kappa(t^{[1]} \circ \Psi_M \circ f)\\ &= u_A \circ (t^{[1]} \circ \Psi_M \circ f)^{[0]} \circ \Phi_A = u_A \circ f^{[0]} \circ \Psi_M^{[0]} \circ t^{[1][0]} \circ \Phi_A\\ &= f^{[1]} \circ u_M \circ \Psi_M^{[0]} \circ \Phi_{M^{[0]}} \circ t = f^{[1]} \circ u_M \circ t = (\theta_{1,M}(u_M \circ t))f\\ &= (\theta_{1,M}((F_A u_M)t))f\ .\end{aligned}$$

(Recall that $u_\kappa = \kappa \circ \_$.) □

The previous results imply that everything we have proved for bilinear forms over rings in the previous sections also applies to arbitrary categories with a double duality. However, precaution should be taken since a bilinear form which is epic in $\mathscr{A}$ might not be epic (i.e. surjective) once transferred to Mod-R_A. Nevertheless, monic bilinear forms over $\mathscr{A}|_A$ are transferred to monic (i.e. injective) bilinear forms over Mod-R_A.

REMARK 2.8.9. Under mild assumptions, we can say quite a lot about the structure of K_A: Assume that there is a right regular bilinear form (A, b_0). Then b_0 induces an anti-endomorphism $*$ of R_A given by $r \mapsto b^{-1} \circ r^{[1]} \circ b$. Let K be the double R_A-modules obtained from R_A by defining $x \odot_0 r = r^* x$ and $x \odot_1 r = xr$ $(x, r \in R_A)$. Then $K \cong K_A$ as double R_A-modules. The isomorphism is given by $k \mapsto b \circ k$. In particular, if b_0 is u-symmetric for some $u : [0] \to [1]$, then $*$ is an involution and the bilinear forms on $\mathscr{A}|_A$ are equivalent to sesquilinear forms over $(R_A, *)$. When restricted to categories with duality, the last observation is just the classical transfer principle.

REMARK 2.8.10. Observe that the transfer principle we have obtained in this section is interesting even for categories with duality. Indeed, the standard transfer in categories with duality (see the end of the previous remark or [**71**], [**86**, Ch. 7]) can be applied only for objects A admitting a regular symmetric or skew-symmetric bilinear form b_0. We have dropped this condition, as well as the dependency in b_0, which is inherent in the classical transfer.

2.9. Rings That Are Morita Equivalent to Their Opposites

In this section, we use our new notion of bilinear forms to partially answer a problem that was suggested to the author by David Saltman (to whom the author is grateful). Consider the following three properties that a ring R might posses:

(1) There is a ring with an involution $(S, *)$ and S is Morita equivalent to R.
(2) There is a ring with an anti-automorphism $(S, *)$ and S is Morita equivalent to R.
(3) R is Morita equivalent to R^{op}.

While (1)$\Longrightarrow$(2)$\Longrightarrow$(3) is obvious, one could ask whether there are other implications between (1), (2) and (3). Indeed, in [**82**], Saltman proves (2)$\Longrightarrow$(1) in case R is an Azumaya algebra over some commutative ring, and the following conditions are well known to be equivalent when R is a f.d. simple algebra (e.g. [**2**, Ch. X]):

(1′) R has an involution of the first kind, (2′) R has an anti-automorphism fixing $\mathrm{Cent}(R)$.

(However, (2′)$\not\Longrightarrow$(1′) for Azumaya algebras.) We will show below that (3)$\Longrightarrow$(2) for a large family of rings and against expectations, (2)$\not\Longrightarrow$(1) even for f.d. algebras. *The results to follow, as well as some improvements such as a new proof of Saltman's result, can also be found in* [**40**].

It will be useful to introduce some general notation for this section: For a ring R, let proj-R denote the category of finite projective right R-modules and let Iso(proj-R) denote the isomorphism classes of proj-R. The isomorphism class of $P \in$ proj-R will be denoted by $[P]$. Let R and S be rings. By saying M is an (S,R)*-progenerator* we mean that M is an (S,R)-bimodule, M_R and ${}_SM$ are progenerators (of the appropriate categories), $R = \mathrm{End}({}_SM)$ and $S = \mathrm{End}(M_R)$. Recall that an (S,R)-progenerator exists precisely when R is Morita equivalent to S. For a detailed discussion of Morita equivalence, see [**58**, §18], [**80**, §4.1] and also [**72**, Ch. 4].

We begin by proving (3)$\Longrightarrow$(2) for certain rings.

Definition 2.9.1. *Let $(M,+)$ be an abelian monoid. An element $x \in M$ is called* indecomposable *if $x = y + z$ implies $y = 0$ or $z = 0$. We say $(M,+)$ is* strongly finitely generated *if M is spanned as a monoid by a finite set of indecomposable elements.*

A ring R is said to be of (right) finite projective representation type *(abbrev.: FPRT) if* (Iso(proj-R), $\oplus$) *is strongly f.g., i.e. if R admits finitely many indecomposable finite projectives (up to isomorphism) and any finite projective module is a direct sum of finite number of indecomposables.*

Note that if $(M,+)$ is a monoid and S is a generating set for M consisting of indecomposable elements, then S is the only generating set consisting of indecomposable elements and it consists of *all* indecomposable elements. In particular, any automorphism of M permutes S.

Example 2.9.2. Any semiperfect ring has FPRT since (Iso(proj-R), $\oplus$) $\cong$ $(\mathbb{N}^k,+)$ for some $k \in \mathbb{N}$; see [**80**, §2.9]. More generally, if R_R^n has a Krull-Schmidt decomposition (i.e. a representation as a sum of indecomposables which is unique up to isomorphism and reordering) for all $n \in \mathbb{N}$, then R has FPRT (since (Iso(proj-R), $\oplus$) is spanned by the indecomposable components of R). For example, this holds when R is homogeneous semilocal (i.e. $R/\mathrm{Jac}(R)$ is simple artinian), as follows from [**26**]. Other examples of rings with FPRT include maximal orders in f.d. simple algebras over global fields. This follows from [**72**, Th. 26.4, §35-36].

Theorem 2.9.3. *Let R be a ring with FPRT that is Morita equivalent to its opposite, then there exists a ring with anti-isomorphism $(S,*)$ such that S is Morita equivalent to R.*

Proof. Let P be an (R^{op},R)-progenerator. We make P into a double R-module by letting $\odot_1$ be the standard right action of R on P and $\odot_0$ be the right action of R on P obtained by twisting the left action of R^{op}. Observe that for $i \in \{0,1\}$, $R^{[i]} = \mathrm{Hom}(R_R, P_{1-i}) \cong P_i$ and hence $R^{[i][1-i]} \cong \mathrm{Hom}(P_i,P_i) \cong R_R$. It is now routine to verify that R is reflexive and since being reflexive is preserved under finite direct sums and passes to summands, any finite projective right R-module is reflexive. For $i \in \{0,1\}$ define $\varphi_i : \mathrm{Iso}(\text{proj-}R) \to \mathrm{Iso}(\text{proj-}R)$ by $\varphi_i[P] = [P^{[i]}]$. Then φ_i is well-defined and the previous discussion implies $\varphi_i = \varphi_{1-i}^{-1}$. Moreover, since $[i]$ preserves direct sums, φ_i is a monoid isomorphism.

Since (Iso(proj-R), $\oplus$) is strongly f.g., there exists indecomposable $P_1,\dots,P_t \in$ proj-R such that $S := \{[P_1],\dots,[P_t]\}$ generates Iso(proj-R). Let $M = P_1 \oplus \dots \oplus P_t$.

Then φ_1 permutes S, hence $[M] \cong [M^{[1]}]$. This gives rise to a right regular bilinear space (M, b, P) (take Ad_b^r would be the isomorphism $M \cong M^{[1]}$). Since M is finite projective, it is left reflexive, so by Proposition 2.5.4(ii) b is also left regular. Therefore, there is an anti-automorphism $* : \mathrm{End}_R(M) \to \mathrm{End}_R(M)$, namely the one that corresponds to b (see Propositions 2.3.3 and 2.3.4). As S generates $\mathrm{Iso}(\text{proj-}R)$, M must be a progenerator, hence $\mathrm{End}_R(M)$ is Morita equivalent to R and we are through. □

REMARK 2.9.4. Define an equivalence relation on Iso(proj-R) by $[P] \sim [Q] \iff$ there exists $n \in \mathbb{N}$ such that $[P^n] = [Q^n]$. Then Iso(proj-$R/\sim, \oplus$) is a monoid. It is easy to see that Theorem 2.9.3 also holds when Iso(proj-$R/\sim, \oplus$) is strongly finitely generated.[13] The proof is similar, but one obtains a module M for which $[M] \sim [M^{[1]}]$. By replacing M with M^n for n sufficiently large, we may assume $M \cong M^{[1]}$ and proceed with the proof.

Other finiteness assumptions on proj-R also imply the existence of M with $M \cong M^{[1]}$. For example, if Iso(proj-R) is finite (see [**4**] and related papers for such examples), then one can take $M = Q_1 \oplus \cdots \oplus Q_t$ where $Q_1, \dots, Q_t$ are representatives for the isomorphism classes of proj-R.

The proof of Theorem 2.9.3 had two stages. The first was to show that any (R^{op}, R)-progenerator gives rise to a duality from proj-R to itself and the second consisted of finding a generator $M \in \text{proj-}R$ with $M \cong M^{[1]}$, or equivalently, a right regular bilinear space (M, b, P) (with P as above). We will now show that any ring with anti-isomorphism $(S, *)$ for which S is Morita equivalent to R is obtained via this principle (compare with [**82**, Th. 4.2]).

PROPOSITION 2.9.5. *Let R be a ring that is Morita equivalent to R^{op} and let M be an R-progenerator. Then any anti-isomorphism $*$ of $S := \mathrm{End}_R(M)$ is induced from a right regular bilinear space (M, b, P), where P is obtained from* some *(R^{op}, R)-progenerator. Moreover, if $*$ is an involution, then P admits an involution κ and b is κ-symmetric.*

PROOF. Consider M as an (S, R)-bimodule and observe that M can be made into an (R^{op}, S)-bimodule by defining $r^{\mathrm{op}} \cdot m \cdot s = s^{*^{-1}} m r$. Define $P = {}_{R^{\mathrm{op}}}P_R = {}_{R^{\mathrm{op}}}M_S \otimes_S {}_S M_R$ and make it into a double R-module by letting:

$$(x \otimes_S y) \odot_0 r = r^{\mathrm{op}} x \otimes_S y, \qquad (x \otimes_S y) \odot_1 r = x \otimes yr \qquad \forall x, y \in M,\ r \in R\ .$$

It is now clear that $b : M \times M \to P$ defined by $b(x, y) = x \otimes_S y$ is a bilinear form. In addition, for all $s \in S$, $b(sx, y) = (sx) \otimes_S y = (x \cdot s^*) \otimes_S y = x \otimes_S s^* y = b(x, s^* y)$, hence the corresponding anti-endomorphism of b is $*$, provided b is regular. However, we postpone the proof of the latter fact to Chapter 3 (Theorem 3.5.5), where we shall generalize the construction of P. If $*$ is an involution, then the map $\kappa : P \to P$ defined by $(x \otimes y)^\kappa = y \otimes x$ $(x, y \in M)$ is well-defined and it is easy to check that it is an involution of P and b is κ-symmetric. Finally, ${}_{R^{\mathrm{op}}}P_R$ is an (R^{op}, R)-progenerator because it is the tensor product of an (R^{op}, S)-progenerator (namely, ${}_{R^{\mathrm{op}}}M_S$) and an (S, R)-progenerator $({}_S M_R)$. (This fact is a consequence of Morita's Third Theorem; see [**58**, §18D].) □

Note that Proposition 2.9.5 only promises us *some* (R^{op}, R)-progenerator, but in general, a given (R^{op}, R)-progenerator P need not admit a regular bilinear form (M, b, P) with M a progenerator (hence the proof of Theorem 2.9.3 does not work for arbitrary rings). This is demonstrated in the next example. Can it be that all (R^{op}, R)-progenerators P are "bad" (in the sense of not having a regular bilinear

[13] The author does not know if this condition is implied from R having FPRT, but this is true in case R is right noetherian.

space (M, b, P))? We believe that the answer is yes. In particular, we conjectures that (3)$\not\Rightarrow$(2) in general.

EXAMPLE 2.9.6. Let F be a field and let $R = \varinjlim \{\mathrm{M}_2(F)^{\otimes n}\}_{n\in\mathbb{N}}$. Then any finite projective right R-module is obtained by scalar extension from a finite projective over $\mathrm{M}_2(F)^{\otimes n} \hookrightarrow R$. It now not hard (but tedious) to show that the monoid $(\mathrm{Iso}(\text{proj-}R), \oplus)$ is isomorphic to $(\mathbb{Z}[\frac{1}{2}] \cap [0,\infty), +)$. (If V_n is the unique indecomposable right projective over $\mathrm{M}_2(F)^{\otimes n}$, then $V_n \otimes R$ is mapped to 2^{-n}.)

Let T denote the transpose involution on $\mathrm{M}_2(F)$. Then $\widehat{T} = \varinjlim \{T^{\otimes n}\}_{n\in\mathbb{N}}$ is an involution of R. Now let $P = R^2 \in \text{proj-}R$. Then $\mathrm{End}_R(P) \cong \mathrm{M}_2(R) \cong R$ and using $\widehat{T}$, we can identify $\mathrm{End}_R(P)$ with R^{op}, thus making P into an (R^{op}, R)-progenerator. We claim that there is no regular bilinear form (M, b, P) with $0 \neq M \in \text{proj-}R$. To see this, identify $\mathrm{Iso}(\text{proj-}R)$ with $\mathbb{Z}[\frac{1}{2}] \cap [0,\infty)$ and observe that monoid isomorphisms $\varphi_0, \varphi_1 : \mathbb{Z}[\frac{1}{2}] \cap [0,\infty) \to \mathbb{Z}[\frac{1}{2}] \cap [0,\infty)$ of Theorem 2.9.3 satisfy $\varphi_0(1) = \frac{1}{2}$ and $\varphi_1(1) = 2$, hence $\varphi_0(x) = \frac{1}{2}x$ and $\varphi_1(x) = 2x$ for all $x \in \mathbb{Z}[\frac{1}{2}] \cap [0,\infty)$. But this means $\varphi_1(x) \neq x$ for all $0 \neq x \in \mathbb{Z}[\frac{1}{2}] \cap [0,\infty)$, so $M \ncong M^{[1]}$ for all $0 \neq M \in \text{proj-}R$.

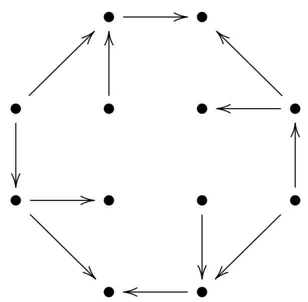
The next example shows that (2)$\not\Rightarrow$(1). The example we bring was suggested by Scharlau in [**85**] as an example of a ring with an anti-automorphism but without involution. However, it turns out that any ring that is Morita equivalent to this example does not have an involution.

EXAMPLE 2.9.7. Recall that a poset consists of a finite set I equipped with a transitive reflexive relation which we denote by $\leq$. For a field F and a poset I, the *incidence algebra* $A = F(I)$ is defined to be the subalgebra of the I-indexed matrices over F spanned as an F-vector space by $\{e_{ij} \,|\, i,j \in I,\ i \leq j\}$.

The poset I can be recovered (up to isomorphism) from A as follows: The ring $B = A/\mathrm{Jac}(A)$ is a semisimple ring. Let $e_1, \dots, e_t$ denote the set of central idempotents in B for which e_iBe_i is simple and let $\ell_i = \mathrm{length}(e_iBe_i)$. Since $\mathrm{Jac}(A)$ is nil, $e_1, \dots, e_t$ can be lifted to orthogonal idempotents $f_1, \dots, f_t \in B$ (the f-s are uniquely determined up to conjugation). Define $I' = \{x_{ij} \,|\, 1 \leq i \leq t, 1 \leq j \leq \ell_i\}$, and let $x_{ij} \leq x_{kl} \iff f_iAf_k \neq 0$. Then $A \cong F(I')$. This implies that two incidence algebras are isomorphic (as rings) if and only if their underlying posets are isomorphic. Moreover, any anti-automorphism (resp. involution) of A permutes $e_1, \dots, e_t$, preserves $\ell_1, \dots, \ell_t$, reverse the order in I', and thus induces an anti-automorphism (resp. involution) on I'. It follows that A has an anti-automorphism (resp. involution) if and only if I has one.

Any poset $(I, \leq)$ gives rise to an equivalence relation $\sim$ on I defined by $i \sim j \iff i \leq j$ and $j \leq i$. The quotient set $I/\!\sim$ can be made into a poset by defining $[x] \leq [y] \iff x \leq y$ (where $[x]$ is the equivalence class of x). It is well known that two incidence algebras $F(I)$ and $F(J)$ are Morita equivalent if and only if $I/\!\sim\, \cong J/\!\sim$ as posets. The converse is also true, any ring that is Morita equivalent to $F(I)$ is an incidence algebra $F(J)$ with $I/\!\sim\, \cong J/\!\sim$.

Now observe that if I admits an involution, then so is $I/\!\sim$. Therefore, by the previous paragraphs, if we can find I such that $I = I/\!\sim$ (i.e. $\leq$ is anti-symmetric) and I admits an anti-automorphism but no involution, then any ring that is Morita equivalent to $F(I)$ does not have an involution. (Otherwise, this would imply that $I = I/\!\sim$ has an involution). Such an example was given in [**85**] by Scharlau (for other purposes); I is the 12-element poset whose Hasse diagram is:



(Using Scharlau's words, it is "the simplest example I could find".) The anti-automorphism of I is given by rotating the diagram by ninety degrees clockwise.

REMARK 2.9.8. Incidence algebras are a good source of examples for rings without anti-automorphisms that are Morita equivalent to their opposites — just take I such that $I \not\cong I^{\mathrm{op}}$ but $I/\sim\, \cong I^{\mathrm{op}}/\sim$. The simplest such example is $I = \{1,2,3\}$ with the relation $I \times I \setminus \{(3,2),(3,1)\}$. In this case, the poset algebra is the algebra of 3×3 matrices of the form

$$\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}.$$

Note that in this case $I/\sim$ has an involution, hence $F(I/\sim)$ has an involution although $F(I)$ does not even have an anti-automorphism. Moreover, by Proposition 2.9.5 this means that there is a *symmetric* regular bilinear form over $F(I)$ defined over a faithful $F(I)$-module, despite the fact that $F(I)$ does not admit an anti-automorphism.

REMARK 2.9.9. Call a semiperfect ring R *basic* if R_R is a direct sum of *non-isomorphic* indecomposable projectives. Every semiperfect ring has a unique basic ring that is Morita equivalent to it (see [**58**, Prp. 18.37] and the preceding discussion). For instance, in case $R = F(I)$ for some field F and a poset I, $F(I/\sim)$ is the basic ring that is Morita equivalent to R. The proof of Theorem 2.9.3 now implies that if a semiperfect ring is Morita equivalent to its opposite, then then the basic ring which is Morita equivalent to it has an anti-automorphism. Indeed, the basic ring that is Morita equivalent to R is just $\mathrm{End}_R(P_1 \oplus \cdots \oplus P_t)$ where $P_1, \dots, P_t$ are the indecomposables in proj-R (up to isomorphism). In addition, Example 2.9.7 was based on the observation that if $F(I)$ has an involution, then so does its basic ring $F(I/\sim)$. The author believes that the this claim actually holds for other families of semiperfect rings.

2.10. Addendum

In Example 2.4.12, we have defined M to be the free monoid over $\{x_0, x_1, x_2, \dots\}$ subject to the relations:

$$x_{2k+1}x_{2k} = 1 = x_{2k+1}x_{2k+2}$$

$$x_{n+2+2k}x_{2k} = x_{2k}x_{n+2k}, \qquad x_{2k+1}x_{n+2k+3} = x_{n+2k+1}x_{2k+1}$$

for all $n, k \geq 0$. The example relies on the fact that $x_0 \neq x_2$ and this addendum is dedicated to verify that. In fact, we will solve the word problem in M and show that any element of M admits a unique canonical form.

We start by recalling the Bergman-Bokut Diamond Lemma. For proof, see [**18**].

Proposition 2.10.1 (Diamond Lemma). *Let X be a set and $\to$ a reduction relation on X satisfying:*

(i) *Any element of X can be reduced only a finite number of times.*
(ii) *If $a \to b$ and and $a \to c$ $(a, b, c \in X)$, then b and c have a common reduction $d \in X$ (i.e., $b \xrightarrow{*} d$ and $c \xrightarrow{*} d$, where $\xrightarrow{*}$ is the transitive closure of $\to$).*

Then any element of X has a unique irreducible reduction.

Let F denote the free word monoid on the letters $\{x_0, x_1, x_2, \ldots\}$. Inspired by the relations of M, we define four families of reduction rules on the words of F (denoted I,II,III and IV) given by:

I: $x_{2k+1}x_{2k} \longrightarrow 1$
II: $x_{2k+1}x_{2k+2} \longrightarrow 1$
III: $x_{n+2+2k}x_{2k} \longrightarrow x_{2k}x_{n+2k}$ ("even indices move left")
IV: $x_{2k+1}x_{n+2k+3} \longrightarrow x_{n+2k+1}x_{2k+1}$ ("odd indices move right")

(where $n, k \geq 0$.) Let $\to$ denote the union of all these reduction relations. We will now prove that the conditions of the Diamond Lemma hold.

Lemma 2.10.2. *In the previous notation:*

(i) *Any word in F can be reduced only a finite number of times.*
(ii) *Let $a \in F$. If $a \to b$ and $a \to c$ then b and c has a common subreduction.*

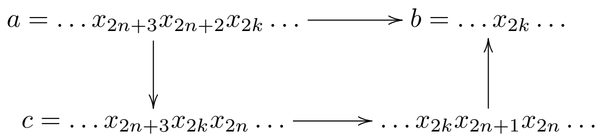
Proof. (i) Each reduction decreases the sum of the indices of the letters in the word. Therefore, only finitely many reductions can be applied on a given word.

(ii) Let a, b, c be given. Denote by t_1 and t_2 the type of the reductions $a \to b$ and $a \to c$ respectively (recall that there are four such types: I,II,III and IV). We now split into cases, checking separately all possible pairs (t_1, t_2). By symmetry, we may assume $t_1 \leq t_2$. We may also assume that the letters exchanged in the reduction $a \to b$ overlap those exchanged in $a \to c$. (Otherwise, it clear that b and c have a common reduction.) We also ignore the case $b = c$ for obvious reasons.

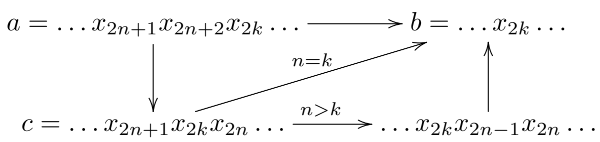
In the rest of the proof, our notation will consist of diagrams of reductions. To avoid extra notation, a condition on an arrow means that the reduction is valid (only) when the condition holds. We let $*$ be the anti-endomorphism of F obtained by sending x_n to x_{n+1}. Observe that it preserves the reduction relation. We will always assume m, n, k are non-negative integers.

(t_1, t_2) is one of (I,I), (I,II), (II,II): Here either $b = c$ or the letters exchanged in a do not overlap.

(t_1, t_2)=(I,III): The case is resolved as in the following diagram ($n \geq k$):

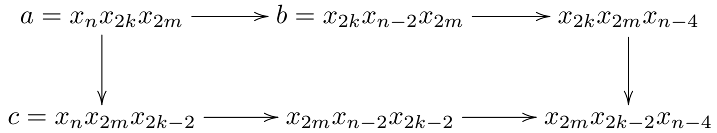


(t_1, t_2)=(II, III): ($n \geq k$):

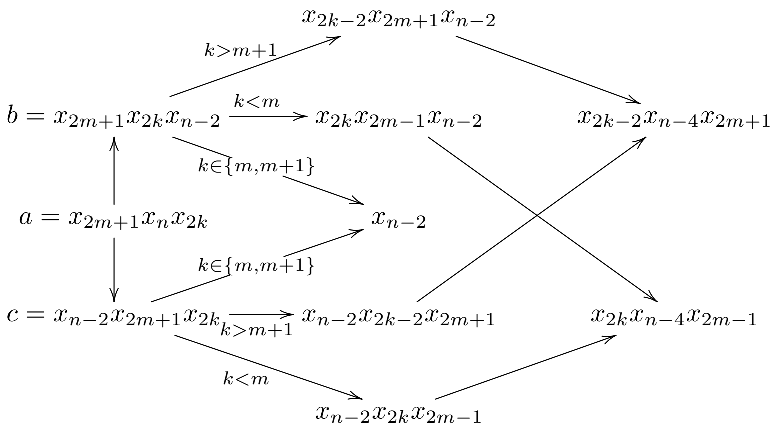


(t_1, t_2)=(I,IV): Apply $*$ on the diagram of case (t_1, t_2) =(II,III).
(t_1, t_2)=(II,IV): Apply $*$ on the diagram of case (t_1, t_2) =(I,III).

(t_1, t_2)=(III,III): ($n \geq 2k+2 \geq 2m+4$; we removed all the dots; it suffices to check the overlaps.)



(t_1, t_2)=(IV,IV): Apply $*$ on the diagram of case (t_1, t_2) =(III,III).
(t_1, t_2)=(III,IV): ($n \geq 2m+3, 2k+2$; Again, we removed the dots.)



We conclude that in all cases, b and c have a common reduction. □

We can now assert by the Diamond Lemma that any element of F has a unique irreducible reduction w.r.t. $\to$. This implies that two words in F are equal in M if and only if the have the same irreducible reduction. (This solves the word problem in M.) In particular, $x_0 \neq x_2$ in M since both words are irreducible.

CHAPTER 3

Bilinear Forms and Anti-Endomorphisms

The following theorem is a classical result about bilinear forms over fields that lies at the heart of the connection between quadratic forms and involutions; for proof and generalizations see [**57**, Ch. 1].

THEOREM 3.0.1. *Let F be a field and let V be a f.d. vector space. Then there is a one-to-one correspondence between regular bilinear forms $b : V \times V \to F$, considered up to scalar multiplication, and anti-automorphisms of* $\mathrm{End}_F(V)$ *preserving F. The correspondence is given by sending each form b to its corresponding anti-automorphism $*$, i.e. the anti-automorphism satisfying*

$$b(\sigma x, y) = b(x, \sigma^* y) \qquad \forall\ x, y \in V,\ \sigma \in \mathrm{End}_F(V)\ .$$

Moreover, under this correspondence, symmetric and anti-symmetric forms correspond to orthogonal and symplectic involutions, respectively.

Our goal in this chapter is to generalize Theorem 3.0.1 to bilinear forms over rings, as defined in the previous chapter. That is, we would like to show that the map sending a right regular bilinear form to its corresponding anti-endomorphism induces a one-to-one correspondence between the right regular bilinear forms on a given module $M \in \text{Mod-}R$, considered up to a certain equivalence, and the anti-endomorphisms of $\mathrm{End}(M)$. We will show that: (1) the correspondence fails over arbitrary rings, and in particular over f.d. algebras, (2) under mild assumptions on the module M (e.g. being finite projective) or on the base ring R, the correspondence holds in its original setting and (3) in some cases the correspondence holds under a slight adjustment, namely the anti-endomorphisms correspond to certain right stable bilinear forms rather than to right regular bilinear forms. For example, when R is a semiprime Goldie ring (e.g. an semiprime noetherian ring), the adjusted correspondence holds when M is a f.g., faithful and torsion-free.

Byproducts of the work include several results about *general rings of quotients* and *pseudo-Frobenius* (abbrev.: PF) rings, such as:

(1) Let R be a ring and let S be a (two-sided) *denominator set* such that RS^{-1} is right PF ring. Then for any faithful f.g. S-torsion-free $M \in \text{Mod-}R$, $\mathrm{End}(MS^{-1})$ is the *maximal symmetric quotient ring* of $\mathrm{End}(M)$.
(2) For any faithful right module M over a right PF ring, $\mathrm{End}(M)$ coincides with its maximal symmetric ring of quotients.
(3) Suppose R is an *Ore domain* and D is the division ring of fractions of R. Then $\mathrm{End}(M \otimes_R D)$ is the (two-sided) *classical fractions ring* of $\mathrm{End}(M)$ for any torsion-free f.g. $M \in \text{Mod-}R$.

In addition, the adjusted correspondence holds for M in each of these cases.

The contents of each section are described at the end of section 3.1, which also serves as a preface. *Some results of this chapter can also be found at* [**39**].

3.1. The Correspondence

Let R be a ring and let M be a fixed right R-module. Set $W = \mathrm{End}_R(M)$ and let $\mathrm{End}^-(W)$ ($\mathrm{Aut}^-(W)$) denote the set of anti-endomorphisms (anti-automorphisms) of W. Recall that a bilinear space (M,b,K) is *right regular* if $\mathrm{Ad}_b^r : M \to M^{[1]}$ is bijective (see section 2.1 for all relevant definitions). In this case b is *right stable*, i.e. for every $w \in W$, there exists unique $w' \in W$ such that

$$b(wx,y) = b(x,w^\alpha y) \qquad \forall x,y \in M \ .$$

We denote the map $w \mapsto w'$ by $\alpha = \alpha(b)$. It is routine to verify that it lies in $\mathrm{End}^-(W)$. Furthermore, if b is κ-symmetric w.r.t. to some involution κ of K, then α is an involution. Indeed,

$$b(x,wy) = b(wy,x)^\kappa = b(y,w^\alpha x)^\kappa = b(w^\alpha x,y) = b(x,w^{\alpha\alpha} y)$$

for all $x,y \in M$ and $w \in W$, so the right-stability of b implies $w = w^{\alpha\alpha}$ (see Proposition 2.2.9).

Denote by $\mathrm{Bil}_{\mathrm{reg}}(M)$ the *class* of all right regular bilinear forms over M (the forms can take values in any double R-module, hence this is not a set; we will soon make this class into a category). In this section, we will explain in detail how to make the map $b \mapsto \alpha(b) : \mathrm{Bil}_{\mathrm{reg}}(M) \to \mathrm{End}^-(W)$ into the ideal correspondence described in the preamble.

Our first step is to introduce an "inverse" to $b \mapsto \alpha(b)$. While this requires most of the work in Theorem 3.0.1 and its generalizations (using tools such as the Skolem-Noether Theorem), our new notion of bilinear forms allows an easy and explicit construction of such an inverse.

Let $\alpha \in \mathrm{End}^-(W)$ and let A, B be two *left* W-modules. Define:

$$A \otimes_\alpha B = \frac{A \otimes_{\mathbb{Z}} B}{\langle wa \otimes b - a \otimes w^\alpha b \,|\, a \in A, b \in B, w \in W \rangle} \ .$$

For $a \in A$ and $b \in B$, we let $a \otimes_\alpha b$ denote the image of $a \otimes_{\mathbb{Z}} b$ in $A \otimes_\alpha B$ (the subscript α will be dropped when obvious from the context).

Remark 3.1.1. For any $B \in W\text{-Mod}$ and $\alpha \in \mathrm{End}^-(W)$, let B^α denote the right W-module obtained by twisting B via α. Namely, $B^\alpha = B$ as sets, but B^α is equipped with a right action $\diamond_\alpha : B \times W \to B$ given by $x \diamond_\alpha w = w^\alpha x$ for all $x \in B$ and $w \in W$. Then the abelian group $A \otimes_\alpha B$ can be identified with $B^\alpha \otimes_W A$. Therefore, $\otimes_\alpha$ is an additive bifunctor and $W^n \otimes_\alpha B \cong B^n$.

Now consider M as a left W-module and let $\alpha \in \mathrm{End}^-(W)$. Define $K_\alpha = M \otimes_\alpha M$ and note that K_α is a double R-module w.r.t. the operations

$$(x \otimes_\alpha y) \odot_0 r = xr \otimes_\alpha y \qquad \text{and} \qquad (x \otimes_\alpha y) \odot_1 r = x \otimes_\alpha yr$$

($x,y \in M$, $r \in R$). It is now clear that the map $b_\alpha : M \times M \to K_\alpha$ defined by $b_\alpha(x,y) = x \otimes_\alpha y$ is a bilinear form and

$$b_\alpha(wx,y) = wx \otimes_\alpha y = x \otimes_\alpha w^\alpha y = b_\alpha(x, w^\alpha y) \tag{12}$$

for all $x,y \in M$ and $w \in W$, hence $\alpha(b_\alpha) = \alpha$, provided b_α is right regular. In fact, the pair (b_α, K_α) is universal w.r.t. satisfying (12) in sense that if $b : M \times M \to K$ is another bilinear form satisfying (12), then there is a unique double R-module homomorphism $f : K_\alpha \to K$ such that $b = f \circ b_\alpha$. Moreover, assume α is an involution. Then K_α admits an involution κ_α given by $x \otimes y \mapsto y \otimes x$ and b_α is κ_α-symmetric, so every involution corresponds to a symmetric form!

Example 3.1.2. Let F be a field and let α be an anti-automorphism of $\mathrm{M}_n(F) \cong \mathrm{End}(F^n)$ preserving F. We will show below that K_α is just F with $\odot_0$ and $\odot_1$ being the standard action of F on itself. Moreover, if α is an involution, then $\kappa_\alpha = \mathrm{id}_F$ if α is orthogonal and $\kappa_\alpha = -\,\mathrm{id}_F$ if α is symplectic.

Let us summarize what we have done so far: For every bilinear form $b \in \mathrm{Bil}_{\mathrm{reg}}(M)$, we have defined the anti-endomorphism $\alpha = \alpha(b) \in \mathrm{End}^-(W)$ to be the unique anti-endomorphism of W satisfying.

$$b(wx, y) = b(x, w^\alpha y) \qquad \forall x, y \in M,\, w \in W$$

(namely, α is the corresponding anti-endomorphism of b). In addition, for every $\alpha \in \mathrm{End}^-(W)$, we constructed a bilinear space (M, b_α, K_α) and showed that it is universal w.r.t. satisfying (12). The maps $b \mapsto \alpha(b)$ and $\alpha \mapsto b_\alpha$ will induce, after suitable adjustments, the desired correspondence.

Our next step is to define some equivalence relation on $\mathrm{Bil}_{\mathrm{reg}}(M)$. Call two bilinear forms $b : M \times M \to K$ and $b' : M \times M \to K'$ *similar* if there is an isomorphism $f \in \mathrm{Hom}_{\mathrm{DMod}\text{-}R}(K, K')$ such that $b' = f \circ b$. In this case, f is called a *similarity* from b to b' and we write $b \sim b'$. The class $\mathrm{Bil}_{\mathrm{reg}}(M)$ can be made into a category by taking the similarities as morphisms, and we let $\mathrm{Iso}(\mathrm{Bil}_{\mathrm{reg}}(M))$ denote its isomorphism classes. Clearly any two similar right regular forms b, b' satisfy $\alpha(b) = \alpha(b')$.

EXAMPLE 3.1.3. Let F be a field and let V be a f.d. F-vector space. Then two bilinear forms $b, b' : V \times V \to F$ are similar if and only if they are the same up to (non-zero) scalar multiplication.

We conclude the previous paragraphs by stating that we would like to have a 1-1 correspondence as in the following diagram

$$\text{(13)} \qquad \mathrm{Iso}(\mathrm{Bil}_{\mathrm{reg}}(M)) \underset{\alpha \mapsto b_\alpha}{\overset{b \mapsto \alpha(b)}{\rightleftarrows}} \mathrm{End}^-(W) \quad .$$

One can easily verify that this description agrees with the correspondence of Theorem 3.0.1.

However, it turns out that the correspondence in (13) fails in general, and for two gaps, which the reader might have already spotted:

(a) b_α is not always right regular (e.g. see Example 3.4.2 below).
(b) $b_{\alpha(b)}$ need not be similar to b, even when both b and $b_{\alpha(b)}$ are regular (see Example 3.4.8).

In addition, it is still open whether that b is right regular implies that so is $b_{\alpha(b)}$. We note that the problems (a) and (b) occur even when considering bilinear forms over f.d. algebras.

REMARK 3.1.4. For a bilinear space (M, b, K), let $\mathrm{im}(b)$ denote the additive group spanned by $\{b(x, y) \,|\, x, y \in M\}$.[1] It is easy to see that $\mathrm{im}(b)$ is a sub-double-R-module of K. We will say b is *onto* if $\mathrm{im}(b) = K$. It might look as if problem (b) would be solved if we insisted on considering only forms that are onto, but this is not the case. The forms constructed in Example 3.4.8 are onto, thus demonstrating that problem (b) is inherent.

Problem (a) is solved when restricting to special cases, e.g. when M is finite projective or a generator (see section 3.5). However, these cases are not so common. Another way to approach (a) is to replace $\mathrm{Bil}_{\mathrm{reg}}(M)$ with $\mathrm{Bil}_{\mathrm{st}}(M)$ in (13), where $\mathrm{Bil}_{\mathrm{st}}(M)$ is the category of *right stable* bilinear forms on M (with similarities as morphisms). Note that b_α is right stable if and only if it is right semi-stable (see section 2.1 for definitions). In particular, if b_α is right injective, then it is right stable (in contrast to arbitrary right injective forms; e.g. Example 2.4.9). We will

[1] Caution: in general $\mathrm{im}(b)$ is not the image of b in the usual sense.

show below that this adjustment is indeed crucial sometimes. In particular, there are anti-automorphisms α such that b_α is right stable but not right regular (and not even right injective; see Example 3.4.5).

While extending the domain of $b \mapsto \alpha(b)$ to $\mathrm{Bil}_{\mathrm{st}}(M)$ only worsens problem (b), it turns out that it can be solved completely by restricting the domain of $b \mapsto \alpha(b)$ to the image of $\alpha \mapsto b_\alpha$ (up to similarity). This calls for the following definition:

DEFINITION 3.1.5. *A bilinear form $b : M \times M \to K$ is called* generic *if it is right stable and b is similar to $b_{\alpha(b)}$.*[2]

Since $\alpha(b_\alpha) = \alpha$ (provided b_α is right stable), b_α is always generic, and by definition, any generic form is obtained this way, up to similarity. As implied from our previous comments, generic does not imply right regular nor does right regular imply generic (unless special assumptions are made on the module M).

PROPOSITION 3.1.6. *Let (M, b, K) and (M, b', K') be two right stable bilinear spaces. Then:*

(i) *If b and b' are generic, then $\alpha(b) = \alpha(b')$ implies $b \sim b'$.*
(ii) *If b is generic, then it is onto (in the sense of Remark 3.1.4).*
(iii) *If b is generic and $\alpha(b) = \alpha(b')$, then there exists a unique double R-module homomorphism f such that $b' = f \circ b$.*
(iv) *If b is generic and $\alpha(b)$ is an involution, then K has an involution κ and b is κ-symmetric.*[3]
(v) *$b_{\alpha(b)}$ is generic. (In particular, b_α is right stable.)*

PROOF. (i) By definition, $b \sim b_{\alpha(b)} \sim b_{\alpha(b')} \sim b'$.

(ii) Clearly $b_{\alpha(b)}$ is onto and since being onto is preserved under similarity, b is onto.

(iii) The universal property of $b_{\alpha(b)}$ implies that there is a unique double R-module homomorphism $g : K_{\alpha(b)} \to K'$ such that $b' = g \circ b$ (define $g(x \otimes_{\alpha(b)} y) = b'(x, y)$ for all $x, y \in M$). Let h be a similarity from b to $b_{\alpha(b)}$. Then $f = g \circ h$ is the required morphism. The uniqueness of f is easy to prove and is left to the reader.

(iv) We can identify K with K_α and b with $b_{\alpha(b)}$. Then $b_{\alpha(b)}$ is $\kappa_{\alpha(b)}$-symmetric, as explained above.

(v) We only need to check that $b_{\alpha(b)}$ is right semi-stable. By the universal property of $b_{\alpha(b)}$ there is $f \in \mathrm{Hom}_{\mathrm{DMod}\text{-}R}(K_{\alpha(b)}, K)$ such that $b = f \circ b_{b(\alpha)}$. We are now done by the following Lemma. □

LEMMA 3.1.7. *Let (M, b, K) and (M, b', K') be two bilinear spaces and let $f \in \mathrm{Hom}_{\mathrm{DMod}\text{-}R}(K, K')$ such that $b' = f \circ b$. If b' is right (left) semi-stable then so is b.*

PROOF. We treat only the right case. Assume $\sigma \in \mathrm{End}(M)$ is such that $b(x, \sigma y) = 0$ for all $x, y \in M$. By applying f on both sides we get $b'(x, \sigma y) = 0$, hence $\sigma = 0$, as required. □

Let $\mathrm{Bil}_{\mathrm{gen}}(M)$ stand for the category of generic bilinear forms over M with similarities as morphisms. Then the last proposition implies:

---

[2] Generic forms as defined now should have been called *right* generic. However, we will not consider *left* generic forms in this chapter. Moreover, we shall see in section 3.9 below that the left and right definitions can be united into a left-right symmetric definition.

[3] If b is not generic, then this is false even when b is regular; see Example 3.4.8 below. Example 2.4.5 above already demonstrates that in case b is only right regular.

Corollary 3.1.8. *In the previous notation, provided b_α is right stable for all $\alpha \in \mathrm{End}^-(W)$, there is a one-to-one correspondence:*

$$\mathrm{Iso}(\mathrm{Bil}_{\mathrm{gen}}(M)) \underset{\alpha \mapsto b_\alpha}{\overset{b \mapsto \alpha(b)}{\rightleftarrows}} \mathrm{End}^-(W) \quad . \tag{14}$$

Note that Proposition 3.1.6(v) implies that any right stable form b can be turned into a generic form by replacing it with $b_{\alpha(b)}$. This process is called *generization* and it is a useful tool for studying bilinear forms.

Remark 3.1.9. Call two right stable bilinear forms *weakly similar* (denoted $\sim_w$) if they have similar generizations. Then under the assumptions of Corollary 3.1.8, there is a one-to-one correspondence between $\mathrm{Bil}_{\mathrm{st}}(M)/\sim_w$ and $\mathrm{End}^-(W)$. However, we could not find a natural way to make $\mathrm{Bil}_{\mathrm{st}}(M)$ into a category whose isomorphism classes are the equivalence classes of $\sim_w$, i.e. defining *weak similarities*. As a thumb rule, a definition of weak similarities would be appropriate if it applied to arbitrary bilinear forms, rather than just right stable forms.

In section 3.2 we present some of the basic properties of b_α, such as when it admits an asymmetry. We also show that provided b_α and b_β are regular, $K_\alpha \cong K_\beta$ if and only if $\alpha \circ \beta^{-1}$ is an inner automorphism. Section 3.3 explains how the map $b \mapsto b_\alpha$ interacts with orthogonal sums. (Namely, assume $e \in \mathrm{E}(W)$ is such that $e^\alpha = e$. Then $\alpha_1 := \alpha|_{eWe} \in \mathrm{End}^-(eWe) = \mathrm{End}^-(\mathrm{End}_R(eM))$ and we can from $b_{\alpha_1} : eM \times eM \to K_{\alpha_1}$. How does b_{α_1} relate to b_α?) The results obtained are used to give an explicit description of K_α in case M is a generator. Section 3.4 present various examples. In particular, problems (a) and (b) are demonstrated. In section 3.5, we provide sufficient conditions for b_α to be right regular. For example, b_α is right regular when M is a finite projective and regular when M is a generator and $\alpha \in \mathrm{Aut}^- W$. Sections 3.6 and 3.7 present conditions that insure b_α is right injective, e.g. those described in the preamble. In addition, in section 3.7 we obtain several results about general quotient rings and right PF rings that are of interest in their own right (e.g. Theorem 3.7.10, Corollary 3.7.22). In section 3.9, we show how to generalize the generization process to non-stable bilinear forms.

We note that most of our results about regularity or injectivity of b_α assume α is an anti-*auto*morphism; we will usually get a one-to-one correspondence between (left and right) stable generic forms, considered up to similarity, and anti-automorphisms of W.

3.2. Basic Properties

Let R, M and W be as in the previous section and let $\alpha \in \mathrm{End}^-(W)$. In this section we present some basic properties of K_α and b_α. In particular, we discuss when is $K_\alpha \cong K_\beta$ (for $\beta \in \mathrm{End}^-(W)$) and when b_α has an asymmetry. Throughout, $\mathrm{Inn}(W)$ denotes the group of inner automorphisms of W (i.e. those given by conjugation with an invertible element of W).

Proposition 3.2.1. *Let $\alpha \in \mathrm{End}^-(W)$ and assume b_α is right stable. Then b_α is left stable (semi-stable) $\iff$ α is bijective (injective).*

Proof. This follows from Proposition 2.3.4. □

Proposition 3.2.2. *Let $\alpha \in \mathrm{End}^-(W)$ and assume there exists $\lambda \in W$ such that $w^{\alpha\alpha}\lambda = \lambda w$ for all $w \in W$ and $\lambda^\alpha \lambda \in W^\times$ (e.g. if $\alpha^2 \in \mathrm{Inn}(W)$). Then the map $\kappa : K_\alpha \to K_\alpha$ defined by $(x \otimes_\alpha y)^\kappa = y \otimes_\alpha \lambda x$ is well-defined, it is an anti-isomorphism of K_α and λ is a right κ-asymmetry of b_α. Moreover, if $\lambda^\alpha\lambda = 1$,* *then κ is an involution. Conversely, if b_α is right regular and K_α has an anti-isomorphism (or involution) κ, then there exists $\lambda \in W$ as above and κ is induced from λ.*

PROOF. This is similar to the proof of Proposition 2.4.1(ii) (which is actually a special case of this proposition — take $M = R_R$). Nevertheless, we will repeat the argument since there are additional details to be added. Throughout, $w \in W$, $r \in R$ and $x, y \in M$.

The map κ is well-defined since

$$(wx \otimes_\alpha y)^\kappa = y \otimes_\alpha \lambda wx = y \otimes_\alpha w^{\alpha\alpha}\lambda x = w^\alpha y \otimes_\alpha \lambda x = (x \otimes_\alpha w^\alpha y)^\kappa \ .$$

To see that κ is invertible, it is enough to check that κ^2 is invertible. This holds since

$$(x \otimes_\alpha y)^{\kappa\kappa} = \lambda x \otimes_\alpha \lambda y = x \otimes_\alpha \lambda^\alpha \lambda y \ ,$$

and the map $x \otimes_\alpha y \mapsto x \otimes_\alpha \lambda^\alpha \lambda y$ has an inverse given by $x \otimes_\alpha y \mapsto x \otimes_\alpha (\lambda^\alpha\lambda)^{-1} y$. (The latter is well-defined since $(\lambda^\alpha\lambda)^{-1}$ commutes with $\mathrm{im}(\alpha)$; see Remark 2.4.2.) That $(k \odot_i r)^\kappa = k^\kappa \odot_{1-i} r$ for all $k \in K$ is straightforward and hence κ is an anti-isomorphism. In addition, the last equation also implies that κ is an involution if $\lambda^\alpha\lambda = 1$. That λ is a right κ-asymmetry of b_α is routine.

If b_α is right regular and K_α has an anti-isomorphism κ, then b_α has a right κ-asymmetry λ, and by Proposition 2.3.9(i) and Lemma 2.3.12, λ satisfy all the requirements. The anti-isomorphism κ is necessarily induced from λ because

$$(x \otimes_\alpha y)^\kappa = b_\alpha(x, y)^\kappa = b_\alpha(y, \lambda x) = y \otimes_\alpha \lambda x \ . \qquad \square$$

We do not know if the second part of the last proposition holds under the weaker assumption that b_α is right stable.

COROLLARY 3.2.3. *If $\alpha \in \mathrm{End}^-(W)$ and α^2 is inner, then b_α is right regular if and only if b_α is left regular.*

PROOF. Proposition 3.2.2 implies K_α has an involution. In addition, α is bijective (since α^2 is). Therefore, we are done by Proposition 2.3.13. $\square$

PROPOSITION 3.2.4. *Let $\alpha \in \mathrm{End}^-(W)$ and $\varphi \in \mathrm{Inn}(W)$. Then $K_\alpha \cong K_{\varphi\circ\alpha}$ as double R-modules. Conversely, if $\alpha, \beta \in \mathrm{End}^-(W)$ are such that b_α and b_β are right regular and $K_\alpha \cong K_\beta$ as double R-modules, then there exists $\varphi \in \mathrm{Inn}(W)$ such that $\beta = \varphi \circ \alpha$.*

PROOF. Let $u \in W^\times$ be such that $\varphi(w) = u^{-1}wu$ for all $w \in W$. Define $f : K_\alpha \to K_{\alpha\circ\varphi}$ by $f(x \otimes_\alpha y) = x \otimes_{\varphi\circ\alpha} uy$. Then f is well-defined since

$$f(wx \otimes_\alpha y) = wx \otimes_{\varphi\circ\alpha} uy = x \otimes_{\varphi\circ\alpha} (uw^\alpha u^{-1})uy = x \otimes_{\varphi\circ\alpha} uw^\alpha y = f(x \otimes w^\alpha y)\,,$$

and it is easy to see that f is an isomorphism of double R-modules (its inverse is given by $x \otimes_{\varphi\circ\alpha} y \mapsto x \otimes_\alpha u^{-1}y$). Therefore, $K_\alpha \cong K_{\varphi\circ\alpha}$.

To prove the second part of the proposition, it is enough to show that if $b, c : M \times M \to K$ are two right regular bilinear forms, with corresponding anti-endomorphisms α and β, then there exists $\varphi \in \mathrm{Inn}(W)$ s.t. $\beta = \varphi \circ \alpha$. Indeed, define $u = (\mathrm{Ad}^r_b)^{-1} \circ \mathrm{Ad}^r_c \in W^\times$. Then for all $x, y \in M$, $c(x, y) = (\mathrm{Ad}^r_c y)x = (\mathrm{Ad}^r_b(uy))x = b(x, uy)$. Therefore, for all $w \in W$:

$$c(x, w^\beta y) = c(wx, y) = b(wx, uy) = b(x, w^\alpha uy) = c(x, u^{-1}w^\alpha uy)$$

and it follows that $w^\beta = u^{-1}w^\alpha u$, as required. $\square$

We finish this section by presenting a *left* analogue of b_α. Assume $A, B \in W$-Mod. In section 3.1 we have defined $A \otimes_\alpha B$ and we now similarly define

$$A_\alpha\!\otimes B = \frac{A \otimes_{\mathbb{Z}} B}{\langle a \otimes wb - w^\alpha a \otimes b \,|\, a \in A, b \in B, w \in W \rangle} \ .$$

In addition, we define ${}_\alpha K = M_\alpha\!\otimes M$ and ${}_\alpha b : M \times M \to {}_\alpha K$ by $b(x,y) = x_\alpha\!\otimes y$. All the results of this chapter have *left versions* obtained by replacing b_α, K_α with ${}_\alpha b$, ${}_\alpha K$ and every right property with its left analogue.

We also note that if α is bijective, then $A \otimes_\alpha B$ is naturally isomorphic to $A_{\alpha^{-1}}\!\otimes B$ (via $x \otimes_\alpha y \leftrightarrow x_{\alpha^{-1}}\!\otimes y$) and b_α is similar to ${}_{\alpha^{-1}}b$, hence both right and left versions of our results apply. We will use freely the fact that $A \otimes_\alpha B \cong B^\alpha \otimes_W A$ and $A_\alpha\!\otimes B \cong A^\alpha \otimes_W B$. (Recall that for $A \in W$-Mod and $\alpha \in W$-Mod, A^α denotes the right W-module obtained by twisting A via α; see Remark 3.1.1.)

3.3. Relation to Orthogonal Sums

Let R, M and W be as in the previous section and let $\alpha \in \mathrm{End}^-(W)$. In this section, we shall examine how the map $\alpha \mapsto b_\alpha$ interacts with orthogonal sums. We shall then use our results to describe K_α explicitly in case M is a generator of Mod-R. This in turn is then used to show that b_α is regular for all $\alpha \in \mathrm{Aut}^- W$ (provided M is an R-generator) and to justify the assertions made in Example 3.1.2.

Let $\alpha \in \mathrm{End}^-(W)$ and assume there are orthogonal idempotents $e_1, \dots, e_t \in W$ such that $1_W = \sum e_i$ and $e_i^\alpha = e_i$ for all i. Then $\alpha_i := \alpha|_{e_iWe_i}$ is an anti-endomorphism of e_iWe_i, hence we can form $b_{\alpha_i} : e_iM \times e_iM \to K_{\alpha_i}$. It is now natural to ask what is the connection between b_α, K_α and $\{b_{\alpha_i}, K_{\alpha_i}\}_{i=1}^t$.

To make this less obscure, let $M_i = e_iM \in$ Mod-R. Then $M = \bigoplus_{i=1}^t M_i$ and for all $i \neq j$:

$$b_\alpha(M_i, M_j) = b_\alpha(e_iM_i, M_j) = b_\alpha(M_i, e_i^\alpha M_j) = b_\alpha(M_i, e_iM_j) = b_\alpha(M_i, 0) = 0 \ ,$$

hence $b_\alpha = b_1 \perp \cdots \perp b_t$ where $b_i = b_\alpha|_{M_i \times M_i}$. As clearly $b_i(wx, y) = b_i(x, w^{\alpha_i}y)$ for all $w \in e_iWe_i$ and $x, y \in M_i$, there is a unique double R-module homomorphism $f_i : K_{\alpha_i} \to K_\alpha$ such that $b_i = f_i \circ b_{\alpha_i}$. It is given by $f_i(x \otimes_{\alpha_i} y) = x \otimes_\alpha y$. Our question thus becomes whether f_i is an isomorphism, or at least injective. In general, the answer is "no" (even when all forms involved are right regular). However, in special cases, a positive answer can be guaranteed.

Example 3.3.1. The maps f_i are neither injective nor surjective in general: Let F be a field, let R be the ring of upper-triangular 2×2 matrices over F and let $M = R_R$. We identify $\mathrm{End}(M_R) = \mathrm{End}(R_R)$ with R in the standard way. Define $\alpha : R \to R$ by $\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]^\alpha = \left[\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right]$ and let $e_i = e_{ii}$ (where $\{e_{ij}\}$ are the standard matrix units). Then α is an anti-endomorphism satisfying $e_i^\alpha = e_i$ for $i = 1, 2$. Define M_i, α_i, b_i and f_i as above. We shall now compute f_1 and f_2 explicitly.

Firstly, we claim $K_\alpha \cong R$ via $x \otimes_\alpha y \mapsto x^\alpha y$ where the double R-module structure on R is given by $x \odot_0 r = r^\alpha x$ and $x \odot_1 r = xr$ for all $x, r \in R$. This is easily seen once noting $({}_RR)^\alpha \otimes_R ({}_RR) \cong R$ via $x \otimes_R y \mapsto x \diamond_\alpha y = y^\alpha x$. Next, make $K := \mathrm{M}_2(F)$ into a double R-module by defining $x \odot_0 r = r^T x$ and $x \odot_1 r = xr$ (where r^T is the transpose of $r \in R$). It is easy to see that the map $K_{\alpha_i} \to K$ given by $x \otimes_{\alpha_i} y \mapsto x^T y$ is an injection of double R-modules. (Indeed, $\alpha_i = \mathrm{id}_{e_iRe_i}$ and $e_iRe_i \cong F$, hence $K_{\alpha_i} = M_i \otimes_{\alpha_i} M_i \cong M_i \otimes_F M_i$ via $x \otimes_\alpha y \mapsto x \otimes_F y$ and $M_i \otimes_F M_i$ embeds in $\mathrm{M}_2(F)$ via $x \otimes_F y \mapsto x^T y$.) We can thus identify K_{α_1} with K

and K_{α_2} with $\{[\begin{smallmatrix}0&0\\0&a\end{smallmatrix}]\in K \mid a\in F\}\subseteq K$. The isomorphisms are given by:

$$\begin{bmatrix} a & b\\ 0 & 0\end{bmatrix}\otimes_{\alpha_1}\begin{bmatrix} a' & b'\\ 0 & 0\end{bmatrix} \quad\mapsto\quad \begin{bmatrix} aa' & ab'\\ ba' & bb'\end{bmatrix}$$
$$\begin{bmatrix} 0 & 0\\ 0 & c\end{bmatrix}\otimes_{\alpha_2}\begin{bmatrix} 0 & 0\\ 0 & c'\end{bmatrix} \quad\mapsto\quad \begin{bmatrix} 0 & 0\\ 0 & cc'\end{bmatrix}$$

This allows us to compute f_1 and f_2 explicitly — under the previous identifications they are given by:

$$f_1\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right) = \begin{bmatrix} a & b\\ 0 & 0\end{bmatrix}$$
$$f_2\left(\begin{bmatrix} 0 & 0\\ 0 & x\end{bmatrix}\right) = \begin{bmatrix} 0 & 0\\ 0 & x\end{bmatrix}$$

(Indeed, the first formula easily follows from $f_1([\begin{smallmatrix}aa'&ab'\\ba'&bb'\end{smallmatrix}]) = f_1(b_1([\begin{smallmatrix}a&b\\0&0\end{smallmatrix}],[\begin{smallmatrix}a'&b'\\0&0\end{smallmatrix}])) = b_\alpha([\begin{smallmatrix}a&b\\0&0\end{smallmatrix}],[\begin{smallmatrix}a'&b'\\0&0\end{smallmatrix}]) = [\begin{smallmatrix}a&b\\0&0\end{smallmatrix}]^\alpha[\begin{smallmatrix}a'&b'\\0&0\end{smallmatrix}] = [\begin{smallmatrix}aa'&ab'\\0&0\end{smallmatrix}]$ and the second is shown via similar computation.) In particular, f_1 is neither injective nor surjective and f_2 is not surjective. Note that b_1, b_2, b_{α_1}, b_{α_2} and b_α are all right regular. This easy fact is left to the reader. (Alternatively, that b_{α_1}, b_{α_2} and b_α are right regular follows from Theorem 3.5.5 below, because M, M_1 and M_2 are finite projective, and b_1, b_2 are right regular because they are summands of b_α, see Proposition 2.6.2(i).)

LEMMA 3.3.2. *Let $N\in$ Mod-W, $M\in W$-Mod and let $e\in \mathrm{E}(W)$. Define $\varphi: Ne\otimes_{eWe} eM\to N\otimes_W M$ by $x\otimes_{eWe} y\mapsto x\otimes_W y$. Then:*
- (i) *$WeM = M \Longrightarrow \varphi$ is onto.*
- (ii) *$WeW = W \Longrightarrow \varphi$ is an isomorphism.*

PROOF. (i) Let $x\in N$, $y\in M$. Then there is $y'\in M$ and $w\in W$ such that $y = wey'$. Thus, $x\otimes_W y = x\otimes_W wey' = xwe\otimes_W ey' = \varphi(xwe\otimes_{eWe} ey')$, so φ is onto.

(ii) Write $1_W = \sum_i u_iu_i'$ where $u_1,\dots,u_t\in We$ and $u_1',\dots,u_t'\in eW$ and define $\psi: N\otimes_W M\to Ne\otimes_{eWe} eM$ by $\psi(x\otimes_W y) = \sum_i xu_i\otimes_{eWe} u_i'y$. Then ψ is well-defined because

$$\begin{aligned}\psi(xw\otimes_W y) &= \sum_i xwu_i\otimes_{eWe} u_i'y = \sum_{i,j} xu_ju_j'wu_i\otimes_{eWe} u_i'y\\ &= \sum_{i,j} xu_j\otimes_{eWe} u_j'wu_iu_i'y = \sum_j xu_j\otimes_{eWe} u_j'wy = \psi(x\otimes_W wy)\,,\end{aligned}$$

and it is straightforward to check that $\psi = \varphi^{-1}$. □

REMARK 3.3.3. An idempotent $e\in\mathrm{E}(W)$ satisfying $WeW = W$ is called *full*. This condition is equivalent to eW_W (or ${}_WWe$) being a progenerator (so eWe is Morita equivalent to W in this case).

PROPOSITION 3.3.4. *In the notation prior to Example 3.3.1:*
- (i) *$K_\alpha = \sum f_i(K_{\alpha_i})$.*
- (ii) *If b_α is right stable, then so is b_{α_i}.*
- (iii) *If $WM_i = M$, then f_i is onto.*
- (iv) *If e_i is full (i.e. $We_iW = W$), then f_i is an isomorphism.*

PROOF. (i) It is enough to prove $x\otimes_\alpha y\in\sum f_i(K_{\alpha_i})$ for all $x,y\in M$. Indeed, $x\otimes_\alpha y = \sum_{i,j} e_ix\otimes_\alpha e_jy = \sum_{i,j} e_ix\otimes_\alpha e_i^\alpha e_jy = \sum_i e_ix\otimes_\alpha e_iy = \sum_i f_i(e_ix\otimes_{\alpha_i} e_iy)$.

(ii) It is enough to prove b_{α_i} is right semi-stable. Indeed, observe that $b_\alpha = f_i\circ b_{\alpha_i}$, so this follows from the proof of Lemma 3.1.7 (note we have identified $\mathrm{End}(M_i)$ with $e_iWe_i\subseteq\mathrm{End}(M)$).

To see (iii) and (iv), let $e = e_i$. Identify K_α with $M^\alpha \otimes_W M$ and K_{α_i} with $M_i^{\alpha_i} \otimes_{eWe} M_i = M^\alpha e \otimes_{eWe} eM$. Then under these identifications, f_i is the map φ of Lemma 3.3.2, hence we are through. □

COROLLARY 3.3.5. *Let $M \in$ Mod-R, $W = \mathrm{End}(M_R)$, $\alpha \in \mathrm{End}^-(W)$ and $n \in \mathbb{N}$. Identify $\mathrm{End}(M_R^n)$ with $\mathrm{M}_n(W)$ and let $\beta \in \mathrm{End}^-(\mathrm{M}_n(W))$ be defined by*

$$\begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix}^\beta = \begin{bmatrix} w_{11}^\alpha & \dots & w_{n1}^\alpha \\ \vdots & \ddots & \vdots \\ w_{1n}^\alpha & \dots & w_{nn}^\alpha \end{bmatrix}.$$

Then $n \cdot b := \underbrace{b_\alpha \perp \dots \perp b_\alpha}_{n \text{ times}}$ is similar to b_β (and in particular, $K_\alpha \cong K_\beta$).

PROOF. Let $\{e_{ij}\}$ be the standard matrix units of $U := \mathrm{M}_n(W)$, let $\beta_i = \beta|_{e_{ii}Ue_{ii}}$ and let $\psi_i : M \to M^n$ be the embedding of M as the i-th component of M^n. Let $1 \le i \le n$. Then $g_i : K_\alpha \to K_{\beta_i}$ defined by $g_i(x \otimes_\alpha y) = \psi_i x \otimes_{\beta_i} \psi_i y$ is an isomorphism (this is straightforward), and since $Ue_{ii}U = U$, $f_i : K_{\beta_i} \to K_\beta$ defined by $f_i(x \otimes_{\beta_i} y) = x \otimes_\beta y$ is an isomorphism (Proposition 3.3.4(iii)). Thus $h := f_i \circ g_i : K_\alpha \to K_\beta$ is an isomorphism and it is independent of i since $h(x \otimes_\alpha y) = \psi_i x \otimes_\beta \psi_i y = e_{ij}e_{ji}\psi_i x \otimes_\beta \psi_i y = e_{ji}\psi_i x \otimes_\beta e_{ij}^\beta \psi_i x = \psi_j x \otimes \psi_j y$ for all $x, y \in M$. It is now routine to verify that h is a similarity from $n \cdot b_\alpha$ to b_β. □

COROLLARY 3.3.6. *Let $b : M \times M \to K$ be a bilinear form and let $n \in \mathbb{N}$. Then b is generic $\iff$ $n \cdot b$ is generic.*

PROOF. By Corollary 2.6.6, b is right stable if and only if $n \cdot b$ is right stable. Assume this holds and let α, β be the corresponding anti-endomorphisms of b, $n \cdot b$, respectively. Then it is easy to check that β is obtained from α as in Corollary 3.3.5 and hence $b_\beta \sim n \cdot b_\alpha$. Now, if b is generic then $b \sim b_\alpha$, hence $n \cdot b \sim n \cdot b_\alpha \sim b_\beta$. On the other hand, if $n \cdot b$ is generic then $n \cdot b \sim b_\beta \sim n \cdot b_\alpha$. Let $M_1 = M \times 0 \times \dots \times 0 \subseteq M^n$. Then the previous similarity induces a similarity $(n \cdot b)|_{M_1 \times M_1} \sim (n \cdot b_\alpha)|_{M_1 \times M_1}$ and this clearly implies $b \sim b_\alpha$. □

The previous corollary leads to the following question, which is still open.

QUESTION 2. *Let $b_1 : M_1 \times M_1 \to K$ and $b_2 : M_2 \times M_2 \to K$ be two* generic *bilinear forms. Is $b_1 \perp b_2$ always right stable? Provided it is, is it always generic?*

We shall now exploit Proposition 3.3.4 to provide an explicit description of K_α in case M is an R-generator.[4] We first recall the following definition.

DEFINITION 3.3.7. *Let M be a right R-module and $W = \mathrm{End}_R(M)$. The module M is called* faithfully balanced *if the standard map $R \to \mathrm{End}_W(M)$ is an isomorphism.*

EXAMPLE 3.3.8. It is well known that any generator of Mod-R is faithfully balanced (e.g., see [**80**, Exer. 4.1.14]).

Let M be a generator of Mod-R. Then R_R is a summand of M^n for some $n \in \mathbb{N}$. Let $e : M^n \to R$ be the projection from M^n to R_R. Then e is an idempotent in $\mathrm{End}(M_R^n)$ which we identify with $U := \mathrm{M}_n(\mathrm{End}(M_R)) = \mathrm{M}_n(W)$. Observe that ${}_UUe \cong {}_UM^n$ via $ue \mapsto u(1_R)$. (Here 1_R is the unity of R, considered as an element of M^n. The inverse of this isomorphism is given by $x \mapsto [y \mapsto x \cdot e(y)] \in U$.) Identify Ue with M^n. Then, $\mathrm{End}({}_UM^n) = \mathrm{End}({}_UUe) = eUe$ and since M_R^n is faithfully

[4] Recall that a module $M \in$ Mod-R is called a *generator* (or an R-generator for brevity) if for all $A, B \in$ Mod-R and $0 \neq f \in \mathrm{Hom}(A, B)$ there is $g : M \to A$ such that $f \circ g \neq 0$. This is equivalent to R_R being a summand of M^n for some $n \in \mathbb{N}$; see [**58**, §18B].

balanced (it is a generator), it follows that $R \cong eUe$ as rings, so we may assume $R = eUe$. In particular, Ue and M^n coincide as (U, R)-bimodules and $e_{ii}Ue$ is just the i-th copy of M in M^n (where $\{e_{ij}\}$ are the standard matrix units in U).

PROPOSITION 3.3.9. *Keeping the previous notation, let $\alpha \in \mathrm{End}^-(W)$ and define $\beta \in \mathrm{End}^-(U)$ as in Corollary 3.3.5. Make $e^\beta Ue$ into a double R-module by letting*

$$u \odot_0 r = r^\beta u, \qquad u \odot_1 r = ur \qquad \forall\, r \in R = eUe,\ u \in e^\beta Ue$$

and define $b : M \times M = e_{11}Ue \times e_{11}Ue \to e^\beta Ue$ by $b(x, y) = x^\beta y$. Then:

(i) *$b_\alpha \sim b$. The similarity is given by $x \otimes_\alpha y \mapsto x^\beta y$ ($x, y \in M = e_{11}Ue$).*
(ii) *Assume κ is an involution. Then, when identifying K_α with eUe^β, κ_α is just $\beta|_{e^\beta Ue}$.*

PROOF. (i) We can understand K_α as K_{β_1} where $\beta_1 = \beta|_{e_{11}Ue_{11}}$. By Proposition 3.3.4(iii), the map $f_1 : K_{\beta_1} \to K_\beta$, given by $x \otimes_{\beta_1} y \mapsto x \otimes_\beta y$, is an isomorphism (because $Ue_{11}U = U$). Consider K_β as $(Ue)^\beta \otimes_U Ue$. Then the latter is isomorphic to $(Ue)^\beta \diamond_\beta e = e^\beta Ue$ via $x \otimes_U u \mapsto x \diamond_\beta u = u^\beta x$ (this is a general fact; for any $A \in \text{Mod-}U$, $A \otimes_U Ue \cong Ae$). Part (i) now follows by composing the isomorphisms $K_\alpha \to K_\beta$ and $K_\beta \to e^\beta Ue$. This is illustrated in the following:

$$\begin{array}{ccccccc} e_{11}Ue \otimes_{\beta_1} e_{11}Ue & \cong & Ue \otimes_\beta Ue & \cong & (Ue)^\beta \otimes_U Ue & \cong & e^\beta Ue \\ x \otimes_{\beta_1} y & \mapsto & x \otimes_\beta y & \mapsto & y \otimes_U x & \mapsto & y \diamond_\beta x = x^\beta y\ . \end{array}$$

(ii) Assume α is an involution and identify K_α with $e^\beta Ue$. Then for all $x, y \in e_{11}Ue$, $(x \otimes_\alpha y)^\kappa = y \otimes_\alpha x$, so under the identification we get $(x^\beta y)^{\kappa_\alpha} = y^\beta x$ and the latter equals $(x^\beta y)^\beta$ since β is also an involution. Thus, κ_α coincides with β on $e^\beta Ue$. □

COROLLARY 3.3.10. *Assume $M \in \text{Mod-}R$ is free of rank $n \in \mathbb{N}$, let $W = \mathrm{End}(M_R)$ and let $\alpha \in \mathrm{Aut}^-(W)$. Then $(K_\alpha)_1^n \cong R^n$ as right R-modules. (Recall that $(K_\alpha)_1$ means "K_α considered as a right R-module w.r.t. $\odot_1$".)*

PROOF. Assume $M = R^n$ and identify W with $\mathrm{M}_n(R)$. Let $\{e_{ij}\}$ be the standard matrix units of W. Then by Proposition 3.3.9, we may assume $K_\alpha = e_{11}^\alpha We_{11}$ (take $e = e_{11}$). Consider $K_i := e_{ii}^\alpha We_{11}$ a right R-module. Then $K_i \cong K_j$ for all i, j (the isomorphism being multiplication on the left by e_{ij}^α). Thus, $(K_\alpha)_1^n \cong K_1 \oplus \cdots \oplus K_n = (\sum_i e_{ii}^\alpha)We_{11} = We_{11} \cong R_R^n$ as right R-modules. □

COROLLARY 3.3.11. *Assume $M \in \text{Mod-}R$ is a generator, let $W = \mathrm{End}(M_R)$ and let $\alpha \in \mathrm{End}^-(W)$. If α is injective, then b_α is left injective. If α is bijective, then b_α is regular.*

PROOF. By Corollary 3.3.5 and Proposition 2.6.2(i), we can replace b_α with $n \cdot b_\alpha$, thus assuming $n = 1$, $U = \mathrm{M}_1(W) = W$, $e_{11} = 1$ and $\beta = \alpha$ in previous computations. (This step is not really necessary, but it simplifies the arguments to follow.) Let b be as in the last proposition. Then it is enough to prove b is injective/regular. Indeed, $b(x, M) = 0$ implies $x^\alpha \in \mathrm{ann}^\ell\, Ue$. Since $U = \mathrm{End}(M_R) = \mathrm{End}(Ue_{eUe})$, ${}_UUe$ is faithful, so $x^\alpha = 0$. Thus, if α is injective, $x = 0$, and thus b_α is left injective.

Now assume α is bijective. We claim that b is left surjective. This is easily seen to be equivalent to the fact that any $f \in \mathrm{Hom}_R(Ue, e^\alpha Ue)$ is induced by left multiplication with an element of $(Ue)^\beta = e^\beta U$. Indeed, viewing f is an endomorphism of Ue, we see that $f(x) = ux$ for some $u \in U$ (because $U = \mathrm{End}(M_R^n) = \mathrm{End}(Ue_R)$). Replacing u with $e^\beta u$ if needed (it is not needed), we may assume $u \in e^\beta U$, as required. Thus, b is left surjective and hence left regular by the previous paragraph. That b right regular follows by symmetry. □

We will give another proof for the second assertion of Corollary 3.3.11 in section 3.5. We finish with the following example that proves the claims posed in Example 3.1.2.

Example 3.3.12. (i) Let F be a field, let V be a nonzero f.d. vector space, let $W = \mathrm{End}_F(V)$ and let $\alpha \in \mathrm{End}^-(W)$. Note that $\alpha(\mathrm{Cent}(W)) \subseteq \mathrm{Cent}(W)$. Thus, α induces an anti-endomorphism of $\mathrm{Cent}(W)$, which we freely identify with F. Let e be any primitive idempotent of $W := \mathrm{End}(V_F)$. Then e induces a projection from V to F_F, hence Proposition 3.3.9 implies $K_\alpha \cong e^\alpha We$ as double F-modules. Since isomorphism between F and eWe is given by $a \mapsto ae$ ($a \in F$), the double F-module structure on $e^\alpha We$ is given by

$$(e^\alpha we) \odot_0 a = (ae)^\alpha (e^\alpha we) = e^\alpha a^\alpha e^\alpha we = (e^\alpha we)a^\alpha$$

(the last equality holds since $a^\alpha \in F = \mathrm{Cent}(W)$) and

$$(e^\alpha we) \odot_1 a = (e^\alpha we)a \ .$$

Therefore, $k \odot_0 a = k \odot_1 a^\alpha$ for all $a \in F$. In addition, by Corollary 3.3.10, $e^\alpha We_F \cong F_F$ (but $\dim_F e^\alpha We$ might be larger than 1!). Moving the 0-product along this isomorphism we get that $K_\alpha \cong F$ where F is considered as a double F-module via the actions $k \odot_0 a = a^\alpha k$ and $k \odot_1 a = ka$. In particular, if α is an F-algebra isomorphism, then $\odot_0 = \odot_1$ and $b_\alpha : V \times V \to F$ is just a "classical" bilinear form (i.e. a standard (non-symmetric) bilinear form over a field). Since any classical regular bilinear form $b : V \times V \to F$ gives rise to such α, it follows that all classical regular bilinear forms are generic.

The previous argument still works if we replace F with any commutative ring C and take e to be e_{ii} for some i. The only exception is the fact that $(K_\alpha)_1$ need not be isomorphic to C_C, but rather $(K_\alpha)_1^n \cong C^n$ where $V \cong C^n$, i.e. $(K_\alpha)_1$ is a rank-1 projective.

(ii) Keeping the notation of (i), assume α is an involution of the first kind. By Proposition 3.3.9(ii), κ_α is just α restricted to $e^\alpha We$, which we henceforth identify with K_α. We will now compute κ_α by carefully choosing the idempotent e.

Assume first α is orthogonal. Then α is obtained from a classical regular non-alternating[5] symmetric bilinear form $b : V \times V \to F$. In this case, it is well known that V admits a basis $x_1, \dots, x_n$ such that $b = (b|_{x_1F\times x_1F}) \perp \dots \perp (b|_{x_nF\times x_nF})$ (even when $\mathrm{char}\, F = 2$; see [**86**, Th. 3.5] for $\mathrm{char}\, F \neq 2$ and [**1**] for arbitrary characteristic). Take e to be projection from V to x_1F with kernel $x_2F \oplus \dots \oplus x_nF$. Then it is easy to see that $e^\alpha = e$ (see the proof of Proposition 2.6.2(ii)). Now, $e^\alpha We = e^\alpha Fe = eF$ so it is clear that $\kappa_\alpha = \mathrm{id}_{K_\alpha}$.

Now assume α is symplectic, i.e. α is the corresponding anti-endomorphism of an a classical regular alternating form $b : V \times V \to F$. Then it is well known that $\dim V$ is even and V admits a basis $x_1, \dots, x_n$ such that

$$b(x_i, x_j) = \begin{cases} 1 & i + j = n + 1 \text{ and } i > j \\ -1 & i + j = n + 1 \text{ and } i < j \\ 0 & \text{otherwise} \end{cases}$$

Take $x_1, \dots, x_n$ to be the standard basis of V and identify W with $\mathrm{M}_n(F)$. Then $b(e_{11}x_i, x_j) = b(x_i, e_{nn}x_j)$ for all i, j, hence $b(e_{11}x, y) = b(x, e_{nn}y)$ for all $x, y \in V$. This implies $e_{11}^\alpha = e_{nn}$ and similarly, one obtains $e_{n1}^\alpha = -e_{n1}$. Take $e = e_{11}$. Then $e^\alpha We = e_{11}^\alpha We_{11} = e_{nn}We_{11} = e_{n1}F$. Since $e_{n1}^\alpha = -e_{n1}$, it follows that $\kappa_\alpha = -\mathrm{id}_{K_\alpha}$.

[5] A bilinear form $b : V \times V \to F$ is *alternating* if its associated quadratic form is 0, i.e. $b(x, x) = 0$ for all $x \in V$. Symmetric alternating bilinear forms exists only when $\mathrm{char}\, F = 2$.

A similar argument would show that when α is an involution of the second kind, κ_α is just $\alpha|_F$, where K_α is identified with F as in (i). (Thus, b_α is just an $\alpha|_F$-hermitian form.)

3.4. Examples

Before we turn to prove our main results about the regularity or injectivity of b_α, we present a series of examples with explicit computations of b_α and K_α. In particular, the examples demonstrate problems (a) and (b) of section 3.1.

We begin with a positive example in which b_α is always right regular.

Example 3.4.1. Let R be a ring and let α be an anti-endomorphism of R. Identify R with $\mathrm{End}(R_R)$ in the standard way. Then $K_\alpha \cong R$ via $x \otimes_\alpha y \mapsto x^\alpha y$. Here R is a considered as a double R-module by

$$x \odot_0 r = r^\alpha x, \qquad x \odot_1 r = xr, \qquad \forall x, r \in R\ .$$

The proof is similar to the argument in the second paragraph of Example 3.3.1 above. Identifying K_α with R, b_α is given by $b_\alpha(x,y) = x^\alpha y$. Thus, by Example 2.1.4, b_α is right regular, so there is a one-to-one correspondence between generic bilinear forms on R_R and anti-endomorphism of R, as in (14). (We will show below that this still holds if we replace R_R with any finite projective R-module.)

The next two examples demonstrate that the correspondence in (14) fails in general, even over f.d. algebras.

Example 3.4.2. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. It is well known that $\mathrm{End}(M_{\mathbb{Z}}) = \mathbb{Z}_p$ where $\mathbb{Z}_p$ are the p-adic integers. (This follows from Matlis' Duality Theory; see [**63**] or [**58**, §3I]). Take $\alpha = \mathrm{id}_{\mathbb{Z}_p} \in \mathrm{End}^-(M_{\mathbb{Z}})$ and note that $K_\alpha = M \otimes_\alpha M \cong M \otimes_{\mathbb{Z}_p} M$. The module M is p-divisible, hence for all $x, y \in M$,

$$x \otimes y = x \otimes p^n(p^{-n})y = \alpha(p^n)x \otimes p^{-n}y = p^n x \otimes p^{-n} y\ .$$

(The "quotient" $p^{-n}y$ is not uniquely determined, but this does not matter to us.) As $p^n x = 0$ for sufficiently large n, it follows that $x \otimes y = 0$. This implies $K_\alpha = 0$, hence $b_\alpha = 0$ (!). Moreover, the universal property of b_α implies that there is no bilinear form $0 \neq b' : M \times M \to K'$ satisfying $b'(wx, y) = b'(x, w^\alpha y)$ for all $w \in \mathbb{Z}_p$ and $x, y \in M$. In particular, α does not correspond to a right stable form on M.

Example 3.4.3. Let F be a field and let R be the commutative subring of $\mathrm{M}_3(F)$ consisting of matrices of the form:

$$\begin{bmatrix} a & & \\ b & a & \\ c & & a \end{bmatrix}.$$

Let $x = e_{21}$ and $y = e_{31}$ (where $\{e_{ij}\}$ are the standard matrix units of $\mathrm{M}_3(F)$). Then $\{1, x, y\}$ is an F-basis of R. Consider the elements of $M = F^3$ as row vectors and let $\{e_1, e_2, e_3\}$ be the standard F-basis of M. Then M is naturally a right R-module (the action of R being matrix multiplication on the right) and a straightforward computation shows that $\mathrm{End}(M_R) \cong R$, i.e. all R-linear maps $f : M \to M$ are of the form $m \mapsto mr$ for some $r \in R$. Let $\alpha = \mathrm{id}_R \in \mathrm{Aut}^-(R)$. Then we may assume $M \otimes_\alpha M = M \otimes_R M$. Now:

$$\begin{aligned} b_\alpha(e_1, e_1) &= e_1 \otimes e_1 = xe_2 \otimes e_1 = e_2 \otimes xe_1 = 0 \\ b_\alpha(e_2, e_1) &= e_2 \otimes e_1 = e_2 \otimes ye_3 = ye_2 \otimes e_3 = 0 \\ b_\alpha(e_3, e_1) &= e_3 \otimes e_1 = e_3 \otimes xe_2 = xe_3 \otimes e_2 = 0\ . \end{aligned}$$

Therefore, $b_\alpha(M, e_1) = 0$, hence b_α is not right injective. Moreover, let $\sigma : M \to M$ be defined by $\sigma(x, y, z) = (y, 0, 0)$. Then $\sigma \in \mathrm{End}(M_R)$ and $b_\alpha(x, \sigma y) = 0$ for all $x, y \in M$, implying b_α is not right semi-stable (and hence not right stable). Similarly, $b_\alpha(e_1, M) = 0$ and b_α is not left semi-stable. In addition, a detailed computation would show that $\{e_i \otimes e_j \,|\, i, j \in \{2,3\}\}$ is an F-basis for K_α, and this is easily seen to imply that b is not right nor left surjective. In particular, the correspondence (14) fails for M.

The next examples demonstrate that b_α can be stable even when it is not regular.

EXAMPLE 3.4.4. Let R be a commutative ring admitting a proper nonzero ideal $A \trianglelefteq R$ with the following properties:

(a) A_R is flat and $\mathrm{ann}(A) = 0$.
(b) $A^2 = A$.
(c) $\mathrm{End}(A_R) \cong R$, i.e. all R-linear maps $f : A \to A$ are of the form $a \mapsto ar$ for some $r \in R$.

Let $\alpha = \mathrm{id}_{\mathrm{End}(A_R)} = \mathrm{id}_R$. Then we can identify $A \otimes_\alpha A$ with $A \otimes_R A$. Since A_R is flat, the latter is isomorphic to $A^2 = A$ via $x \otimes y \mapsto xy$ (see [**58**, §4A]). This is clearly a double R-module isomorphism, where the actions $\odot_0$ and $\odot_1$ on A are the standard action of R on A. Thus, b_α is similar to $b : A \times A \to A$ defined by $b(x, y) = xy$. Moreover, $b(A, x) = 0$ implies $x \in \mathrm{ann}\, A = 0$, hence b is right injective, thus right stable. However, b is not right regular since $\mathrm{id}_A \neq \mathrm{Ad}^r_b(a)$ for all $a \in A$. (Indeed, if $\mathrm{id}_A = \mathrm{Ad}^r_b(a)$, then $1 - a \in \mathrm{ann}(A) = 0$, hence $1 = a \in A$ which contradicts our assumptions.) Similarly, b is left stable but not right regular.

It is left to provide an explicit example of R and A. Let F be a field. Then any of the following satisfies (a), (b) and (c):

(1) $R = F[x^q \,|\, 0 < q \in \mathbb{Q}]$ and $A = \langle x^q \,|\, 0 < q \in \mathbb{Q} \rangle$.
(2) $R = \prod_{\aleph_0} F$ and $A = \bigoplus_{\aleph_0} F$.

In (1) any ideal of R is flat since R is a Prüfer domain, and in (2) any ideal of R is flat since R is von-Neumann regular; see [**58**, §4B]. The rest of the details are left to the reader.

We also note that in case (1), the *stable* generic bilinear forms on A correspond to anti-automorphism of R as in (14) (but there is no correspondence between *regular* generic forms on A and $\mathrm{Aut}^-(R)$).[6] This follows from Theorem 3.7.19 below (take Q to be the fraction field of R.)

EXAMPLE 3.4.5. Let F be a field and let $T = F[x^r \,|\, 0 < r \in \mathbb{R}]$. For any set S of non-negative real numbers, let I_S denote the ideal of S generated by $\{x^s \,|\, s \in S\}$. Define $R = T/I_{[1,\infty)}$ and let $M = I_{(0,\infty)}/I_{(1,\infty)}$. Then M is a right R-module and it is routine to check that $\mathrm{End}(M_R)$ can be understood as the ring W of formal power series $\sum_{n=1}^{\infty} \alpha_n x^{\varepsilon_n}$ with $\{\alpha_n\}_{n=1}^{\infty} \subseteq F$, $0 \leq \varepsilon_1 < \varepsilon_2 < \varepsilon_3 \cdots < 1$ and $\varepsilon_n \xrightarrow{n \to \infty} 1$ subject to the relation $x^\varepsilon = 0$ for all $\varepsilon \geq 1$. (The element $x^\varepsilon \in W$ acts on M like $x^\varepsilon + I_{[1,\infty)} \in R$).

Let $\alpha = \mathrm{id}_W$ and identify K_α with $M \otimes_W M$. We make M into a double R-module by letting both $\odot_0$ and $\odot_1$ be the standard action of R on M. To simplify the notation, let $\overline{r} := r + I_{(1,\infty)} \in M$ for $r \in I_{(0,\infty)}$. We claim that $K_\alpha \cong M$ as double R-modules via $\overline{a} \otimes_W \overline{b} \mapsto \overline{ab}$. Indeed, as F-vector spaces:

$$M \otimes_W M = \frac{M \otimes_F M}{\mathrm{span}_F\{w\overline{x^\varepsilon} \otimes_F \overline{x^{\varepsilon'}} - \overline{x^\varepsilon} \otimes_F w\overline{x^{\varepsilon'}} \,|\, w \in W,\ \varepsilon, \varepsilon' \in (0,1]\}} \,.$$

[6] Moreover, there are no regular bilinear forms on A. This follows from the fact that $A^{[1]} \cong A^{[0]} \cong R_R \not\cong A_R$.

The set $\{\overline{x^\varepsilon}\otimes_F\overline{x^{\varepsilon'}}\,|\,\varepsilon,\varepsilon'\in(0,1]\}$ is an F-basis for the nominator, and the denominator is easily seen to be spanned by

$$\left\{\overline{x^{\varepsilon_1}}\otimes_F\overline{x^{\varepsilon_2}}-\overline{x^{\varepsilon_3}}\otimes_F\overline{x^{\varepsilon_4}}\;\middle|\;\begin{array}{c}\varepsilon_1,\dots,\varepsilon_4\in(0,1],\\ \varepsilon_1+\varepsilon_2=\varepsilon_3+\varepsilon_4\end{array}\right\}\bigcup\left\{\overline{x^{\varepsilon_1}}\otimes_F\overline{x^{\varepsilon_2}}\;\middle|\;\begin{array}{c}\varepsilon_1,\varepsilon_2\in(0,1],\\ \varepsilon_1+\varepsilon_2>1\end{array}\right\}.$$

It is now straightforward to check that the map $f:M\otimes_W M\to M$ sending $\overline{x^\varepsilon}\otimes_R\overline{x^{\varepsilon'}}$ to $\overline{x^{\varepsilon+\varepsilon'}}$ is a well-defined F-vector space isomorphism. As f is easily seen to be a double R-module homomorphism, we conclude that $M\otimes_W M\cong M$, as required. Thus, b_α is similar to $b:M\times M\to M$ defined by $b(\overline{a},\overline{b})=\overline{ab}$. Observe that $b(\overline{x},M)=b(M,\overline{x})=0$, hence b is right and left degenerate (i.e. not injective). However, b is right stable since any $w\in W$ satisfying $b(\overline{a},w\overline{b})=0$ for all $\overline{a},\overline{b}\in M$ satisfies $w\overline{x^\varepsilon}=0$ for all $\varepsilon>0$ (take $a=b=x^{\varepsilon/2}$) and this implies $w=0$. Therefore, b_α is right semi-stable, hence right stable. Similarly, b_α is also left stable.

Now consider the ring homomorphism $\beta:R\to R$ defined by $\beta(g(x)+I_{[1,\infty)})=g(x^2)+I_{[1,\infty)}$. Then a similar argument would show that $K_\beta\cong K$, where K is M equipped with double R-module structure given by:

$$\overline{s}\odot_0 r=\beta(r)\overline{s}\qquad \overline{s}\odot_1 r=\overline{s}r\qquad \forall\overline{s}\in M,\ r\in R\ ,$$

and b_α is similar to $b:M\times M\to K$ defined by $b(\overline{a},\overline{b})=\overline{\beta(a+I_{[1,\infty)})b}$. Then b is again right and left degenerate and right stable, but b is not left semi-stable since $b(wM,M)=0$ for $w=x^{1/2}\in W$.

The next two examples demonstrate what might happen when M is a generator, but $\alpha\in\mathrm{End}^-(\mathrm{End}(M))$ is not bijective. In particular, they imply that the injectivity of α in Corollary 3.3.11 is essential.

Example 3.4.6. Let N be any nonzero torsion $\mathbb{Z}$-module and let $M=\mathbb{Z}\oplus N\in$ Mod-$\mathbb{Z}$. We consider the elements of M as column vectors. Then

$$W:=\mathrm{End}_{\mathbb{Z}}(M)=\left[\begin{array}{cc}\mathrm{End}(\mathbb{Z}_{\mathbb{Z}}) & \mathrm{Hom}(N,\mathbb{Z})\\ \mathrm{Hom}(\mathbb{Z},N) & \mathrm{End}(N_{\mathbb{Z}})\end{array}\right]=\left[\begin{array}{cc}\mathbb{Z} & 0\\ N & \mathrm{End}(N_{\mathbb{Z}})\end{array}\right].$$

Note that M is a generator and $e:=\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\in W$ is a projection from M to $\mathbb{Z}_{\mathbb{Z}}$. Thus, we can identify M with

$$We=\left[\begin{array}{cc}\mathbb{Z} & 0\\ N & 0\end{array}\right].$$

Define $\alpha\in\mathrm{End}^-(W)$ by $\left[\begin{smallmatrix}a&0\\b&c\end{smallmatrix}\right]^\alpha=\left[\begin{smallmatrix}a&0\\0&\overline{a}\end{smallmatrix}\right]$ where $\overline{a}$ is the image of $a\in\mathbb{Z}$ in $\mathrm{End}(N_{\mathbb{Z}})$. Then by Proposition 3.3.9, b_α is similar to $b:M\times M\to e^\alpha We=We=M$ defined by $b(\left[\begin{smallmatrix}x&0\\y&0\end{smallmatrix}\right],\left[\begin{smallmatrix}z&0\\w&0\end{smallmatrix}\right]))=\left[\begin{smallmatrix}x&0\\y&0\end{smallmatrix}\right]^\alpha\left[\begin{smallmatrix}z&0\\w&0\end{smallmatrix}\right]=\left[\begin{smallmatrix}x&0\\0&x\end{smallmatrix}\right]\left[\begin{smallmatrix}z&0\\w&0\end{smallmatrix}\right]\left[\begin{smallmatrix}xz&0\\xw&0\end{smallmatrix}\right]$. It is now easy to see that b is right injective but not left injective. In addition, b is not right regular. To see this, let $0\neq f\in\mathrm{End}(N_{\mathbb{Z}})$ and note that the homomorphism $\left[\begin{smallmatrix}x&0\\y&0\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&0\\f(y)&0\end{smallmatrix}\right]\in\mathrm{Hom}_{\mathbb{Z}}(M,(K_\alpha)_0)$ does not lie in $\mathrm{im}(\mathrm{Ad}^r_b)$.

Example 3.4.7. View $N:=\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ as a $\mathbb{Z}_p$-module as in Example 3.4.2. Then $\mathrm{End}(N_{\mathbb{Z}_p})=\mathbb{Z}_p$. Define $M=\mathbb{Z}_p\oplus N\in$ Mod-$\mathbb{Z}_p$ and consider the elements of M as column vectors. Then

$$W:=\mathrm{End}_{\mathbb{Z}}(M)=\left[\begin{array}{cc}\mathrm{End}(\mathbb{Z}_p) & \mathrm{Hom}(N,\mathbb{Z}_p)\\ \mathrm{Hom}(\mathbb{Z}_p,N) & \mathrm{End}(N)\end{array}\right]=\left[\begin{array}{cc}\mathbb{Z}_p & 0\\ N & \mathbb{Z}_p\end{array}\right].$$

Let $e\in W$ be as in the previous example and identify M with We. Define $\alpha\in\mathrm{End}^-(W)$ by $\left[\begin{smallmatrix}x&0\\y&z\end{smallmatrix}\right]=\left[\begin{smallmatrix}z&0\\0&z\end{smallmatrix}\right]$. Then by Proposition 3.3.9, K_α is isomorphic to $e^\alpha We=0We=0$, so b_α is the zero form!

Our last example demonstrates that $b_{\alpha(b)}$ need not be similar to b.

EXAMPLE 3.4.8. Let $1 < n \in \mathbb{N}$ and let F be a field. Denote by T_n the ring of upper-triangular matrices over F. For $0 \le i \le n$, let M_i denote the right T_n-module consisting of row vectors

$$(0,\dots,0,\underbrace{*,\dots,*}_{i}) \in F^n$$

with T_n acting by matrix multiplication on the right. It is not hard to verify that $\dim_F \mathrm{Hom}_{T_n}(M_n, M_n/M_m) = 1$ for all $0 \le i < n$. In particular, $\mathrm{End}_{T_n}(M_n/M_m) = F$ for all $0 \le m < n$. We need the following fact which easily follows from the Krull-Schmidt Theorem (see Theorem 1.1.1 above or [**80**, Th. 2.9.17]): Assume $a_0 + \dots + a_n = b_0 + \dots + b_n$ and $\bigoplus_{i=0}^n (M_n/M_i)^{a_i} \cong \bigoplus_{i=0}^n (M_n/M_i)^{b_i}$. Then $a_i = b_i$ for all $0 \le i \le n$. (The assumption $a_0 + \dots + a_n = b_0 + \dots + b_n$ was needed because M_n/M_n is the zero module.)

Make $K = \mathrm{M}_n(F)$ into a double T_n-module by defining

$$A \odot_0 B = B^T A \qquad A \odot_1 B = AB$$

for all $A \in K$ and $B \in T_n$. Then $b : M_n \times M_n \to K$ defined by $b(x,y) = x^T y$ is a bilinear form. For $0 \le u, v \le n$, let $K_{u,v}$ denote the matrices $A = (A_{ij}) \in K$ for which $A_{ij} = 0$ if $i \le u$ or $j \le v$. For example, when $n = 3$, $K_{1,2}$ and $K_{2,0}$ consist of matrices of the forms:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} \quad , \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}$$

respectively. Then $K_{u,v}$ is a sub-double-T_n-module of K, hence $K/K_{u,v}$ is a double T_n-module in its own right and $b_{u,v} : M_n \times M_n \to K/K_{u,v}$ defined by $b(x,y) = x^T y + K_{u,v}$ is a bilinear form.

We claim that $b_{u,v}$ is right regular when $u > 0$ and (left and right) stable if $(u,v) \neq (0,0)$. Indeed, it is easy to check that $b_{u,v}$ is right injective if $u > 0$. Moreover, in this case $(K/K_{u,v})_0 \cong M_n^v \oplus (M_n/M_{n-u})^{n-v}$ as right T_n-modules (the summands are the columns of $K/K_{u,v} = \mathrm{M}_n(F)/K_{u,v}$) and hence

$$\dim_F \mathrm{Hom}_{T_n} M^{[1]} = \dim_F \mathrm{Hom}_{T_n}(M_n, (K/K_{u,v})_0)$$

$$= v \dim_F \mathrm{Hom}_{T_n}(M_n, M_n) + (n-v) \dim_F \mathrm{Hom}_{T_n}(M_n, M_n/M_{n-u}) = n \ .$$

Therefore, dimension considerations imply $\mathrm{Ad}^r_{b_{u,v}}$ is bijective, i.e. $b_{u,v}$ is right regular. To see that $b_{u,v}$ is right stable when $(u,v) \neq 0$, observe that $\ker \mathrm{Ad}^r_{b_{u,v}}$ is always contained in M_{n-1} (which a unique maximal submodule of M_n). As $\mathrm{Hom}_{T_n}(M_n, M_{n-1}) = 0$, it follows that $\mathrm{Hom}(M_n, \ker \mathrm{Ad}^r_{b_{u,v}}) = 0$, hence $b_{u,v}$ is right semi-stable. Now observe that $\mathrm{End}_{T_n}(M_n) \cong F$ (i.e. all endomorphisms of M_n are given by $x \mapsto xa$ for some $a \in F$) and $b_{u,v}(ax, y) = b_{u,v}(x, ay)$ for all $x, y \in M_n$ and $a \in F$. Thus, $b_{u,v}$ is right stable with corresponding anti-endomorphism id_F. The form $b_{u,v}$ is left stable by symmetry.

We have thus shown that the forms $\{b_{u,v} \,|\, 0 \le u, v \le n,\ (u,v) \neq (0,0)\}$ have the same generization and we now claim that this generization is similar to b. Indeed, let $\alpha = \mathrm{id}_F \in \mathrm{End}^-(F)$. Then $\dim_F K_\alpha = \dim_F M_n \otimes_\alpha M_n = \dim_F M_n \otimes_F M_n = n^2$. The universality of b_α implies that there is a double T_n-module homomorphism $f : K_\alpha \to K$, that must be onto since $\mathrm{im}(b) = K$.[7] Since f is clearly F-linear and $\dim_F K = n^2$, dimension considerations imply that f is an isomorphism. Thus, b is the generization of all the forms $\{b_{u,v} \,|\, 0 \le u, v \le n,\ (u,v) \neq (0,0)\}$.

We now exhibit an interesting phenomena — the forms

$$\{b_{u,v} \,|\, 0 \le u, v < n,\ (u,v) \neq (0,0)\} \cup \{b_{n,n}\}$$

[7] Recall that $\mathrm{im}(b)$ was defined to be the additive group spanned by $\{b(x,y) \,|\, x, y \in M_n\}$.

share the same generization (up to similarity), but they are pairwise non-similar. The reason is that $(K/K_{u,v})_0 \not\cong (K/K_{u',v'})_0$ for distinct pairs $(u,v),(u',v') \in \{0,\dots,n-1\}^2 \cup \{(n,n)\} \setminus \{(0,0)\}$, so there cannot be an isomorphism from $K_{u,v}$ to $K_{u',v'}$. Indeed, $(K/K_{u,v})_0 \cong M_n^v \oplus (M_n/M_{n-u})^{n-v}$ and $M_n^v \oplus (M_n/M_{n-u})^{n-v} \cong M_n^{v'} \oplus (M_n/M_{n-u'})^{n-v'}$ implies $(u,v) = (u',v')$ by the fact stated above. In particular, the forms $\{b_{u,v} \mid (u,v) \in \{1,\dots,n-1\}^2\}$ are non-generic regular forms.

We also point out that if $(u,v),(u',v') \in \{0,\dots,n-1\}^2 \setminus \{(0,0)\}$ are distinct and satisfy $uv = u'v'$, then any double T_n-module homomorphism from $K/K_{u,v}$ to $K/K_{u',v'}$ is not injective nor surjective. For otherwise, it would have to be bijective since $\dim_F K/K_{u,v} = \dim_F K/K_{u',v'}$. This shows that the problem of defining *weak similarities*, posed in Remark 3.1.9, is far from trivial. In particular, one cannot expect weak similarities to merely consist of morphisms between double modules. In addition, we also note that $K/K_{u,v}$ does not have an anti-isomorphism when $u \neq 0$, although $\alpha(b_{u,v}) = \mathrm{id}_F$ is an involution. This is true because

$$(K/K_{u,v})_1 \cong M_n^u \oplus (M_n/M_{n-v})^{n-u} \not\cong M_n^v \oplus (M_n/M_{n-u})^{n-v} \cong (K/K_{u,v})_0$$

when $u \neq v$ (and an anti-isomorphism on $K/K_{u,v}$ clearly induces an isomorphism $(K/K_{u,v})_1 \cong (K/K_{u,v})_0$).

To finish, observe that the form $b_{0,v}$ $(v > 0)$ is right degenerate and can thus be classified as "badly behaved". However, we have seen that its generization is regular, which can be considered as "well behaved". This demonstrates how generization can make badly behaved forms into well behaved forms.

We could neither find nor contradict the existence of:

- An anti-*auto*morphism α such that b_α is right regular but not left regular. (In this case α^2 cannot be inner, as implied by Corollary 3.2.3.)

An example of a f.g. torsion-free module M over a noetherian integral domain and $\alpha \in \mathrm{Aut}^-(\mathrm{End}(M))$ such that b_α is not regular (but necessarily injective, as we shall see at the end of section 3.6), was found after the submission of the dissertation and can be found in [**39**].

3.5. Conditions That Imply b_α Is Right Regular

Let R, M and W be as in section 3.1. In this section we present conditions on R, M, W and α that ensure b_α is right regular, as well as other supplementary results.

Assume momentarily that W and R are arbitrary rings and let Mod-(W,R) denote the category of (W,R)-bimodules. Let $A \in$ Mod-W, $B \in$ Mod-R and $C \in$ Mod-(W,R). Then $\mathrm{Hom}_R(B,C)$ is a right W-module w.r.t. the action $(fw)m = f(wm)$ (where $f \in \mathrm{Hom}_R(B,C)$, $w \in W$ and $m \in M$), and there is a natural map

$$\Gamma = \Gamma_{A,B,C} : A \otimes_W \mathrm{Hom}_R(B,C) \to \mathrm{Hom}_R(B, A \otimes_W C)$$

given by $(\Gamma(a \otimes f))b = a \otimes f(b)$ for all $f \in \mathrm{Hom}_R(B,C)$, $a \in A$ and $b \in B$.

Now assume $M \in$ Mod-R, $W = \mathrm{End}(M_R)$ and $\alpha \in \mathrm{End}^-(W)$. Then M can be viewed as a (W,R)-bimodule. Therefore, we have a map

$$\Gamma = \Gamma_{M^\alpha,M,M} : M^\alpha \otimes_W \mathrm{Hom}_R(M,M) \to \mathrm{Hom}_R(M, M^\alpha \otimes_W M) \ .$$

The following lemma shows that up to certain identifications, $\mathrm{Ad}^r_{b_\alpha}$ is Γ.

LEMMA 3.5.1. *In the previous notation, there is a commutative diagram*

$$\begin{array}{ccc} M^\alpha \otimes_W \mathrm{End}(M_R, M_R) & \xrightarrow{\Gamma} & \mathrm{Hom}_R(M, M^\alpha \otimes_W M) \\ \downarrow \psi & & \downarrow \varphi \\ M & \xrightarrow{\mathrm{Ad}^r_{b_\alpha}} & M^{[1]} \end{array}$$

where $M^{[1]} = \mathrm{Hom}_R(M,(K_\alpha)_0)$ *and* ψ, φ *are bijective.*

PROOF. Let ψ be the identity map $M^\alpha \to M$ (recall that $M^\alpha = M$ as sets) composed on the standard isomorphism

$$M^\alpha \otimes_W \mathrm{End}(M_R, M_R) = M^\alpha \otimes_W W \cong M^\alpha \ .$$

Then ψ is given by $\psi(m \otimes_W w) = m \diamond_\alpha w = w^\alpha m$ and its inverse is $m \mapsto m \otimes 1$. The map φ is defined by $\varphi(f) = \delta \circ f$ where δ is the isomorphism $M^\alpha \otimes_W M \to K_\alpha$ given by $x \otimes_W y \mapsto y \otimes_\alpha x$. The diagram commutes since for all $x, y \in M$ and $w \in W$:

$$(\mathrm{Ad}^r_{b_\alpha}(\psi(x \otimes_W w)))y = (\mathrm{Ad}^r_{b_\alpha}(w^\alpha x))y = b_\alpha(y, w^\alpha x) = y \otimes_\alpha w^\alpha x$$

$$= wy \otimes_\alpha x = \delta(x \otimes_W wy) = \delta((\Gamma(x \otimes_W w))y) = (\varphi(\Gamma(x \otimes_W w)))y \ . \qquad \square$$

It is now of interest to find sufficient conditions for Γ to be bijective (injective, surjective). This is done in the following lemma.

LEMMA 3.5.2. *Let* $A \in \text{Mod-}W$, $B \in R\text{-Mod}$ *and* $C \in \text{Mod-}(W,R)$. *Then:*

(i) *If one of the following holds:*
  (a) *A is finite projective.*
  (b) *A is projective and B is f.g.*
  (c) *B is finite projective.*
  (d) *B is projective and A is f.p.*
  Then Γ is bijective.
(ii) *If A is projective, then Γ is injective.*
(iii) *If B is projective and A is f.g., then Γ is surjective.*
(iv) *If there is an exact sequence $A_1 \to A_0 \to A \to 0$ and B is projective, then:*
  (a) *$\Gamma_{A_0,B,C}$ is surjective $\Longrightarrow \Gamma_{A,B,C}$ is surjective.*
  (b) *$\Gamma_{A_0,B,C}$ is bijective and $\Gamma_{A_1,B,C}$ is surjective $\Longrightarrow \Gamma_{A,B,C}$ is bijective.*
(v) *If there is an exact sequence $0 \to A \to A_0 \to A_1$ and $\mathrm{Hom}_R(B,C)$ is flat (in W-Mod), then:*
  (a) *$\Gamma_{A_0,B,C}$ is injective $\Longrightarrow \Gamma_{A,B,C}$ is injective.*
  (b) *$\Gamma_{A_0,B,C}$ is bijective, $\Gamma_{A_1,B,C}$ is injective and ${}_WC$ is flat $\Longrightarrow \Gamma_{A,B,C}$ is bijective.*
(vi) *If there is an exact sequence $B_1 \to B_0 \to B \to 0$ and A is flat, then:*
  (a) *$\Gamma_{A,B_0,C}$ is injective $\Longrightarrow \Gamma_{A,B,C}$ is injective.*
  (b) *$\Gamma_{A,B_0,C}$ is bijective and $\Gamma_{A,B_1,C}$ is injective $\Longrightarrow \Gamma_{A,B,C}$ is bijective.*

In particular, this implies that:

(vii) *If A embeds in a free module and $\mathrm{Hom}_R(B,C)$ is flat, then Γ is injective.*
(viii) *If A embeds in a flat module, B is f.g. and $\mathrm{Hom}_R(B,C)$ is flat, then Γ is injective.*
(ix) *If A is flat and B is f.p., then Γ is bijective.*

PROOF. We prove (i), (ii) and (iii) together: Since Γ is additive, we may replace projective with free and finite projective with f.g. and free. Assume $A = \bigoplus_{i\in I} W$, then Γ becomes the standard map $\bigoplus_{i\in I} \mathrm{Hom}_R(B,C) \to \mathrm{Hom}_R(B, \bigoplus_{i\in I} C)$. This map is clearly injective and provided I is finite, it is bijective. In addition, it is also easy to verify it is surjective if B is f.g. Now assume $B = \bigoplus_{i\in I} R$. Then Γ becomes the standard map $\varepsilon : A \otimes \prod_{i\in I} C \to \prod_{i\in I}(A \otimes C)$, which is bijective if I is finite. In addition, by [**58**, §4F], ε is surjective if A is f.g. and bijective if A is finitely presented.

(iv) We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} A_1 \otimes \mathrm{Hom}_R(B,C) & \longrightarrow & A_0 \otimes \mathrm{Hom}_R(B,C) & \longrightarrow & A \otimes \mathrm{Hom}_R(B,C) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Gamma_{A_1,B,C}} & & \downarrow{\scriptstyle \Gamma_{A_0,B,C}} & & \downarrow{\scriptstyle \Gamma_{A,B,C}} & & \\ \mathrm{Hom}_R(B, A_1 \otimes C) & \longrightarrow & \mathrm{Hom}_R(B, A_0 \otimes C) & \longrightarrow & \mathrm{Hom}_R(B, A \otimes C) & \longrightarrow & 0 \end{array}$$

(The bottom row is exact because B is projective.) Then (a) and (b) now follow from the Four Lemma and the Five Lemma, respectively.

(v) and (vi) are very similar to (iv) and are left to the reader.

(vii) Let $0 \to A \to A_0 \to A_1$ be an exact sequence with A_0 free. Then $\Gamma_{A_0,B,C}$ is injective by (ii), hence $\Gamma_{A,B,C}$ is injective by (v), since $\mathrm{Hom}_R(B,C)$ is flat.

(viii) Let $0 \to A \to A_0 \to A_1$ be an exact sequence with A_0 flat and let $B_1 \to B_0 \to B \to 0$ be a projective resolution with B_0 finitely generated. Then $\Gamma_{A_0,B_0,C}$ is bijective by (i)-(c), hence $\Gamma_{A_0,B,C}$ is injective (by (vi), since A_0 is flat), so $\Gamma_{A,B,C}$ is injective (by (v), since $\mathrm{Hom}_R(B,C)$ is flat).

(ix) Let $B_1 \to B_0 \to B \to 0$ be an exact sequence with B_1 and B_0 being finite projective. Then $\Gamma_{A,B_1,C}$ and $\Gamma_{A,B_0,C}$ are bijective by (i)-(c), hence $\Gamma_{A,B,C}$ is bijective (by (vi), since A is flat). □

Corollary 3.5.3. *Let $M \in$ Mod-R, $W = \mathrm{End}(M_R)$ and $\alpha \in \mathrm{End}^-(W)$. Then:*

(i) *If M_R or M^α are finite projective, then b_α is right regular.*
(ii) *If M_R is projective and M^α is f.g., then b_α is right surjective.*
(iii) *If M^α embeds in a free right W-module, then b_α is right injective.*
(iv) *If M^α embeds in flat right W-module and M_R is f.g., then b_α is right injective.*
(v) *If M^α is flat and M_R is f.p., then b_α is right regular.*

Proof. By Lemma 3.5.1, that b_α is bijective (injective, surjective) is equivalent to $\Gamma_{M^\alpha,M,M}$ being bijective (injective, surjective). Observe that $\mathrm{Hom}_R(M_R, {}_W M_R) \cong W_W$ and hence $\mathrm{Hom}_R(M,M)$ is flat. Parts (i)-(v) of the corollary now follow from parts (i), (iii), (vii), (viii) and (ix) of Lemma 3.5.2, respectively. □

Remark 3.5.4. Let α be an anti-*auto*morphism of W, then M^α is (resp.: embeds in) a free/projective W-module if and only if M is. Since any flat module is a direct limit of f.g. free modules (see [**60**]) and twisting commutes with direct limits, the previous assertion holds upon replacing “free” with “flat”.

We now get the following remarkable result.

Theorem 3.5.5. *If M_R is finite projective then there exists a one-to-one correspondence between* $\mathrm{Iso}(\mathrm{Bil}_{\mathrm{gen}}(M))$ *and* $\mathrm{End}^-(W)$ *as in* (14). *Moreover, all generic forms on M are right regular.*[8]

Proof. This is clear from Corollary 3.5.3(i). □

In addition, we now have another proof for the second part of Corollary 3.3.11.

Theorem 3.5.6. *If M_R is a generator and α is an anti-*automorphism* of W, then b_α is regular. In particular, there is a one-to-one correspondence between regular generic forms on M, considered up to similarity, and anti-*automorphisms* of W.*

[8] However, generic forms on M need not be left regular! Just take α to be non-bijective.

PROOF. It is well known that ${}_W M$ is finite projective (see [**80**, Exer. 4.1.14]; this follows from the discussion before Proposition 3.3.9). Thus, M^α_W is finite projective, so we are done by Corollary 3.5.3(i) (b_α is left regular by symmetry, as explained at the end of section 3.2). □

The following example includes additional cases when b_α is right regular or right injective for all $\alpha \in \mathrm{Aut}^-(W)$.

EXAMPLE 3.5.7. (i) If W is semisimple, then any W-module is projective. In particular, M^α is projective, hence by Corollary 3.5.3(iii), b_α is right injective for all $\alpha \in \mathrm{End}^-(M)$. For example, it is well known that W is semisimple if M is semisimple and f.g. (hence b_α is actually right regular in this case). In addition, by [**58**, Thms. 13.1, 13.3], W is semisimple if M is quasi-injective[9] (abbrev.: QI) non-singular[10] of finite uniform dimension.[11]

(ii) More generally, if W is von-Neumann regular,[12] then any W-module is flat ([**58**, Th. 4.21]) and in particular M^α. It follows that if M_R is f.g. (f.p.), then b_α is right injective (regular). For example, $W = \mathrm{End}_R(M)$ is von Neumann regular if M is QI and non-singular by [**58**, Th. 13.1].

(iii) A ring R is called right *pseudo-Frobenius* (abbrev.: PF) if any faithful right R-module is a generator (this is equivalent to R being right self-injective, semilocal and $\mathrm{soc}(R_R) \subseteq_e R$; see [**58**, Th. 19.25]). Hence, provided $M \in \text{Mod-}R$ is faithful, we can apply Theorem 3.5.6 to assert that b_α is regular for all $\alpha \in \mathrm{Aut}^-(W)$. Any semisimple ring or a local artinian ring with a simple right socle is a two-sided PF.

(iv) If R is a Dedekind domain and M_R is f.g. then M_R is generator over $R/\mathrm{ann}(M)$. This follows from classification of f.g. modules over Dedekind domains (e.g. see [**72**, Th. 4.14]). Therefore, as in (iii), b_α is regular for all $\alpha \in \mathrm{Aut}^-(W)$.

3.6. Conditions That Imply b_α Is Right Injective – Commutative Localization

Let M, R and $W = \mathrm{End}_R(M)$ be as in the previous section. In the following two sections, we will provide conditions on R, M, W and $\alpha \in \mathrm{End}^-(W)$ ensuring that b_α is right injective.

The results of both sections will be based on the following lemma.

LEMMA 3.6.1. *Let M, R, W and α be as above. Assume that there are rings $R' \supseteq R$, $W' \supseteq W$ and a (W', R')-bimodule M' containing M as a (W, R)-bimodule such that $W' = \mathrm{End}_{R'}(M')$. Furthermore, assume α extends to an anti-endomorphism of W', denoted α'. Then $b_{\alpha'}$ is right injective implies is b_α right injective.*

---

[9] A module M_R is QI if any homomorphism from a submodule of M to M can be extended to an endomorphism of M. Any injective module is QI, but not vice versa. For example, $\mathbb{Z}/p^n \in \text{Mod-}\mathbb{Z}$ is QI but not injective for any prime $p \in \mathbb{N}$. See [**58**, §6G]

[10] A module M_R is called non-singular if $\mathrm{ann}_R(m)$ is not essential in R for all m. For example, the non-singular $\mathbb{Z}$-modules are precisely the torsion free modules. See [**58**, §7].

[11] The uniform dimension of a module M_R, denoted $\mathrm{u.dim}\, M_R$ is defined to be the largest n s.t. M contains a direct sum of n non-zero modules. For example, any module containing an essential noetherian submodule has a finite uniform dimension. See [**58**, §6].

[12] A ring W is von-Neumann regular if for all $x \in W$ there exists $y \in W$ such that $xyx = x$. The endomorphism ring of a semisimple module is always von-Neumann regular.

Proof. Since M is a (W,R)-submodule of M', M^α is a (right) W-submodule of $(M')^{\alpha'}$. Therefore, we have a commutative square:

$$\begin{array}{ccc} M^\alpha \otimes_W \mathrm{Hom}_R(M, {}_W M) & \longrightarrow & (M')^{\alpha'} \otimes_{W'} \mathrm{Hom}_{R'}(M', {}_{W'} M') \\ \downarrow \Gamma & & \downarrow \Gamma \\ \mathrm{Hom}_R(M, M^\alpha \otimes_W M) & \longrightarrow & \mathrm{Hom}_{R'}(M', (M')^{\alpha'} \otimes_{W'} M') \end{array}$$

Since the top arrow is injective (it is just the inclusion $M^\alpha \to (M')^{\alpha'}$), that $\Gamma_{(M')^{\alpha'},M',M'}$ is injective implies $\Gamma_{M^\alpha,M,M}$ is injective, so we are through by Lemma 3.5.1. □

Our strategy in this section will be to take R' of Lemma 3.6.1 to be a localization of R. For that purpose, we shall now briefly recall the basic properties of (classical non-commutative) localization (also known as *Ore localization*). For a detailed discussion, see [**80**, §3.1] or [**58**, §10].

Let $S \subseteq R$ be a multiplicative monoid. A *classical right fractions ring of R (w.r.t. S)* is a ring R' equipped with a homomorphism $\varphi : R \to R'$ such that

(1) $\varphi(S) \subseteq (R')^\times$.
(2) $R' = \{\varphi(r)\varphi(s)^{-1} \,|\, r \in R, s \in S\}$.
(3) $\ker \varphi = \{r \in R \,|\, \exists s \in S : rs = 0\}$.

We will usually omit φ from the notation, writing rs^{-1} instead of $\varphi(r)\varphi(s)^{-1}$. The ring R' exists precisely when S is a *right denominator set*, namely (1) for all $s \in S$ and $r \in R$, $sR \cap rS \neq 0$ and (2) if $sr = 0$ for some $s \in S$ and $r \in R$, then there exists $s' \in S$ such that $rs' = 0$. (For example, if $S \subseteq \mathrm{Cent}(R)$, then S is a right denominator set.) In this case, R' is unique up to isomorphism and we write $RS^{-1} := R'$. Furthermore, there is an exact functor $M \mapsto MS^{-1}$ from Mod-R to Mod-RS^{-1} and a natural R-module homomorphism $M \to MS^{-1}$ with kernel $\{m \in M \,|\, \exists s \in S : ms = 0\}$. When this kernel is trivial, M is said to be *S-torsion-free*. We will not bring here the construction of RS^{-1} and MS^{-1}, but instead record the following two useful facts:

- If $x_1, \dots, x_n \in MS^{-1}$, then there exists $m_1, \dots, m_n \in M$ and $s \in S$ such that $x_i = m_i s^{-1}$ for all i.
- If $m_1, m_2 \in M$ and $s_1, s_2 \in S$, then $m_1 s_1^{-1} = m_2 s_2^{-1}$ (in MS^{-1}) if and only if there are $a_1, a_2 \in R$ such that $m_1 a_1 = m_2 a_2$ and $s_1 a_1 = s_2 a_2 \in S$. (Intuitively, this means $m_1 s_1^{-1} = (m_1 a_1)(s_1 a_1)^{-1} = (m_2 a_2)(s_2 a_2)^{-1} = m_2 s_2^{-1}$.)

Note that the standard map $R \to RS^{-1}$ is injective if and only if S consists of regular elements (i.e. non-zero-divisors). If RS^{-1} exists when S is the set of the *all* regular elements, then we say R is *right Ore* and call RS^{-1} the *classical right fractions ring of* R, which is denoted by $Q^r_{\mathrm{cl}}(R)$.

When S is a *left and right* denominator set of R, then rings RS^{-1} and $S^{-1}R$ coincide. We then call S a *denominator set*. In the special case where S consists of all regular elements in R, we get that $Q^r_{\mathrm{cl}}(R)$ coincides with $Q^\ell_{\mathrm{cl}}(R)$ (i.e. the classical *left* factions ring of R), provided both exist.

Our first step will be to establish that under mild assumptions, anti-endomorphisms of R extends to RS^{-1}.

Proposition 3.6.2. *Let S be a right denominator set of a ring W. Assume $\alpha \in \mathrm{End}^-(W)$ is such that $S^\alpha \subseteq S$. Then there is $\alpha' \in \mathrm{End}^-(WS^{-1})$ extending α. Moreover, if α is bijective and $S^\alpha = S$, then α' is bijective (in this case S is necessarily a* two-sided *denominator set).*

Proof. Define $\alpha' : W' \to W'$ by $\alpha'(ws^{-1}) = (s^\alpha)^{-1}w^\alpha$. We need to check that α' is a well-defined anti-automorphism of WS^{-1}.

To see that α' is well-defined, assume $w_1s_1^{-1} = w_2s_2^{-1}$. Then there are $a_1, a_2 \in W$ such that $s_1a_1 = s_2a_2 \in S$ and $w_1a_1 = w_2a_2$. In addition, there are $v \in W$ and $t \in S$ such that $(s_1^\alpha)^{-1}w_1^\alpha = vt^{-1}$, implying $w_1^\alpha t = s_1^\alpha v$. Therefore, $(s_1a_1)^\alpha v = a_1^\alpha s_1^\alpha v = a_1^\alpha w_1^\alpha t = (w_1a_1)^\alpha t$, which means $(s_1^\alpha)^{-1}w_1^\alpha = vt^{-1} = ((a_1s_1)^\alpha)^{-1}(a_1w_1)$ (recall that $s_1a_1 \in S$ and $S^\alpha \subseteq S$). Similarly $(s_2^\alpha)^{-1}w_2 = ((a_2s_2)^\alpha)^{-1}(a_2w_2)$, hence $(w_1s_1^{-1})^\alpha = (w_2s_2^{-1})^\alpha$ (becase $s_1a_1 = s_2a_2$ and $w_1a_1 = w_2a_2$).

Next, we need to verify that α is an anti-endomorphism of W'. That $1^\alpha = 1$ is clear. Let $x, y \in W$. Then there is $s \in S$ and $w_1, w_2 \in W$ such that $x = w_1s^{-1}$ and $y = w_2s^{-1}$. It is now easy to see that $(x+y)^\alpha = x^\alpha + y^\alpha$. In addition, there exist $v \in W$ and $t \in S$ such that $s^{-1}w_2 = vt^{-1} \implies w_2t = sv \implies t^\alpha w_2^\alpha = v^\alpha s^\alpha \implies w_2^\alpha(s^\alpha)^{-1} = (t^\alpha)^{-1}v^\alpha$. We now get:

$$\begin{aligned}(xy)^\alpha &= (w_1s^{-1}w_2s^{-1})^\alpha = (w_1v(st)^{-1})^\alpha = ((st)^\alpha)^{-1}(w_1v)^\alpha \\ &= (s^\alpha)^{-1}(t^\alpha)^{-1}v^\alpha w_1^\alpha = (s^\alpha)^{-1}w_2^\alpha(s^\alpha)^{-1}w_1^\alpha = y^\alpha x^\alpha\ ,\end{aligned}$$

so α reverses order of multiplication.

If α is bijective and $S^\alpha = S$, then we can extend α^{-1} to WS^{-1} as we did with α. It is straightforward to check that this extension is the inverse of α'. □

Keeping our general assumptions on R, M and W, assume S is a right denominator set in R and M is S-torsion free. Then any endomorphism of M naturally extends to an RS^{-1}-endomorphism of $M' := MS^{-1}$, hence we can view W as a subring of $W' := \mathrm{End}_{RS^{-1}}(M')$ (the map $W \to W'$ has trivial kernel because $M \hookrightarrow MS^{-1}$). The following proposition is well known.

Proposition 3.6.3. *In the previous notation, if S is central in R and M is f.g., then $W' = W\widehat{S}^{-1}$ where $\widehat{S} = \{m \mapsto ms \,|\, s \in S\}$.*

Proof. For all $s \in S$, write $\widehat{s} = [m \mapsto ms] \in W$. Then $\widehat{s}$ is clearly invertible in W' (its inverse is the map $x \mapsto xs^{-1} \in W'$). Now let $\{x_1, \dots, x_t\}$ be a set of generators for M and let $w' \in W'$. Then there are $m_1, \dots, m_t \in M$ and $s \in S$ such that $w'x_i = m_is^{-1}$ for all i. Therefore, $w'\widehat{s}(M) \subseteq M$, implying $w'\widehat{s} \in W$. It is left to verify that if $\widehat{s}w' = 0$ for some $s \in S$ and $w' \in W$, then $w'\widehat{t} = 0$ for some $t \in S$. However, this is trivial because $\widehat{S}$ consists of regular elements. (Indeed, $W \hookrightarrow W'$ and $\widehat{S} \subseteq (W')^\times$.) □

Corollary 3.6.4. *Under the assumptions of Proposition 3.6.3, if $\alpha \in \mathrm{End}^-(W)$ is such that $(\widehat{S})^\alpha \subseteq \widehat{S}$, then α extends to an anti-endomorphism $\alpha' : W' \to W'$. Furthermore, if $b_{\alpha'}$ is right injective (e.g., if MS^{-1} is finite projective over RS^{-1}), then so is b_α.*

Proof. This follows from Propositions 3.6.3, 3.6.2 and Lemma 3.6.1. □

The condition $(\widehat{S})^\alpha \subseteq \widehat{S}$ of the last corollary is usually very limiting. Our next results concern special cases in which it can be dropped.

Corollary 3.6.5. *Keep the assumptions of Proposition 3.6.3 and assume all regular central elements of W' are invertible (e.g. if W' is artinian, local or simple). Then any $\alpha \in \mathrm{Aut}^-(W)$ extends to an anti-automorphism $\alpha' : W' \to W'$ and if $b_{\alpha'}$ is right injective, then so is b_α.*

Proof. By Proposition 3.6.3, W' is a localization of W w.r.t. some right denominator set consisting of central regular elements. Therefore, the assumptions on W' imply it is also the localization of R w.r.t. the set T of *all* central regular elements in W. Since α is bijective, $\alpha(T) = T$ and hence Proposition 3.6.2 implies that α extends to some $\alpha' \in \mathrm{Aut}^-(W')$. We are done by Lemma 3.6.1. □

LEMMA 3.6.6. *Let M be a faithfully balanced R-module and let $W = \mathrm{End}(M_R)$. Then* $\mathrm{Cent}(R) \cong \mathrm{Cent}(W)$.

PROOF. Let $r \in \mathrm{Cent}(R)$. Then the map $w_r : m \mapsto mr$ from M to itself clearly lies in $\mathrm{Cent}(W)$. Similarly, for all $w \in W$, the map $r_w : m \mapsto wm$ lies in $\mathrm{Cent}(R)$ (since M_R is faithfully balanced). For all $m \in M$ and $r \in \mathrm{Cent}(R)$ we have $mr = w_r m = mr_{w_r}$, hence $r = r_{w_r}$. Similarly, $w = w_{r_w}$ for all $w \in W$. The map $r \mapsto w_r : \mathrm{Cent}(R) \to \mathrm{Cent}(W)$ is easily seen to be a ring homomorphism, so we are through. □

COROLLARY 3.6.7. *Keep the assumptions of Proposition 3.6.3 and assume that:*

(1) *S is the set of all regular elements in* $\mathrm{Cent}(R)$. *(Caution: This need not be the set of central regular elements in R.)*
(2) *MS^{-1} is a balanced RS^{-1}-module (e.g. if MS^{-1} is an RS^{-1}-generator).*

Then the assertions of Corollary 3.6.5 apply.

PROOF. All regular elements of $\mathrm{Cent}(RS^{-1})$ are invertible, so by the previous lemma all regular elements of $\mathrm{Cent}(W')$ are invertible. Since $W' = W\widehat{S}^{-1}$ and $\widehat{S}$ is a set of regular central elements of W, it follows that $W' = WT^{-1}$ where T is the set of all regular elements in $\mathrm{Cent}(W)$. The set T is preserved under any anti-automorphism of W, so we can argue as in Corollary 3.6.5. □

We conclude some special cases of the previous corollaries in the following theorem.

THEOREM 3.6.8. *Let $M \in$ Mod-R be f.g., let $W = \mathrm{End}(M_R)$ and let S be a central multiplicative submonoid of R such that M is S-torsion-free. Assume that at least one of the following holds:*

(i) *All central regular elements of* $\mathrm{End}(MS^{-1})$ *are invertible and MS^{-1} is either finite projective or a generator in* Mod-RS^{-1}.
(ii) *S consists of all regular elements of* $\mathrm{Cent}(R)$ *and MS^{-1} is a generator.*

Then for any $\alpha \in \mathrm{Aut}^- W$, b_α is injective. In particular, there is a one-to-one correspondence between stable generic forms on M, considered up to similarity, and anti-automorphisms of W. Furthermore, any generic form on M is injective.

PROOF. Let $W' = \mathrm{End}_{RS^{-1}}(MS^{-1})$. Then Corollaries 3.6.5 and 3.6.7 imply that any $\alpha \in \mathrm{Aut}^-(W)$ extends to an $\alpha' \in \mathrm{Aut}^- W'$ (note that when (ii) holds, $MS^{-1}_{RS^{-1}}$ is faithfully balanced because it is a generator). Since MS^{-1} is a generator or finite projective as an RS^{-1}-module, $b_{\alpha'}$ is right regular (Theorems 3.5.5 and 3.5.6). Therefore, b_α is right injective and by symmetry, it is also left injective. □

EXAMPLE 3.6.9. (i) Let C be a integral domain with fractions field F and let A be a f.d. F-algebra. A *C-order* in A is a C-subalgebra $R \subseteq A$ such that $FR = A$. Such orders are extensively studied in the literature, especially when A is semisimple (e.g. see the classical works [**72**], [**92**] and related papers). If R is a C-order in A, then clearly $A = RS^{-1}$ where $S := C \setminus \{0\}$. Furthermore, for any f.g. S-torsion-free right R-module M, the ring $W' := \mathrm{End}_A(MS^{-1})$ is a f.d. F-algebra, hence all regular elements in W' are invertible. Therefore, that if MS^{-1} is projective or a generator as an A-module, then the assertions of Theorem 3.6.8 apply. Also note that when A is semisimple, any f.g. A-module is finite projective and when A is *quasi-Frobenius* (e.g. a finite group algebra), any faithful A-module is a generator.

(ii) Generalizing (i), let R be a ring and let S be a central multiplicative submonoid such that RS^{-1} is π-regular (e.g. a semiprimary ring; see section 1.2). Let

M be a f.p. S-torsion-free R-module. Then MS^{-1} is also f.p., hence by Corollary 1.7.3, $W' := \mathrm{End}(MS^{-1})$ is π-regular. By [**58**, Ex. 11.6(2)], this implies all regular elements in W' are invertible, so we can apply Theorem 3.6.8 if MS^{-1} is finite projective or a generator.

(iii) Let C and F be as in (i), let R be any torsion-free C-algebra and let $A = R\otimes_C F = R(C\setminus\{0\})^{-1}$. Write $S = C\setminus\{0\}$. Then the endomorphism ring of any R-module is a C-algebra. If α is anti-endomorphism of $\mathrm{End}(M)$ that respects the C-algebra structure, then $\alpha(\widehat{S})\subseteq\widehat{S}$. Thus, we can apply Corollary 3.6.4.

Remark 3.6.10. We could not find R, M, W and α as in Theorem 3.6.8 such that b_α is not regular. However, we believe such examples should exist.

The question of what happens when we localize R in a non-central set is more difficult and will be dealt in a more general context in the next section. Roughly speaking, we will show that the assertions of Theorem 3.6.8 hold when S is any two-sided denominator set and MS^{-1} is a *torsionless* RS^{-1}-generator (the latter is always satisfied when RS^{-1} is a right pseudo-Frobenius ring). However, when R is an *Ore domain*, we can provide an answer without introducing additional notation, and we shall now tend to this.

A right Ore domain is a domain that is right Ore (i.e. $Q^r_{\mathrm{cl}}(R)$ exists). It turns out that a *domain* R is right Ore $\iff$ $\mathrm{u.dim}\, R_R<\infty$ $\iff$ $\mathrm{u.dim}\, R_R = 1$ and that in this case $Q^r_{\mathrm{cl}}(R)$ is division ring. Examples of right Ore domains include right noetherian domains, PI domains and twisted polynomial rings over right Ore domains; see [**58**, §10B] for more details.

Proposition 3.6.11. *Let R be a (two-sided) Ore domain (e.g. a noetherian or a PI domain), let $S = R\setminus\{0\}$ and let M be a f.g. S-torsion-free R-module. Then* $\mathrm{End}_{RS^{-1}}(MS^{-1}) = Q^r_{\mathrm{cl}}(\mathrm{End}_R(M)) = Q^\ell_{\mathrm{cl}}(\mathrm{End}_R(M))$ *and b_α is injective for all* $\alpha\in\mathrm{Aut}^-(\mathrm{End}(RS^{-1}))$.

Proof. Let $D = Q^r_{\mathrm{cl}}(R) = Q^\ell_{\mathrm{cl}}(R)$. Then D is a division ring, hence $V := MS^{-1}$ is a f.d. right D-vector space. Let $t = \dim V_D$.

We first claim that for any f.g. module $N_R\leq V_R$ there is a D-basis $\{e_1,\dots,e_t\}$ such that $N\subseteq e_1R+\dots+e_tR$. Indeed, let $x_1,\dots,x_s\in N$ be a set of generators for N and let $\{e_1,\dots,e_t\}$ be an arbitrary D-basis of V. Then we can write $x_j = \sum_{i=1}^t e_id_{ij}$ with $d_{ij}\in D$. Since R is left Ore, there is $s\in S$ and r_{ij}-s in R such that $d_{ij} = s^{-1}r_{ij}$. Therefore, $x_j = \sum_{i=1}^t(e_is^{-1})r_{ij}\in\sum_i e_is^{-1}R$, so $\{e_1s^{-1},\dots,e_ts^{-1}\}$ is the required basis.

For the rest of the proof, let $\{m_1,\cdots,m_s\}$ be a set of generators of M. Clearly $s\geq t$ and w.l.o.g. we may assume $\{m_1,\dots,m_t\}$ is a D-basis of V. In addition, let $W = \mathrm{End}(M)$, $W' = \mathrm{End}(MS^{-1})$ and let T be the set of regular elements in W. We will consider elements of W as elements of W'. Observe that under this identification, an element $w\in W'$ lies in W if and only $w(M)\subseteq M$. In order to show $W' = Q^r_{\mathrm{cl}}(W) = Q^\ell_{\mathrm{cl}}(W)$, we need to prove $T\subseteq(W')^\times$ and

$$W' = \{wu^{-1}\,|\,w\in W, u\in T\} = \{u^{-1}w\,|\,w\in W, u\in T\}\ .$$

Let $u\in W\setminus(W')^\times$. Then there is $0\neq x\in V$ such that $u(x)$. Since $V = MS^{-1}$, there is $s\in S$ such that $0\neq xs\in M$. Let $\{e_1,\dots,e_t\}$ be a D-basis such that $M\subseteq\sum e_iR$ and let $w: V\to V$ be defined by $w(e_i) = xs$. Then $w(M)\subseteq M$, hence $w\in W$. Since clearly $uw = 0$, it follows that $u\notin T$. Therefore, $T\subseteq(W')^\times$

Let $w\in W'$. Then $w(M)+M$ is f.g., hence there is a D-basis $\{e_1,\dots,e_t\}$ such that $w(M)+M\subseteq\sum e_iR$. Define $u: V\to V$ to be the unique D-linear map satisfying $u(e_i) = m_i$. Then u lies in W (because $u(M)\subseteq M$) and it is clearly invertible in W', thus $u\in T$. Furthermore, $uw(M)\subseteq M$, hence $uw\in W$. It now follows that $W' = \{u^{-1}w\,|\,w\in W, u\in T\}$.

Now observe that $w^{-1}(M) \subseteq_e V_R$ (because $M \subseteq_e V_R$), and since the intersection of essential modules is also essential, $N := w^{-1}(M) \cap M \subseteq_e V_R$. This implies $\mathrm{u.dim}\, N_R = \mathrm{u.dim}\, V_R = \mathrm{u.dim}\, V_D = t$ (see [**58**, Cr. 6.10(2), Exer. 10.18(5)]). Therefore, there are $0 \neq e_1, \dots, e_t \in N$ such that $e_1R \oplus \dots \oplus e_tR \subseteq N$. Note that $\{e_1, \dots, e_t\}$ must be a D-basis. Now let $\{e'_1, \dots, e'_t\}$ be a D-basis such that $M \subseteq \sum e'_i R$ and let $u : V \to V$ be defined by $u(\sum e'_i d_i) = \sum e_i d_i$. Then $u(M) \subseteq M$ and $u \in (W')^\times$, hence $u \in T$. Moreover, $wu(M) \subseteq w(N) \subseteq M$, so $wu \in W$. Thus, $W' = \{wu^{-1} \,|\, w \in W, u \in T\}$ and we conclude that $W' = Q^r_{\mathrm{cl}}(W) = Q^\ell_{\mathrm{cl}}(W)$.

To see the final assertion of the proposition, let $\alpha \in \mathrm{Aut}^-(W)$. Then $\alpha(T) = T$, so by Proposition 3.6.2, α extends to an anti-endomorphism $\alpha' \in \mathrm{End}^-(W')$. By Theorem 3.5.5, $b_{\alpha'}$ is right regular by (since V_D is finite projective), hence by Lemma 3.6.1, b_α is right injective. By symmetry, b_α is also left injective. □

3.7. Conditions That Imply b_α Is Right Injective – General Case

This section generalizes the results of the previous section to non-commutative localizations and, more generally, to *rational extensions*. Reading this section requires basic knowledge about rational extensions, maximal rings of quotients and pseudo-Frobenius rings (abbrev.: PF rings). Non-experts are advised to read the preliminaries chapter before continuing, if they have not done so yet. Alternatively, one can skip this section without loss of continuity.

3.7.1. Preliminaries. We begin by recalling some facts about rational extensions of rings and modules. For an extensive discussion and definitions we refer to [**80**] and [**58**]. Our notation and terminology mostly follow the latter reference.

Let $M \in \text{Mod-}R$ and $N \leq M$. Throughout, $N \subseteq_d M$ ($N \subseteq_e M$) means that N is *dense*[13] (*essential*[14]) in M, or equivalently, that M is a *rational (essential) extension* of N. For all $x \in M$, $x^{-1}N$ denotes the right ideal $\{r \in R : xr \in N\}$. It is well known that $N \subseteq_d M$ ($N \subseteq_e M$) implies $x^{-1}N \subseteq_d R_R$ ($x^{-1}N \subseteq_e R_R$). We define the following submodules of M:

$$\mathcal{Z}(M) = \{x \in M \,:\, \mathrm{ann}_R\, x \subseteq_e R_R\}, \qquad \mathcal{T}(M) = \{x \in M \,:\, \mathrm{ann}_R\, x \subseteq_d R_R\}\ .$$

The module $\mathcal{Z}(M)$ is called the *singular radical* of M and M (resp. R) is called (right) nonsingular if $\mathcal{Z}(M) = 0$ (resp. $\mathcal{Z}(R_R) = 0$).[15] The *rational (injective) hull* of M will be denoted by $\widetilde{E}(M)$ ($E(M)$).

A ring Q containing R will be called a *right quotient ring* of R if $R_R \subseteq_d Q_R$.[16] If Q is both a left and right quotient ring of R, then Q will be called a *quotient ring* of R. We let $Q^r_{\max}(R)$ ($Q^\ell_{\max}(R)$) denote the *maximal right (left) quotient ring* of R. Recall that this ring is maximal in the sense that if Q' is any other right quotient ring of R, then there exists a unique embedding of Q' in $Q^r_{\max}(R)$ that fixes R.

We will need the following facts to proceed:

Proposition 3.7.1. *Let $M, N, K \in \text{Mod-}R$.*

(i) *If $f \in \mathrm{Hom}(N, M)$, then $f(\mathcal{Z}(N)) \subseteq \mathcal{Z}(M)$ and $f(\mathcal{T}(N)) \subseteq f(\mathcal{T}(M))$.*
(ii) *If $N \subseteq_e M$ and $\mathcal{T}(N) = 0$ ($\mathcal{Z}(N) = 0$), then $\mathcal{T}(M) = 0$, ($\mathcal{Z}(M) = 0$)*

---

[13] Recall that N is dense in M if for all $x, y \in M$ with $x \neq 0$ there exists $r \in R$ such that $xr \neq 0$ and $yr \in N$. This is equivalent to $\mathrm{Hom}(N'/N, M) = 0$ for all $N \subseteq N' \subseteq M$.

[14] The names "big" and "large" are also used in the literature.

[15] Other texts use $\mathrm{Sing}(M)$ instead of $\mathcal{Z}(M)$.

[16] The term "quotient ring" usually refers to a quotient of the ring by an ideal (i.e. an epimorphic image of the ring) and thus many authors prefer to use "ring of quotients" instead of "quotient ring" (which usually leads to cumbersome phrasing). However, as we do not consider quotients of rings by ideals anywhere in this section, there is no risk of confusion.

(iii) *If $N \subseteq_d M$ and $\mathcal{T}(K) = 0$, then any homomorphism $f \in \mathrm{Hom}(N, K)$ extends uniquely to $f' \in \mathrm{Hom}(M, \widetilde{E}(K))$. In particular, if $f(N) = 0$, then $f' = 0$.*

(iv) *Let Q be a right quotient ring of R and assume M and N have a right Q-module structure extending their right R-module structure. Then provided $\mathcal{T}(M_R) = 0$, $\mathrm{Hom}_Q(N, M) = \mathrm{Hom}_R(N, M)$. Moreover, the Q-module structure on M is the only one extending its R-module structure.*

(v) *Let Q, N, M be as in (iv) and assume $\mathcal{T}(N_R) = 0$. Then $N_Q \subseteq_d M_Q$ if and only if $N_R \subseteq_d M_R$.*

(vi) *If $\mathcal{Z}(M) = 0$ and $N, K \subseteq M$, then $\widetilde{E}(M) = E(M)$ and $N \subseteq_d K$ if and only if $N \subseteq_e K$. In particular, if R is right nonsingular, then $\mathcal{T}(M) = \mathcal{Z}(M)$ for all M.*

(vii) *If M embeds in a free product $\prod_{i\in I} R_R$, then $\mathcal{T}(M) = 0$.*

(viii) *Let Q be a right quotient ring of R and let Q' be a right quotient ring of Q. Then Q' is a right quotient ring of R. In particular, $Q^r_{\max}(Q^r_{\max}(R)) = Q^r_{\max}(R)$.*

Proof. (i) This is immediate since $\mathrm{ann}_R(fx) \supseteq \mathrm{ann}_R\, x$ for all $x \in N$.

(ii) We have $0 = \mathcal{T}(N) = N \cap \mathcal{T}(M)$. Since $N \subseteq_e M$, this implies $\mathcal{T}(M) = 0$. The argument remains valid upon replacing $\mathcal{T}$ with $\mathcal{Z}$.

(iii) Let f' and f'' be two extensions of f. Then there is a nonzero homomorphism $g : M/N \to K$ given by $g(x + N) = f'(x) - f''(x)$. For all $x \in M$, $\mathrm{ann}_R(x + N) = x^{-1}N \subseteq_d R_R$, hence $x + M \in \mathcal{T}(M/N)$ and by (i) $g(x + M) \in \mathcal{T}(K) = 0$. As this holds for all $x \in M$, $g = 0$ and $f' = f''$.

To see that f exists, we may assume $K = \widetilde{E}(K)$. By [**58**, Th. 8.24], it is now enough to prove that $\mathrm{Hom}(M/N, E(K)) = 0$. Indeed, the previous argument implies $\mathcal{T}(M/N) = M/N$ and (ii) implies $\mathcal{T}(E(K)) = 0$, so we are done by (i).

(vi) Let $f \in \mathrm{Hom}_R(N, M)$. We need to prove $f \in \mathrm{Hom}_Q(N, M)$ (the opposite implication is clear). Let $x \in N$ and define $h : Q \to M$ by $h(q) = f(xq) - f(x)q$. Then h is an R-module homomorphism and $h(R) = 0$. Therefore, by (iii), $h = 0$, hence f is Q-linear. To see that the Q-module structure on M is unique, put $N = Q_Q$ and observe that there is a natural isomorphism $M_Q \cong \mathrm{Hom}_Q({}_QQ_Q, M_Q) = \mathrm{Hom}_R({}_QQ_R, M_R)$. Therefore, the Q-module structure on M is induced from the Q-module structure on $\mathrm{Hom}_R({}_QQ_R, M_R)$, which depends only on M_R.[17]

(v) Clearly $N_R \subseteq_d M_R$ implies $N_Q \subseteq_d M_Q$ (this holds for any Q containing R). To see the converse, assume $N_Q \subseteq_d M_Q$ and let $x, y \in M$ with $x \neq 0$. Then there is $q \in Q$ such that $xq \neq 0$ and $xq, yq \in N$. Let $A = q^{-1}R \subseteq_d R_R$ and observe that $xqA \neq 0$ (because $\mathcal{T}(N_R) = 0$). Therefore, there is $a \in A$ such that $x(qa) \neq 0$ and $y(qa) \in N$. We are done because $qa \in R$.

(vi) See [**58**, Ex. 8.18(5) and Prp. 8.7].

(vii) It is enough to prove $\mathcal{T}(R_R) = 0$. Indeed, let $0 \neq u \in R$ and let $x = u$, $y = 1$. Then $yr \in \mathrm{ann}^r\, u$ implies $xr = 0$, hence $\mathrm{ann}^r\, u \not\subseteq_d R_R$.

(viii) By (vii), $\mathcal{T}(R_R) = 0$, hence by (i), $\mathcal{T}(Q_R) = 0$ (because $R \subseteq_d Q$), thus by (v), $Q_R \subseteq_d Q'_R$ (because $Q_Q \subseteq_d Q'_Q$), so $R_R \subseteq_d Q'_R$. □

Part (vii) of Proposition 3.7.1 calls for the following definition, which is taken from [**58**].

Definition 3.7.2. *A right R-module M is called* torsionless *if it embeds in some free product $\prod_{i\in I} R_R$. Equivalently, M is torsionless if for all $x \in M$ there exists $f \in \mathrm{Hom}_R(M, R_R)$ with $f(x) \neq 0$.*

[17] This can also be proved directly: If M admits two Q-module structures $\diamond_1, \diamond_2 : M \times Q \to M$ that extend its R-module structure, then for all $x \in M$ define $h : Q \to M$ by $h(q) = x \diamond_1 q - x \diamond_2 q$. Then h is an R-module homomorphism vanishing on R, so it must be 0.

Recall that a module $M \in \text{Mod-}R$ is called a cogenerator if for all $A, B \in \text{Mod-}R$ and $0 \neq f \in \text{Hom}(A,B)$, there exists $g \in \text{Hom}(B,M)$ such that $g \circ f \neq 0$. All right R-modules are torsionless precisely when R_R is a cogenerator. Such rings are called *right cogenerator rings*.

3.7.2. Localizing at a Quotient Ring. In the previous section, we have considered localization of modules w.r.t. some denominator set. We shall now introduce a generalization of this process that works for arbitrary right quotient ring of R. Namely, for any right quotient ring Q of R, we will construct a functor $M \mapsto MQ$ defined on modules $M \in \text{Mod-}R$ with $\mathcal{T}(M) = 0$. To do this, we first recall the following fact.

PROPOSITION 3.7.3. *For all $M \in \text{Mod-}R$ with $\mathcal{T}(M) = 0$, $\widetilde{E}(M)$ can be endowed with a unique $Q^r_{\max}(R)$-module structure extending its R-module structure.*

PROOF (SKETCH). [18] If we can show existence, then the uniqueness is guaranteed by Proposition 3.7.1(iv).

Let M be an arbitrary right R-module. Consider the set X of pairs (A, f) where $A \subseteq_d R_R$ and $f \in \text{Hom}_R(A, M)$ and define an equivalence relation on X by $(A, f) \sim (B, g)$ if f and g agree on some dense right ideal of R. Let $\widehat{M} = \{[A, f] \,|\, (A, f) \in X\}$ where $[A, f]$ is the equivalence class of (A, f). Then $\widehat{M}$ is a right R-module w.r.t. the operations:

$$[A, f] + [B, g] = [A \cap B, f|_{A\cap B} + g|_{A\cap B}], \qquad [A, f] \cdot r = [r^{-1}A, s \mapsto f(rs)]$$

and there is a natural R-module homomorphism $M \to \widehat{M}$ given by sending $m \in M$ to $[R_R, r \mapsto mr] \in \widehat{M}$. The kernel of this homomorphism is $\mathcal{T}(M)$, hence M embeds in $\widehat{M}$ if $\mathcal{T}(M) = 0$. It is well known that $Q^r_{\max}(R)$ can be understood as $\widehat{R}$ where the multiplication on $\widehat{R}$ is given by:

$$[A, f] \cdot [B, g] = [g^{-1}(A), f \circ g] \qquad \forall\ [A, f], [B, g] \in \widehat{R}\ .$$

By letting $[A, f]$ range over $\widehat{M}$ instead of $\widehat{R}$, the previous formula defines a right $Q^r_{\max}(R)$-module structure on $\widehat{M}$.

We now claim that when $\mathcal{T}(M) = 0$, $\widehat{M} \cong \widetilde{E}(M)$ as R-modules, so we can transfer the $Q^r_{\max}(R)$-module structure on $\widehat{M}$ to $\widetilde{E}(M)$. Indeed, for all $x \in \widetilde{E}(M)$, let $f_x : R \to \widetilde{E}(M)$ be given by $f_x(r) = xr$. Define $\psi : \widetilde{E}(M) \to \widehat{M}$ by $\psi(x) = [x^{-1}M, f_x]$ (note that $x^{-1}M \subseteq_d R_R$ because $M \subseteq_d \widetilde{E}(M)$). Then ψ is easily seen to be an R-module homomorphism. It is injective because if f_x and f_y agree on a dense right ideal then they must be equal by Proposition 3.7.1(iii), and it is surjective because for any $[A, f] \in \widehat{M}$, f extends to some $f' \in \text{Hom}(R_R, \widetilde{E}(M))$ (again by Proposition 3.7.1(iii)) and thus $f' = f_x$ for some $x \in \widetilde{E}(M)$. □

Let M be a right R-module with $\mathcal{T}(M) = 0$ and let Q be a right quotient ring of R. Then Propositions 3.7.3 and 3.7.1(iv) imply that $\widetilde{E}(M)$ has a unique Q-module structure extending its R-module structure. In this case, we define MQ to be the Q-submodule of $\widetilde{E}(M)$ generated by M. Part (iv) of the following proposition shows that MQ is a "correct" generalization of MS^{-1}.

PROPOSITION 3.7.4. *Let $\mathscr{M}_R$ denote the category of right R-modules M with $\mathcal{T}(M) = 0$. Then:*

(i) *The map $M \mapsto MQ$ is an additive functor from $\mathscr{M}_R$ to $\mathscr{M}_Q$.*

[18] We could not find a reference proving Proposition 3.7.3 explicitly (possibly due to a lack of applications); most textbooks on ring theory do not mention this fact (e.g. [**58**]), or prove it only when R is right nonsingular (e.g. [**80**]), or treat it implicitly under the more general (or too general) context of *torsion theories*.

(ii) *Any endomorphism of $M \in \mathscr{M}_R$ extends uniquely to an endomorphism of MQ_Q. In particular,* $\mathrm{End}_R(M)$ *embeds in* $\mathrm{End}_Q(MQ) = \mathrm{End}_R(MQ)$.

(iii) *If $M \in \mathscr{M}_R$ is a generator (resp. torsionless), then so is MQ_Q.*

(iv) *Let S be a right denominator set in R consisting of regular elements. Then any $M \in \mathscr{M}_R$ is S-torsion-free and $MRS^{-1} \cong MS^{-1}$ as RS^{-1}-modules.*

(v) $\widetilde{E}(M_R) = \widetilde{E}(MQ_Q)$.

Proof. (i) If $N, M \in \mathscr{M}_R$, then by Proposition 3.7.1(iii), any $f \in \mathrm{Hom}_R(N,M)$ extends uniquely to an R-homomorphism $f' : \widetilde{E}(N) \to \widetilde{E}(M)$, which is a $Q^r_{\max}(R)$-homomorphism by Proposition 3.7.1(iv). Therefore, f' maps NQ into MQ and it is routine to verify that $M \mapsto MQ$ becomes a functor once defining $fQ := f'|_{NQ}$.

To see that $M \mapsto MQ$ is additive, it is enough to prove that $\widetilde{E}(M_1 \oplus M_2) = \widetilde{E}(M_1) \oplus \widetilde{E}(M_2)$ for all $M_1, M_2 \in \mathscr{M}$.[19] Indeed, by [**58**, Prp. 8.19], $\widetilde{E}(M_1 \oplus M_2)$ embeds in $\widetilde{E}(M_1) \oplus \widetilde{E}(M_2)$ and $M_1 \oplus M_2$ is fixed under this embedding (this holds for arbitrary modules). Therefore, it is enough to prove $M_1 \oplus M_2 \subseteq_d \widetilde{E}(M_1) \oplus \widetilde{E}(M_2)$. Let $(x_1, x_2), (y_1, y_2) \in \widetilde{E}(M_1) \oplus \widetilde{E}(M_2)$ be such that $(x_1, x_2) \neq 0$. Then $y_1^{-1}M_1, y_2^{-1}M_2 \subseteq_d R_R$ and hence $A := y_1^{-1}M_1 \cap y_2^{-1}M_2 \subseteq_d R_R$. W.l.o.g. $x_1 \neq 0$ and therefore $A \not\subseteq_d \mathrm{ann}_R\, x_1$ (otherwise $x_1 \in \mathcal{T}(M_1) = 0$). Take some $r \in A \setminus \mathrm{ann}_R\, x_1$. Then r satisfies $(y_1, y_2)r \in M_1 \oplus M_2$ and $(x_1, x_2)r \neq 0$, so we are done.

(ii) This is immediate from (i) and Proposition 3.7.1(iii) (because $M_R \subseteq_d MQ_R$).

(iii) If M is a generator, then there is an epimorphism $f : M^n \to R_R$ for some n. Therefore, there is a homomorphism $fQ : MQ^n \to Q_Q$. Since $1 \in \mathrm{im}(f) \subseteq \mathrm{im}(fQ)$, fQ is onto, implying MQ is a Q-generator. Now assume M is torsionless. Then for all $m \in M$, there is a homomorphism $f_m : M \to R_R$ such that $f_m(m) \neq 0$. Define $\widehat{f} : MQ \to (Q_Q)^M$ by $\widehat{f}(x) = ((f_mQ)x)_{m\in M}$. Then $\ker \widehat{f} \cap M = 0$, hence $\ker \widehat{f} = 0$ (because $M_R \subseteq_e MQ_R$). Thus, MQ is torsionless.

(iv) Assume that $ms = 0$ for some $m \in M$ and $s \in S$. Then $sR \subseteq \mathrm{ann}_R\, m$. We claim $sR_R \subseteq_d R_R$ and therefore $m = 0$. Indeed, if $x, y \in R$ with $x \neq 0$, then $yS \cap sR \neq \phi$ (because S is a right denominator set) and hence there is $t \in S$ such that $yt \in sR$. As t is regular, $xt \neq 0$, so $sR_R \subseteq_d R_R$, as required.

Observe that $M \subseteq_d MS^{-1}$ as R-modules,[20] hence there is an embedding of R-modules $f : MS^{-1} \to \widetilde{E}(M)$ which is an RS^{-1}-homomorphism by Proposition 3.7.1(iv) (because $R_R \subseteq_d RS^{-1}_R$). It is now easy to see that the image of f is MRS^{-1}. Since $\ker f \cap M = 0$, it follows that $\ker f = 0$, hence $f : MS^{-1} \to MRS^{-1}$ is an isomorphism.

(v) By Proposition 3.7.1(v), any rational extension of MQ_Q is a rational extension of MQ_R and hence of M_R. Therefore, we can view $\widetilde{E}(MQ_Q)$ as an R-submodule of $\widetilde{E}(M)$ and by Proposition 3.7.1(iv), the former is in fact a Q-submodule of the latter. Now, $MQ_R \subseteq_d \widetilde{E}(M)_R$ implies $MQ_Q \subseteq_d \widetilde{E}(M)_Q$, so $\widetilde{E}(MQ_Q)_Q \subseteq_d \widetilde{E}(M_R)_Q$ and thus equality must hold. □

Remark 3.7.5. At this level of generality, we do not know whether $MQ \cong M \otimes_R Q$ or $MQ^r_{\max}(R) = \widetilde{E}(M)$. However, it is well known that $MS^{-1} \cong M \otimes_R RS^{-1}$.

3.7.3. The Maximal Symmetric Quotient Ring. Our next step would be to prove an analogue of Proposition 3.6.2 for right or left quotient rings (rather than

[19] Caution: In general, the module $\widetilde{E}(M_1 \oplus M_2)$ does not coincide with $\widetilde{E}(M_1) \oplus \widetilde{E}(M_2)$; see [**58**, Ex. 8.21].

[20] Indeed, let $x, y \in MS^{-1}$ with $x \neq 0$. Then we can write $y = ms^{-1}$ for some $m \in M$ and $s \in M$. Thus, $ys \in M$ and $xs \neq 0$ (because $xss^{-1} = x \neq 0$).

classical fraction rings). However, this turns out to be impossible, since an anti-automorphisms of R need not extend to $Q^r_{\max}(R)$. To overcome this, we introduce the *maximal symmetric quotient ring* of R.[21] This construction was apparently first suggested in [**87**] and was lengthly studied in [**59**]. As this notion rarely appears in textbooks, we shall include proofs regarding it.

LEMMA 3.7.6. *Let $R \subseteq Q$ be rings such that $R_R \subseteq_e Q_R$. Then Q is a* left *quotient ring of R if and only if for all $q \in Q$ there is a dense left ideal $A \subseteq_d {}_RR$ such that $Aq \subseteq R$.*

PROOF. Assume ${}_RR \subseteq_d {}_RQ$. Then for all $q \in Q$,

$$Rq^{-1} := \{r \in R \,:\, rq \in R\} \subseteq_d R_R \ .$$

Therefore, $A = Rq^{-1}$ is a dense left ideal satisfying $Aq \subseteq R$. To see the converse, let $x, y \in Q$ with $x \neq 0$. Then there is $A \subseteq_d {}_RR$ such that $Ay \subseteq R$. Since $R_R \subseteq_e Q_R$, there is $a \in R$ such that $0 \neq xa \in R$. Now, $Axa \neq 0$ (otherwise, $xa \in \mathcal{T}(R_R) = 0$), implying that there exists $r \in A$ such that $rxa \neq 0$. This r satisfies $rx \neq 0$ and $ry \in R$, so ${}_RR \subseteq_d {}_RQ$. □

PROPOSITION 3.7.7. *Let R be a ring and let $Q = Q^r_{\max}(R)$. Define*

$$Q^s_{\max}(R) := \{q \in Q \mid \exists A \subseteq_d {}_RR : Aq \subseteq R\} \ .$$

Then $Q^s_{\max}(R)$ is a (two-sided) quotient ring of R. Moreover, it is maximal in the sense that any other quotient ring of R, Q', admits a unique embedding into $Q^s_{\max}(R)$ (as extensions of R). Up to isomorphism, $Q^s_{\max}(R)$ is the only maximal quotient ring of R.

PROOF. Provided $Q^s_{\max}(R)$ is a ring, $Q^s_{\max}(R)$ is clearly a right quotient ring of R (because it is contained in $Q^r_{\max}(R)$) and by the previous lemma it is also a left quotient ring of R.

To see that $Q^s_{\max}(R)$ is a ring, let $q, q \in Q^s_{\max}(R)$. Then there are $A, A' \subseteq_d {}_RR$ such that $Aq, A'q' \subseteq R$. This implies $(A \cap A')(q+q') \subseteq R$, hence $q+q' \in Q^s_{\max}(R)$. In addition, $B := \{a \in A \mid aq \in A'\} \subseteq_d {}_RR$ (because $B = f^{-1}(A')$ where $f : A \to {}_RR$ is defined by $f(a) = aq$). This implies $Bqq' \subseteq A'q' \subseteq R$, so $qq' \in Q^s_{\max}(R)$, as required.

Now let Q' be any quotient ring of R. Then Q' is a right quotient ring, hence there exists a unique embedding $\varphi : Q' \to Q^r_{\max}(R)$ that fixes R. By Lemma 3.7.6, any $q \in Q'$ admits a left ideal $A \subseteq_d R_R$ such that $Aq \subseteq R$. This means that $A\varphi(q) \subseteq R$ and hence $\operatorname{im}\varphi \subseteq Q^s_{\max}(R)$, as required. The uniqueness of the embedding follows from Proposition 3.7.1(iii), since the embedding is an R-module homomorphism.

The maximal quotient ring of R is unique up to isomorphism because being a maximal ring of quotients is a universal property. Details are left to the reader. □

The ring $Q^s_{\max}(R)$ of the last proposition is called the *maximal symmetric quotient ring of* R. Observe that it can also be defined as a subring of $Q^\ell_{\max}(R)$ by $Q^s_{\max}(R) = \{q \in Q^\ell_{\max}(W) \mid \exists A \subseteq_d R_R : qA \subseteq R\}$. (In fact, $Q^s_{\max}(R)$ is the largest extension of R that is contained in both $Q^r_{\max}(R)$ and $Q^\ell_{\max}(R)$.) We now have the following:

PROPOSITION 3.7.8. *Let W be any ring. Then any $\alpha \in \operatorname{Aut}^-(W)$ extends uniquely into an anti-isomorphism $\alpha' : Q^r_{\max}(W) \to Q^\ell_{\max}(W)$. If one considers $Q^s_{\max}(W)$ as a subring of both $Q^r_{\max}(W)$ and $Q^\ell_{\max}(W)$, then α' restricts to an anti-automorphism of $Q^s_{\max}(W)$.*

[21] Note: The maximal symmetric quotient ring is not the *Martindale symmetric ring of quotients* which is often used when studying semiprime rings.

PROOF. Consider α as a ring isomorphism $\alpha : W \to W^{\rm op}$. Then it is well known that α extends to an isomorphism $\alpha' : Q^r_{\max}(W) \to Q^r_{\max}(W^{\rm op}) \cong Q^\ell_{\max}(W)^{\rm op}$ (note that the last isomorphism fixes $W^{\rm op}$). This gives rise to an anti-isomorphism $\alpha' : Q^r_{\max}(W) \to Q^\ell_{\max}(W)$ that extends α.

For all $q \in Q^r_{\max}(W)$ and $A \leq {}_W W$, $Aq \subseteq W \iff q^{\alpha'} A^\alpha \subseteq W$ and $A \subseteq_d {}_W W \iff A^\alpha \subseteq_d W_W$. Thus, $q \in Q^s_{\max}(W) \iff q^{\alpha'} \in Q^s_{\max}(W)$ (where the latter $Q^s_{\max}(W)$ is the copy of $Q^s_{\max}(W)$ inside $Q^\ell_{\max}(W)$). □

EXAMPLE 3.7.9. An anti-automorphism $\alpha \in \mathrm{Aut}^-(W)$ need not extend to an anti-endomorphism of $Q^r_{\max}(W)$. Let F be a field and let $V = F^n$ for some $n > 1$. Define $W = \{[\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}] \,|\, a, b \in F, v \in V\}$. Then by [**58**, Ex. 13.26], $Q^r_{\max}(W) \cong \mathrm{M}_{n+1}(F)$ and the embedding $W \hookrightarrow Q^r_{\max}(W)$ is given by

$$\begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \mapsto \begin{bmatrix} & & & v_1 \\ & aI_n & & \vdots \\ & & & v_n \\ 0 & \dots & 0 & b \end{bmatrix}$$

where I_n is the unity of $\mathrm{M}_n(F)$ and $v = (v_1, \dots, v_n)$. Define $\alpha \in \mathrm{Aut}^-(W)$ by $[\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}]^\alpha = [\begin{smallmatrix} b & v \\ 0 & a \end{smallmatrix}]$ and assume by contradiction that α can be extended to an anti-endomorphism of $\mathrm{M}_{n+1}(F)$, which we also denote by α. Let T be the transpose involution. Then $\alpha \circ T$ is an F-algebra endomorphism of $\mathrm{M}_{n+1}(F)$ and thus inner by the Skolem-Noether Theorem. This means that $\alpha = \varphi \circ T$ for some $\varphi \in \mathrm{Inn}(\mathrm{M}_{n+1}(F))$ and therefore, X and X^α has the same characteristic polynomial for all $X \in \mathrm{M}_{n+1}(F)$. But this is absurd (if $n > 1$) because

$$\begin{bmatrix} 1 \cdot I_n & 0 \\ 0 & 0 \end{bmatrix}^\alpha = \begin{bmatrix} 0 \cdot I_n & 0 \\ 0 & 1 \end{bmatrix} .$$

(Note: The ring W just defined seems to have originated in a paper by Zelmanowitz and Li ([**62**, Ex. 2.7]), who only considered the case $F = \mathbb{Q}$ and $n = 2$ for other purposes. The example was then generalized by Lam in [**58**, Ex. 13.26] to arbitrary F and n and was used to demonstate that $Q^r_{\max}(W)$ and $Q^\ell_{\max}(W)$ might be isomorphic as rings, but not as extensions of W.)

3.7.4. Main Result and Consequences. Let M be a right R-module with $\mathcal{T}(M) = 0$. A rational extension of M, M', is said to have the *extension property* if any endomorphism of M extends to a (necessarily unique) endomorphism of M'.[22] In this case, $W := \mathrm{End}_R(M)$ can be considered as a subring of $W' := \mathrm{End}_R(M')$. For example, by Proposition 3.7.4, MQ has the extension property for any right quotient ring Q of R. Our main result is:

THEOREM 3.7.10. *Let Q be a quotient ring of R, let M be a right R-module with $\mathcal{T}(M) = 0$ and let M' be a right Q-module such that $M_R \subseteq_d M'_R$ and M'_R has the extension property w.r.t. M. Assume that:*

(0) *M'_Q is f.g. or $Q = Q^s_{\max}(R)$.*
(1) *M'_Q is a torsionless generator.*
(2) *$W' := \mathrm{End}_R(M')$ is a quotient ring of $W := \mathrm{End}_R(M)$.*

Then there exists $M'' \in \text{Mod-}Q^s_{\max}(R)$ such that:

(i) *$M'_Q \subseteq_d M''_Q$ and $M_R \subseteq_d M''_R$.*
(ii) *M'' has the extension property w.r.t. M_R and M_Q.*
(iii) *$\mathrm{End}_{Q^s_{\max}(R)}(M'') = \mathrm{End}_R(M'') = Q^s_{\max}(W)$.*
(iv) *$M''_{Q^s_{\max}(R)}$ is a torsionless generator.*

[22] This is not to be confused with the extension property of homorphisms used to define injective modules.

Moreover, in this case, b_α *is injective for any* $\alpha \in \mathrm{Aut}^-(W)$.

We shall postpone the proof of Theorem 3.7.10 to the end of this section, bringing first its various applications. However, at this point, we can easily deduce the final assertion of the theorem from (i)-(iv): By Proposition 3.7.8, α' extends to an anti-endomorphism of $Q^s_{\max}(W) = \mathrm{End}_{Q^s_{\max}(R)}(M'')$ and since $M''_{Q^s_{\max}(R)}$ is a generator, $b_{\alpha'}$ is regular. Therefore, by Lemma 3.6.1, b_α is right injective (the left injectiveness follows by symmetry).

The hardest obstruction for applying Theorem 3.7.10 is condition (2), which is by no means easy to verify. Therefore, let us first record some cases in which it is satisfied. (These cases will in fact be used in the proof of Theorem 3.7.10.) To begin with, note that condition (2) is satisfied if M is f.g., $Q = RS^{-1}$ and $M = RS^{-1}$ for some *central* denominator set S consisting of regular elements, as implied by Proposition 3.6.3.

LEMMA 3.7.11. *Let* Q *be a (two-sided) quotient ring of* R*, let* $X, A, B \in \mathrm{Mod}\text{-}R$ *and let* $0 \neq f \in \mathrm{Hom}(A, B)$.

(i) *Assume there is* $n \in \mathbb{N}$ *and a generator* $G \in \mathrm{Mod}\text{-}R$ *such that:*
   (1) $X^n \subseteq_d G$.
   (2) $\mathcal{T}(B) = 0$.
   Then there exists $g \in \mathrm{Hom}(X, A)$ *such that* $f \circ g \neq 0$.

(ii) *Assume that:*
   (1) *B is dense in a f.g. R-module, B_1, and B_1* embeds *(as an R-module) in a torsionless right Q-module, T.*
   (2) *X is faithful and* $\mathcal{T}(B) = 0$.
   Then there exists $g \in \mathrm{Hom}(B, X)$ *such that* $g \circ f \neq 0$. *If $Q = R$, then the assumption that B_1 is f.g. can be dropped.*

(iii) *Assume that:*
   (1) *R is commutative*[23] *or semiprime.*
   (2) *X is dense in a f.g. R-module, X_1, and X_1 is dense (as an R-module) in a generator of* Mod-*Q, G.*
   (3) $\mathcal{T}(X) = 0$ *and* $\mathcal{T}(B) = 0$.
   Then there exists $g \in \mathrm{Hom}(X, A)$ *such that* $g \circ f \neq 0$.

PROOF. (i) Since G is a generator there is $h \in \mathrm{Hom}(G, A)$ such that $f \circ h \neq 0$. By Proposition 3.7.1(iii), $f \circ h|_{X^n} \neq 0$ (otherwise, $f \circ h|_{X^n}$ would be a nonzero extension of the zero map from M^n to B). Therefore, $f \circ h$ must be nonzero on at least one of the copies of X in G. Let g be the restriction of h to that copy. Then g is clearly the required homomorphism.

(ii) Pick some nonzero $b_0 \in \mathrm{im}(f)$. It is enough to find $g \in \mathrm{Hom}(B, X)$ with $g(b_0) \neq 0$. Condition (1) implies that there is $h \in \mathrm{Hom}_Q(T, Q_Q)$ such that $h(b_0) \neq 0$. Let $\{b_1, \dots, b_t\}$ be a set of generators for B_1. Then since ${}_RR \subseteq_d {}_RQ$, there is $r \in R$ such that $rh(b_i) \in R$ for all $0 \leq i \leq t$ and $rh(b_0) \neq 0$. Since X is faithful, there is $x \in X$ such that $xrh(b_0) \neq 0$. Now define $g : B \to X$ by $g(b) = xrh(b)$.

If $Q = R$, then we can take $r = 1$ regardless of the generators of B_1, so the assumption that B_1 is finitely generated is superfluous.

(iii) Since G_Q is a generator, there is $n \in \mathbb{N}$ such that Q_Q is a summand of G^n. Denote by $\pi : G^n \to Q_Q$ the corresponding projection and observe that $M^n \subseteq_d M_1^n \subseteq_d G^n_R$.[24]

Let $B_0 = \mathrm{im}(f)$ and let $U = \mathrm{ann}_R B_0$. Then $U \trianglelefteq R$ and U_R is not dense in R_R (because $\mathcal{T}(B) = 0$). Therefore, by [**58**, Ex. 8.3(4)] there exists $x \in R$ such

[23] This implies Q is commutative; see [**58**, Lm. 14.15].

[24] Caution: In general $M \subseteq_d M'$ and $N \subseteq_d N'$ does not imply $M \oplus N \subseteq_d M' \oplus N'$ (see [**58**, Ex. 8.21]). However, it is routine to check that this is true when $M = N$ and $M' = N'$.

that $xU = 0$. Condition (1) now implies that there is $y \in R$ such that $Uy = 0$. (This is clear if R is commutative. If R is semiprime, then $RxR \cap U = 0$ because $(RxR \cap U)^2 = 0$, hence $Ux \subseteq U \cap RxR = 0$.) Therefore, again by [**58**, Ex. 8.3(4)], ${}_RU \not\subseteq_d {}_RR$.

Let $\{m_1, \ldots, m_t\}$ be a set of generators for M_1^n. Then since ${}_RR \subseteq_d {}_RQ$, $R\pi(m_i)^{-1} := \{r \in R : r\pi(m_i) \in R\} \subseteq_d {}_RR$ and this implies that

$$L := \{r \in R : r\pi(M_1^n) \subseteq R\} = \bigcap_{i=1}^{t} R\pi(m_i)^{-1} \subseteq_d {}_RR \ .$$

Thus, $L \setminus U \neq \phi$ (because ${}_RU \not\subseteq_d {}_RR$). Let $r \in L \setminus U$. Then there is $b \in B_0$ such that $br \neq 0$. Let $a \in A$ be such that $f(a) = b$, let $X_2 = \pi^{-1}(R) + X_1^n \subseteq G_R^n$ and define $h : X_2 \to A$ by $h(x) = ar\pi(x)$. Then for any $x \in X_2$ with $\pi(x) = 1_R$ we have $f(h(x)) = f(ar) = br \neq 0$, so $f \circ h \neq 0$. By definition, $X^n \subseteq X_2$ and since $X^n \subseteq_d G_R^n$, $X^n \subseteq_d X_2$. Therefore, by Proposition 3.7.1(iii), $f \circ h|_{X^n} \neq 0$ and we can proceed as in (i). □

REMARK 3.7.12. (i) In part (iii) of the last lemma (and also of the next theorem), one can replace condition (1) with the weaker assumption:

(1′) For all $J \trianglelefteq R$, $\mathrm{ann}^r J = 0$ implies $\mathrm{ann}^\ell J = 0$ (or equivalently, ${}_RJ \subseteq_d {}_RR$ implies $J_R \subseteq_d R_R$).

This condition fails for nonsingular rings, as implied by the next example.

(ii) There is a f.g. faithful module M over a (prime Goldie) ring R s.t. M is *not* dense in a generator but M^2 is dense in a generator. Indeed, let p be a prime number, let

$$R = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ \middle|\ x \in \mathbb{Z}_{\langle p \rangle},\ a, b, c, d \in p\mathbb{Z}_{\langle p \rangle} \right\}$$

and take M to be the right R-module consisting of matrices of the form $\left[\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right]$ in R. Then M^2 is isomorphic to $\mathrm{Jac}(R)$ which is dense in R. However, R is local and hence must be a direct summand of any generator. This means that if P is a generator containing M, then $\mathrm{u.dim}\, P \geq \mathrm{u.dim}\, R_R = 2$. However, $\mathrm{u.dim}\, M = 1$ and hence M cannot be essential in P (see [**58**, Th. 6.1]).

EXAMPLE 3.7.13. Lemma 3.7.11(iii) fails upon dropping condition (1), even when R is nonsingular. For example, let F be a field, let R be the ring of 2×2 upper-triangular matrices over F, let $X = Q^r_{\max}(R) = Q^\ell_{\max}(R) = \mathrm{M}_2(F)$ (see [**58**, Ex. 13.13]) and let A consist of the matrices in R of the form $\left[\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right]$. Then X is obviously dense in a $Q^r_{\max}(R)$-generator and $Q^r_{\max}(R)$ is a two-sided quotient ring of R. However, $\mathrm{Hom}_R(X, A) = 0$, so the claim of Lemma 3.7.11(iii) fails regardless of B and f.

Despite the last example, we believe that part (iii) of the following theorem (which relies on Lemma 3.7.11(iii)) holds when R is nonsingular.

THEOREM 3.7.14. *Let Q be a (two-sided) quotient ring of R, let M be a right R-module with $\mathcal{T}(M) = 0$ and let M' be a rational extension of M satisfying the extension property. Write $W = \mathrm{End}_R(M)$ and $W' = \mathrm{End}_R(M')$. Then:*

(i) *If M^n is dense in a generator of* Mod-*R for some $n \in \mathbb{N}$, then W' is a general right quotient ring of W.*

(ii) *Assume that:*

   (1) *There exists a f.g. R-module M_1 such that $M \subseteq M_1 \subseteq M'$ and M' embeds (as an R-module) in a torsionless Q-module.*

   (2) *M is faithful.*

Then W' is a general left quotient ring of W. If $Q = R$, then the assumption that M_1 is f.g. can be dropped.

(iii) *Assume that:*
  (1) *R is commutative or semiprime.*
  (2) *M is dense in a f.g. R-module, M_1, and M_1 is dense (as an R-module) in a generator of* Mod-Q *(which need not contain M').*

Then W' is a general right quotient ring of W.

In particular, if the assumptions of (i) and (ii) or (ii) and (iii) are satisfied, then condition (2) of Theorem 3.7.10 is satisfied.

Proof. Throughout, we will freely consider elements of $\mathrm{End}(M)$ as elements of $\mathrm{End}(M')$.

(i) We need to prove that $W_W \subseteq_d W'_W$. That is, for all $u, v \in W'$ with $v \neq 0$, there is $w \in W$ such that $uw \in W$ and $vw \neq 0$. Indeed, let $N = M \cap u^{-1}(M)$. Then by Lemma 3.7.11(i) (take $X = M$, $A = N$, $B = v(N)$ and $f = v|_N$; note that $v|_N \neq 0$ because $N \subseteq_d M'$), there exists $w \in \mathrm{Hom}(M, N)$ such that $v|_N \circ w \neq 0$. Since $uw(M) \subseteq u(N) \subseteq u(u^{-1}(M)) \subseteq M$, $uw \in W$, as required.

(ii) We need to prove that ${}_WW \subseteq_d {}_WW'$. That is, for all $u, v \in W'$ with $v \neq 0$, there is $w \in W$ such that $wu \in W$ and $wv \neq 0$. Indeed, let $B = u(M) + M$. Then $B \subseteq_d u(M_1)+M_1$, which is finitely generated. Therefore, by Lemma 3.7.11(ii) (take $X = M$, $A = v^{-1}(B)$ and $f = v|_A$), there exists $w : B \to M$ such that $w \circ v|_{v^{-1}(B)} \neq 0$. Note that w extends to an endomorphism of M' because $w|_M \in \mathrm{End}(M)$ (and the extension of $w|_M$ to M' must agree with w on B by Proposition 3.7.1(iii)). Thus, we have $wv \neq 0$ and since $wu(M) \subseteq M$, $wu \in W$, as required.

(iii) Argue as in (i) using part (iii) of Lemma 3.7.11 instead of part (i) (take $X = M$, $A = N := M \cap u^{-1}(M)$, $B = v(N)$ and $f = v|_N$). □

Remark 3.7.15. One can prove a "dual" claim to part (i) of the last theorem, namely, if M is a cogenerator with $\mathcal{T}(M) = 0$, then W' is a general left quotient ring of W. However, this boils down to triviality because these assumptions imply $M' = M$. For otherwise, there would be a nonzero homomorphism from M'/M to M, which is impossible by the proof of Proposition 3.7.1(iii).

Corollary 3.7.16. *Let Q be a quotient ring of R and let $M, M_1 \subseteq$ Mod-R be such that $M \subseteq_d M_1$, $\mathcal{T}(M) = 0$ and M_1Q has the extension property w.r.t. M. Then:*

(i) *If M is faithful, M_1 is f.g. and M_1Q_Q is torsionless, then $\mathrm{End}(M_1Q_R)$ is a left quotient ring of $\mathrm{End}(M_R)$. When $Q = R$, the assumption that M_1 is f.g. can be dropped.*

(ii) *If at least one of the following holds:*
  (1) *R is semiprime or commutative, M_1 is f.g. and M_1Q_Q is a generator.*
  (2) *M_1 is a generator.*

then $\mathrm{End}(M_1Q_R)$ is a right quotient ring of $\mathrm{End}(M_R)$.

If both (i) and (ii) are satisfied, then conditions (0)–(2) of Theorem 3.7.10 are satisfied for $M' = M_1Q$ and, in particular, b_α is injective for all $\alpha \in \mathrm{Aut}^-(\mathrm{End}_R(M))$.

Proof. Apply Theorem 3.7.14 with $M' = M_1Q$. □

In case Q of the corollary is a classical localization of R (i.e. $Q = RS^{-1} = S^{-1}R$ with S a (two-sided) denominator set), we can prove a slightly stronger version of Corollary 3.7.16.

Corollary 3.7.17. *Let S be a (two-sided) denominator set of R consisting of regular elements, let $M, M_1 \in$ Mod-R be such that $M \subseteq_d M_1$, $\mathcal{T}(M) = 0$, M_1 is f.g. and M_1S^{-1} has the extension property w.r.t. M. Then:*

(i) *If M is faithful and $M_1S^{-1}_{RS^{-1}}$ is torsionless, then* $\mathrm{End}(M_1S^{-1}_R)$ *is a left quotient ring of* $\mathrm{End}(M_R)$.

(ii) *If M_1S^{-1} is an RS^{-1}-generator, then* $\mathrm{End}(M')$ *is a general right quotient ring of* $\mathrm{End}(M)$ *for any rational extension M' of M with the extension property.*

If both (i) and (ii) are satisfied, then conditions (0)–(2) of Theorem 3.7.10 are satisfied for $M' = M_1S^{-1}$ and, in particular, b_α is injective for all $\alpha \in \mathrm{Aut}^-(\mathrm{End}_R(M))$.

PROOF. Note that $MS^{-1} = MRS^{-1}$ by Proposition 3.7.4(iv), so part (i) is just a special case of Corollary 3.7.16(i). We thus turn to part (ii).

Let $Q = RS^{-1} = S^{-1}R$. Then there is n such that $(M_1S^{-1})^n = eQ \oplus V$ for some $V \in \text{Mod-}Q$ and $e \in (M_1S^{-1})^n$ with $\mathrm{ann}_Q\, e = 0$. Let $\{m_1, \dots, m_t\}$ be a set of generators for M_1^n. Then we can write $m_i = eq_i + v_i$ for unique $q_i \in Q$ and $v_i \in V$. There exist $s \in S$ and $r_1, \dots, r_t \subseteq R$ s.t. $q_i = s^{-1}r_i$ for all i (here we need $Q = S^{-1}R$). Therefore, replacing e with es^{-1}, we may assume $M_1^n \subseteq P := eR \oplus V$. The r.h.s. is clearly a generator of Mod-R and M^n is dense in $(M_1S^{-1})^n$ and hence in P. We are now done by Theorem 3.7.14(i). □

REMARK 3.7.18. Keeping the notation of Corollary 3.7.17, note that if $Q := RS^{-1}$ is right Kasch (i.e. Q_Q contains a copy of any simple right Q-module), then $\mathcal{T}(M) = 0 \iff M$ is S-torsion-free. Indeed, one direction follows from Proposition 3.7.4(iv). To see the converse, assume by contradiction that $A := \mathrm{ann}_R(m) \subseteq_d R_R$ for some $m \in M$. Then a routine argument shows that $AS^{-1} \subseteq_d RS^{-1} = Q$ (as Q-modules). However, Q is right Kasch, so by [**58**, Cr. 8.28], $AS^{-1} = Q$. This means $1 = as^{-1}$ for some $a \in A$ and $s \in S$, thus implying $s \in A$. But then $ms = 0$, so $m = 0$ since M is S-torsion-free.

In general, that M is S-torsion-free need does not imply $\mathcal{T}(M) \neq 0$ even when S consists of all regular elements in R. For example, let F be a field and take $R = Q = \prod_{\aleph_0} F$, $S = R^\times$ and $M = R/\bigoplus_{\aleph_0} F$.

The previous corollaries become much sharper when Q or RS^{-1} is right Pseudo Frobenius (abbrev. PF). Recall that a ring Q is said to be right PF if all faithful right Q-modules are generators. This turns out to be equivalent to Q_Q being an injective cogenerator (see [**54**, Ch. 12] and [**58**, Th. 19.25]; also see [**67**], [**94**]), hence all right Q-modules are torsionless and all faithful right Q-modules are cogenerators.

THEOREM 3.7.19. *Let Q be a quotient ring of R and let M be a faithful right R-module satisfying $\mathcal{T}(M) = 0$ which is dense in a f.g. R-module M_1. Assume Q is right PF. Then $MQ = \widetilde{E}(M_R)$. Furthermore, if one of the following holds:*

(1) *there is a (two-sided) denominator set S consisting of regular elements such that $Q = RS^{-1}$,*

(2) *M_1 is dense in an R-generator,*

(3) *R is commutative or semiprime,*

then $\mathrm{End}(MQ_R) = Q^s_{\max}(\mathrm{End}(M_R))$ *and b_α is injective for all $\alpha \in \mathrm{Aut}^-(\mathrm{End}(M_R))$.*

PROOF. Proposition 3.7.4(v) implies $\widetilde{E}(M_R) = \widetilde{E}(MQ_Q)$, so it is enough to prove $\widetilde{E}(MQ_Q) = MQ_Q$. Indeed, MQ_Q is faithful, so the preceding discussion implies MQ_Q is a cogenerator, and since $\mathcal{T}(MQ_Q) \subseteq \mathcal{T}(MQ_R) = 0$, $\widetilde{E}(MQ_Q) = MQ_Q$ by Remark 3.7.15. Therefore, $MQ = \widetilde{E}(M_R)$. In particular, MQ must coincide with M_1Q_R.

Since MQ_Q is a torsionless generator, Corollaries 3.7.16 and 3.7.17 imply that we can apply Theorem 3.7.10 with $M' = MQ$ to obtain a module M'' as in that theorem. As M''_R is a rational extension of $MQ = \widetilde{E}(M_R)$, necessarily $M'' = MQ$, so we are done. □

EXAMPLE 3.7.20. Recall that a ring R is said to be a *right order* in a ring Q containing R if $Q = Q^r_{\mathrm{cl}}(R)$ (and the latter exists). (Equivalently, R is a right order in Q if $S \subseteq Q^\times$ and $Q = \{rs^{-1} \mid r \in R,\, s \in S\}$ where S is the set or regular elements in R.) Two-sided orders in right PF rings satisfy condition (1) of Theorem 3.7.19 and these turn out to be quite common. For example, Goldie's Theorem ([**45**]) characterizes the right orders in semisimple rings as the semiprime *right Goldie* rings (i.e. rings with ACC on right annihilators and finite right uniform dimension; see [**58**, §11A] or [**80**, §3.2]), so noetherian semiprime rings are orders in semisimple rings (and thus in right PF rings).

More generally, sufficient and necessary conditions for R to be a right order in a QF or a right PF ring were given by Shock in [**89**], [**90**] and [**91**]. We also note that if R is an order in a QF ring Q, then so is RG for any finite group G. Indeed, it is easy to see that RG is an order in QG and the latter is noetherian (clear) and self-injective by [**74**].

REMARK 3.7.21. Condition (3) of Theorem 3.7.19 implies condition (1) in many cases, namely when R is semiprime or when Q is commutative with ACC on right annihilators (e.g. if Q is QF). Indeed, if R is commutative and Q has ACC on right annihilators, then R has ACC on right annihilators (this is straightforward), hence by [**58**, Cr. 13.16], $Q = Q^r_{\max}(R) = Q^r_{\mathrm{cl}}(R)$ ($Q = Q^r_{\max}(R)$ because Q_Q is injective). To see the semiprime case, recall that any right PF ring is semilocal and satisfies $\mathrm{soc}(R_R) \subseteq_e R_R$ (see [**58**, Th. 19.25]). Now, Q is semiprime by [**58**, Exer. 13.8], hence $\mathrm{Jac}(Q) \cap \mathrm{soc}(Q_Q) = 0$ (because this is a nilpotent ideal). Since $\mathrm{soc}(Q_Q) \subseteq_e Q_Q$, $\mathrm{Jac}(Q) = 0$, which implies Q is semisimple (because Q is semilocal). This means R has ACC on annihilators (because Q has) and $\mathrm{u.dim}\, R_R = \mathrm{u.dim}\, Q_R = \mathrm{u.dim}\, Q_Q < \infty$ (because $R_R \subseteq_e Q_R$), hence R is a semiprime right Goldie. The same argument implies R is also a left Goldie ring and thus $Q^r_{\max}(R) = Q^\ell_{\max}(R) = Q^\ell_{\mathrm{cl}}(R) = Q^r_{\mathrm{cl}}(R)$.

In general, a commutative PF ring need not have ACC on annihilators. See [**58**, Ex. 19.24] (the example is due to Osofsky).

Theorem 3.7.10 also has interesting consequences in the "trivial case" $R = Q = Q^s_{\max}(R)$ and $M = M'$. Many examples of rings with $R = Q^s_{\max}(R)$ can be found in [**59**], [**24**] and related papers, and the following corollary enables one to find even more.

COROLLARY 3.7.22. *Let R be a ring satisfying $R = Q^s_{\max}(R)$. Then:*

(i) *For every torsionless generator $M \in$ Mod-R, there is a torsionless generator $G \in$ Mod-R such that $M \subseteq_d G$, G has the extension property w.r.t. M and* $\mathrm{End}(G) = Q^s_{\max}(\mathrm{End}(M))$.

(ii) *If R_R is a cogenerator, then any generator $M \in$ Mod-R satisfies* $\mathrm{End}(M) = Q^s_{\max}(\mathrm{End}(M))$. *If R is right PF, then any faithful module $M \in$ Mod-R satisfies* $\mathrm{End}(M) = Q^s_{\max}(\mathrm{End}(M))$.

PROOF. (i) Apply Theorem 3.7.10 with $Q = R$ and $M_1 = M$. Take G to be M''. Conditions (0)-(2) of the theorem are automatically satisfied.

(ii) When R_R is a cogenerator, M is torsionless and a cogenerator (because R_R is a summand of M^n). Thus, by Remark 3.7.15, $\widetilde{E}(M) = M$, so if we apply (i), G must necessarily be M. If R is right PF, then R_R is a cogenerator and M is a generator, so the previous argument implies $\mathrm{End}(M) = Q^s_{\max}(\mathrm{End}(M))$. □

To demonstrate the non-triviality of the last corollary, we note that it is well known that if P is a finite projective over a QF algebra, then $\mathrm{End}(P)$ need not be QF (and thus not PF). (In this case P cannot be faithful, for otherwise it would be a progenerator, and since being QF is a categorical property, this would imply $\mathrm{End}(P)$ is QF.) The problem of determining when $\mathrm{End}(P)$ is QF (when P is projective) and its obvious generalization to PF rings were considered in [**77**], [**97**] and related papers.

3.7.5. Proof of the Main Result. We finally turn to prove Theorem 3.7.10.

LEMMA 3.7.23. *Let Q be a right quotient ring of R and let $e \in \mathrm{E}(R)$ be such that $\mathrm{ann}^\ell_R Re = 0$. Then eQe is a right quotient ring of eRe and $Re_{eRe} \subseteq_d Qe_{eRe}$.*

PROOF. Let $x, y \in eQe$ be such that $x \neq 0$. Then there is $r \in R$ such that $xr \neq 0$ and $xr, yr \in R$. Thus $xrRe \neq 0$, hence there is $s \in Re$ such that $xrs \neq 0$. This implies $x(erse) = xrs \neq 0$ and $y(erse) = yrs \in R$, hence eQe is a right quotient ring of eRe. The second assertion is shown in the same manner. □

LEMMA 3.7.24. *Let W be a ring, let $e \in \mathrm{E}(W)$ and consider We as a right eWe-module. If $\mathrm{ann}^r eW = 0$, then We is torsionless. The converse holds when ${}_W We$ is faithfully balanced (i.e. the standard map $W \to \mathrm{End}(We_{eWe})$ is an isomorphism).*

PROOF. Assume $\mathrm{ann}^r eW = 0$ and let $0 \neq x \in We$. Then there exists $y \in eW$ such that $yx \neq 0$. Now, the map $f : We \to eWe$ defined by $f(w) = yw$ is an eWe-module homomorphism satisfying $f(x) \neq 0$. Conversely, assume We_{eWe} is torsionless and $\mathrm{End}(We_{eWe}) = W$. Then ${}_W We$ is faithful, hence for all $0 \neq x \in W$ there is $z \in We$ such that $xz \neq 0$. Since $xz \in We$, there is an eWe-module homomorphism $f : We \to eWe$ such that $f(xz) \neq 0$. As $\mathrm{End}(We_{eWe}) = W$, the homomorphism f is given by $f(w) = yw$ ($w \in We$) for some $y \in eW$. Thus, $yxz \neq 0$, hence $x \notin \mathrm{ann}^r eW$ (because $yx \neq 0$). Therefore, $\mathrm{ann}^r eW = 0$. □

We are now ready to prove Theorem 3.7.10. This is done in several steps.

PROOF OF THEOREM 3.7.10. STEP 1. Without loss of generality, we may assume $R = Q$ and $M = M'$ (use Proposition 3.7.1 to see that this is indeed allowed). We thus drop M' and Q from our notation henceforth (in Step 2, M' will be redefined as a different module). Note that M_R is now a torsionless generator.

Next, we claim that we may assume $R = Q^s_{\max}(R)$. Indeed, just replace M with $MQ^s_{\max}(R)$. The latter is a torsionless $Q^s_{\max}(R)$-generator by Proposition 3.7.4(iii) and part (ii) of that proposition implies $MQ^s_{\max}(R)$ has the extension property w.r.t. M_R. In addition, parts (i) and (ii) of of Theorem 3.7.14 imply that $\mathrm{End}(MQ^s_{\max}(R))$ is a two-sided quotient ring of $\mathrm{End}(M)$. (Apply the theorem with $R, Q^s_{\max}(R), M, MQ^s_{\max}(R)$ in place of R, Q, M, M'. In part (i) take $G = M$ and in part (ii) take $M_1 = M$.)

STEP 2. Let $W = \mathrm{End}(M_R)$ and let $W' = Q^s_{\max}(W)$. Since M_R is a generator, there is $n \in \mathbb{N}$ such that $M^n \cong R_R \oplus N$. Repeating the argument in the comment before Proposition 3.3.9, we may assume that there is $e \in \mathrm{E}(U)$, where $U := \mathrm{M}_n(W)$, such that $R = eUe$, $M = e_{11}Ue$ (as (W,R)-bimodules) and $M^n = Ue$ (as (U,R)-bimodules). Define $U' = \mathrm{M}_n(W')$ and, abusing the notation, let $M' = e_{11}U'e$. Then M' is a (W', R)-bimodule and we claim it is the required R-module M'', i.e. M'_R satisfies (i)-(iv).

STEP 3. We start by observing $\mathrm{End}({}_{W'}M') = eU'e$. Indeed, let $U' = \mathrm{M}_n(W')$ act on $(M')^n$ from the left in the standard way. Then ${}_{U'}(M')^n \cong {}_{U'}U'e$ and it is well known that $\mathrm{End}({}_{U'}U'e) = eU'e$ (where $eU'e$ acts on the right by standard multiplication). It is now routine to verify that $\mathrm{End}({}_{U'}(M')^n) \cong \mathrm{End}({}_{W'}M')$, so our claim is proved.

Since $M^n = Ue$, we may identify $\mathrm{End}(Ue_R)$ with $\mathrm{M}_n(W) = U$. As $\mathrm{End}({}_U Ue) = eUe = R$, it follows that Ue is faithfully balanced (in both Mod-R and U-Mod). In addition, Ue_R is a torsionless generator (since M_R is). Thus, by Lemma 3.7.24,

$\operatorname{ann}^r eU = 0$. Moreover, ${}_U Ue$ is faithfully balanced, thus faithful, so $\operatorname{ann}^\ell_U Ue = \operatorname{ann}_U({}_U Ue) = 0$. Now, Lemma 3.7.23 implies that $eU'e$ is a two-sided quotient ring of $R = eUe$ and $(M')^n = U'e_R$ is a rational extension of $M^n = Ue_R$, hence $M_R \subseteq_d M'_R$ (this is straightforward). As $R = Q^s_{\max}(R)$, it must coincide with $eU'e$. Therefore, R_R is a summand of $(M')^n_R$, hence M'_R is a generator. Furthermore, $\operatorname{ann}^r_U(eU') \subseteq \operatorname{ann}^r_U(eU) = 0$ and $\operatorname{ann}^r_U(eU') = U \cap \operatorname{ann}^r_{U'}(eU')$, so $\operatorname{ann}^r_{U'}(eU') = 0$ because $U_U \subseteq_e U'_U$. This implies $(M')^n_R$ is torsionless (Lemma 3.7.24) and hence so is M'_R. We have thus shown that $M_R \subseteq_d M'_R$ and M'_R is a torsionless generator with $\operatorname{End}({}_{W'}M') = R$. In addition, M'_R clearly has the extension property w.r.t. M_R. This settles (i), (ii) and (iv).

STEP 4. We finish by showing (iii), i.e. $W' = \operatorname{End}(M'_R)$. (The last assertion of Theorem 3.7.10 was verified immediately after its statement, so this concludes the proof.) First observe that W' acts on $M'_R = e_{11}U'e$ on the right by left multiplication. This action is faithful (for otherwise the action of U' on $U'e = (M')^n$ would be non-faithful), so we can consider W' as a subring of $\operatorname{End}(M'_R)$ that contains W. Since M'_R is torsionless and M_R is a generator, parts (i) and (ii) of Theorem 3.7.14 imply $\operatorname{End}(M'_R)$ is a two-sided quotient ring of W. As $W' = Q^s_{\max}(W)$, we must have $W' = \operatorname{End}(M'_R)$, as required. □

REMARK 3.7.25. We do not know if the module M' constructed in step 2 of the last proof coincides with M (this would imply $M'' = M'Q^s_{\max}(R)$ in Theorem 3.7.10). Part (ii) of Corollary 3.7.22 presents special cases in which this can be guaranteed. Furthermore, we do not know if M' is unique w.r.t. to being a rational extension of M satisfying $Q^s_{\max}(\operatorname{End}(M)) = \operatorname{End}(M')$. However, the map $M \mapsto M'$ is a closure operation (i.e. $(M')' = M'$) defined for all torsionless generators over rings R with $R = Q^s_{\max}(R)$.

3.8. An Easy Proof for a Result of Osborn

As an application of the previous theory, we present an easy proof for a special case of a result of Osborn:

THEOREM 3.8.1 (Osborn). *Let (W, α) be a ring with involution such that $2 \in W^\times$ and every element $w \in W$ with $w^\alpha = w$ is either a unit or nilpotent. Let α' denote the induced involution on $W/\operatorname{Jac}(W)$. Then $\operatorname{Jac}(W) \cap \{w \in W : w^\alpha = w\}$ consists of nilpotent element and one of the following holds:*

(i) *$W/\operatorname{Jac}(W)$ is a division ring.*
(ii) *$W/\operatorname{Jac}(W) \cong D \times D^{\mathrm{op}}$ for some division ring D and under that isomorphism α' exchanges D and D^{op}.*
(iii) *$W/\operatorname{Jac}(W) \cong \mathrm{M}_2(F)$ for some field F and under that isomorphism α' is a symplectic involution (i.e. it is induced by a classical alternating bilinear form).*

PROOF. See [**66**, §4]. □

Osborn's result has several generalizations (see papers related to [**66**]) and his proof is based on Jordan algebras. Theorem 3.5.5 allows us to give a new proof for the special case where W is semisimple (Osborn's result can be deduced for semilocal rings using this special case; see [**66**, §4]). Our assumptions are milder, though.

THEOREM 3.8.2. *Let (W, α) be a ring with involution such that W is semisimple and the only $*$-invariant idempotents in W are 0 and 1. Then one of the following holds:*

(i) *W is a division ring.*

(ii) $W \cong D \times D^{\mathrm{op}}$ *for some division ring* D *and under that isomorphism* α *exchanges* D *and* D^{op}.
(iii) $W \cong \mathrm{M}_2(F)$ *for some field* F *and under that isomorphism* α *is a symplectic involution.*

PROOF. We may assume W is not the zero ring. Let $\{e_1, \dots, e_n\}$ be the primitive idempotents of $\mathrm{Cent}(W)$. Then α permutes $e_1, \dots, e_n$. Assume $n > 1$. Then we have $e_i \neq e_i^\alpha$ for all i. This implies $e_1 + e_1^\alpha$ is a non-zero α-invariant idempotent, hence $e_1 + e_1^\alpha = 1$. Thus, $n = 2$ and $e_1^* = e_2$. Write $W_i = e_i W$. Then $W \cong W_1 \times W_2$ and α exchanges W_1 and W_2. If $0 \neq e \in W_1$ is an idempotent, then $e^\alpha \in W_2$, hence $e + e^\alpha$ is a non-zero α-invariant idempotent, implying $e + e^\alpha = 1$ and $e = 1_{W_1}$. This means W_1 is a simple artinian ring with no non-trivial idempotents, hence it is a division ring. As $W_2 \cong W_1^{\mathrm{op}}$ via α, (ii) holds.

Now assume $n = 1$. Then W is simple artinian, hence we can write $W = \mathrm{End}(D_D^k)$ for some division ring D. Let $b = b_\alpha$, $K = K_\alpha$ and $\kappa = \kappa_\alpha$. Then $b : D^k \times D^k \to K$ is a regular κ-symmetric bilinear form by Theorem 3.5.5. Moreover, by Corollary 3.3.10, $\dim(K_1)_D = 1$.

We claim that if $D^k = U_1 \oplus U_2$ with $b(U_1, U_2) = b(U_2, U_1) = 0$, then $U_1 = 0$ or $U_2 = 0$. Indeed, let e be the projection from D^k to U_1 with kernel U_2. Then it is straightforward to check that $b(ex, y) = b(ex, ey) = b(x, ey)$ and hence $e^\alpha = e$. Therefore $e = 1$ or $e = 0$, so $U_1 = D^k$ or $U_1 = 0$.

Assume there is $x \in D^k$ such that $b(x, x) \neq 0$. Define $L = x^\perp = \{y \in D^k \mid b(x, y) = 0\}$. We claim that $D^k = L \oplus xD$. Clearly $x \notin L$, hence $xD \cap L = 0$. On the other hand, for all $v \in D^k$, there exists $d \in D$ such that $b(x, x) \odot_1 d = b(x, v)$ (because $\dim(K_1)_D = 1$), hence $b(x, v - xd) = b(x, v) - b(x, x) \odot_1 d = 0$ and this implies $v = xd + (v - xd) \in xD + L$. Now, since b is κ-symmetric $b(L, xD) = b(xD, L)^\kappa = 0$, so by the previous paragraph, $L = 0$. But this means $k = 1$. Therefore, $W = \mathrm{End}(D_D^1)$ is a division ring and (i) holds.

We may now assume that $b(x, x) = 0$ for all $x \in D^k$. Then $\kappa = -\mathrm{id}_K$ since $0 = b(x + y, x + y) = b(x, y) + b(y, x) = b(x, y) + b(x, y)^\kappa$ for all $x, y \in D^k$. Furthermore, for all $x, y \in D^k$ and $a \in D$ we have $b(x, y) \odot_0 a = b(xa, y) = -b(y, xa) = -b(y, x) \odot_1 a = b(x, y) \odot_1 a$, hence $\odot_0 = \odot_1$. This implies that for any $0 \neq k \in K$ and $a, b \in D$, we have $k \odot_1 (ab) = (k \odot_1 a) \odot_1 b = (k \odot_0 a) \odot_1 b = (k \odot_1 b) \odot_0 a = (k \odot_1 b) \odot_1 a = k \odot_1 (ba)$, hence $ab = ba$. Therefore, D is a field and K is isomorphic as a double R-module to D, with $\odot_0, \odot_1$ being the standard right action of D on itself. As $b(x, x) = 0$ for all $x \in D^k$, b is a classical alternating bilinear form. We are thus finished if we prove that $k = 2$ (as this would imply $W = \mathrm{End}(D_D^2) \cong \mathrm{M}_2(D)$, as in (iii)). However, this follows from the well-known fact that every regular alternating form is the orthogonal sum of 2-dimensional alternating forms (and b cannot be the orthogonal sum of two non-trivial forms as argued above). □

3.9. Generization of Arbitrary Forms

We finish this chapter by suggesting a way to define non-stable generic forms.

Let (M, b, K) be a bilinear space over a ring R and let $W = \mathrm{End}(M_R)$. A pair $(\sigma, \sigma') \in W \times W$ will be called *b-adjoint* if $b(\sigma x, y) = b(x, \sigma' y)$ for all $x, y \in M$. For example, all pairs in $\{0\} \times \mathrm{Hom}(M, \ker \mathrm{Ad}_b^r)$ are b-adjoint. Let $P(b)$ denote the set of b-adjoint pairs in $W \times W$ and define:

$$K_b = \frac{M \otimes_{\mathbb{Z}} M}{\langle \sigma x \otimes y - x \otimes \sigma' y \mid x, y \in M,\ (\sigma, \sigma') \in P(b) \rangle} .$$

The image of $x\otimes_{\mathbb{Z}} y$ in K_b will be denoted by $x\otimes_b y$. We make K_b into a double R-module by letting

$$(x\otimes_b y)\odot_0 r = xr\otimes_b y, \qquad (x\otimes_b y)\odot_1 r = x\otimes_b yr,$$

for all $x,y\in M$ and $r\in R$. The *generization* of (M,b,K) is defined to be (M,b_{gen},K_b) where b_{gen} is the bilinear form defined by $b_{\mathrm{gen}}(x,y)=x\otimes_b y$. The form b_{gen} has the following universal property: If (M,b',K') is another bilinear space such that $P(b)\subseteq P(b')$, then there exists a unique $f\in\mathrm{Hom}_{\mathrm{DMod}\text{-}R}(K_b,K')$ such that $b'=f\circ b_{\mathrm{gen}}$. (In particular, taking $b'=b$ yields that b can be recovered from b_{gen}.)

When b is right stable with *right* corresponding anti-endomorphism α, b_α is easily seen to be similar to b_{gen}. (Indeed, $P(b)=\{(\sigma,\sigma^\alpha)\,|\,\sigma\in W\}$ so $K_\alpha\cong K_b$ via by $x\otimes_\alpha y\mapsto x\otimes_b y$.) Thus, the map $b\mapsto b_{\mathrm{gen}}$ is a generalization of the generization defined in section 3.1. However, in contrast to the one-sided definition of section 3.1, the generization just defined is a left-right symmetric process. In particular, if b is left stable with *left* corresponding anti-endomorphism α, then ${}_\alpha b$ is also similar to b_{gen}.

PROPOSITION 3.9.1. *Let (M,b,K) be a bilinear space. Then:*

(i) $P(b_{\mathrm{gen}})=P(b)$.
(ii) $(b_{\mathrm{gen}})_{\mathrm{gen}}=b_{\mathrm{gen}}$.
(iii) *b is right (semi-)stable $\iff$ b_{gen} is right (semi-)stable.*
(iv) *If $P(b)$ is a symmetric relation on $\mathrm{End}(M)$, then the map $\kappa_b:K_b\to K_b$ defined by $(x\otimes_b y)^{\kappa_b}=y\otimes_b x$ is an involution, and b_{gen} is κ_b-symmetric.*

PROOF. (i) That $P(b)\subseteq P(b_{\mathrm{gen}})$ is clear. To see the converse, let $f:K_b\to K$ be the double R-module homomorphism satisfying $b=f\circ b_{\mathrm{gen}}$. Then $(\sigma,\sigma')\in P(b_{\mathrm{gen}})$ implies $b_{\mathrm{gen}}(\sigma x,y)=b_{\mathrm{gen}}(x,\sigma' y)$ for all $x,y\in M$. Applying f to both sides yields $b(\sigma x,y)=b(x,\sigma' y)$, so $(\sigma,\sigma')\in P(b)$.

(ii) This is follows from (i) and the definition of b_{gen}.

(iii) This also follows from (i), since being right stable or right semi-stable can be phrased as a property of $P(b)$ (e.g.: b is right stable precisely when $P(b)$ is a function).

(iv) This is straightforward. □

It is now natural to call a bilinear form $b:M\times M\to K$ *generic* when $b\sim b_{\mathrm{gen}}$. Furthermore, call two bilinear forms b, b' on M *weakly similar* if $P(b)=P(b')$. By part (i) of the last proposition, this is equivalent to $b_{\mathrm{gen}}=b'_{\mathrm{gen}}$. Proposition 3.9.1(iv) also calls for a new notion of symmetry — call b *pre-symmetric* if $P(b)$ is a symmetric relation. While all these new notions deserve further attention, the author could not pursue them further due to time and space limitations.

EXAMPLE 3.9.2. Let $b:\mathbb{Z}^2\times\mathbb{Z}^2\to\mathbb{Z}$ be the bilinear form defined in Example 2.4.9, namely, $b(x,y)=x^T\left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right]y$. Then b is not left nor right stable, but it is generic. Indeed, the analysis done in Example 2.4.9 shows that

$$P(b)=\{(\left[\begin{smallmatrix}a&2b\\c&d\end{smallmatrix}\right],\left[\begin{smallmatrix}a&2c\\b&d\end{smallmatrix}\right])\,|\,a,b,c,d\in\mathbb{Z}\}$$

(we consider $\mathrm{End}(\mathbb{Z}^2)$ as $\mathrm{M}_2(\mathbb{Z})$). Let

$$e_1=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\otimes_{\mathbb{Z}}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\quad e_2=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\otimes_{\mathbb{Z}}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right],\quad e_3=\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\otimes_{\mathbb{Z}}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\quad e_4=\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\otimes_{\mathbb{Z}}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$$

and let $\overline{e}_i$ denote the image of e_i in K_b. Then $\{e_1,\dots,e_4\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^2\otimes_{\mathbb{Z}}\mathbb{Z}^2$ and it not hard (but tedious) to verify that

$$K_b=\frac{e_1\mathbb{Z}+e_2\mathbb{Z}+e_3\mathbb{Z}+e_4\mathbb{Z}}{e_2\mathbb{Z}+e_3\mathbb{Z}+(2e_1-e_4)\mathbb{Z}}\ .$$

Thus, $K_b \cong \mathbb{Z}$ via $\sum \overline{e}_i a_i \mapsto a_1 + 2a_4$. Denoting this map by f, one sees that f is a similarity from b_{gen} to b. Indeed:

$$\begin{aligned} f(b_{\text{gen}}([\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}], [\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}])) &= f(x_1 y_1 \overline{e}_1 + x_1 y_2 \overline{e}_2 + x_2 y_1 \overline{e}_3 + x_2 y_2 \overline{e}_4) \\ &= x_1 y_1 + 2 x_2 y_2 = b([\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}], [\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}]) \ . \end{aligned}$$

CHAPTER 4

Isometry and Decomposition

In a paper from 1974 ([**76**]), C. Riehm, basing on the work of Wall, solved the isometry problem of classical (non-symmetric) regular bilinear forms over fields, where solving means reducing it to isometry of hermitian forms over other fields. Extensions of the solution to degenerate forms ([**44**]) and to sesquilinear forms ([**75**], [**84**]) soon followed and similar techniques were used to study pairs of symmetric bilinear forms (e.g. see [**88**]). While this topic was somewhat ignored in the 80's and the 90's, it has regained considerable interest in the last decade, the main problems now being providing canonical representatives for isometry classes (e.g. [**25**], [**51**], [**50**], [**52**], [**43**]), various decompositions of forms (e.g. [**31**], [**38**], [**93**]), determining conjugation classes w.r.t. special matrices (such as unitary matrices; e.g. [**30**], [**28**]) and other topics (e.g. [**61**], [**29**]). Many of these papers consider pairs of bilinear and sesquilinear forms as well.

In this final chapter, we present a method for generalizing the work of Riehm and its predecessors to bilinear forms over rings. Moreover, we will show that there is a canonical way of translating the theory of arbitrary non-symmetric forms into the theory of regular symmetric forms and many of the previously mentioned references turn out to "factor" through it. Strictly speaking, we show that the category of arbitrary bilinear forms over a *category with a double duality* (see section 2.7) is isomorphic to the category of symmetric regular bilinear forms over some *category with duality*.[1] This allows us to apply results originally designed for symmetric bilinear forms over categories with duality to non-symmetric or non-regular forms over rings. The applications are numerous and include:

(1) Witt's Cancelation Theorem: Let b_1, b_2, b_3 be (not-necessarily symmetric) bilinear forms over a *good* ring (e.g. artinian ring in which 2 is a unit). Then $b_1 \perp b_2 \cong b_1 \perp b_3 \iff b_2 \cong b_3$ (where "$\cong$" denotes isometry).
(2) The isometry problem of bilinear forms over *good* rings can be reduced to isometry of hermitian forms over division rings.
(3) The notion of *isotypes* (see below) can be suitably generalized to bilinear forms over a *good* ring. Any form over such a ring is the orthogonal sum of isotypes of different *types* and these isotypes are uniquely determined up isometry. (This also applies to degenerate forms!)
(4) Classification of the indecomposable bilinear forms over *good* rings (generalizing [**93**] and [**38**]).
(5) If R is a f.d. algebra over an algebraically closed field F and b is an F-linear (not-necessarily-symmetric) bilinear form over R, then there exists an exact sequence of F-algebraic groups $1 \to U \to O(b) \to G \to 1$ where U is unipotent, $O(b)$ is the isometry group of b and G is a product of copies of $O_n(F)$, $\mathrm{GL}_m(F)$ and $\mathrm{Sp}_k(F)$. The terms in the product are determined by the *types* of the *isotypes* appearing in the decomposition of b into isotypes (compare with [**14**]).

[1] Categories with duality are also called *hermitian categories*.

Furthermore, these results hold for (arbitrarily large) systems of bilinear forms. (Additional applications were not included due to space and time limitations.)

The first two sections of this chapter are not mandatory and provide a survey of Riehm's solution to the isometry problem of classical regular bilinear forms over a field and a (somewhat selective) survey of the theory of hermitian categories, respectively. In section 4.3, we prove the categorical equivalence mentioned above and discuss some of its implications. However, in order to apply the equivalence properly, we need to make sure that the category with duality we obtain satisfies several properties. This is done in section 4.4. Section 4.5 explains how to extend the previous results to *systems of bilinear forms*. As the equivalence is a very powerful, yet very unexplicit tool, sections 4.6–4.10 are concerned with providing an explicit approach to study bilinear forms over rings. The results obtained are merely a "pull-back along the equivalence" of known results on symmetric regular bilinear forms over categories with duality, but they are proved explicitly, with no trace of categories. Section 4.6 covers the basics of *Kronecker modules* of bilinear forms and their connection with the *asymmetry* of the form, section 4.7 defines and discusses hyperbolic forms, section 4.8 presents a "dictionary" for translating claims on bilinear forms to ring theoretic claims and section 4.9 shows how to "lift" some of these claims from an epimorphic image of the ring to the ring itself. Section 4.10 deals with additional technical issues. The rest of the chapter is concerned with applications: section 4.11 classifies indecomposable bilinear spaces, section 4.12 is devoted to *isotypes* and in section 4.13 we prove Witt's Cancelation Theorem (for non-symmetric non-regular systems of bilinear forms) and show how to reduce the isometry problem of bilinear forms over *good* rings to isometry of hermitian forms over division rings. Finally, section 4.14 uses the previous results to prove some strong structural results about bilinear forms and isometry groups, provided the base ring is a f.d. algebra over an algebraically closed field.

4.1. Survey: Isometry of Classical Bilinear Forms

We begin with a short overview on how to solve the isomorphism problem of *regular* bilinear forms, where by solving one (always) means reduction to the isomorphism problem of hermitian forms, and possibly other "easy" problems. We have included this procedure because it presents many basic tools and concepts, such as *isotypes*. In addition, it demonstrates how decomposition of bilinear forms into orthogonal sums is essential to solve the isomorphism problem. Our exposition roughly follows Riehm ([**76**]), Scharlau ([**84**]) and also the author's M.Sc. Thesis ([**38**]); proofs can be found in these references. For simplicity, we shall *not* consider the case $\operatorname{char} F = 2$.

Let F be a field with $\operatorname{char} F \neq 2$. For every monic polynomial $f(x) \in F[x]$ with $f(0) \neq 0$, define $f^*(x) = f(0)^{-1}x^{\deg f} f(x^{-1})$. Then $f^{**} = f$ and $(fg)^* = f^*g^*$ for all monic $f, g \in F[x]$ with $f(0), g(0) \neq 1$.

Let V be a f.d. F-vector space and let $b : V \times V \to F$ be a regular bilinear form. Then b admits a (right id_F-)asymmetry, namely, a map $\lambda \in \operatorname{End}(V)$ satisfying

$$b(u, v) = b(v, \lambda u) \qquad \forall u, v \in V \ .$$

Let f_λ denote the minimal polynomial of λ. It turns out that $f_\lambda^* = f_\lambda$, hence we can write $f_\lambda = (p_1p_1^*)^{m_1} \cdots (p_tp_t^*)^{m_t} q_1^{n_1} \cdots q_s^{n_s}$ where $p_1, p_1^*, \dots, p_t, p_t^*, q_1, \dots, q_s$ are distinct monic primes and $q_i = q_i^*$ for all $1 \leq i \leq s$.

Theorem 4.1.1. *In the previous notation, let*

$$\begin{aligned}\mathcal{P} &= \{(p_ip_i^*)^m \,|\, 1 \leq i \leq t, 1 \leq m \leq m_i\} \\ \mathcal{Q} &= \{q_i^n \,|\, 1 \leq i \leq s, 1 \leq n \leq n_i\} \ .\end{aligned}$$

Consider V as an $F[x]$-module by letting x act as λ. Then V can be expressed as a direct sum of $F[x]$-modules

$$V = \bigoplus_{g\in\mathcal{P}\cup\mathcal{Q}} V_g$$

such that:

(i) *$V_g \cong (F[x]/g)^k$ as $F[x]$-modules for some $k \in \mathbb{N}\cup\{0\}$,*
(ii) *the spaces $\{V_g\}_{g\in\mathcal{P}\cup\mathcal{Q}}$ are pairwise orthogonal, i.e. $b(V_g, V_{g'}) = 0$ for every distinct $g, g' \in \mathcal{P}\cup\mathcal{Q}$ (hence $b = \perp_{g\in\mathcal{P}\cup\mathcal{Q}} (b|_{V_g\times V_g})$).*

Moreover, if $V = \bigoplus_{g\in\mathcal{P}\cup\mathcal{Q}} V'_g$ is another such decomposition, then $(V_g, b|_{V_g\times V_g}) \cong (V'_g, b|_{V'_g\times V'_g})$ for all $g \in \mathcal{P}\cup\mathcal{Q}$.

The existence of a decomposition satisfying (i) easily follows for the classification of f.g. modules over principal ideal domains. Part (ii), as well as the uniqueness of the decomposition up to isometry, are not trivial. The previous theorem calls for the following definition:

Definition 4.1.2. *The regular bilinear space (V, b) is called an* isotype *(or an* f_λ-isotype*) if $V \cong (F[x]/f_\lambda)^k$ as $F[x]$-modules for some $k \in \mathbb{N}$ and $f_\lambda = (pp^*)^m$ for some monic prime $p \in F[x]$ with $p \neq p^*$ or $f_\lambda = q^n$ for some monic prime $q \in F[x]$ with $q = q^*$.*

Theorem 4.1.1 thus reduces the isomorphism problem of bilinear forms into the isomorphism problem of g-isotypes. This process is a basic step in many of the papers mentioned earlier.

It turns out the the parameter g strongly affects the diversity of isometry classes of g-isotypes.

Theorem 4.1.3. *Let $g \in F[x]$ and let (V, b) be a g-isotype. Assume that at least one of the following holds:*

(1) *$g = (pp^*)^n$ for some $n \in \mathbb{N}$ and prime $p \in F[x]$ with $p \neq p^*$.*
(2) *$g = (x - (-1)^n)^n$ for some $n \in \mathbb{N}$.*

Then V is the direct sum of $F[x]$-modules $V_1 \oplus V_2$ such that:

(i) *V_1, V_2 are totally isotropic (i.e. $b(V_1, V_1) = b(V_2, V_2) = 0$).*
(ii) *If (1) holds, then $V_1 \cong (F[x]/p^n)^k$ and $V_2 \cong (F[x]/(p^*)^n)^k$ (as $F[x]$-modules) for some $k \in \mathbb{N}$.*
(iii) *If (2) holds, then $V_1 \cong V_2 \cong (F[x]/g)^k$ (as $F[x]$-modules) for some $k \in \mathbb{N}$.*

In any case, the isometry class of b is determined by $\dim V$ *and g.*

Proof (partial, sketch). We only show the existence of V_1 and V_2 in case (1) holds. Provided V_1 and V_2 exists, we show that b is uniquely determined up to isometry by g and $\dim V$.

Assume (1) holds and view V as an $F[\lambda] := F[x]/g$-module. Then the Chinese Remainder Theorem implies $F[\lambda] \cong F[x]/p^n \times F[x]/(p^*)^n$. Thus, any $F[\lambda]$-module U decompose into a direct sum of an $F[x]/p^n$-module and an $F[x]/(p^*)^n$-module, namely $U = \operatorname{ann}_U(p^n) \oplus \operatorname{ann}_U((p^*)^n)$ (and $F[\lambda]$ acts on each component via the isomorphism $F[\lambda] \cong F[x]/p^n \times F[x]/(p^*)^n$). Let $V_1 = \operatorname{ann}_V(p^n) = \ker p^n(\lambda)$ and $V_2 = \operatorname{ann}_V((p^*)^n) = \ker(p^*)^n(\lambda)$. Since $V_{F[\lambda]}$ is free, $(V_1)_{F[x]/p^n}$ and $(V_2)_{F[x]/(p^*)^n}$ are also free of the same rank, so (ii) holds.

To see (i), let α be the corresponding anti-endomorphism of b. Then it is easy to see that $\lambda^\alpha = \lambda^{-1}$ (Lemma 2.3.12(ii), $\gamma = \mathrm{id}_V$ in our case). Also observe that the image of $(p^*)^n$ in $F[x]/p^n$ is a unit, hence $(p^*)^n(\lambda|_{V_1}) \in \mathrm{GL}(V_1) := \mathrm{Aut}_F(V_1)$. Let $a = p^*(0)^n$. Then $(p^*)^n(\lambda^\alpha) = (p^n)^*(\lambda^{-1})^n = a((p^n)^{**})(\lambda) = ap^n(\lambda)$. Now let $u, v \in V_1$. Then there is $u_0 \in V_1$ such that $u = (p^*)^n(\lambda)u_0$. Therefore, $b(u, v) = b((p^*)^n(\lambda)u_0, v) = b(u_0, (p^*)^n(\lambda^\alpha)v) = b(u_0, ap^n(\lambda)v) = b(u_0, 0) = 0$, as required.

Now assume (V', b') is another g-isotype. Write $V' = V_1' \oplus V_2'$ for V_1', V_2' satisfying (i) and (ii) and let λ' be the asymmetry of b'. Then $\lambda|_{V_1}$ and $\lambda'|_{V_1'}$ are conjugate (since they have the same *canonical rational form* by (ii)). Thus, $b \cong b'$ by Proposition 4.7.6 below (the proof is a well-known argument). □

In light of the previous theorem, we need to solve the isomorphism problem for g-isotypes only when $g = q^n$ for some prime $q \in F[x]$ with $x - (-1)^n \neq q = q^*$. The latter turns out to be equivalent to the isomorphism problem of hermitian forms over $F[x]/q$.

Let K/F be a f.d. field extension admitting an F-linear involution α. Assume that there is $\lambda_0 \in K$ such that $\lambda_0 \lambda_0^\alpha = 1$ and let U be a K-vector space. Recall that a biadditive map $h : U \times U \to K$ is called an (α, λ_0)-*hermitian form* if

$$h(xa, y) = a^\alpha h(x, y), \qquad h(x, ya) = h(x, y)a, \qquad h(x, y) = h(y, x)^\alpha \lambda_0$$

for all $x, y \in U$ and $a \in F$. Let $\mathrm{Tr} : K \to F$ be a non-zero F-linear map such that $\mathrm{Tr}(a) = \mathrm{Tr}(a^\alpha)$ for all $a \in K$. Such a map always exist. We now have:

Proposition 4.1.4 (Riehm). *Assume $K = F[\lambda_0]$ and let U be a f.d. K-vector space. Let $H(\alpha, \lambda_0)$ denote the set of (α, λ_0)-hermitian forms defined on U and let $B(\lambda_0)$ be the set of bilinear forms on U having λ_0 as a (right id_F-)asymmetry.*[2] *Then $H(\alpha, \lambda_b) \cong B(\lambda_0)$ via $h \mapsto \mathrm{Tr} \circ h$.*

Assume (V, b) is a g-isotype, where $g = q^n$ as above. Let λ be the asymmetry of b and define

$$(\pi, \varepsilon) = \begin{cases} (\lambda^{\deg q/2} q(\lambda), 1) & g(x) \neq (x + (-1)^n)^n \\ (\lambda - \lambda^{-1}, -1) & g(x) = (x + (-1)^n)^n \end{cases} .$$

(Note that $\deg q$ is even if $q(1), q(-1) \neq 0$.) It is easy to verify that

$$b(\pi u, v) = \varepsilon b(u, \pi v) \qquad \forall u, v \in V \ , \tag{15}$$

and π generates the unique maximal ideal of $F[\lambda]$ (which is $\langle q(\lambda) \rangle$). Let $U = V/\pi V$ and define a bilinear form $b_{\mathrm{red}} : U \times U \to F$ by $b_{\mathrm{red}}(\overline{u}, \overline{v}) = (\pi^{n-1} u, v)$ (this is well-defined by (15)).[3] Consider U as a vector space over $K := F[\lambda]/\pi = F[x]/q$ and let $\lambda_0 := \varepsilon^{n-1} \overline{\lambda}$. Then K admits an involution α sending $\overline{x}$ (and hence λ_0) to its inverse. As λ_0 is easily seen to be the asymmetry of b_{red}, Proposition 4.1.4 implies that there is an (α, λ_0)-hermitian form $h : U \times U \to K$ such that $b_{\mathrm{red}} = \mathrm{Tr} \circ h$. The following theorem states that the isometry class of b is determined by the isometry class of h.

Theorem 4.1.5 (Riehm). *Let (V, b), (V, b') be two g-isotypes having the same asymmetry λ and assume $g = q^n$ as above. Let h, h' be the (α, λ_0)-hermitian forms induced by b and b' as just explained. Then $b \cong b'$ if and only if $h \cong h'$.*

To finish, we note that Riehm's solution was extended by Gabriel to degenerate forms ([**44**]). Roughly speaking, Gabriel proved that every bilinear space (V, b) can be decomposed as a sum of a regular part and an *essentially degenerate* space and that decomposition is unique up to isometry. While the isomorphism problem of the regular part is solved as above, Gabriel showed that the degenerate part is determined up to isometry by its *Kronecker module* (see below). We will not describe Gabriel's results here, but rather state them later in section 4.11, where we shall provide a new easy proof. Somewhat ironically, the techniques used by Gabriel (i.e. the Kronecker modules) are essential to deal with bilinear forms over rings, regular or not.

[2] In [**76**], Riehm also requires the forms in $B(\lambda_0)$ to satisfy $b(xa, y) = b(x, ya^\alpha)$ for all $x, y \in U$ and $a \in K$. However, since we assume $K = F[\lambda_0]$, Riehm's extra condition is automatically satisfied because $b(\lambda_0 x, y) = b(x, \lambda_0^\alpha y)$.

[3] The letters "red" in b_{red} stand for "reduced".

4.2. Survey: Categories with Duality (Hermitian Categories)

In this section, we briefly survey the theory of categories with duality (also called hermitian categories). This theory, designed to handle hermitian forms over rings with involution, has initiated in 70's and was developed by various authors including Bak ([**6**]), Knebusch ([**55**]), Quebemmann Scharlau and Schulte ([**71**]) and others (see also [**86**, Ch. 7] and [**7**]). We will recall most of the definitions and bring some of the fundamental results in this area. All proofs can be found in [**71**] or [**86**].

DEFINITION 4.2.1. *A* category with duality *is a triplet* $(\mathscr{H},*,\omega)$ *such that* $\mathscr{H}$ *is a category,* $*:\mathscr{H}\to\mathscr{H}$ *is a contravariant functor and* $\omega:\mathrm{id}_{\mathscr{H}}\to **$ *is a natural transformation satisfying*

$$\omega_M^*\circ\omega_{M^*}=\mathrm{id}_{M^*}$$

for all $M\in\mathscr{H}$*. If* $\mathscr{A}$ *is an additive category, then we require* $*$ *to be additive.*

A bilinear form over $\mathscr{H}$ *is a pair* (M,b) *such that* $M\in\mathscr{H}$ *and* $b\in\mathrm{Hom}(M,M^*)$*. The form* (M,b) *is called* symmetric *if* $b=\widetilde{b}:=b^*\circ\omega_M$*. In this case,* b *is called* regular *if it is bijective.*

An isometry *from a bilinear form* (M,b) *to a bilinear form* (M',b') *is an isomorphism* $\sigma:M\to M'$ *such that* $\sigma^*\circ b'\circ\sigma=b$*. If* $\mathscr{H}$ *is additive, then we define* $(M,b)\perp(M',b'):=(M\oplus M',b\oplus b')$ *(we identify* $(M\oplus M')^*$ *with* $M^*\oplus M'^*$*).*

We will usually omit $*$ and ω from the notation, writing $\mathscr{H}$ instead of $(\mathscr{H},*,\omega)$.

REMARK 4.2.2. It is customary to assume ω is a natural *isomorphism*, but as in section 2.7, we will not enforce it. Note that if (M,b) is a regular symmetric form over $\mathscr{H}$, then $\omega_M=(b^*)^{-1}\circ b$ is bijective.

EXAMPLE 4.2.3. Let (R,α) be a ring with involution and let $\lambda\in R$ be a central element satisfying $\lambda^\alpha\lambda=1$. For all $M\in$ Mod-R, let $M^*=\mathrm{Hom}(M,R_R)$. Then M^* is a right R-module w.r.t. $(f\cdot r)m=r^\alpha\cdot(fm)$ (where $r\in R$, $m\in M$, $f\in M^*$). Furthermore, there exists a natural transformation $\omega_M:M\to M^{**}$ given by $(\omega_M m)f=\lambda\cdot(fm)^\alpha$. Then (Mod-$R,*,\omega$) is a category with duality.

The bilinear forms on Mod-R correspond to *(α-)sesquilinear forms* over R. Indeed, for every sesquilinear form $b:M\times M\to R$, the pair (M,Ad_b^r) (which clearly determines b) is easily seen to be a bilinear form over Mod-R. Furthermore, b is (α,λ)-hermitian (i.e. $b(x,y)=\lambda b(y,x)^\alpha$) if and only if (M,Ad_b^r) is symmetric. In fact, the category of sesquilinear (λ-hermitian) forms over R can be understood as the category of (symmetric) bilinear forms over Mod-R.

REMARK 4.2.4. Let $\mathscr{H}$ be an additive category with duality. It turns out that *locally* $\mathscr{H}$ looks like the previous example. More precisely, every object in $\mathscr{H}$ is contained in a full subcategory that is isomorphic (as a category with duality) to a full suchcategory of proj-R (the category of finite projective right R-modules), considered as a category with duality w.r.t. some involution α of R and $\lambda\in\mathrm{Cent}(R)$ with $\lambda^\alpha\lambda=1$. This is the principle of transfer and we refer the reader to [**71**] or section 2.8 above for a detailed discussion.

In order to proceed, we need to assume that 2 is invertible in $\mathscr{H}$.[4] That is, $\mathscr{H}$ satisfies the following condition:

(C0) $2\,\mathrm{id}_M$ is an isomorphism for all $M\in\mathscr{H}$.

Furthermore, we also consider the following conditions:

[4] This can be avoided by replacing *bilinear* forms with *quadratic* forms (see [**71**]). We have omitted the definition and the details since we are only interested in bilinear forms.

(C1) All idempotents in $\mathscr{H}$ split.[5]

(C2) End(M) is complete semilocal for all $M \in \mathscr{H}$.

In addition, it will be sometimes more convenient to consider the following stronger version of (C2):

(C2′) End(M) is semiprimary for all $M \in \mathscr{H}$.

While conditions (C0) and (C1) are easy to satisfy, conditions (C2) and (C2′) are very strong, so let us exhibit some examples in which they are satisfied.

Example 4.2.5. (i) Let F be a field. An F-category is a preadditive category $\mathscr{C}$ such that for every $M, N \in \mathscr{C}$, Hom(M, N) is endowed with a f.d. F-vector space structure and composition is F-bilinear. In this case, End(M) is a f.d. F-algebra for every $M \in \mathscr{C}$, hence condition (C2′) holds.

(ii) By Theorem 1.7.3 (resp. Corollary 1.8.5), the category of f.p. right R-modules satisfies (C2′) (resp. (C2)) whenever R is semiprimary (resp. complete semilocal with Jacobson radical f.g. as a right ideal).

Conditions (C1) and (C2) imply that every object in $\mathscr{H}$ has a *Krull-Schmidt decomposition* (see Theorem 1.1.1) and that the endomorphism ring of every indecomposable object is local (and complete by (C2)). If Σ is a set of isomorphism classes of indecomposable objects in $\mathscr{H}$, we say that an object $M \in \mathscr{H}$ is of *type*-Σ if for any indecomposable summand A of M, we have $[A] \in \Sigma$ (where $[A]$ is the isomorphism class of A). We also let Σ_M denote the set of isomorphism classes of indecomposable summands of M.

The following three theorems imply that regular symmetric bilinear forms over categories with duality satisfying conditions (C0)–(C2) have a very special structure.

Theorem 4.2.6 (Decomposition into Isotypes). *Assume conditions (C0)–(C2) hold. Let (M, b) be a bilinear form over $\mathscr{H}$ and let $\overline{\Sigma}_M := \{\{[A],[A^*]\} \mid [A] \in \Sigma_M\}$. Then there exists a decomposition*

$$(M, b) \cong \operatorname*{\perp}_{\zeta \in \overline{\Sigma}_M} (M_\zeta, b_\zeta)$$

such that M_ζ is of type-ζ. The summand (M_ζ, b_ζ) is uniquely determined up to isometry by (M, b) and ζ.

Let $A \in \mathscr{H}$ be an indecomposable object and let $\zeta = \{[A],[A^*]\}$. A bilinear form (M, b) is called a ζ-isotype if M is of type-ζ. The previous theorem thus asserts that every bilinear space over a category with duality satisfying (C0), (C1) and (C2) is a sum of isotypes, which are uniquely determined up to isometry.

Definition 4.2.7. *Let $M \in \mathscr{H}$. The hyperbolic form $\mathbb{H}(M)$ is defined to be $(M \oplus M^*, b_M)$ where*

$$b_M = \left[\begin{smallmatrix} 0 & \mathrm{id}_{M^*} \\ \omega_M & 0 \end{smallmatrix}\right] \in \operatorname{Hom}(M \oplus M^*, (M \oplus M^*)^*) = \operatorname{Hom}(M \oplus M^*, M^* \oplus M^{**}) \; .$$

A bilinear form (M, b) is called hyperbolic *if $(M, b) \cong \mathbb{H}(N)$ for some $N \in \mathscr{H}$.*

Theorem 4.2.8. *Assume conditions (C0), (C1) and (C2) hold. Let A be an indecomposable object such that there is no regular symmetric bilinear form on A (e.g. if $A \not\cong A^*$). Then every $\{[A],[A^*]\}$-isotype is hyperbolic. Moreover, an $\{[A],[A^*]\}$-isotype (M, b) is determined up to isometry by $[M]$.*

[5] Let $\mathscr{A}$ be a category and $A \in \mathscr{A}$. An idempotent $e \in \mathrm{End}_{\mathscr{A}}(A)$ is split if there exist $B \in \mathscr{A}$, $i \in \mathrm{Hom}(B, A)$ and $p \in \mathrm{Hom}(A, B)$ such that $i \circ p = e$ and $p \circ i = \mathrm{id}_B$. If $\mathscr{A}$ is additive, then condition (C1) means that every idempotent $e \in \mathrm{End}_{\mathscr{A}}(A)$ corresponds to a summand of A (such a summand need not exist in general).

THEOREM 4.2.9. *Assume conditions (C0), (C1) and (C2) hold. Let A be an indecomposable object such that there exists a symmetric regular bilinear form (A,h). Let $L = \mathrm{End}(A)$ and let α be the involution of L corresponding to h (i.e. $\alpha : x \mapsto h^{-1} \circ x^* \circ h$). Then $D = L/\operatorname{Jac}(L)$ is a division ring and α induces an involution on D, which we continue denoting by α. Consider* mod-D *(the category of f.g. D-modules) as a category with duality w.r.t. α and $\lambda = 1$ as in Example 4.2.3 and let $\mathscr{H}|_A$ be the full subcategory of $\mathscr{H}$ consisting of objects of type $\{[A]\} = \{[A],[A^*]\}$. Then there exists a* homomorphism of categories with duality

$$\overline{} : \mathscr{H}|_A \to \text{mod-}D$$

such that $\overline{A} = D_D$, and for every two bilinear forms $(M,b),(M',b')$ over $\mathscr{H}$ and every isometry $\sigma_0 : \overline{(M,b)} \to \overline{(M',b')}$ there exists an isometry $\sigma : (M,b) \to (M',b')$ with $\overline{\sigma} = \sigma_0$. In particular, the isometry problem of $\{[A]\}$-isotypes can be reduced to isometry of 1*-hermitian forms over D.*

The last three theorems show that the isometry problem of symmetric regular bilinear forms in a category with duality satisfying (C0)–(C2) can be reduced to: (1) isomorphism and decomposition problems in $\mathscr{H}$ and (2) isometry of hermitian forms over division rings. The applications of this principle are numerous; for instance, we get:

COROLLARY 4.2.10 (Witt's Cancelation Theorem). *Let (M_1,b_1), (M_2,b_2), (M_3,b_3) be symmetric regular bilinear spaces over a hermitian category $\mathscr{H}$ satisfying (C0)–(C2). Then $(M_1,b_1) \perp (M_2,b_2) \cong (M_1,b_1) \perp (M_3,b_3) \iff (M_2,b_2) \cong (M_3,b_3)$.*

PROOF. This holds since Witt's Cancelation Theorem holds for hermitian forms over division rings of characteristic not 2. (Moreover, Witt's Cancelation Theorem holds for symmetric regular forms over semilocal rings; see [**73**].) □

COROLLARY 4.2.11. *Let F be an algebraically closed field and assume $\mathscr{H}$ is an F-category such that $*$ is F-linear. Then the isometry class of a regular symmetric bilinear form (M,b) is determined by $[M]$.*

PROOF. Reduce to isometry of 1-hermitian forms over f.d. division F-algebras with F-linear involution α. Since F is algebraically closed, the only such division ring is F and $\alpha = \mathrm{id}_F$. As F is algebraically closed, a hermitian form is determined up to isometry but the dimension of its underlying vector spaces. The corollary follows immediately. □

REMARK 4.2.12. (i) The assumption that the conditions (C0)–(C2) hold for *all* objects in $\mathscr{A}$ is usually superfluous. In most of the previous results, it is enough to assume that the bilinear form under question, (M,b), satisfies $2 \in \mathrm{End}(M)^\times$, all idempotents of $\mathrm{End}(M)$ split and $\mathrm{End}(M)$ is complete semilocal.

(ii) All previous results hold when replacing (C2) with the slightly milder condition:[6]

(C2$''$) $\mathrm{End}(M)$ is semiperfect pro-semiprimary (w.r.t. some ring topology) for all $M \in \mathscr{H}$.

The proofs remain almost the same. (In addition, we give a detailed proof of the critical arguments in section 4.9 below.) This is important since by Theorem 1.8.3, the category of Hausdorff f.p. right R-modules satisfies (C2$''$) when R is first countable semiperfect pro-semiprimary (e.g. a complete semilocal ring). Furthermore,

[6] To writing these words, we do not know whether (C2$''$) is indeed weaker than (C2); see section 1.10.

condition (C2$''$) passes from an object to its summands (Proposition 1.2.3(ii)), while it is not clear to us if the same applies to (C2).[7]

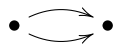
The reader may have spotted some similarity between the results of this section and the results of Riehm from the previous section. In the next section we will show that this is no coincidence — there is a canonical way to translate the theory of non-symmetric forms into the theory of symmetric forms, and the results of Riehm, Gabriel and others "factor" through it.

4.3. From Non-Symmetric to Symmetric

Until now, we have stated some known results which are due to various authors. Beginning from this section, we describe our own work.

In this section, we will prove a deep result showing that the isometry problem of bilinear forms is equivalent to the isometry problem of *regular symmetric* forms. More explicitly, we will prove that the category of (arbitrary) bilinear forms over a *category with a double duality* (see section 2.7 or the summary below) is isomorphic to the category of regular symmetric bilinear forms over another category with duality.[8] The latter category will be the category of *Kronecker modules*. Once that is achieved, the rest of this chapter will be dedicated to reduce this result into "down-to-earth" results.[9]

Let us begin by first recalling the classical definition of Kronecker modules. In the sense of [**44**], a Kronecker module over a field F consists of a quartet (U, f, g, V) where V, U are vector spaces and $f, g \in \mathrm{Hom}(U, V)$. Kronecker modules correspond to modules over the *Kronecker algebra*, $\mathbb{K}(F) = \left[\begin{smallmatrix} F & F\oplus F \\ 0 & F \end{smallmatrix}\right]$, which is the path algebra of the quiver:



(the vector spaces U, V correspond to the vertices and f, g correspond to the arrows). In the second half of the nineteenth centaury, Kronecker gave an explicit description of the indecomposable Kronecker modules (of finite dimension) and moreover, provided an algorithm for decomposition of a given module. As the Krull-Schmidt Theorem implies that a Kronecker module is determined up to isomorphism by its indecomposable factors (with multiplicities; see section 1.1), the isomorphism problem of Kronecker modules can be considered as solved or at least very well-understood (see [**44**, §3] for more details).

Any bilinear form $b : V \times V \to F$ gives rise to a Kronecker module

$$(V, \mathrm{Ad}^{\ell}_b, \mathrm{Ad}^{r}_b, V^*)$$

(where $V^* = \mathrm{Hom}_F(V, F)$). As mentioned above, such Kronecker modules were used by Gabriel and others (see [**44**] and related papers) to reduce the isomorphism problem of arbitrary classical bilinear forms to nondegenerate forms. However, in this chapter we shall consider Kronecker modules for completely different purposes.

Throughout, $(\mathscr{A}, [0], [1], \Phi, \Psi)$ is a category with a double duality (see section 2.7). Recall that this means $[0], [1] : \mathscr{A} \to \mathscr{A}$ are contravariant functors (written

---

[7] This boils down to the question whether for every complete semilocal ring R and $e \in \mathrm{E}(R)$, eRe is also complete semilocal. This is not trivial since $\mathrm{Jac}(eRe)^n = (e\,\mathrm{Jac}(R)e)^n$ might be strictly smaller than $e\,\mathrm{Jac}(R)^n e$ in general (but equality holds for $n = 1$).

[8] Categories with duality are also known as "hermitian categories".

[9] A similar result, applying to *arbitrary symmetric* bilinear forms over a category with duality was obtained in [**16**]. Furthermore, several days before submitting this work, I was introduced with the unpublished (and recent) work [**17**], which proves a similar principle for *arbitrary* bilinear forms over categories with duality. Both of these references assume all objects in the given category with duality are reflexive, which is not necessary for the result obtained here. See Remarks 4.3.7 and 4.3.8 below for more details. (*Eventually, the authors of* [**17**] *and I have combined our results and submitted them together in* [**11**].)

exponentially) and $\Phi : \mathrm{id}_{\mathscr{A}} : [1][0]$, $\Psi : \mathrm{id}_{\mathscr{A}} \to [0][1]$ are natural transformations satisfying $\Psi_M^{[0]} \circ \Phi_{M^{[0]}} = \mathrm{id}_{M^{[0]}}$ and $\Phi_M^{[1]} \circ \Psi_{M^{[1]}} = \mathrm{id}_{M^{[1]}}$ for all $M \in \mathscr{A}$. These identities induce a natural isomorphism

$$I_{A,B} : \mathrm{Hom}(B, A^{[1]}) \to \mathrm{Hom}(A, B^{[0]})$$

given by $I_{A,B}(f) = f^{[0]} \circ \Phi_A$; its inverse is given by $I_{A,B}^{-1}(g) = g^{[1]} \circ \Psi_B$. If $\mathscr{A}$ is additive (or preadditive), then we require [0] and [1] to be additive.

A bilinear form over $(\mathscr{A}, [0], [1], \Phi, \Psi)$ (or just $\mathscr{A}$ for brevity) is a pair (M, b) such that $M \in \mathscr{A}$ and $b \in \mathrm{Hom}(M, M^{[1]})$. In this case we define $\widetilde{b} = I_{M,M}(b) = b^{[0]} \circ \Phi_M \in \mathrm{Hom}(M, M^{[0]})$. A bilinear forms is right (resp. left) regular if b (resp. $\widetilde{b}$) is bijective. An isometry from (M, b) to (M', b') is an isomorphism $\sigma \in \mathrm{Hom}(M, M')$ such that $\sigma^{[1]} \circ b' \circ \sigma = b$.

Henceforth, $(\mathscr{A}, [0], [1], \Phi, \Psi)$ is a fixed category with a double duality.

Definition 4.3.1. *A* Kronecker module *over* $\mathscr{A}$ *(or* $(\mathscr{A}, [0], [1], \Phi, \Psi)$*) is a quartet* (M, f_0, f_1, N) *such that* $M, N \in \mathscr{A}$ *and*

$$f_0 \in \mathrm{Hom}(M, N^{[0]}), \qquad f_1 \in \mathrm{Hom}(M, N^{[1]}) \ .$$

A homomorphism between two Kroneker modules (M, f_0, f_1, N) *and* (M', f_0', f_1', N') *is a formal pair* $(\sigma, \tau^{\mathrm{op}})$ *with* $\sigma \in \mathrm{Hom}(M, M')$ *and* $\tau \in \mathrm{Hom}(N', N)$ *such that:*

$$f_0' \circ \sigma = \tau^{[0]} \circ f_0 \qquad \text{and} \qquad f_1' \circ \sigma = \tau^{[1]} \circ f_1 \ .$$

The composition of two morphisms of Kronecker modules $(\sigma, \tau^{\mathrm{op}}) : (M, f_0, f_1, N) \to (M', f_0', f_1', N')$ *and* $(\sigma', \tau'^{\mathrm{op}}) : (M', f_0', f_1', N') \to (M'', f_0'', f_1'', N'')$ *is given by*

$$(\sigma', \tau'^{\mathrm{op}}) \circ (\sigma, \tau^{\mathrm{op}}) = (\sigma' \circ \sigma, (\tau \circ \tau')^{\mathrm{op}}) \ .$$

This makes the class of Kronecker modules into a category which we denote by $\mathrm{Kr}(\mathscr{A})$*.*

Example 4.3.2. Let R be a ring and let K be a double R-module. Then Mod-R admits a standard structure of category with a double duality induced by K, hence we can consider Kronecker modules over Mod-R. These would be quartets (M, f_0, f_1, N) with $M, N \in$ Mod-R and $f_i \in \mathrm{Hom}(M, N^{[i]})$ for $i \in \{0, 1\}$. Note that precaution should be taken as these Kronecker modules do not naturally correspond to (ring theoretic) modules over the Kronecker algebra $\mathbb{K}(R) := \left[\begin{smallmatrix} R & R\oplus R \\ 0 & R \end{smallmatrix}\right]$. The latter correspond to quartets (M, f, g, N) such that $M, N \in$ Mod-R and $f, g \in \mathrm{Hom}(M, N)$ (and we shall stick to this description henceforth).

Nevertheless, when K admits an anti-isomorphism κ, there is a functor from Kr(Mod-R) to Mod-$\mathbb{K}(R)$ given by $F : (M, f_0, f_1, N) \mapsto (M, u_{\kappa,N} \circ f_0, f_1, N^{[1]})$ (see Proposition 2.2.7 for the definition of u_κ; recall that $u_{\kappa,N} : N^{[0]} \to N^{[1]}$ is a natural isomorphism). A morphism $(\sigma, \tau^{\mathrm{op}})$ in Kr(Mod-R) will be mapped by F to $(\sigma, \tau^{[1]})$.

In general, the functor just defined is neither faithful nor full. However, if $Z := (M, f_0, f_1, N)$ and $Z' := (M', f_0', f_1', N')$ are Kronecker modules such that N, N' are *reflexive* (see section 2.5), then the map $[1] : \mathrm{Hom}(N', N) \to \mathrm{Hom}(N'^{[1]}, N^{[1]})$ is bijective (Proposition 2.5.6(iii)), implying that $F : \mathrm{Hom}_{\mathrm{Kr}(\text{Mod-}R)}(Z, Z') \to \mathrm{Hom}_{\text{Mod-}\mathbb{K}(R)}(FZ, FZ')$ is bijective. Thus, once restricted to the category of Kronecker modules (M, f_0, f_1, N) with N reflexive, F becomes faithful and full.

For example, if R is a field and K is the double R-module obtained from R by letting $\odot_0$ and $\odot_1$ be the standard right actions of R on itself, then every f.d. R-module is reflexive. Thus, the category of "finite dimensional" Kronecker modules is equivalent to the category of f.g. $\mathbb{K}(R)$-modules. As the latter is equivalent to the category of Kronecker modules *in the sense of* [**44**] (see above), we get that at least in the f.d. case, our new definition of Kronecker modules agrees (modulo equivalence of categories) with the definition of Gabriel in [**44**].

The category $\mathrm{Kr}(\mathscr{A})$ inherits some of the properties of $\mathscr{A}$. This is demonstrated in the following proposition.

PROPOSITION 4.3.3. *In the previous assumptions:*

(i) *If $\mathscr{A}$ preadditive*[10] *(in which case we assume* $[0]$ *and* $[1]$ *are additive), then so is* $\mathrm{Kr}(\mathscr{A})$*. The sum of two morphisms* $(\sigma_1, \tau_1^{\mathrm{op}}), (\sigma_1, \tau_1^{\mathrm{op}}) \in \mathrm{Hom}(Z, Z')$ *is given by* $(\sigma_1 + \sigma_2, (\tau_1 + \tau_2)^{\mathrm{op}})$*.*

(ii) *If $\mathscr{A}$ is additive, then so is* $\mathrm{Kr}(\mathscr{A})$*; the direct sum of two Kronecker modules* $(M, f_0, f_1, N), (M', f_0', f_1', N') \in \mathrm{Kr}(\mathscr{A})$ *is given by*

$$(M \oplus M', f_0 \oplus f_0', f_1 \oplus f_1', N \oplus N')\ .$$

(iii) *If $\mathscr{A}$ is preadditive and all idempotent morphisms in $\mathscr{A}$ split, then all idempotent morphisms in* $\mathrm{Kr}(\mathscr{A})$ *split.*

(iv) *Let F be a field. If $\mathscr{A}$ is an F-category and* $[0], [1]$ *are F-linear, then* $\mathrm{Kr}(\mathscr{A})$ *is an F-category.*

PROOF. This is routine. □

For every Kronecker module $Z = (M, f_0, f_1, N) \in \mathrm{Kr}(\mathscr{A})$, define its *dual* by

$$Z^* = (N, I_{N,M}(f_1), I_{M,N}^{-1}(f_0), M) = (N, f_1^{[0]} \circ \Phi_N, f_0^{[1]} \circ \Psi_N, M)\ .$$

The map $Z \mapsto Z^*$ is a contravariant functor from $\mathrm{Kr}(\mathscr{A})$ to itself, where the dual of a morphism $(\sigma, \tau^{\mathrm{op}}) \in \mathrm{Hom}(Z, Z')$ is defined to be $(\tau, \sigma^{\mathrm{op}}) \in \mathrm{Hom}(Z'^*, Z^*)$. (Indeed, if $Z' = (M', f_0', f_1', N')$, then $\sigma^{[0]} \circ (f_1'^{[0]} \circ \Phi_{N'}) = (f_1' \circ \sigma)^{[0]} \circ \Phi_{N'} = (\tau^{[1]} \circ f_1)^{[0]} \circ \Phi_{N'} = f_1^{[0]} \circ \tau^{[1][0]} \circ \Phi_{N'} = (f_1^{[0]} \circ \Phi_N) \circ \tau$ and similarly $\sigma^{[1]} \circ (f_0'^{[1]} \circ \Psi_{N'}) = (f_0^{[1]} \circ \Phi_N) \circ \tau$, so $(\tau, \sigma^{\mathrm{op}})$ lies in $\mathrm{Hom}(Z'^*, Z^*)$.) In addition, clearly $** = \mathrm{id}_{\mathscr{A}}$. Therefore, $(\mathrm{Kr}(\mathscr{A}), *, \mathrm{id})$ is a category with duality, where id is the identity isomorphism from the identity functor $\mathrm{id}_{\mathrm{Kr}(\mathscr{A})}$ to $** = \mathrm{id}_{\mathrm{Kr}(\mathscr{A})}$.[11]

Our newly defined duality structure on $\mathrm{Kr}(\mathscr{A})$ allows us to consider symmetric bilinear forms over Kronecker modules. Such forms consist of pairs $(Z, (\sigma, \tau^{\mathrm{op}}))$ with $Z = (M, f_0, f_1, N) \in \mathrm{Kr}(\mathscr{A})$ and $(\sigma, \tau^{\mathrm{op}}) \in \mathrm{Hom}(Z, Z^*)$ (which implies $\sigma, \tau \in \mathrm{Hom}(M, N)$). The form $(Z, (\sigma, \tau^{\mathrm{op}}))$ is *symmetric* if $(\sigma, \tau^{\mathrm{op}}) = (\sigma, \tau^{\mathrm{op}})^* \circ \mathrm{id}_Z$, i.e. if $\sigma = \tau$, and regular if $(\sigma, \tau^{\mathrm{op}})$ is invertible, i.e. σ, τ are invertible. We let $\mathrm{Sym}_{\mathrm{reg}}(\mathrm{Kr}(\mathscr{A}))$ denote the category of *regular symmetric* bilinear forms over $\mathrm{Kr}(\mathscr{A})$ and $\mathrm{Bil}(\mathscr{A})$ denote the category of *arbitrary* bilinear forms over $\mathscr{A}$. The morphisms in both categories are isometries.

Observe that any bilinear space $(M, b) \in \mathrm{Bil}(\mathscr{A})$ induces a Kronecker module, namely

$$Z(M, b) = (M, \widetilde{b}, b, M)\ .$$

(For example, in the special case where $\mathscr{A}$ is obtained from a ring R and a double R-module K, the Kronecker module of $b : M \times M \to K$ will be $(M, \mathrm{Ad}_b^{\ell}, \mathrm{Ad}_b^{r}, M)$). The following proposition characterizes the Kronecker modules obtained from bilinear spaces.

PROPOSITION 4.3.4. *In the previous assumptions:*

(i) $Z = Z(M, b)$ *for some bilinear form* $(M, b) \in \mathrm{Bil}(\mathscr{A}) \iff Z = Z^*$*.*

(ii) *The following conditions are equivalent:*

  (a) $Z \cong Z(M, b)$ *for some bilinear form* $(M, b) \in \mathrm{Bil}(\mathscr{A})$*.*

  (b) *There exists an isomorphism* $\eta : M \to N$ *s.t.* $(f_1^{[0]} \circ \Phi_N) \circ \eta = \eta^{[0]} \circ f_0$*.*

---

[10] The category $\mathscr{A}$ is preadditive if for all $N, M \in \mathscr{A}$, $\mathrm{Hom}(M, N)$ is equipped with an additive group structure such that composition is biadditive.

[11] The appropriate notation for id above should have been $\mathrm{id}_{\mathrm{id}_{\mathrm{Kr}(\mathscr{A})}}$, but this is somewhat incomprehensible, not to mention that it looks peculiar.

(c) *There exists an isomorphism* $\eta : M \to N$ *s.t.* $(f_0^{[1]} \circ \Psi_N) \circ \eta = \eta^{[1]} \circ f_1$.
(d) *There exists an isomorphism* $\eta : M \to N$ *s.t.* $(\eta, \eta) \in \mathrm{Hom}(Z, Z^*)$.
(e) *There exists a symmetric regular bilinear form (over* $\mathrm{Kr}(\mathscr{A})$*) on* Z.

PROOF. (i) $Z = Z^* \iff M = N$, $f_0 = I_{M,M}(f_1)$ and $f_1 = I_{M,M}^{-1}(f_0) \iff M = N$ and $f_0 = I_{M,M}(f_1) \iff Z = Z(M, f_1)$.

(ii) First, observe that the preceding discussion implies that $(Z, (\eta, \eta^{\mathrm{op}}))$ is a symmetric bilinear form over $\mathrm{Kr}(\mathscr{A})$ and it is regular if and only if η is invertible. Thus, (d) $\iff$ (e).

We next prove (b) $\iff$ (c). Assume (b) holds, i.e. $(f_1^{[0]} \circ \Phi_N) \circ \eta = \eta^{[0]} \circ f_0$. Then this implies $\eta^{[1]} \circ \Phi_N^{[1]} \circ f_1^{[0][1]} = f_0^{[1]} \circ \eta^{[0][1]}$. Composing on the right with Ψ_M yields $f_0^{[1]} \circ \eta^{[0][1]} \circ \Psi_M = \eta^{[1]} \circ \Phi_N^{[1]} \circ f_1^{[0][1]} \circ \Psi_M = \eta^{[1]} \circ \Phi_N^{[1]} \circ \Psi_{N^{[1]}} \circ f_1 = \eta^{[1]} \circ f_1$ (in the second equality we used the fact that Ψ is natural and in the last equality we used Proposition 2.2.1). As the l.h.s. equals $f_0^{[1]} \circ \Psi_N \circ \eta$ (since Ψ is natural), (c) holds. That (c) implies (b) follows by symmetry.

By definition, (d) is equivalent to "(b) and (c)". As (b) $\iff$ (c), we get (b) $\iff$ (c) $\iff$ (d). Assume (d) holds. Define $Z' = (M, \eta^{[0]} \circ f_0, \eta^{[1]} \circ f_1, M)$. Then (id_M, η) is clearly an isomorphism from Z to Z'. In addition, by (c), $(\eta^{[0]} \circ f_0)^{[1]} \circ \Psi_M = f_0^{[1]} \circ \eta^{[0][1]} \circ \Psi_M = f_0^{[1]} \circ \Psi_N \circ \eta = \eta^{[1]} \circ f_1$ and similarly, $(\eta^{[1]} \circ f_1)^{[0]} \circ \Phi_M = \eta^{[0]} \circ f_0$. Therefore, $Z' = Z'^*$ and by (ii), $Z \cong Z' = Z(b)$ for some bilinear form b, i.e. (a) holds.

Finally, assume (a) holds, i.e. there exists an isomorphism $(\sigma, \tau) : Z \cong Z(b)$ for some bilinear form b. Write $Z(b) = (A, g_0, g_1, A)$. Then $\tau^{[0]} \circ f_0 = g_0 \circ \sigma$ and $\tau^{[1]} \circ f_1 = g_1 \circ \sigma$. By (ii), $Z(b)^* = Z(b)$, hence $g_0 = g_1^{[0]} \circ \Phi_A$. This and the previous equations imply $f_1^{[0]} \circ \Phi_N \circ (\tau \circ \sigma) = f_1^{[0]} \circ \tau^{[1][0]} \circ \Phi_A \circ \sigma = (\tau^{[1]} \circ f_1)^{[0]} \circ \Phi_A \circ \sigma = (g_1 \circ \sigma)^{[0]} \circ \Phi_A \circ \sigma = \sigma^{[0]} \circ g_1^{[0]} \circ \Phi_A \circ \sigma = \sigma^{[0]} \circ g_0 \circ \sigma = \sigma^{[0]} \circ \tau^{[0]} \circ f_0 = (\tau \circ \sigma)^{[0]} \circ f_0$, hence $\eta = \tau \circ \sigma$ satisfies (b). □

REMARK 4.3.5. (i) Caution: $Z \cong Z^*$ does not imply that Z is isomorphic to a Kronecker module of a bilinear form. This is demonstrated later and suggests the following hierarchy:

(1) Z is *of bilinear type* (or just *bilinear*, for brevity) if $Z \cong Z(M, b)$ for some bilinear form $(M, b) \in \mathrm{Bil}(\mathscr{A})$,
(2) Z is *self-dual* if $Z \cong Z^*$,

with (1) obviously implying (2). If $Z \ncong Z^*$, then Z is called *non-self-dual.*

(ii) For every Kronecker module $Z = (M, f_0, f_1, N)$, there exists a bilinear form (M_Z, b_Z) such that $Z \oplus Z^* \cong Z(M_Z, b_Z)$. It is given by (M_Z, b_Z) where $M_Z := M \oplus N$ and

$$b_Z := \begin{bmatrix} 0 & I^{-1}(f_0) \\ f_1 & 0 \end{bmatrix} \in \mathrm{Hom}(M \oplus N, M^{[1]} \oplus N^{[1]}) \ .$$

In the special case where $\mathscr{A}$ is obtained from a ring R and a double R-module K, the form b_Z is given by the formula $b((x, y), (x', y')) = (f_1 x')y + (f_0 x)y'$ for all $x, x' \in M$ and $y, y' \in N$. (The form b_Z is precisely the form induced from a Kronecker module in the sense of [**44**].)

The highlight of this section (and one of the main highlights of this chapter) is the following theorem which, roughly speaking, reduces the study of arbitrary bilinear forms into regular symmetric bilinear forms.

THEOREM 4.3.6. *There is an equivalence of categories*

$$F : \mathrm{Bil}(\mathscr{A}) \to \mathrm{Sym}_{\mathrm{reg}}(\mathrm{Kr}(\mathscr{A}))$$

given by

$$F(M,b) = (Z(M,b), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})) \qquad \textit{and} \qquad F\sigma = (\sigma, (\sigma^{-1})^{\mathrm{op}})$$

for all $(M,b) \in \mathrm{Bil}(\mathscr{A})$ *and any morphism* σ *in* $\mathrm{Bil}(\mathscr{A})$.

Proof. Let $(M,b),(M',b') \in \mathrm{Bil}(\mathscr{A})$ and $\sigma \in \mathrm{Hom}_{\mathrm{Bil}(\mathscr{A})}((M,b),(M',b'))$. Then $(\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})^* = (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})$, hence $F(M,b)$ is indeed a symmetric regular form over $\mathrm{Sym}_{\mathrm{reg}}(\mathrm{Kr}(\mathscr{A}))$. In addition, $b = \sigma^{[1]} \circ b' \circ \sigma$, hence $(\sigma^{-1})^{[1]} \circ b = b' \circ \sigma$. Applying [0] to the first equation yields $b^{[0]} = \sigma^{[0]} \circ b'^{[0]} \circ \sigma^{[0][1]}$. Composing this with Φ_M on the right and $(\sigma^{-1})^{[0]}$ on the left, we get $(\sigma^{-1})^{[0]} \circ \widetilde{b} = (\sigma^{-1})^{[0]} \circ b^{[0]} \circ \Phi_M = b'^{[0]} \circ \sigma^{[0][1]} \circ \Phi_M = b'^{[0]} \circ \Phi_{M'} \circ \sigma = \widetilde{b'} \circ \sigma$. Thus, $(\sigma, (\sigma^{-1})^{\mathrm{op}}) \in \mathrm{Hom}(Z(M,b), Z(M',b'))$. In addition,

$$(F\sigma)^* \circ (\mathrm{id}_{M'}, \mathrm{id}_{M'}^{\mathrm{op}}) \circ F\sigma = (\sigma^{-1}, \sigma^{\mathrm{op}}) \circ (\mathrm{id}_{M'}, \mathrm{id}_{M'}^{\mathrm{op}}) \circ (\sigma, (\sigma^{-1})^{\mathrm{op}}) = (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}}),$$

so $F\sigma$ is an isometry from $F(M,b)$ to $F(M',b')$. That F preserves composition is straightforward.

We now define an inverse of F. Let $(Z, (\sigma, \tau^{\mathrm{op}})) \in \mathrm{Sym}_{\mathrm{reg}}(\mathrm{Kr}(\mathscr{A}))$ with $Z = (M, f_0, f_1, N)$. Then the discussion preceding Proposition 4.3.4 implies that $\sigma = \tau$ and $\sigma : M \to N$ is an isomorphism. In addition, Proposition 4.3.4 implies that $(f_1^{[0]} \circ \Phi_N) \circ \sigma = \sigma^{[0]} \circ f_0$. Define

$$G(Z, (\sigma, \sigma^{\mathrm{op}})) = (M, \sigma^{[1]} \circ f_1)$$

and for any isometry $(\eta, \theta^{\mathrm{op}}) : (Z, (\sigma, \sigma^{\mathrm{op}})) \to (Z', (\sigma', \sigma'^{\mathrm{op}}))$, define

$$G(\eta, \theta^{\mathrm{op}}) = \eta \ .$$

Write $Z' = (M', f_0', f_1', N')$. Let us check that η is indeed an isometry from $G(Z)$ to $G(Z')$, which amounts to $\eta^{[1]} \circ (\sigma'^{[1]} \circ f_1') \circ \eta = \sigma^{[1]} \circ f_1$. First note that $f_1' \circ \eta = \theta^{[1]} \circ f_1$ and $(\eta, \theta^{\mathrm{op}})^* \circ (\sigma', \sigma'^{\mathrm{op}}) \circ (\eta, \theta^{\mathrm{op}}) = (\sigma, \sigma^{\mathrm{op}})$ which means that $\theta \circ \sigma' \circ \eta = \sigma$. We now get $\eta^{[1]} \circ \sigma'^{[1]} \circ f_1' \circ \eta = \eta^{[1]} \circ \sigma'^{[1]} \circ \theta^{[1]} \circ f_1 = (\theta \circ \sigma' \circ \eta)^{[1]} \circ f_1 = \sigma^{[1]} \circ f_1$, as required. That G preserves composition is routine.

It is fairly easy to see that $GF = \mathrm{id}_{\mathrm{Bil}(\mathscr{A})}$. On the other hand, keeping the above notation, we have

$$FG(Z, (\sigma, \sigma^{\mathrm{op}})) = ((M, (\sigma^{[1]} \circ f_1)^{[0]} \circ \Phi_M, \sigma^{[1]} \circ f_1, M), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})) \ .$$

As $(f_0^{[1]} \circ \Psi_M) \circ \sigma = \sigma^{[1]} \circ f_1$, we get that $(\sigma^{[1]} \circ f_1)^{[0]} \circ \Phi_M = f_1^{[0]} \circ \sigma^{[1][0]} \circ \Phi_M = f_1^{[0]} \circ \Phi_N \circ \sigma = \sigma^{[0]} \circ f_0$, so

$$FG(Z, (\sigma, \sigma^{\mathrm{op}})) = ((M, \sigma^{[0]} \circ f_0, \sigma^{[1]} \circ f_1, M), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})) \ . \tag{16}$$

Define a natural transformation $i : \mathrm{id}_{\mathrm{Sym}_{\mathrm{reg}}(\mathrm{Kr}(\mathscr{A}))} \to FG$ by $i = i(Z, (\sigma, \sigma^{\mathrm{op}})) = (\mathrm{id}_M, \sigma^{\mathrm{op}})$. It is easy to see from (16) that i is an isomorphism. To see that i is natural, let $Z', \sigma', \eta, \theta$ be as above. We need to show that $i(Z', (\sigma', \sigma'^{\mathrm{op}})) \circ (\eta, \theta^{\mathrm{op}}) = FG(\eta, \theta^{\mathrm{op}}) \circ i(Z, (\sigma, \sigma^{\mathrm{op}}))$. Indeed, $FG(\eta, \theta) = F\eta = (\eta, (\eta^{-1})^{\mathrm{op}})$ hence $i \circ (\eta, \theta^{\mathrm{op}}) = (\mathrm{id}_{M'}, \sigma'^{\mathrm{op}}) \circ (\eta, \theta^{\mathrm{op}}) = (\eta, (\theta \circ \sigma')^{\mathrm{op}}) = (\eta, (\sigma \circ \eta^{-1})^{\mathrm{op}}) = (\eta, (\eta^{-1})^{\mathrm{op}}) \circ (\mathrm{id}_M, \sigma^{\mathrm{op}}) = FG(\eta, \theta^{\mathrm{op}}) \circ i$, as required. We thus conclude that F is an equivalence of categories. □

Remark 4.3.7. A result of the same flavor was obtained by E. Bayer-Fluckiger and L. Fainsilber in [**16**]. They showed that there is an equivalence between the category of *arbitrary symmetric* bilinear forms over a category with duality $(\mathscr{H}, *, \omega)$ (here ω must be an isomorphism!) and the category of symmetric regular bilinear forms over $\mathrm{Mor}(\mathscr{H})$, the category of morphisms in $\mathscr{H}$. The latter consists of triplets (A, h, B) where $A, B \in \mathscr{H}$ and $h \in \mathrm{Hom}(A, B)$ with obvious morphisms.

The contravariant functor $* : (A,h,B) \mapsto (B^*, h^*, A^*)$ together with the natural isomorphism $\widehat{\omega} : \mathrm{id} \to **$ given by $\widehat{\omega}_{(A,h,B)} = (\omega_A, \omega_B)$ makes $\mathrm{Mor}(\mathscr{H})$ into a category with duality. The equivalence is given by $F' : (M,b) \mapsto ((M,b,M^*), (\omega_M, \mathrm{id}_{M^*}))$.

To understand how the result in [**16**] relates to Theorem 4.3.6, assume that $(\mathscr{A}, [0], [1], \Phi, \Psi) = (\mathscr{H}, *, *, \omega, \omega)$. Then $\mathrm{Kr}(\mathscr{A})$ admits a full subcategory $\mathscr{S}$ consisting of Kronecker modules $Z = (M, f_0, f_1, N)$ with $f_0 = f_1$ (this is possible since $[0] = * = [1]$). Observe that any symmetric bilinear form (M,b) over $\mathscr{A} = \mathscr{H}$ satisfies $Z(M,b) \in \mathscr{S}$. Now, there is a functor $T : \mathscr{S} \to \mathrm{Mor}(\mathscr{H})$ given by

$$T(M,f,f,N) = (M,f,N^*) \qquad \text{and} \qquad T(\sigma, \tau^{\mathrm{op}}) \mapsto (\sigma, \tau^*)$$

for all $(M,f,f,N) \in \mathscr{S}$ and any morphism $(\sigma, \tau^{\mathrm{op}})$ in $\mathscr{S}$. It can be checked that T induces an isomorphism of categories with duality from $\mathscr{S}$ to $\mathrm{Mor}(\mathscr{H})$ (see [**71**] or section 2.8); the natural isomorphism $i : T* \to *T$ is given by $i(M,f,f,N) = (\omega_N, \mathrm{id}_M)$ (here we need ω to be invertible!). Thus, $\mathrm{Mor}(\mathscr{H})$ can be identified as a full sub-category-with-duality of $\mathrm{Kr}(\mathscr{A})$. This isomorphism also induces an equivalence of categories $\mathrm{Sym}_{\mathrm{reg}}(\mathscr{S}) \to \mathrm{Sym}_{\mathrm{reg}}(\mathrm{Mor}(\mathscr{H}))$ given by

$$((M,f,f,N),(\sigma,\sigma^{\mathrm{op}})) \mapsto ((M,f,N^*),(\omega_N \circ \sigma, \sigma^*))$$

and $T(\eta, \theta^{\mathrm{op}}) = (\eta, \theta^*)$. The functor $F' : \mathrm{Sym}(\mathscr{H}) \to \mathrm{Sym}_{\mathrm{reg}}(\mathrm{Mor}(\mathscr{H}))$ of [**16**] is the composition of the functor $\mathrm{Sym}_{\mathrm{reg}}(\mathscr{S}) \to \mathrm{Sym}_{\mathrm{reg}}(\mathrm{Mor}(\mathscr{H}))$ just defined with F of Theorem 4.3.6, restricted to symmetric bilinear forms on $\mathscr{A} = \mathscr{H}$.

We note that since the assumption that ω is invertible is essential in [**16**], Theorem 4.3.6 is more general than [**16**] even for symmetric forms. We will exploit this later to work with systems of bilinear forms, rather than a single form.

Remark 4.3.8. Several days before submitting this dissertation, we were introduced with the (still unpublished) work of E. Bayer-Fluckiger and D. Moldovan ([**17**], [**64**]). Independently of us, they obtained results which are very similar to Theorem 4.3.6 by using a very similar construction, which also applies to systems of bilinear forms. Explicitly, they have shown that the category of I-indexed systems of arbitrary bilinear forms over a category with duality $(\mathscr{H}, *, \omega)$ *for which* ω *is a natural isomorphism* is equivalent to the category of regular symmetric bilinear forms over another category with duality. (We should note that Theorem 4.3.6 can also be applied to systems of bilinear forms; see section 4.5.) *E. Bayer-Fluckiger, D. Moldovan and I eventually published some of our results as a joint work; see* [**11**].

To make Theorem 4.3.6 fully applicable, we need to know whether conditions (C0), (C1), (C2), (C2′), (C2″) of the previous section hold for $\mathrm{Kr}(\mathscr{A})$. Indeed, conditions (C0) and (C1) clearly pass from $\mathscr{A}$ to $\mathrm{Kr}(\mathscr{A})$ (Proposition 4.3.3), and in section 4.4 below we will see that the same applies to (C2′) (Corollary 4.4.2). However, in general, not much can be said about when $\mathrm{Kr}(\mathscr{A})$ satisfies (C2) or (C2″); sufficient conditions appear in the following two sections.

Provided $\mathrm{Kr}(\mathscr{A})$ satisfies conditions (C0)–(C2), the results of section 4.2 imply that bilinear forms over $\mathscr{A}$ have:

- Decomposition into isotypes (Theorem 4.2.6).
- Witt's Cancelation Theorem (Corollary 4.2.10).
- The isomorphism problem can be reduced to isomorphism and decomposition of objects in $\mathrm{Kr}(\mathscr{A})$ and isometry of hermitian forms over division rings.
- If $\mathscr{A}$ is an F-category with F algebraically closed and $[0], [1]$ F-linear, then the isometry class of a bilinear form (M,b) over $\mathscr{A}$ is determined by $[Z(M,b)]$.

Moreover, under mild assumptions, these results apply to *systems of bilinear forms* over categories with a double duality. This will be demonstrated in section 4.5.

While this is quite impressive, the reader might now ask questions like: what are the isotypes? how does the reduction of the isometry problem over $\mathscr{A}$ work in practice? what does being hyperbolic in $\mathrm{Kr}(\mathscr{A})$ mean? etc. With the exception of sections 4.4–4.5, the rest of this chapter is dedicated to answer these questions, i.e. to decipher the isomorphism of Theorem 4.3.6. Step by step, we will define hyperbolic forms and isotypes, and reduce the isometry problem of bilinear forms to isometry of hermitian forms. The discussion and proofs will be "category-free" except some sporadic comments.

At the moment, we can answer the following question: How does Riehm and its predecessors' work relates to Theorem 4.3.6? The answer is that the isotypes of section 4.1 become isotypes in $\mathrm{Kr}(\text{Mod-}F)$ after applying the isomorphism of Theorem 4.3.6. Furthermore, the isotypes discussed in Theorem 4.1.3 are hyperbolic in $\mathrm{Kr}(\text{Mod-}F)$ and actually correspond to the isotypes of Theorem 4.2.8. In general, a bilinear space (V,b) becomes hyperbolic in $\mathrm{Kr}(\text{Mod-}F)$ if and only if there are totally isotropic subspaces $V_1,V_2\subseteq V$ such that $V=V_1\oplus V_2$. In addition, Theorem 4.1.5 is just Theorem 4.2.9 applied to $\mathrm{Kr}(\text{Mod-}F)$. The proof of all these statements is technical and thus left to the reader; parts of the proof can be found in the remainder of the chapter.

4.4. Conditions (C2), (C2′) and (C2″)

Throughout, $(\mathscr{A},[0],[1],\Phi,\Psi)$ is an additive category with a double duality. In this section we will show that under mild assumptions, the conditions (C2), (C2′) and (C2″) pass from $\mathscr{A}$ to $\mathrm{Kr}(\mathscr{A})$. Recall that these conditions are:

(C2) $\mathrm{End}(M)$ is complete semilocal for all $M\in\mathscr{A}$.

(C2′) $\mathrm{End}(M)$ is semiprimary for all $M\in\mathscr{A}$.

(C2″) $\mathrm{End}(M)$ is semiperfect and pro-semiprimary (w.r.t. some topology) for all $M\in\mathscr{A}$.

Note that (C2′)$\Longrightarrow$(C2)$\Longrightarrow$(C2″).

The results of Chapter 1 will play an essential role in this section, mainly because of the following result.

Proposition 4.4.1. *Let* $Z=(M,f_0,f_1,N)\in\mathrm{Kr}(\mathscr{A})$*,* $W=\mathrm{End}(M)$ *and* $U=\mathrm{End}(N)$*. Then* $\mathrm{End}(Z)$ *is a semi-invariant subring of* $W\times U^{\mathrm{op}}$*.*

Proof. First observe that $\mathrm{End}(Z)$ can be understood as a subring of $W\times U^{\mathrm{op}}$ since it consists of pairs $(\sigma,\tau^{\mathrm{op}})$ with $\sigma\in W$ and $\tau\in U$.

For $i\in\{0,1\}$, view $H_i:=\mathrm{Hom}(M,N^{[i]})$ as a (U^{op},W)-bimodule by letting $u^{\mathrm{op}}h=u^{[i]}\circ h$ and $hw=h\circ w$ for all $h\in H_i$, $w\in W$ and $u\in U$. Let

$$S=\begin{bmatrix} W & 0\\ H_0 & U^{\mathrm{op}}\end{bmatrix}\times\begin{bmatrix} U^{\mathrm{op}} & H_1\\ 0 & W\end{bmatrix}\ .$$

Then we can consider $W\times U^{\mathrm{op}}$ as a subring of S via

$$(w,u^{\mathrm{op}})\mapsto\left(\begin{bmatrix} w & 0\\ 0 & u^{\mathrm{op}}\end{bmatrix},\begin{bmatrix} u^{\mathrm{op}} & 0\\ 0 & w\end{bmatrix}\right)\ .$$

Observe that $f_0\in H_0$ and $f_1\in H_1$. We claim that under the previous embedding, $\mathrm{End}(Z)=\mathrm{Cent}_{W\times U^{\mathrm{op}}}(t)$, where $t=\left(\begin{bmatrix} 0 & 0\\ f_0 & 0\end{bmatrix},\begin{bmatrix} 0 & f_1\\ 0 & 0\end{bmatrix}\right)$. Indeed, $\left(\begin{bmatrix} w & 0\\ 0 & u^{\mathrm{op}}\end{bmatrix},\begin{bmatrix} u^{\mathrm{op}} & 0\\ 0 & w\end{bmatrix}\right)$ commutes with t $\iff$ $u^{[0]}\circ f_0=f_0\circ w$ and $u^{[1]}\circ f_1=f_1\circ w$ $\iff$ $(w,u^{\mathrm{op}})\in\mathrm{End}(Z)$. Thus, W_b is a *semi-centralizer* subring of $W\times W^{\mathrm{op}}$, so we are done by Proposition 1.3.1(b). □

As an immediate corollary, we get:

Corollary 4.4.2. *If* $\mathscr{A}$ *has (C2′), then* $\mathrm{Kr}(\mathscr{A})$ *has (C2′).*

Proof. This follows from Proposition 4.4.1 and Theorem 1.4.6 (because (C2′) implies $W \times U^{\mathrm{op}}$ is semiprimary). □

Recall that a topological ring R is called *linearly topologized* (abbrev.: LT) if it admits a local basis consisting of two-sided ideals. In this case, we let $\mathcal{I}_R$ denote the set of all open ideals (see section 1.5 for further details). We would like to get an analogue of Proposition 4.4.1 to T-semi-invariant subrings, so that we could apply it to conditions (C2) and (C2′′). In case Φ and Ψ are injective, there is a general method to do this, but it is hard to deduce explicit results from it.

Keeping the notation of Proposition 4.4.1, observe that if Φ_N and Ψ_N are monic, then the map $U^{\mathrm{op}} \to U_i := \mathrm{End}(N^{[i]})$ given by $u^{\mathrm{op}} \mapsto u^{[i]}$ is injective (Proposition 2.5.6(ii)). Thus, we get the following embedding:

$$S \subseteq \begin{bmatrix} W & 0 \\ H_0 & U^{\mathrm{op}} \end{bmatrix} \times \begin{bmatrix} U^{\mathrm{op}} & H_1 \\ 0 & W \end{bmatrix} \hookrightarrow \begin{bmatrix} \mathrm{End}(M) & 0 \\ \mathrm{Hom}(M,N^{[0]}) & U_0 \end{bmatrix} \times \begin{bmatrix} U_1 & \mathrm{Hom}(M,N^{[1]}) \\ 0 & \mathrm{End}(M) \end{bmatrix}$$

$$\subseteq \mathrm{End}(M \oplus N^{[0]}) \times \mathrm{End}(N^{[1]} \oplus M)$$

Now, if $\mathrm{End}(M \oplus N^{[0]})$ and $\mathrm{End}(N^{[1]} \oplus M)$ are endowed with some *Hausdorff* linear ring topologies, we can pull the product of these topologies back to S and the copy of $W \times U^{\mathrm{op}}$ inside S, thus making $\mathrm{End}(Z)$ into a T-semi-centralizer subring of $W \times U^{\mathrm{op}}$ (and T-semi-centralizer subrings are T-semi-invariant by Proposition 1.5.4(b)). The problem is that it is very hard to say something about the structure of $W \times U^{\mathrm{op}}$ as a topological ring. Indeed, if the topologies on $\mathrm{End}(M \oplus N^{[0]})$ and $\mathrm{End}(N^{[1]} \oplus M)$ are denoted by τ_1 and τ_2, respectively, then the topology induced on $W \times U^{\mathrm{op}}$ is $\tau_W \times \tau_U$ where τ_W and τ_U are defined as follows:

(1) Pullback τ_1 and τ_2 to W along the injections $w \mapsto \begin{bmatrix} w & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{End}(M \oplus N^{[0]})$ and $w \mapsto \begin{bmatrix} 0 & 0 \\ 0 & w \end{bmatrix} \in \mathrm{End}(N^{[1]} \oplus M)$, respectively. Then τ_W is the supremum of the two topologies obtained in this manner.

(2) Pullback τ_1 and τ_2 to U^{op} along the injections $u^{\mathrm{op}} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & u^{[0]} \end{bmatrix} \in \mathrm{End}(M \oplus N^{[0]})$ and $u^{\mathrm{op}} \mapsto \begin{bmatrix} u^{[1]} & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{End}(N^{[1]} \oplus M)$, respectively. Then τ_U is the supremum of the two topologies obtained in this manner.

However, in special cases, explicit statements can be shown. (This is perhaps the place to note that we do not know if a ring which is pro-semiprimary w.r.t. to two given ring topologies is pro-semiprimary w.r.t. to their supermum. A positive answer would allow some improvement of the results that follow.)

Proposition 4.4.3. *Assume that* $\mathscr{A}$ *satisfies (C2′′) and for every* $M \in \mathscr{A}$, $\mathrm{End}(M)$ *is right or left noetherian. Then:*

(i) $\mathscr{A}$ *satisfies (C2).*

(ii) *If* Φ *and* Ψ *are bijective, then* $\mathrm{Kr}(\mathscr{A})$ *satisfies (C2′′).*

Proof. (i) By Proposition 1.9.10, every pro-semiprimary LT ring which is right or left noetherian is complete semilocal and its topology is the Jacobson topology (i.e. the ring topology is spanned by powers of the Jacobson radical). Thus, (C2) holds.

(ii) Assume Φ and Ψ are bijective. By (C2′′), $\mathrm{End}(M \oplus N^{[0]})$ and $\mathrm{End}(N^{[1]} \oplus M)$ are pro-semiprimary and semiperfect w.r.t. some linear ring topologies. Keeping the previous setting, choose τ_1 and τ_2 above to be these topologies. Since Φ and Ψ are bijective, the maps $u^{\mathrm{op}} \mapsto u^{[0]}$ and $u^{\mathrm{op}} \mapsto u^{[1]}$ are bijective (Proposition 2.5.6(iii)). Therefore, each of the two topologies induced on U^{op} in (2) makes it into a pro-semiprimary ring (since if R is pro-semiprimary, then so is eRe for every $e \in \mathrm{E}(R)$; see Proposition 1.2.3). As $U^{\mathrm{op}} = \mathrm{End}(N)^{\mathrm{op}}$ is right or left noetherian, Proposition 1.9.10 implies each of these topologies is the Jacobson topology. Since the same argument applies to W, we get that the topology induced on $W \times U^{\mathrm{op}}$

is the Jacobson topology and $W \times U^{\mathrm{op}}$ is complete semilocal. As $\mathrm{End}(Z)$ is a T-semi-invariant subring of $W \times U^{\mathrm{op}}$ w.r.t. this topology, Theorem 1.5.15 implies that $\mathrm{End}(Z)$ is pro-semiprimary and semiperfect. □

EXAMPLE 4.4.4. Let C be a commutative *noetherian* pro-semiprimary ring and assume that for every $M \in \mathscr{A}$, $\mathrm{End}(M)$ is a C-algebra which is f.g. as a module over C. (For example, if R is a C-algebra which is f.g. as a module over C, then the category of f.g. right R-modules has this property). Then $\mathrm{End}(M)$ is pro-semiprimary (by Corollary 1.9.12(ii)) and noetherian (since $\mathrm{End}(M)_C$ is f.g.) for all $M \in \mathscr{A}$, so the assertions of the previous proposition apply to $\mathscr{A}$.

The following proposition applies only for categories of modules, but it does not assume Φ and Ψ are bijective.

PROPOSITION 4.4.5. *Let R be an LT ring and let K be a double R-module. Make* Mod-R *into a category with a double duality in the standard way and assume that*

$$\bigcap_{J\in\mathcal{I}_R}(K_0J+K_1J)=0$$

(here K_i denotes K considered as a right R-module via $\odot_i$). For every $M \in$ Mod-R, let τ_M be the ring topology on $\mathrm{End}(M)$ spanned by the local basis[12]

$$\{\mathrm{Hom}(M,MJ)\,|\,J\in\mathcal{I}_R\}\ .$$

Then for every Kronecker module $Z = (M, f_0, f_1, N) \in \mathrm{Kr}(\text{Mod-}R)$ for which τ_M and τ_N are Hausdorff, $\mathrm{End}(Z)$ is a T-semi-invariant subring of $\mathrm{End}(M) \times \mathrm{End}(N)^{\mathrm{op}}$. In fact, this holds for any linear ring topology on $\mathrm{End}(M) \times \mathrm{End}(N)^{\mathrm{op}}$ that contains $\tau_M \times \tau_N^{\mathrm{op}}$ (where $\tau_N^{\mathrm{op}} = \{\{x^{\mathrm{op}}\,|\,x \in X\}\,|\,X \in \tau_N\}$).

PROOF. Let us assume first that $\mathrm{End}(M)\times\mathrm{End}(N)^{\mathrm{op}}$ is endowed with $\tau_M\times\tau_N^{\mathrm{op}}$. Set $W = \mathrm{End}(M)$ and $U = \mathrm{End}(N)$. We will use the notation of the proof of Proposition 4.4.1.

It is enough to endow S with a linear ring topology such that the embedding $W\times U^{\mathrm{op}} \hookrightarrow S$ is a topological embedding (where the l.h.s. is endowed with $\tau_M\times\tau_N^{\mathrm{op}}$). For all $J \in \mathcal{I}_R$, let $K^J = K_0J + K_1J$. Then K^J is a double R-module. Also define $H_i^J = \mathrm{Hom}(M, \mathrm{Hom}(N, K^J_{1-i})) \subseteq H_i$ ($i = 0, 1$), $W^J = \mathrm{Hom}(M, MJ)$ and $U^J = \mathrm{Hom}(N, NJ)^{\mathrm{op}}$. Then, $\bigcap_{J\in\mathcal{I}_R} H_i^J = 0$ (since $\bigcap_{J\in\mathcal{I}_R} K^J = 0$), $\bigcap_{J\in\mathcal{I}_R} W^J = 0$ (since τ_M is Hausdorff) and $\bigcap_{J\in\mathcal{I}_R} U^J = 0$ (since τ_N is Hausdorff). Let

$$Q^J=\begin{bmatrix} W^J & 0\\ H_0^J & U^J\end{bmatrix}\times\begin{bmatrix} U^J & H_1^J\\ 0 & W^J\end{bmatrix}\subseteq S\ .$$

We claim Q^J is an ideal of S. Once we have proved that, it is easy to see that the local basis $\{Q^J\,|\,J \in \mathcal{I}_R\}$ induces a topology on S as required. Indeed, checking that $Q^J \trianglelefteq S$ amounts to checking that $W^J \trianglelefteq W$, $U^J \trianglelefteq U^{\mathrm{op}}$, $U^JH_i + H_iW^J \subseteq H_i$ and $UH_i^J + H_i^JW \subseteq H_i^J$ ($i \in \{0,1\}$). The first two assertions are straightforward. As for the others, let $f^{\mathrm{op}} \in U^J$, $f' \in W^J$ and $h, h' \in H_i$. Then for all $x, y \in M$:

$$((f^{\mathrm{op}}h+h'f')x)y=(f^{[i]}(hx))y+(h'f'x)y=(hx)(fy)+(h'(f'x))y\in$$
$$(hx)(MJ)+(h'(MJ))y\subseteq((hx)(M))\odot_0 J+((h'(M))y)\odot_1 J\subseteq K_0J+K_1J=K^J,$$

hence $f^{\mathrm{op}}h + h'f' \in \mathrm{Hom}(M, \mathrm{Hom}(M, K^J_{1-i})) = H_i^J$. Next, let $f^{\mathrm{op}} \in U^{\mathrm{op}}$, $f' \in W$ and $h, h' \in H_i^J$. Then for all $x, y \in M$:

$$((f^{\mathrm{op}}h+h'f')x)y=(f^{[i]}(hx))y+(h'f'x)y=(hx)(fy)+(h'(f'x))y\in$$
$$\mathrm{Hom}(M,K^J_{1-i})(fy)+\mathrm{Hom}(M,K^J_{1-i})(y)\subseteq K^J+K^J=K^J,$$

so $f^{\mathrm{op}}h + h'f' \in \mathrm{Hom}(M, \mathrm{Hom}(M, K^J_{1-i})) = H_i^J$ and we are through.

[12] This is the topology τ_1^M of section 1.8.

Now assume τ is a linear ring topology on $W \times U^{\mathrm{op}}$ that contains $\tau_M \times \tau_N^{\mathrm{op}}$. Observe that every ideal of $W \times U^{\mathrm{op}}$ is a product of an ideal of W and an ideal of U^{op}. Let $\mathcal{B}$ be a local basis of τ consisting of ideals. For every $A \times B^{\mathrm{op}} \in \mathcal{B}$, define $S^{A,B^{\mathrm{op}}}$ to be the S-ideal generated by

$$\begin{bmatrix} A & 0 \\ 0 & B^{\mathrm{op}} \end{bmatrix} \times \begin{bmatrix} B^{\mathrm{op}} & 0 \\ 0 & A \end{bmatrix}$$

It is easy to check that $S^{A,B^{\mathrm{op}}} \cap (W \times U^{\mathrm{op}}) = A \times B^{\mathrm{op}}$ and that $S^{W^J,U^J} \subseteq Q^J$ (note that $W^J \times U^J \in \tau$ by assumption). Therefore, the linear ring topology on S spanned by $\{S^{A,B^{\mathrm{op}}} \mid A \times B^{\mathrm{op}} \in \mathcal{B}\}$, denoted τ_S, restricts to τ on $W \times U^{\mathrm{op}}$ and contains the ring topology spanned by $\{Q^J \mid J \in \mathcal{I}_R\}$. Thus, τ_S is Hausdorff and we are through. □

Corollary 4.4.6. *Let R be an LT ring which is first countable semiperfect and pro-semiprimary and let K be a double R-module such that $\bigcap_{J \in \mathcal{I}_R}(K_0J+K_1J) = 0$. Consider* Mod-$R$ *as a category with a double duality in the standard way. Then for every Kronecker module $Z = (M, f_0, f_1, N) \in$* Kr(Mod-R) *for which M, N are f.p. and τ_M, τ_N of Proposition 4.4.5 are Hausdorff,* End(Z) *is semiperfect and pro-semiprimary. The assumption that τ_M, τ_N are Hausdorff can be dropped if R is strictly pro-right-artinian (e.g. if R is right noetherian; see Chapter 1).*

Proof. Recall that τ_M is just τ_1^M of section 1.8. By Theorem 1.8.3, End(M) and End(N) are semiperfect and pro-semiprimary w.r.t. τ_2^M and τ_2^N, which contain τ_M and τ_N, respectively. Thus, Proposition 4.4.5 implies End(Z) is a T-semi-invariant subring of End(M) $\times$ End(N)$^{\mathrm{op}}$, when endowed with $\tau_2^M \times (\tau_2^N)^{\mathrm{op}}$. As End($M$) $\times$ End(N)$^{\mathrm{op}}$ is semiperfect and pro-semiprimary w.r.t. this topology, we get that End(Z) is semiperfect and pro-semiprimary w.r.t. the induced topology, by Theorem 1.5.15. The assumption that τ_M, τ_N are Hausdorff can be dropped when R is strictly pro-right-artinian because in this case the Hausdorffness follows from Corollary 1.9.8. □

Example 4.4.7. Let R be a *Hausdorff* LT ring and let α be an anti-endomorphism. Let K be the double R-module obtained from R by defining $k \odot_0 r = r^\alpha k$ and $k \odot_1 r = kr$. Assume α is continuous. Then for all $J \in \mathcal{I}_R$, there is $I \in \mathcal{I}_R$ such that $I \subseteq \alpha^{-1}(J)$ and this implies $K_0(I \cap J) + K_1(I \cap J) \subseteq I^\alpha R + RJ \subseteq J$. As $I \cap J \in \mathcal{I}_R$, this means $\bigcap_{J \in \mathcal{I}_R}(K_0J + K_1J) \subseteq \bigcap_{J \in \mathcal{I}_R} J = 0$. In particular, if R is first countable semiperfect and pro-right-artinian, then the corollary implies that the endomorphism ring any Kronecker module (M, f_0, f_1, N) over Mod-R with M and N finitely presented is semiperfect and pro-semiprimary. (Here we endow Mod-R with the the double duality structure induced by K.) Roughly speaking, this means (C2$''$) applies to the category of Kronecker modules with “f.p. support”.

4.5. Systems of Bilinear Forms

Most of the theory of this chapter applies to systems of bilinear forms and not only to single forms. In this short section we will show how to obtain this. Throughout, $\mathscr{A}$ is an additive category.

Let I be a nonempty set and assume that for every $i \in I$, $\mathscr{A}$ admits a structure of a category with a double duality $(\mathscr{A}, [0]_i, [1]_i, \Phi_i, \Psi_i)$. A *system of bilinear forms* over $(\mathscr{A}, [0]_i, [1]_i, \Phi_i, \Psi_i)_{i \in I}$ is a pair $(M, \{b_i\}_{i \in I})$ such that $M \in \mathscr{A}$ and $b_i \in \mathrm{Hom}(M, M^{[1]_i})$. If $(M', \{b'_i\}_{i \in I})$ is another system of bilinear forms, then an isometry $\sigma : (M, \{b_i\}_{i \in I}) \to (M', \{b'_i\}_{i \in I})$ is an isomorphism σ from M to M' satisfying $\sigma^{[1]_i} \circ b'_i \circ \sigma = b_i$ for all $i \in I$.

It turns out that under mild assumptions, systems of bilinear forms can be treated as a single form. This is demonstrated in the following proposition.

PROPOSITION 4.5.1. *Keeping the previous assumptions, assume the direct products* $\prod_{i\in I} M^{[0]_i}$ *and* $\prod_{i\in I} M^{[1]_i}$ *exist for all* $M \in \mathscr{A}$. *Then there exists a structure of a category with a double duality on* $\mathscr{A}$, $(\mathscr{A}, [0], [1], \Phi, \Psi)$, *such that the category of bilinear forms over* $(\mathscr{A}, [0], [1], \Phi, \Psi)$ *is isomorphic to the category of systems of bilinear forms over* $(\mathscr{A}, [0]_i, [1]_i, \Phi_i, \Psi_i)_{i\in I}$. *The functors* $[0]$ *and* $[1]$ *are given by:*

$$M^{[0]} = \prod_{i\in I} M^{[0]_i}, \qquad M^{[1]} = \prod_{i\in I} M^{[1]_i} \ .$$

(The functors $[0]$ *and* $[1]$ *act on morphisms in the obvious way.)*

PROOF. We need to define Φ and Ψ. Let $p_{i,M}$ (resp. $q_{i,M}$) denote the projection from $M^{[0]}$ to $M^{[0]_i}$ (resp. $M^{[1]}$ to $M^{[1]_i}$). Observe that every morphism $f : M \to N^{[0]}$ is determined by the I-indexed set $\{p_{i,N} \circ f\}_{i\in I} \in \prod_{i\in I} \mathrm{Hom}(M, N^{[0]_i})$ and every such set gives rise to a morphism $M \to N^{[0]}$. Using this, we define Φ_M to be the unique morphism from M to $M^{[1][0]} = \prod_{i\in I}(M^{[1]})^{[0]_i}$ satisfying $p_{i_0,M^{[1]}} \circ \Phi_M = q_{i_0,M}^{[0]_{i_0}} \circ \Phi_{i_0,M}$ for all $i_0 \in I$. The map $\Psi : \mathrm{id}_{\mathscr{A}} \to [0][1]$ is defined in the same manner. We leave to the reader the (very long) technical check that Φ and Ψ are natural and satisfy $\Phi_M^{[1]} \circ \Psi_{M^{[1]}} = \mathrm{id}_{M_{[1]}}$ and $\Psi_M^{[0]} \circ \Phi_{M^{[0]}} = \mathrm{id}_{M_{[0]}}$ for all $M \in \mathscr{A}$.

It is now easy to check that there is a one-to-one correspondence between bilinear forms over $(\mathscr{A}, [0], [1], \Phi, \Psi)$ and systems of bilinear forms over $(\mathscr{A}, [0]_i, [1]_i, \Phi_i, \Psi_i)_{i\in I}$ given by $(M, b) \mapsto (M, \{q_{i,M} \circ b\}_{i\in I})$. This map can be made into a functor by sending all isometries to themselves. The details are left to the reader. □

We will keep using the maps $p_{i,M}$ and $q_{i,M}$ throughout the section. In addition, we will also write $(\mathscr{A}, [0], [1], \Phi, \Psi) = \prod(\mathscr{A}, [0]_i, [1]_i, \Phi_i, \Psi_i)_{i\in I}$.

EXAMPLE 4.5.2. Let R be a ring and let $\{K_i\}_{i\in I}$ be a system of double R-modules. Then each K_i induces a structure of a category with a double duality on Mod-R, which we denote by (Mod-$R, [0]_i, [1]_i, \Phi_i, \Psi_i$). If we apply Proposition 4.5.1 to (Mod-$R, [0]_i, [1]_i, \Phi_i, \Psi_i)_{i\in I}$, then the resulting structure (Mod-$R, [0], [1], \Phi, \Psi$) would be the one induced by the double R-module $K := \prod_{i\in I} K_i$. Indeed, it is fairly easy to see that a system of bilinear spaces $\{(M, b_i, K_i)\}_{i\in I}$ can be understood as a bilinear form on M taking values in K. If π_i is the projection from K to K_i, then the maps $p_{i,M} : M^{[0]} \to M^{[0]_i}$ (resp. $q_{i,M} : M^{[1]} \to M^{[1]_i}$) are given by $p_{i,M}(f) = \pi_i \circ f$ (resp. $q_{i,M}(f) = \pi_i \circ f$).

Let $(\mathscr{A}, [0]_i, [1]_i, \Phi_i, \Psi_i)_{i\in I}$ be categories with duality and let $(\mathscr{A}, [0], [1], \Phi, \Psi) = \prod(\mathscr{A}, [0]_i, [1]_i, \Phi_i, \Psi_i)_{i\in I}$. In order to apply the conclusions at the end of of section 4.3 to systems of bilinear forms over $(\mathscr{A}, [0]_i, [1]_i, \Phi_i, \Psi_i)_{i\in I}$, we need to know whether one of the conditions (C2), (C2′) or (C2″) holds for $\mathrm{Kr}(\mathscr{A}, [0], [1], \Phi, \Psi)$. Indeed, by Corollary 4.4.2, the condition (C2′) is guaranteed to pass from $\mathscr{A}$ to $\mathrm{Kr}(\mathscr{A}, [0], [1], \Phi, \Psi)$, regardless of $[0], [1], \Phi, \Psi$. However, we cannot apply Proposition 4.4.3 to $\mathrm{Kr}(\mathscr{A}, [0], [1], \Phi, \Psi)$ as it is likely that Φ and Ψ would not be bijective. This gap is treated in the following proposition.

LEMMA 4.5.3. *Keeping the previous notation, let* $Z \in \mathrm{Kr}(\mathscr{A}, [0], [1], \Phi, \Psi)$ *and write* $Z = (M, f_0, f_1, N)$. *Then for all* $i \in I$, $Z_i := (M, p_{i,N} \circ f_0, q_{i,N} \circ f_1, N) \in \mathrm{Kr}(\mathscr{A}, [0]_i, [1]_i, \Phi_i, \Psi_i)$ *and* $\mathrm{End}(Z) = \bigcap_{i\in I} \mathrm{End}(Z_i)$.

PROOF. This is straightforward. □

PROPOSITION 4.5.4. *Keeping the previous notation, assume that $\mathscr{A}$ satisfies (C2″),* $\mathrm{End}(M)$ *is right or left noetherian for all $M \in \mathscr{A}$ and Φ_i, Ψ_i are bijective for all i. Then $\mathscr{A}$ satisfies (C2″).*

PROOF. Let $Z = (M, f_0, f_1, N) \in \mathrm{Kr}(\mathscr{A}, [0], [1], \Phi, \Psi)$ and let $\{Z_i\}_{\in I}$ be defined as in the lemma. Endow $T := \mathrm{End}(M) \times \mathrm{End}(N)^{\mathrm{op}}$ with the Jacobson topology. Then by the proof of Proposition 4.4.3, T is semiperfect, pro-semiprimary and $\mathrm{End}(Z_i)$ is a T-semi-invariant subring of T. Now, Proposition 1.5.4(e) implies that the intersection of T-semi-invariant subrings is again T-semi-invariant, so by Lemma 4.5.3, $\mathrm{End}(Z)$ is a T-semi-invariant subring of T. As the latter is semiperfect and pro-semiprimary, Theorem 1.5.15 implies that so is $\mathrm{End}(Z)$. □

EXAMPLE 4.5.5. Let R be a right noetherian pro-semiprimary ring with $2 \in R^{\times}$ and let $\{\alpha_i\}_{i \in I}$ be a family of anti-*auto*morphisms of R. Let K_i denote the double R-module obtained from R by defining $r \odot_0 a = a^{\alpha_i} r$ and $r \odot_1 a = ra$. Then Witt's Cancelation Theorem applies to systems of bilinear forms $\{(M, b_i, K_i)\}_{i \in I}$, provided M is finite projective. Moreover, the isometry problem of such systems can be rendered to isometry of hermitian forms over division rings. Indeed, let $\mathscr{P}$ be the category of finite projective R-modules and let $(\text{Mod-}R, [0]_i, [1]_i, \Phi_i, \Psi_i)$ be the category with double duality induced by K_i. Then by Example 2.5.3, $\Phi_{i,R}$ and $\Psi_{i,R}$ are bijective, hence Φ_i and Ψ_i are bijective on $\mathscr{P}$. Moreover, since $(K_i)_0 \cong (K_i)_1 \cong R_R$, $R^{[0]_i} \cong (K_i)_0 \cong R_R$ and $R^{[1]_i} \cong (K_i)_1 \cong R_R$. Thus, $[0]_i, [1]_i$ map $\mathscr{P}$ into (and also onto) itself. It follows that $(\mathscr{P}, [0]_i, [1]_i, \Phi_i, \Psi_i)$ is a category with a double duality for which Φ_i and Ψ_i are bijective. Therefore, $(\mathscr{P}, [0]_i, [1]_i, \Phi_i, \Psi_i)_{i \in I}$ satisfies the assumptions of the previous proposition, hence the assertions at the end of section 4.3 apply to systems of bilinear forms $\{(M, b_i, K_i)\}_{i \in I}$ with $M \in \mathscr{P}$. (That $\mathrm{End}(M)$ is right noetherian pro-semiprimary for all $M \in \mathscr{P}$ is a straightforward argument; see Proposition 1.2.3.)

4.6. The Kronecker Module of a Bilinear Form

The time has come to explain how the equivalence of Theorem 4.3.6 works in practice. In the sections to follow, we will restrict to bilinear forms over rings (rather than categories with a double duality) and explain what are hyperbolic forms, what are the isotypes and how to reduce the isometry problem to isometry of hermitian forms over division rings.

Throughout, R is a ring and K is a double R-module. By a *Krnocker Module*, we mean a Kronecker module over Mod-R, considered as a category with a double duality w.r.t. K. That is, a Kronecker module is a quartet (M, f_0, f_1, M) with $M, N \in \text{Mod-}R$ and $f_i \in \mathrm{Hom}(M, N^{[i]})$. Recall that every bilinear space (M, b, K) gives rise to a Kronecker module $(M, \mathrm{Ad}_b^r, \mathrm{Ad}_b^{\ell}, N)$, which we denote by $Z(b)$.

This section is dedicated solely to the study of Kronecker modules. The facts obtained will be used to show that there is a strong connection between the asymmetry of a bilinear form (when exists) and its Kronecker module. This connection explains the role of the asymmetry in the definition of the isotypes in section 4.1.[13] In addition, we will also provide a description of the endomorphism ring of the Kronecker module of a bilinear form in terms of the form.

We begin our discussion by generalizing several properties of bilinear forms to Kronecker modules.

[13] Historical note: What originally led us to consider Kronecker modules was the need to have a replacement for the asymmetry in case K does not have an anti-isomorphism. In particular, until the discovery of Theorem 4.3.6, our point of view on Kronecker modules was that they are generalizations of asymmetries.

DEFINITION 4.6.1. *Let* $Z = (M, f_0, f_1, N)$, $Z' = (M', f'_0, f'_1, N')$ *be Kronecker modules.*

(a) Z *is called* right stable *if for all* $\tau : N \to N$ *there exists unique* $\sigma : M \to M$ *such that* $f_1 \circ \sigma = \tau^{[1]} \circ f_1$.

(b) Z, Z' *are called* right joinable *if* $Z \oplus Z' = (M \oplus M', f_0 \oplus f'_0, f_1 \oplus f'_1, N \oplus N')$ *is right stable.*

(c) *A* left quasi-asymmetry *of* Z *is a map* $q \in \mathrm{Hom}(N^{[1]}, N^{[0]})$ *such that* $q \circ f_1 = f_0$.

(d) *Let* κ *be an anti-isomorphism of* K. *A* left κ-asymmetry *of* Z *is a map* $\lambda \in \mathrm{End}(N)$ *such that* $u_{\kappa,N}^{-1} \circ \lambda^{[1]}$ *is a left quasi-asymmetry, i.e.* $u_{\kappa,N}^{-1} \circ \lambda^{[1]} \circ f_1 = f_0$.[14]

PROPOSITION 4.6.2. *Let* (M, b, K), (M', b', K) *be bilinear spaces. Then:*

(i) b *is right stable* $\iff$ $Z(b)$ *is right stable.*

(ii) b, b' *are right joinable* $\iff$ $Z(b), Z(b')$ *are right joinable.*

(iii) *Let* κ *be an anti-isomorphism of* K. *Then* $\lambda \in \mathrm{End}(M)$ *is a left* κ-*asymmetry of* b $\iff$ λ *is a left* κ-*asymmetry of* $Z(b)$.

PROOF. (i) follows from Proposition 2.2.6 and (ii) follows from (i) because $Z(b \perp b') = Z(b) \oplus Z(b')$. To see (iii), observe that $u_{\kappa,M}^{-1} \circ \lambda^{[1]} \circ \mathrm{Ad}_b^r = \mathrm{Ad}_b^\ell \iff b(\lambda x, y)^{\kappa^{-1}} = b(y, x)$ for all $x, y \in M$, or deduce (iii) directly from Proposition 2.2.8. □

PROPOSITION 4.6.3. *For* $i = 1, 2$, *let* $Z_i = (M_i, g_i, h_i, N_i)$ *be Kronecker modules. Then:*

(i) Z_1, Z_2 *are right joinable* $\iff$ *for all* $i, j \in \{1, 2\}$ *and* $\tau : N_j \to N_i$ *there exists unique* $\sigma : M_i \to M_j$ *such that* $\tau^{[1]} \circ h_i = h_j \circ \sigma$. *In particular,* Z *and* Z' *are right stable.*

(ii) *Let* κ *be an anti-isomorphism of* K. *If* $Z_1 \oplus Z_2$ *has a unique left* κ-*asymmetry* λ, *then* Z_1, Z_2 *have unique* κ-*asymmetries* λ_1, λ_2 *and* $\lambda = \lambda_1 \oplus \lambda_2$.[15]

PROOF. The proof of (i) is similar to the proof of Proposition 2.6.5 and the proof of (ii) is similar to the proof of Proposition 2.6.2(iii). As the latter is not phrased in terms of adjoint maps, we bring it here in full detail. We consider elements of $\mathrm{End}(N_1 \oplus N_2)$, $\mathrm{Hom}(M_1 \oplus M_2, N_1^{[1]} \oplus N_2^{[1]})$, etc. as 2×2 matrices in the standard way.

Let λ be the unique asymmetry of $Z_1 \oplus Z_2$. Write $\lambda = \left[\begin{smallmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{smallmatrix}\right]$, $\lambda^{[1]} = \left[\begin{smallmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{smallmatrix}\right]$ (with $q_{ij} = \lambda_{ji}^{[1]}$) and $u_{\kappa,N_1 \oplus N_2}^{-1} = \left[\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right]$ where $u_i = u_{\kappa,N_i}^{-1}$. Then

$$\begin{bmatrix} u_1 q_{11} h_1 & u_1 q_{12} h_2 \\ u_2 q_{21} h_1 & u_2 q_{22} h_2 \end{bmatrix} = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}.$$

But this implies $u_i \lambda_{ii}^{[1]} h_i = g_i$, hence λ_{ii} is a left κ-asymmetry of Z_i. Therefore, $\lambda_{11} \oplus \lambda_{22}$ is a left κ-asymmetry of $Z_1 \oplus Z_2$, so the uniqueness of λ implies $\lambda = \lambda_{11} \oplus \lambda_{22}$. The latter equality also implies λ_{11} and λ_{22} are unique. □

For the following proposition, recall that two endomorphisms $\sigma_1 \in \mathrm{End}(M_1)$ and $\sigma_2 \in \mathrm{End}(M_2)$ are *conjugate*, denoted $\sigma_1 \cong \sigma_2$, if there exists a module isomorphism $f : M_1 \to M_2$ such that $f \circ \sigma_1 = \sigma_2 \circ f$.[16]

---

[14] See Proposition 2.2.7 for the definition of u_κ.

[15] The converse is not true for bilinear forms (Example 2.6.4) and hence not true for Kronecker modules.

[16] The notation $\sigma_1 \cong \sigma_2$ means that σ_1, σ_2 are isomorphic in the category of endomorphisms of right R-modules.

PROPOSITION 4.6.4. *Let* $Z = (M, f_0, f_1, N)$, $Z' = (M', f'_0, f'_1, N')$ *be Kronecker modules.*

(i) *Assume* Z, Z' *have unique left quasi-asymmetries* q, q' *respectively. Then* $Z \cong Z'$ *implies that there exists an isomorphism* $\tau : N' \to N$ *such that* $\tau^{[0]} \circ q = q' \circ \tau^{[1]}$. *The converse holds when* Z, Z' *are right joinable.*

(ii) *Let* κ *be an anti-isomorphism of* K *and assume* Z, Z' *have unique left* κ*-asymmetries* λ, λ' *respectively. Then* $Z \cong Z' \Longrightarrow \lambda \cong \lambda'$. *The converse holds when* Z, Z' *are right joinable.*

PROOF. We only prove (ii); the proof of (i) is similar (and easier). Assume $(\sigma, \tau^{\mathrm{op}}) : Z \to Z'$ is an isomorphism. We claim $\tau^{-1} \circ \lambda \circ \tau$ is a left κ-asymmetry of Z' and thus must coincide with λ'. This would prove $\lambda \cong \lambda'$. Indeed, $u_{\kappa,N'}^{-1} \circ (\tau^{-1} \circ \lambda \circ \tau)^{[1]} \circ f'_1 = u_{\kappa,N'}^{-1} \circ \tau^{[1]} \circ \lambda^{[1]} \circ (\tau^{[1]})^{-1} \circ f'_1 = \tau^{[0]} \circ u_{\kappa,N}^{-1} \circ \lambda^{[1]} \circ f_1 \circ \sigma^{-1} = \tau^{[0]} \circ f_0 \circ \sigma^{-1} = \tau^{[0]} \circ (\tau^{[0]})^{-1} \circ f'_0 = f'_0$.

Conversely, assume Z, Z' are right joinable and there is an isomorphism $\tau : N' \to N$ such that $\tau \circ \lambda' = \lambda \circ \tau$. The fact Z, Z' are right joinable implies that there are *unique* $\sigma \in \mathrm{Hom}(M, M')$ and $\sigma' \in \mathrm{Hom}(M', M)$ such that $\tau^{[1]} \circ f_1 = f'_1 \circ \sigma$ and $(\tau^{[1]})^{-1} \circ f'_1 = f_1 \circ \sigma'$. This is easily seen to imply $(\mathrm{id}_M)^{[1]} \circ f_1 = f_1 \circ (\sigma' \circ \sigma)$ and $(\mathrm{id}_{M'})^{[1]} \circ f'_1 = f'_1 \circ (\sigma \circ \sigma')$. By Proposition 4.6.3(i), Z and Z' are right stable and hence, $\sigma' \circ \sigma = \mathrm{id}_M$ and $\sigma \circ \sigma' = \mathrm{id}_{M'}$, i.e. σ is invertible with $\sigma^{-1} = \sigma'$. We now claim that $(\sigma, \tau^{\mathrm{op}})$ is an isomorphism from Z to Z'. Indeed, $\tau^{[1]} \circ f_1 = f'_1 \circ \sigma$ follows from the definition of σ, which in turn implies $f'_0 \circ \sigma = u_{\kappa,N'}^{-1} \circ \lambda'^{[1]} \circ f'_1 \circ \sigma = u_{\kappa,N'}^{-1} \circ \lambda'^{[1]} \circ \tau^{[1]} \circ f_1 = u_{\kappa,N'}^{-1} \circ (\tau \circ \lambda')^{[1]} \circ f_1 = u_{\kappa,N'}^{-1} \circ (\lambda \circ \tau)^{[1]} \circ f_1 = u_{\kappa,N'}^{-1} \circ \tau^{[1]} \circ \lambda^{[1]} \circ f_1 = \tau^{[0]} \circ u_{\kappa,N}^{-1} \circ \lambda^{[1]} \circ f_1 = \tau^{[0]} \circ f_0$. We are done since σ, τ are isomorphisms. □

Let (M, b, K) be a bilinear space. The following corollary shows that under mild assumptions, the conjugacy class of the right asymmetry of b (when exists and unique) determines the isomorphism class of $Z(b)$ and vice versa. This explains why the isomorphism of Theorem 4.3.6 takes the isotypes of section 4.1 ("Riehm's isotypes") to the isotypes of section 4.2 ("isotypes of categories with duality").

COROLLARY 4.6.5. *Let* (M, b, K), (M', b', K) *be bilinear spaces and let* κ *be an anti-isomorphism of* K. *Assume* b, b' *have unique left* κ*-asymmetries* λ, λ' *respectively. Then* $Z(b) \cong Z(b') \Longrightarrow \lambda \cong \lambda'$. *The converse holds when* b, b' *are right joinable (e.g. when* b, b' *are right regular).*

PROOF. This follows from the proposition and Proposition 4.6.2(iii). □

We now turn our attention to homomorphisms between Kronecker modules obtained from bilinear forms. Let (M, b, K) and (M', b', K) be two bilinear spaces. Then $(\sigma, \tau^{\mathrm{op}})$ is a homomorphism from $Z(b)$ to $Z(b')$ if and only if

$$\mathrm{Ad}^{\ell}_{b'} \circ \sigma = \tau^{[0]} \circ \mathrm{Ad}^{\ell}_{b} \qquad \text{and} \qquad \mathrm{Ad}^{r}_{b'} \circ \sigma = \tau^{[1]} \circ \mathrm{Ad}^{r}_{b} \,.$$

A straightforward computation shows that this is equivalent to

$$b'(\sigma x, y') = b(x, \tau y') \qquad \text{and} \qquad b'(x', \sigma y) = b(\tau x', y) \tag{17}$$

for all $x, y \in M$ and $x', y' \in M$. It follows that if σ is an isometry from b to b', then (σ, σ^{-1}) is an isomorphism from $Z(b)$ to $Z(b')$, hence the isomorphism class of $Z(b)$ is invariant under isometry. We will write $b \sim_{\mathrm{Kr}} b'$ to denote that b and b' have isomorphic Kronecker modules.

Now consider the endomorphism ring of $Z(b)$, denoted W_b. This ring will turn out to be of great importance, hence the explicit notation. By (17), W_b consists of formal pairs $(\sigma, \tau^{\mathrm{op}})$ such that $\sigma, \tau \in W := \mathrm{End}(M)$ and

$$b(\sigma x, y) = b(x, \tau y) \qquad \text{and} \qquad b(x, \sigma y) = b(\tau x, y) \tag{18}$$

for all $x, y \in M$. Thus, we can consider W_b as a subring of $W \times W^{\mathrm{op}}$. The equation (17) implies that $(\sigma, \tau^{\mathrm{op}}) \in W_b \iff (\tau, \sigma^{\mathrm{op}}) \in W_b$. Therefore, the map $(\sigma, \tau^{\mathrm{op}}) \mapsto (\tau, \sigma^{\mathrm{op}})$, denoted by $\beta = \beta(b)$, is a well-defined involution of W_b.

REMARK 4.6.6. One can also understand β as the map $f \mapsto f^*$ from $\mathrm{End}(Z(b))$ to $\mathrm{End}(Z(b)^*) = \mathrm{End}(Z(b))$. Furthermore, β is the corresponding involution of the bilinear form $(Z(b), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})) \in \mathrm{Sym}_{\mathrm{reg}}(\mathrm{Kr}(\text{Mod-}R))$ (see Theorem 4.3.6).

REMARK 4.6.7. The ring W_b as defined here was defined in the literature for systems of quadratic and bilinear forms over a field. See [**12**] and [**15**], for instance.

PROPOSITION 4.6.8. *Keeping the previous notation, define the* radical *and* quasi-radical *of b to be*

$$\begin{aligned} \mathrm{rad}(b) &:= \{x \in M \mid b(x, M) = b(M, x) = 0\} , \\ \mathrm{qrad}(b) &:= \{w \in W \mid b(wM, M) = b(M, wM) = 0\} , \end{aligned}$$

respectively. Then:

(i) $\mathrm{rad}(b) = \ker \mathrm{Ad}_b^\ell \cap \ker \mathrm{Ad}_b^r$ *and* $\mathrm{qrad}(b) = \mathrm{Hom}(M, \mathrm{rad}(b))$.
(ii) $\mathrm{qrad}(b) \times \mathrm{qrad}(b)^{\mathrm{op}}$ *is an ideal of* W_b.
(iii) *If* $\mathrm{qrad}(b) = 0$*, then* W_b *embeds in* W *via* $(\sigma, \tau^{\mathrm{op}}) \mapsto \sigma$. *(The image of this embedding is the set of elements $\sigma \in W$ for which there is $\tau \in W$ satisfying* (18)*.)*

PROOF. (i) This is straightforward.

(ii) Let $w, w' \in \mathrm{qrad}(b)$. Then $b(wx, y) = 0 = b(x, w'y)$ and $b(x, wy) = 0 = b(w'x, y)$ for all $x, y \in M$, hence $(w, w'^{\mathrm{op}}) \in W_b$. Thus, $\mathrm{qrad}(b) \times \mathrm{qrad}(b)^{\mathrm{op}} \subseteq W_b$. Next, if $(\sigma, \tau^{\mathrm{op}}) \in W_b$, then $b(\sigma w x, y) = b(wx, \tau y) = 0 = b(x, w'\tau y)$ and similarly, $b(x, \sigma w y) = 0 = b(w'\tau x, y)$. This means $\sigma w, w'\tau \in \mathrm{qrad}(b)$, hence $(\sigma, \tau^{\mathrm{op}})(w, w'^{\mathrm{op}}) \in \mathrm{qrad}(b) \times \mathrm{qrad}(b)^{\mathrm{op}}$. Therefore, $\mathrm{qrad}(b) \times \mathrm{qrad}(b)^{\mathrm{op}}$ is a left ideal of W_b and a similar argument shows it is a right ideal as well.

(iii) Assume $\mathrm{qrad}(b) = 0$. If $(0, \sigma^{\mathrm{op}}) \in W_b$, then $b(\sigma x, y) = b(x, 0y) = 0$ and $b(x, \sigma y) = b(0y, x) = 0$, hence $\sigma \in \mathrm{qrad}(b)$, which implies $\sigma = 0$. Thus, the homomorphism $W_b \hookrightarrow W \times W^{\mathrm{op}} \to W$ is one-to-one. □

A bilinear space (M, b, K) is called *reduced* if $\mathrm{rad}(b) = 0$ and *quasi-reduced* if $\mathrm{qrad}(b) = 0$. Every right or left semi-stable bilinear space is quasi-reduced, but not necessarily reduced; see Example 2.4.4. In addition, $(M/\mathrm{rad}(b), \bar{b}, K)$ is always reduced, where $\bar{b}$ is defined by $\bar{b}(x + \mathrm{rad}(b), y + \mathrm{rad}(b)) = b(x, y)$.

Now assume (M, b, K) is right stable. Then Proposition 4.6.8 implies W_b embeds in $W = \mathrm{End}(M)$. The following proposition explains the connection between the involution $\beta(b)$ on W_b and the corresponding anti-endomorphism of b.

PROPOSITION 4.6.9. *Let (M, b, K) be a right stable bilinear space with corresponding anti-endomorphism α and let $W = \mathrm{End}(M)$. Then $\varphi : (\sigma, \tau^{\mathrm{op}}) \mapsto \sigma$ is an isomorphism of rings with involution from (W_b, β) to $(W^{\{\alpha^2\}}, \alpha)$, where $W^{\{\alpha^2\}} := \{w \in W \mid w^{\alpha\alpha} = w\}$.*

PROOF. Observe that $b(\sigma x, y) = b(x, \tau y) \iff \tau = \sigma^\alpha$. Thus, $\varphi((\sigma, \tau^{\mathrm{op}})^\beta) = \varphi(\tau, \sigma^{\mathrm{op}}) = \tau = \sigma^\alpha = (\varphi(\sigma, \tau^{\mathrm{op}}))^\alpha$, i.e. φ is a homomorphism of rings with anti-endomorphism. By Proposition 4.6.8(iii), φ is injective ($\mathrm{qrad}(b) = 0$ since b is right stable) and $\mathrm{im}(\varphi) = W^{\{\alpha^2\}}$ follow by a straightforward argument, hence we are done. □

REMARK 4.6.10. The ring $W^{\{\alpha^2\}}$ of the last proposition can be given a different description in case b has an invertible κ-asymmetry λ — it just $\mathrm{Cent}_W(\lambda)$. This follows from Proposition 2.3.9(i) which states that $w^{\alpha\alpha} = \lambda w \lambda^{-1}$ for all $w \in W$.

(In this case, W_b can be understood as a T-semi-invariant subring of W by Proposition 1.5.4(b).)

4.7. Hyperbolic Forms

Recall that a regular *symmetric* bilinear space (V, b) over a field F is called *hyperbolic* if V is the direct sum of two totally isotropic subspaces, i.e. $V = V_1 \oplus V_2$ with $b(V_1, V_1) = b(V_2, V_2) = 0$. The bilinear space (V, b) is *metabolic* if V admits a subspace $U \subseteq V$ such that $U = \{x \in V \,|\, b(x, U) = 0\}$. Clearly hyperbolic implies metabolic and the converse holds (for symmetric forms) when $\operatorname{char} F \neq 2$.

In this section, we extend the definition of hyperbolic forms to (non-symmetric or non-regular) bilinear forms over rings as defined in Chapter 2 and study their properties. Throughout, R is a ring and K is a fixed double R-module.

Definition 4.7.1. *A bilinear space (M, b, K) is* hyperbolic *if M is the direct sum of two totally isotropic submodules, namely there are $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$ and $b(M_1, M_1) = b(M_2, M_2) = 0$.*

Remark 4.7.2. The regular *symmetric* hyperbolic bilinear spaces over a field F with $\operatorname{char} F \neq 2$ are precisely those isometric to $(V, b) \perp (V, -b)$ for some regular symmetric bilinear space (V, b) and many books define hyperbolic forms in this manner. However, the obvious extension of this definition to arbitrary (non-symmetric) forms is not equivalent to the hyperbolic forms defined here.

Example 4.7.3. (i) Let F be a field, let $n, m \in \mathbb{N}$ and let $B \in \mathrm{M}_{n\times m}(F)$, $C \in \mathrm{M}_{m\times n}(F)$. Consider the block matrix $A = \left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right] \in \mathrm{M}_{n+m}(F)$ and let $b : F^{n+m} \times F^{n+m} \to F$ be defined by $b(x, y) = x^T A y$. Then b is hyperbolic since $F^{n+m} = (F^n \times \{0\}^m) \oplus (\{0\}^n \times F^m)$ and both summands are totally isotropic. The form b is regular precisely when A is invertible, namely when $m = n$ and B, C are invertible.

(ii) The isotypes discussed in Theorem 4.1.3 are hyperbolic.

(iii) Let b be the *zero form* on $M \in \text{Mod-}R$, namely $b(x, y) = 0$ for all $x, y \in M$. Then b is hyperbolic since $M = M \oplus 0$ and M, 0 are totally isotropic.

(iv) Let $Z = (M, f_0, f_1, N)$ be a Kronecker module. In Remark 4.3.5(ii), we have defined the bilinear form $b_Z : M \oplus N \times M \oplus N \to K$ by

$$b_Z((x, y), (x', y')) = (f_1 x')y + (f_0 x)y' \qquad \forall\, x, x' \in M,\ y, y' \in N\ .$$

Then b_Z is hyperbolic since $b_Z(M, M) = b_Z(N, N) = 0$.

It turns out that every hyperbolic form is of the form b_Z for some Kronecker module Z.

Proposition 4.7.4. *Assume (M, b, K) is hyperbolic. Then there exists a Kronecker module Z such that $b = b_Z$.*

Proof. Let M_1, M_2 be totally isotropic submodules of M such that $M_1 \oplus M_2 = M$. Identify $M^{[i]}$ with $M_1^{[i]} \oplus M_2^{[i]}$ via $f \mapsto (f|_{M_1}, f|_{M_2})$. Then since $\mathrm{Ad}^r_b(M_1)(M_1) = b(M_1, M_1) = 0$ and $\mathrm{Ad}^\ell_b(M_1)(M_1) = b(M_1, M_1) = 0$, we have $\mathrm{Ad}^r_b(M_1) \subseteq M_2^{[1]}$ and $\mathrm{Ad}^\ell_b(M_1) \subseteq M_2^{[0]}$. Thus, $Z := (M_1, \mathrm{Ad}^\ell_b|_{M_1}, \mathrm{Ad}^r_b|_{M_1}, M_2)$ is a Kronecker module. We claim $b = b_Z$. Indeed, for all $x, x' \in M_1$ and $y, y' \in M_2$, we have $b(x + y, x' + y') = b(y, x') + b(x, y') = (\mathrm{Ad}^r_b|_{M_1} x')y + (\mathrm{Ad}^\ell_b|_{M_1} x)y' = b_Z(x + y, x' + y')$. $\square$

The last proposition implies that the isometry class of a hyperbolic form should be almost completely determined by its Kronecker module. This is verified in the following proposition.

Proposition 4.7.5. *Let (M, b, K) and (M', b', K) be two hyperbolic bilinear spaces and let $M_1, M_2 \subseteq M$ and $M'_1, M'_2 \subseteq M'$ be totally isotropic modules such that $M = M_1 \oplus M_2$ and $M' = M'_1 \oplus M'_2$. For $i \in \{1, 2\}$ let*

$$\begin{aligned} Z_i &:= (M_i, \mathrm{Ad}^{\ell}_b|_{M_i}, \mathrm{Ad}^{r}_b|_{M_i}, M_{3-i}) \\ Z'_i &:= (M'_i, \mathrm{Ad}^{\ell}_b|_{M'_i}, \mathrm{Ad}^{r}_b|_{M'_i}, M'_{3-i}) \end{aligned}$$

Then:

(i) *Z_1, Z'_1, Z_2, Z'_2 are Kronecker modules, $Z_1 = Z_2^*$, $Z'_1 = Z'^*_2$ and $(\sigma, \tau^{\mathrm{op}}) \in \mathrm{Hom}(Z_1, Z'_1) \iff (\tau, \sigma^{\mathrm{op}}) \in \mathrm{Hom}(Z'_2, Z_2)$.*

(ii) *Assume $Z_1 \cong Z'_1$. Then $Z_2 \cong Z'_2$ and $b \cong b'$.*

Proof. (i) That Z_1, Z'_1, Z_2, Z'_2 are Kronecker modules was shown in Proposition 4.7.4. To see that $Z_1 = Z_2^*$, observe that $(\mathrm{Ad}^{\ell}_b)^{[1]} \circ \Psi_M = \mathrm{Ad}^r_b$ and $(\mathrm{Ad}^r_b)^{[0]} \circ \Phi_M = \mathrm{Ad}^{\ell}_b$ (Corollary 2.2.5). Recall that we identify $M^{[1]}$ with $M_1^{[1]} \oplus M_2^{[1]}$ and under that identification Ad^r_b maps M_i into $M_{3-i}^{[1]}$; similar statements hold for M' and/or [0]. As $\Psi_M = \Psi_{M_1} \oplus \Psi_{M_2}$ and $\Phi_M = \Phi_{M_1} \oplus \Phi_{M_2}$, we get that $(\mathrm{Ad}^{\ell}_b|_{M_2})^{[1]} \circ \Psi_{M_1} = \mathrm{Ad}^r_b|_{M_1}$ and $(\mathrm{Ad}^r_b|_{M_2})^{[0]} \circ \Phi_{M_1} = \mathrm{Ad}^{\ell}_b|_{M_1}$, hence $Z_1 = Z_2^*$ and similarly, $Z'_1 = Z'^*_2$. The last assertion follows since $(\sigma, \tau^{\mathrm{op}}) \in \mathrm{Hom}(Z_1, Z'_1)$ implies $(\tau, \sigma^{\mathrm{op}}) = (\sigma, \tau^{\mathrm{op}})^* \in \mathrm{Hom}(Z'^*_1, Z_1^*) = \mathrm{Hom}(Z'_2, Z_2)$. The converse follows by symmetry.

(ii) Let $(\sigma, \tau^{\mathrm{op}}) : Z_1 \to Z'_1$ be an isomorphism. Then σ and τ are invertible and therefore by (i), $(\tau, \sigma^{\mathrm{op}}) : Z'_2 \to Z_2$ is an isomorphism. Define $\eta = \sigma \oplus \tau^{-1} : M \to M'$. Then η is clearly an isomorphism. We claim that η is an isometry from b to b', i.e. $b'(\eta x, \eta y) = b(x, y)$ for all $x, y \in M$. Since M_1, M_2, M'_1, M'_2 are totally isotropic, it is enough to check the cases $(x, y) \in M_1 \times M_2$ and $(x, y) \in M_2 \times M_1$. Indeed, in the first case

$$b'(\eta x, \eta y) = b'(\sigma x, \tau^{-1} y) = b(x, \tau\tau^{-1} y) = b(x, y) \ ,$$

and in the second case

$$b'(\eta x, \eta y) = b'(\tau^{-1} x, \sigma y) = b(\tau\tau^{-1} x, y) = b(x, y) \ ,$$

as required. □

The proposition has a weaker analogue phrased in terms of asymmetry maps. (This should be of no surprise given Proposition 4.6.4.) This analogue, stated and proved below, was noted by several authors in less general scenarios (e.g. [**76**], [**75**]).

Proposition 4.7.6. *Let κ be an anti-isomorphism of K and let (M, b, K), (M', b', K) be two bilinear spaces with unique* left *κ-asymmetries λ, λ', respectively. Assume $M = M_1 \oplus M_2$, $M' = M'_1 \oplus M'_2$ and $b(M_i, M_i) = 0$, $b'(M'_i, M'_i) = 0$ $(i = 1, 2)$. Then:*

(i) *$\lambda(M_i) \subseteq M_i$ and $\lambda'(M'_i) \subseteq M'_i$ for $i \in \{1, 2\}$.*

(ii) *If $\lambda|_{M_1} \cong \lambda|_{M'_1}$ and b, b' are right joinable, then $\lambda'|_{M_2} \cong \lambda|_{M'_2}$ and $(M, b, K) \cong (M', b', K)$.*

Proof. (i) Let Z_1, Z'_1, Z_2, Z'_2 be as in Proposition 4.7.5. Then by Proposition 4.6.2(iii), λ is a unique left κ-asymmetry of $Z(b)$. As $Z(b) = Z_1 \oplus Z_2$, Proposition 4.6.3(ii) implies that $\lambda|_{M_i}$ is a unique left κ-asymmetry of Z_{3-i} and $\lambda = \lambda|_{M_1} \oplus \lambda|_{M_2}$. A similar claim holds for Z'_1, Z'_2 and in particular, this implies $\lambda(M_i) \subseteq M_i$ and $\lambda'(M'_i) \subseteq M'_i$.

(ii) By Proposition 4.6.4(ii), $\lambda|_{M_1} \cong \lambda|_{M'_1}$ implies $Z_2 \cong Z'_2$, so by Proposition 4.7.5, $Z_1 \cong Z'_1$ and $b \cong b'$. Finally, again by Proposition 4.6.4(ii), $Z_1 \cong Z'_1$ implies $\lambda'|_{M_2} \cong \lambda|_{M'_2}$. □

We will now show that the hyperbolicity of a bilinear form b can be described in terms of the ring W_b and its involution β. The following definition is taken from [**13**].

Definition 4.7.7. *An involution $*$ on a ring S is called* hyperbolic *if there exists an idempotent $e \in \mathrm{E}(S)$ such that $e + e^* = 1$.*

Proposition 4.7.8. *Let (M, b, K) be a bilinear space and let $\beta = \beta(b)$. There is a one-to-one correspondence between decompositions $M = M_1 \oplus M_2$ such that M_1 and M_2 are totally isotropic and idempotents $e \in \mathrm{E}(W_b)$ satisfying $e + e^\beta = 1$. In particular, b is hyperbolic if and only if β is hyperbolic.*

Proof. Given M_1, M_2 as above, define e_1 (resp. e_2) to be projection $M \to M_1$ (resp. $M \to M_2$) with kernel M_2 (resp. M_1). Then for all $x, y \in M$ we have

$$b(e_1x, y) = b(e_1x, e_1y + e_2y) = b(e_1x, e_2y) = b(e_1x + e_2y, e_2y) = b(x, e_2y)\ ,$$

and similarly, $b(x, e_1y) = b(e_2x, y)$. Thus, $e := (e_1, e_2^{\mathrm{op}}) \in W_b$. It is now clear that e is an idempotent of W_b satisfying $e + e^\beta = 1$. Conversely, given $e = (e_1, e_2^{\mathrm{op}}) \in \mathrm{E}(W_b)$ with $1 = e + e^\beta = (e_1 + e_2, e_1^{\mathrm{op}} + e_2^{\mathrm{op}})$, define $M_1 = e_1M$, $M_2 = e_2M$. Then $b(M_1, M_1) = b(e_1M, e_1M) = b(M, e_2e_1M) = b(M, 0) = 0$ and similarly $b(M_2, M_2) = 0$. The rest of the details are left to the reader. □

Remark 4.7.9. The definition of hyperbolic forms given in this section is the "correct one" in the sense that the hyperbolic bilinear forms over R are precisely those taken to hyperbolic forms over Kr(Mod-R) under the isomorphism of Theorem 4.3.6 (symmetric hyperbolic forms over categories with duality were defined in section 4.2). That is, (M, b, K) is hyerbolic $\iff$ $(Z(b), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}}))$ is hyperbolic over Kr(Mod-R). The easiest way to see this is to use that fact that $(W_b, \beta(b))$ is the ring with involution corresponding to $(Z(b), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}}))$ (see Remark 4.6.6) with the previous proposition. As we did not prove Proposition 4.7.8 in the general context of categories with a double duality, let us verify directly that $(Z(b), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}}))$ is hyperbolic when b is.

By Proposition 4.7.4, we may assume $b = b_Z$ with $Z = (M, f_0, f_1, N) \in$ Kr(Mod-R). Consider

$$Z_1 = (M \oplus 0, \left[\begin{smallmatrix} 0 & 0 \\ f_0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ f_1 & 0 \end{smallmatrix}\right], 0 \oplus N)$$

$$Z_2 = (0 \oplus N, \begin{bmatrix} 0 & I_{M,N}^{-1}(f_1) \\ 0 & 0 \end{bmatrix}, \left[\begin{smallmatrix} 0 & I_{N,M}(f_0) \\ 0 & 0 \end{smallmatrix}\right], M \oplus 0)$$

(recall that $I_{N,M}$ is the natural isomorphism $\mathrm{Hom}(M, N^{[0]}) \to \mathrm{Hom}(N, M^{[1]})$). Then $Z_1 \oplus Z_2 = (M \oplus N, \begin{bmatrix} 0 & I_{M,N}^{-1}(f_1) \\ f_0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{N,M}(f_0) \\ f_1 & 0 \end{bmatrix}, M \oplus N) = Z(b_Z)$. In addition, we can identify Z_2 with Z_1^*. Now, if we consider the map $(\mathrm{id}, \mathrm{id}^{\mathrm{op}}) : Z(b_Z) \to Z(b_Z)^* = Z(b_Z)$ as a map from $Z_1 \oplus Z_2 = Z_1 \oplus Z_1^* \to Z_1^* \oplus (Z_1)^{**} = Z_1^* \oplus Z_1$, it is given by $\begin{bmatrix} 0 & \mathrm{id}_{Z_1^*} \\ \mathrm{id}_{Z_1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{id}_{Z_1}^* \\ \mathrm{id}_{Z_1} & 0 \end{bmatrix}$, hence $(Z(b_Z), (\mathrm{id}_{M\oplus N}, \mathrm{id}_{M\oplus N}^{\mathrm{op}}))$ is a hyperbolic bilinear form over Kr(Mod-R).

4.8. A Dictionary

Throughout, R is a ring, K is a double R-module and (M, b, K) is a bilinear space. Let $W = \mathrm{End}(M_R)$ and let $\beta = \beta(b)$ be the involution of W_b (as defined in section 4.6). In the previous section we have seen a first example of how properties of b can be translated into properties of W_b, namely that b is hyperbolic if and only if β is. In this section we shall extend this approach by showing that various properties of b can be phrased in terms of W, W_b and β. This will result in a "dictionary" enabling us to prove claims on b by proving their analogue statements about W, W_b, β and vice versa. In particular, we will see that the ring W_b holds a lot of information on b and forms related to it. (The results of this section are expected, given Theorem 4.3.6 and Remark 4.6.6. However, for the applications, it will be beneficial to work things out explicitly.)

4.8.1. Decomposition. By a *decomposition* of (M, b, K) (or just b) we mean a representation of (M, b, K) as an (inner) orthogonal sum

$$(M_1, b_1, K) \perp \cdots \perp (M_t, b_t, K)$$

with each M_i nonzero. Clearly this induces a decomposition of M, namely $M = \bigoplus_{i=1}^t M_i$. A bilinear space is called *indecomposable* if all its decompositions has length 1 (i.e. $t = 1$).

A *unital decomposition* of a ring S is an ordered set of nonzero pairwise orthogonal idempotents $\{e_1, \dots, e_t\}$ whose sum is 1_S.[17] If S has an involution $*$ and $e_i^* = e_i$ for all i, then $\{e_1, \dots, e_t\}$ is called $*$*-invariant*. It is well known that decompositions of M correspond to unital decompositions of W. Similarly, it turns out that decompositions of b correspond to β*-invariant* unital decompositions of W_b.

Proposition 4.8.1. *There is a one-to-one correspondence between decompositions of b and β-invariant unital decompositions of W_b. In particular, b is indecomposable precisely when W_b does not contain non-trivial β-invariant idempotents.*

Proof. Observe that a β-invariant idempotent in W_b consists of a pair (e, e^{op}) with $e \in \mathrm{E}(W)$ (but $e \in \mathrm{E}(W)$ need not imply $(e, e^{\mathrm{op}}) \in W_b$). Given a β-invariant unital decomposition $\{(e_i, e_i^{\mathrm{op}})\}_{i=1}^t$, let $M_i = e_i M$ and $b_i = b|_{M_i \times M_i}$. Then $\{(M_i, b_i, K)\}_{i=1}^t$ is a decomposition of b. (Indeed, $b(M_i, M_j) = b(e_i M, e_j M) = b(M, e_i e_j M) = b(M, 0) = 0$ for $i \neq j$.) Conversely, if $(M, b, K) = (M_1, b_1, K) \perp \cdots \perp (M_t, b_t, K)$, let e_i be the projection from M to M_i with kernel $\sum_{j \neq i} M_j$. Then for all $x, y \in M$, $b(e_i x, y) = b_i(e_i x, e_i y) = b(x, e_i y)$, hence $(e_i, e_i^{\mathrm{op}}) \in W_b$. Thus, $\{(e_i, e_i^{\mathrm{op}})\}_{i=1}^t$ is a β-invariant unital decomposition of W_b. The rest of the details are left to the reader. □

Definition 4.8.2. *A* subspace *of (M, b, K) is a bilinear space (M_1, b_1, K) such that $M_1 \subseteq M$ and $b_1 = b|_{M_1 \times M_1}$. In this case, b_1 is called a* subform *of b. The subspace (M_1, b_1, K) (or just b_1) is a* summand *of (M, b, K) (or b) if there exists a subspace (M_2, b_2, K) of (M, b, K) such that $(M, b, K) = (M_1, b_1, K) \perp (M_2, b_2, K)$. In this case, (M_2, b_2, K) is called a* complement *of (M_1, b_1, K).*

When b is quasi-reduced, any summand of b admits a unique complement and these summands correspond to β-invariant idempotents in W_b. This is verified in the following propositions.

Proposition 4.8.3. *Assume b is quasi-reduced. Then any summand of b admits a unique complement.*

Proof. Let (M_1, b_1, K) be a summand of (M, b, K) and let (M_2, b_2, K) and (M_2', b_2', K) be complements of (M_1, b_1, K). It is enough to prove $M_2 = M_2'$. Let $\{(e_1, e_1^{\mathrm{op}}), (e_2, e_2^{\mathrm{op}})\}$ (resp. $\{(e_1', e_1'^{\mathrm{op}}), (e_2', e_2'^{\mathrm{op}})\}$) be the β-invariant unital decomposition corresponding to $b = b_1 \perp b_2$ (resp. $b = b_1 \perp b_2'$). Then clearly $e_1 e_1' = e_1'$ and $e_1' e_1 = e_1$. We now have:

$$\begin{aligned} b(x, e_1 y) &= b(e_1 x, y) = b(e_1' e_1 x, y) = b(e_1 x, e_1' y) \\ &= b(x, e_1 e_1' y) = b(x, e_1' y) = b(e_1' x, y) \ , \end{aligned}$$

and similarly $b(e_1 x, y) = b(x, e_1' y)$. Thus, $(e_1, e_1'^{\mathrm{op}}) \in W_b$ which in turn implies $(0, (e_1 - e_1')^{\mathrm{op}}) = (e_1, e_1^{\mathrm{op}}) - (e_1, e_1'^{\mathrm{op}}) \in W_b$. The form b is quasi-reduced, so by

[17] The set $\{e_1, \dots, e_t\}$ is also called a *complete set of orthogonal idempotents*. We have changed it into *unital decomposition* for brevity.

Proposition 4.6.8(iii), we must have $e_1 - e_1' = 0$, hence $e_2 = e_2'$ and $M_2 = e_2 M = e_2' M = M_2'$. □

REMARK 4.8.4. With the notation of the last proof, note that M_2 need *not* be the orthogonal complement of M_1, namely:

$$M_1^{\perp} = \{x \in M \,:\, b(x, M_1) = b(M_1, x) = 0\}$$

(but we always have $M_2 \subseteq M_1^{\perp}$). For example, take any right stable form (b_0, M, K) with $N := \ker \mathrm{Ad}_b^r \cap \ker \mathrm{Ad}_b^{\ell} \neq 0$ (e.g. the form of Example 2.4.4). Then $b = b_0 \perp b_0$ is right stable (Corollary 2.6.6) and, abusing the notation, b_0 is clearly the complement of itself. However, $(M \oplus 0)^{\perp} = N \oplus M \neq 0 \oplus M$. Nevertheless, it is straightforward to check that when b is right injective, $M_2 = M_1^{\perp}$.

PROPOSITION 4.8.5. *Assume b is quasi-reduced. Then there is a one-to-one correspondence between summands of b and β-invariant idempotents in W_b.*

PROOF. Let $e_1 \in \mathrm{E}(W_b)$ be a β-invariant idempotent. Then $\{e_1, 1 - e_1\}$ is a unital decomposition of W_b, and hence it gives rise to a decomposition $(M, b, K) = (M_1, b_1, K) \perp (M_2, b_2, K)$. Let e_1 be the idempotent corresponding to b_1. Conversely, given a summand b_1 of b, let b_2 be its *unique* complement and let $\{e_1, e_2\}$ be the β-invariant unital decomposition corresponding to $b = b_1 \perp b_2$. Then b_1 corresponds to e_1. The rest of the details are left to the reader. □

REMARK 4.8.6. Recall that by Proposition 4.7.8, idempotents $e \in W_b$ with $e + e^{\beta} = 1$ correspond to representations of M as a direct sum to two totally isotropic submodules. However, in contrast to the last proposition, totally isotropic submodules $M_1 \subseteq M$ admitting a totally isotropic $M_2 \leq M$ s.t. $M = M_1 \oplus M_2$ do not correspond to idempotents $e \in W_b$ satisfying $e + e^{\beta} = 1$, even when b is regular. In particular, M_2 is not uniquely determined by M_1.

For example, let F be a field and let $b : F^2 \times F^2 \to F$ be the regular alternating form defined by $b((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$. By Proposition 4.6.9, we may identify (W_b, β) with $(W^{\{\alpha^2\}}, \alpha)$ where α is the corresponding anti-endomorphism of b. It is easy to check that α is given by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{\alpha} = \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$ for all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{M}_2(F) = W$ and hence $W^{\{\alpha^2\}} = W^{\{\mathrm{id}\}} = W$. Now, any 1-dimensional subspace of F^2 is totally isotropic, hence M_2 above cannot be uniquely determined. In addition, $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $e' = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ are idempotents in $W = W_b = \mathrm{M}_2(F)$ such that $e + e^{\alpha} = e' + e'^{\alpha} = 1$ and $eF^2 = e'F^2 = F \times 0$. Thus, $M_1 = F \times 0$ does not correspond to a specific idempotent $e_0 \in W_b$ with $e_0 + e_0^{\alpha} = 1$.

4.8.2. Isometry of Summands of M and b. Let (M_1, b_1, K), (M_2, b_2, K) be two summands of (M, b, K) (we do not assume $b = b_1 \perp b_2$). We shall now present necessary and sufficient conditions (in terms of W_b and β) for b_1 and b_2 to be isometric. By Proposition 4.8.1, we may assume that $M_1 = e_1 M$ and $M_2 = e_2 M$ for some β-invariant idempotents $(e_1, e_1^{\mathrm{op}}), (e_2, e_2^{\mathrm{op}}) \in W_b$ (the idempotents e_1, e_2 are uniquely determined when b is quasi-reduced, as implied by Proposition 4.8.5).

DEFINITION 4.8.7. *Let $(S, *)$ be a ring with involution. Two $*$-invariant idempotents $e, e' \in \mathrm{E}(S)$ are called* isometric *if there exists $s \in e'Se$ such that*

$$s^* s = e, \qquad \text{and} \qquad s s^* = e' \,.$$

In this case, s is called an isometry *from e to e'.*

PROPOSITION 4.8.8. *Let M_i, b_i, e_i be as above. Identify* $\mathrm{Hom}(M_i, M_j)$ *with $e_j W e_i$ ($i, j \in \{1, 2\}$) in the standard way. Then isometries from b_1 to b_2 correspond to isometries from (e_1, e_1^{op}) to (e_2, e_2^{op}) (in W_b) via $\sigma \leftrightarrow (\sigma, (\sigma^{-1})^{\mathrm{op}})$. (Here σ^{-1} stands for the unique element of $e_1 W e_2$ satisfying $\sigma^{-1}\sigma = e_1$ and $\sigma\sigma^{-1} = e_2$).*

PROOF. Let $w \in \mathrm{Hom}(M_1, M_2) = e_2We_1$ be an isometry from b_1 to b_2 and let $w' \in \mathrm{Hom}(M_2, M_1) = e_1We_2$ be its inverse. We claim that $(w, w'^{\mathrm{op}}) \in W_b$. Indeed, for all $x, y \in M$, $b(wx, y) = b_2(e_2wx, e_2y) = b_1(w'e_2wx, w'e_2y) = b_1(w'wx, w'y) = b_1(e_1x, w'y) = b(x, w'y)$ and similarly, $b(x, wy) = b(w'x, y)$. The element (w, w'^{op}) is an isometry from (e_1, e_1^{op}) to (e_2, e_2^{op}) since $(w, w'^{\mathrm{op}})^\beta(w, w'^{\mathrm{op}}) = (w'w, w^{\mathrm{op}}w'^{\mathrm{op}}) = (e_1, e_1^{\mathrm{op}})$ and $(w, w'^{\mathrm{op}})(w, w'^{\mathrm{op}})^\beta = (ww', w'^{\mathrm{op}}w^{\mathrm{op}}) = (e_2, e_2^{\mathrm{op}})$.

Conversely, if $(w, w'^{\mathrm{op}}) \in W_b$ is an isometry from (e_1, e_1^{op}) to (e_2, e_2^{op}), then $(e_1, e_1^{\mathrm{op}}) = (w, w^{\mathrm{op}})^\beta(w, w'^{\mathrm{op}}) = (w'w, w^{\mathrm{op}}w'^{\mathrm{op}})$, hence $e_1 = w'w$ and similarly $e_2 = ww'$. This means that w induces an invertible map from M_1 to M_2 (its inverse is w'). The element w is an isometry from b_1 to b_2 since $b_2(we_1x, we_1y) = b(wx, wy) = b(x, w'wy) = b(x, e_1y) = b(x, e_1e_1y) = b(e_1x, e_1y) = b_1(e_1x, e_1y)$ for all $x, y \in M$. □

REMARK 4.8.9. Taking $e_1 = e_2 = 1_W$ in the previous proposition implies that W_b contains a copy of the *isometry group* of b, namely the group of isometries from b to itself. The proposition also implies that this group corresponds to the elements $u \in W_b^\times$ satisfying $u^{-1} = u^\beta$, as one would expect.

Now let M_1, M_2 be two summands of M (we do not assume $b|_{M_1\times M_1}, b|_{M_2\times M_2}$ are summands of b). *Assume b is right stable* with corresponding anti-endomorphism α. The following proposition shows that it is possible to express the fact that $b_1 := b|_{M_1\times M_1}$ and $b_2 := b|_{M_2\times M_2}$ are isometric in terms of α and W. We may of course assume that $M_1 = e_1M$ and $M_2 = e_2M$ for some $e_1, e_2 \in \mathrm{E}(W)$ (but e_1, e_2 need not be α-invariant nor unique).

PROPOSITION 4.8.10. *In the previous notation, the following are equivalent:*

(i) $b_1 \cong b_2$.

(ii) *There are $w_1 \in e_2We_1$ and $w_2 \in e_1We_2$ such that:*[18]

$$w_1^\alpha w_1 = e_1^\alpha e_1, \qquad w_2^\alpha w_2 = e_2^\alpha e_2, \qquad w_2w_1 = e_1, \qquad w_1w_2 = e_2\ .$$

Furthermore, there is a one-to-one correspondence between isometries from b_1 to b_2 and pairs (w_1, w_2) as in (ii).

PROOF. Throughout, we identify $\mathrm{Hom}(M_i, M_j)$ with e_jWe_i for all $i, j \in \{1, 2\}$. Let $\sigma : M_1 \to M_2$ be an isometry from b_1 to b_2. Then there are $w_1 \in e_2We_1$, $w_2 \in e_1We_2$ such that $\sigma(x) = w_1x$ and $\sigma^{-1}(y) = w_2y$ for all $x \in M_1$ and $y \in M_2$. As w_2w_1 induces the identity homomorphism on M_1, we have $w_2w_1 = e_1$ and similarly $w_1w_2 = e_2$. Now let $x, y \in M$. Then $b(x, w_1^\alpha w_1y) = b(w_1x, w_1y) = b(w_1e_1x, w_1e_1y) = b(\sigma(e_1x), \sigma(e_1y)) = b(e_1x, e_1y) = b(x, e_1^\alpha e_1y)$, hence $w_1^\alpha w_1 = e_1^\alpha e_1$ and similarly $w_2^\alpha w_2 = e_2^\alpha e_2$.

Conversely, assume w_1, w_2 as above are given. Define $\sigma : M_1 \to M_2$ and $\tau : M_2 \to M_1$ by $\sigma(x) = w_1x$ and $\tau(y) = w_2y$. Then it is straightforward to check $\tau \circ \sigma = \mathrm{id}_{M_1}$ and $\sigma \circ \tau = \mathrm{id}_{M_2}$. In addition, for all $x, y \in M_1$, $b(\sigma(x), \sigma(y)) = b(w_1x, w_1y) = b(x, w_1^\alpha w_1y) = b(x, e_1^\alpha e_1y) = b(e_1x, e_1y) = b(x, y)$, hence σ is an isometry from b_1 to b_2. □

REMARK 4.8.11. The element w_2 is uniquely determined by w_1 in the sense that it is the only element in e_1We_2 satisfying $w_2w_1 = e_1$ and $w_1w_2 = e_2$. Indeed, if w_2' also satisfies these relations, then $w_2' = w_2'e_2 = w_2'w_1w_2 = e_1w_2 = w_2$. (A less explicit yet more intuitive explanation for this is that w_2 is induced from $\sigma^{-1} : M_2 \to M_1$ where $\sigma : M_1 \to M_2$ is defined by $\sigma(x) = w_1x$).

[18] The second equality is in fact superfluous since the other three imply $w_2^\alpha w_2 = w_2^\alpha e_1^\alpha e_1 w_2 = w_2^\alpha w_1^\alpha w_1 w_2 = (w_1w_2)^\alpha(w_1w_2) = e_2^\alpha e_2$.

4.8.3. Isometry of Forms Related to b. Recall that for two bilinear spaces (M, b, K) and (M', b', K), we write $b \sim_{\mathrm{Kr}} b'$ to denote that $Z(b) \cong Z(b')$. That is, there are isomorphisms $\sigma \in \mathrm{Hom}(M, M')$ and $\tau \in \mathrm{Hom}(M', M)$ satisfying (17). In this subsection, we will show that isometry classes of bilinear forms b' with $b \sim_{\mathrm{Kr}} b'$ correspond to *congruence classes* of invertible *β-symmetric* elements in W_b.

Let $(S, *)$ be a ring with involution. Two elements $x, y \in S$ are called $*$-congruent if there is $s \in S^\times$ such that $x = s^* y s$. This is an equivalence relation which we denote by $\sim_*$, and its equivalence classes are called congruence classes (w.r.t. $*$). An element $x \in S$ is called *$*$-symmetric* if $x^* = x$. Being $*$-symmetric is preserved under the relation $\sim_*$. The set of $*$-symmetric elements in S will be denoted by $\mathrm{Sym}(S, *)$. For example, $\mathrm{Sym}(W_b, \beta)$ consists of elements of the form (w, w^{op}) in W_b.

Henceforth, we will use $[\,\cdot\,]$ to denote both isometry classes and congruence classes w.r.t. β. In case of ambiguity, the latter will be denoted by $[\,\cdot\,]_\beta$.

Proposition 4.8.12. *There is a one-to-one correspondence between isometry classes bilinear spaces (M', b', K) with $b' \sim_{\mathrm{Kr}} b$ and congruence classes of elements in* $\mathrm{Sym}(W_b, \beta) \cap W_b^\times$.

Proof. Let (M', b', K) be such that $b' \sim_{\mathrm{Kr}} b$ and let $(\sigma, \tau^{\mathrm{op}}) : Z(b) \to Z(b')$ be an isomorphism. The correspondence is given by sending $[b']$ to $[(\tau\sigma, (\tau\sigma)^{\mathrm{op}})]$. However, we need to prove several things before we can assert this is indeed a correspondence.

First, we need to show that $\tau\sigma \in \mathrm{Sym}(W_b, \beta) \cap W_b^\times$. Indeed, by (17), for all $x \in M$ and $x' \in M'$, we have $b'(x', \sigma x) = b(\tau x', x)$ and $b'(\sigma x, x') = b(x, \tau x')$. Thus, for all $x, y \in M$, $b(\tau\sigma y, x) = b'(\sigma y, \sigma x) = b(y, \tau\sigma y)$, implying $(\tau\sigma, (\tau\sigma)^{\mathrm{op}}) \in W_b$. Repeating this argument with $(\tau^{-1}, (\sigma^{-1})^{\mathrm{op}})$, which is also an isomorphism from $Z(b)$ to $Z(b')$ (since $(\tau, \sigma^{\mathrm{op}}) = (\sigma, \tau^{\mathrm{op}})^*$ is an isomorphism from $Z(b')^* = Z(b')$ to $Z(b)^* = Z(b)$), yields that $\sigma^{-1}\tau^{-1} \in W_b$, hence $\sigma\tau \in W_b^\times$.

Next, we need to show that $[(\tau\sigma, (\tau\sigma)^{\mathrm{op}})]$ is independent of b', σ and τ. Let (b'', M'', K) be another bilinear space with $[b''] = [b']$ and let $(\eta, \theta^{\mathrm{op}}) : Z(b) \to Z(b'')$ be an isomorphism. We need to prove that $[(\tau\sigma, (\tau\sigma)^{\mathrm{op}})] = [(\theta\eta, (\theta\eta)^{\mathrm{op}})]$. Let $\zeta : b' \to b''$ be an isometry. Then for all $x, y \in M$,

$$b(\theta\zeta\tau^{-1}x, y) = b''(\zeta\tau^{-1}x, \eta y) = b'(\tau^{-1}x, \zeta^{-1}\eta y) = b(x, \sigma^{-1}\zeta^{-1}\eta y)$$

and similarly $b(x, \theta\zeta\tau^{-1}y) = b(\sigma^{-1}\zeta^{-1}\eta x, y)$. Thus, $s := (\sigma^{-1}\zeta^{-1}\eta, (\theta\zeta\tau^{-1})^{\mathrm{op}})$ lies in W_b. As $(\theta\zeta\tau^{-1}) \cdot (\tau\sigma) \cdot (\sigma^{-1}\zeta^{-1}\eta) = \theta\eta$, it follows that $s^*(\tau\sigma, (\tau\sigma)^{\mathrm{op}})s = (\theta\eta, (\theta\eta)^{\mathrm{op}})$, hence $[(\tau\sigma, (\tau\sigma)^{\mathrm{op}})] = [(\theta\eta, (\theta\eta)^{\mathrm{op}})]$.

Now drop the assumption $[b''] = [b']$ and assume $[(\tau\sigma, (\tau\sigma)^{\mathrm{op}})] = [(\theta\eta, (\theta\eta)^{\mathrm{op}})]$ instead. We need to show that $b'' \cong b'$. Let $s = (u, w^{\mathrm{op}}) \in W_b$ be an element satisfying $s^*(\tau\sigma, (\tau\sigma)^{\mathrm{op}})s = (\theta\eta, (\theta\eta)^{\mathrm{op}})$. Then u, w are automorphisms of M satisfying $w\tau\sigma u = \theta\eta$. Define $\zeta = \theta^{-1}w\tau \in \mathrm{Hom}(M', M'')$. Then $\zeta = \theta^{-1}(w\tau) = \theta^{-1}(\theta\eta u^{-1}\sigma^{-1}) = \eta u^{-1}\sigma^{-1}$ and for all $x, y \in M'$, we have

$$\begin{aligned} b''(\zeta x, \zeta y) &= b''(\theta^{-1}w\tau x, \eta u^{-1}\sigma^{-1}y) = b(w\tau x, \eta^{-1}\eta u^{-1}\sigma^{-1}y) \\ &= b(\tau x, u\eta^{-1}\eta u^{-1}\sigma^{-1}y) = b'(x, \sigma u\eta^{-1}\eta u^{-1}\sigma^{-1}y) = b'(x, y)\ . \end{aligned}$$

Thus, $\zeta : b' \to b''$ is an isometry and $[b'] = [b'']$.

To finish, we prove that every $w \in \mathrm{Sym}(W_b) \cap W_b^\times$ is of the form $\tau\sigma$ for some b', σ, τ as above. Define $b' : M \times M \to K$ by $b'(x, y) = b(wx, y) = b(x, wy)$ (the latter equality holds since $(w, w^{\mathrm{op}}) \in W_b$). Then $(\sigma, \tau^{\mathrm{op}}) := (1, w^{\mathrm{op}})$ is clearly an isomorphism from $Z(b)$ to $Z(b')$ satisfying $[\tau\sigma] = [w]$. □

The previous proposition reduces the isomorphism problem of semi-stable bilinear forms to (1) congruence of elements in W_b and (2) isomorphism of Kronecker modules.

EXAMPLE 4.8.13. It might seem a little surprising that Proposition 4.8.12 holds even for non-quasi-reduced forms, so let us exhibit the correspondence in a non-quasi-reduced example. Assume b is the zero form. Then all bilinear forms b' with $b' \sim_{\mathrm{Kr}} b$ must also be zero and hence isometric to b. We thus expect only one congruence class in $\mathrm{Sym}(W_b,\beta)\cap W_b^\times$. Indeed, in this case $W_b = W\times W^{\mathrm{op}}$ and for all $(w,w^{\mathrm{op}}),(w',w'^{\mathrm{op}})\in \mathrm{Sym}(W_b,\beta)\cap W_b^\times$ we have $(w,w^{\mathrm{op}})\sim_\beta (w',w'^{\mathrm{op}})$ because $(1,(w'w^{-1})^{\mathrm{op}})^\beta(w,w^{\mathrm{op}})(1,(w'w^{-1})^{\mathrm{op}}) = (w',w'^{\mathrm{op}})$.

REMARK 4.8.14. If b is right regular and K has an anti-isomorphism κ, then b has a unique right κ-asymmetry, λ. In this case, the bilinear forms b' with $b'\sim_{\mathrm{Kr}} b$ are the right regular forms admitting a (unique) κ-asymmetry which is conjugate to λ. (This follows from Proposition 4.6.4(ii) since any two right regular forms are right joinable. In addition, it easy to see that if $Z(b)\cong Z(b')$, then b is right regular if and only if b' is.)

4.8.4. Summary. The correspondences presented in this section and in section 4.7 are summarized in the following table:

	Property or Object	**Corresponds To**
1.	decompositions of (M,b,K)	β-invariant unital decompositions of W_b
2.	summands of b (provided b is quasi-reduced)	β-invariant idempotents in W_b
3.	b is indecomposable	W_b does not contain β-invariant idempotents other than 0 and 1
4.	isometries between summands b_1, b_2 of b	isometries from e_1 to e_2 in W_b, where e_i is the *a* β-invariant idempotent such that $b_i = b\vert_{e_iM\times e_iM}$ (e_i is uniquely determined if b is quasi-reduced).
5.	isometries between $b_1 := b\vert_{e_1M\times e_1M}$ and $b_2 := b\vert_{e_2M\times e_2M}$ where $e_1,e_2\in \mathrm{E}(W)$	pairs $(w_1,w_2)\in e_2We_1\times e_1We_2$ satisfying $w_2w_1=e_1$, $w_1w_2=e_2$, $w_1^\alpha w_1 = e_1^\alpha e_1$ and $w_2^\alpha w_2 = e_2^\alpha e_2$
6.	representations $M = M_1\oplus M_2$ with M_1, M_2 totally isotropic	idempotents $e\in \mathrm{E}(W_b)$ such that $e + e^\beta = 1$
7.	b is a hyperbolic form	β is a hyperbolic involution
8.	isometry classes of forms b' with $b'\sim_{\mathrm{Kr}} b$	congruence classes in $\mathrm{Sym}(W_b,\beta)\cap W_b^\times$

The table implies that the ring W_b and its involution β hold a lot of information about the form b and other forms related to it. However, the ring W_b is still far too complicated to allow an immediate usage of our "dictionary". In the next section we will show that under mild assumptions, most properties mentioned in the right column of the table can be "lifted" from $W_b/\mathrm{Jac}(W_b)$ to W_b. Furthermore, we shall later see that $W_b/\mathrm{Jac}(W_b)$ is often semisimple. Once that is achieved, we will have the tools to provide direct proofs to the consequences of Theorem 4.3.6, as well as other applications.

4.9. Lifting Along the Jacobson Radical

In this section, $(R,*)$ denotes a ring with involution and J is an ideal of R such that $J^* = J$ (usually J would be $\mathrm{Jac}(R)$). The involution $*$ induces an involution on $\overline{R} := R/J$ which we also denoted by $*$. The image of $r \in R$ in $\overline{R}$ will be denoted by $\overline{r}$.

Motivated by the previous section, this section is concerned with presenting sufficient conditions on $R, J, *$ to allow the lifting of various properties of $(\overline{R}, *)$ to $(R, *)$. In particular, we will consider lifting of $*$-invariant idempotents, isometries between them, $*$-*isotropic* idempotents and $*$-congruences.

DEFINITION 4.9.1. *The ideal J is called* idempotent lifting *if $J \subseteq \mathrm{Jac}(R)$ and for every $\varepsilon \in \mathrm{E}(\overline{R})$ there exists $e \in \mathrm{E}(R)$ such that $\overline{e} = \varepsilon$.*

EXAMPLE 4.9.2. (i) Any nil ideal is idempotent lifting (see [**80**, Cr. 1.1.28]).

(ii) A semilocal ring is semiperfect if and only it its Jacobson radical is idempotent lifting (by definition).

The following well-known facts will be used throughout the section.

PROPOSITION 4.9.3. *If J is idempotent lifting, then any unital decomposition of $\overline{R}$, $\{\varepsilon_i\}_{i=1}^n$, can be lifted to R, i.e. there is a unital decomposition of R, $\{e_i\}_{i=1}^n$, such that $\overline{e}_i = \varepsilon_i$.*

PROPOSITION 4.9.4. *Let $e \in \mathrm{E}(R)$. Then:*

(i) $\mathrm{Jac}(eRe) = e\,\mathrm{Jac}(R)e$.

(ii) *If $J \trianglelefteq R$ is idempotent lifting, then eJe is idempotent lifting in eRe.*

4.9.1. Lifting $*$-Invariant Idempotents.

Henceforth, J is idempotent lifting.

LEMMA 4.9.5. *For a right ideal $I \leq R_R$, the following are equivalent:*

(a) *$I = eR$ for some $e \in \mathrm{E}(R)$ with $e = e^*$.*

(b) *$R = I \oplus (\mathrm{ann}^\ell I)^*$.*

(c) *$R = I \oplus \mathrm{ann}^r(I^*)$.*

The idempotent e of (a) is unique, i.e. if $e' \in \mathrm{E}(R)$ is such that $e'^* = e'$ and $I = e'R$, then $e = e'$.[19]

PROOF. (a)$\Longrightarrow$(b) and (a)$\Longrightarrow$(c) easily follows from the fact that for all $e \in \mathrm{E}(R)$, $\mathrm{ann}^\ell(eR) = R(1-e)$ and $\mathrm{ann}^r(Re) = (1-e)R$.

(b)$\Longrightarrow$(a): We can write $1 = e + (1-e)$ where $e \in I$ and $(1-e) \in (\mathrm{ann}^\ell I)^*$. It is well known that e and $1-e$ are idempotents and $I = eR$. As $1 - e \in \mathrm{ann}^\ell eR = \mathrm{ann}^\ell I$, we get $(1-e)^*e = 0$, implying $e = e^*e$. But $e^*e = (e^*e)^* = e^*$, so $e = e^*$.

(c)$\Longrightarrow$(a) follows by repeating the previous argument with $e^*(1-e)$ instead of $(1-e)e^*$.

To finish, assume $I = e'R$ and $e'^* = e'$. Then, $e' = ee' = e^*e'^* = (e'e)^* = e^* = e$ (the first and next to last equalities hold since $eR = e'R$). □

THEOREM 4.9.6. *Assume $J \trianglelefteq R$ is idempotent lifting. Then for any $*$-invariant idempotent $\varepsilon \in \mathrm{E}(\overline{R})$ there is a $*$-invariant idempotent $e \in \mathrm{E}(R)$ such that $\overline{e} = \varepsilon$.*[20]

PROOF. Take some $f \in \mathrm{E}(R)$ with $\overline{f} = \varepsilon$. Since $\varepsilon = \varepsilon^*$, $f + (1-f)^* - 1$ lies in J and hence $f + (1-f)^*$ is invertible (because $J \subseteq \mathrm{Jac}(R)$). Therefore, $R = fR + (1-f)^*R$. On the other hand, if $r \in fR \cap (1-f)^*R$, then $(1-f)r = 0$ (because $r = fr$) and $f^*r = 0$ (because $(1-f)^*r = r$), hence $(1 - f + f^*)r = 0$, which implies $r = 0$ since $(1-f) + f^* \in R^\times$. Therefore, $R = fR \oplus (1-f)^*R$.[21]

---

[19] It is worth pointing that this property is special for $*$-invariant idempotents and fails for arbitrary idempotents.

[20] Compare with [**100**, Lm. 3], which proves the same claim when J is nil.

[21] This can also be shown using the fact fR and $(1-f)^*R$ are projective covers of $\varepsilon\overline{R}$ and $(1-e)\overline{R}$, respectively.

Now, since $(\operatorname{ann}^\ell fR)^* = (1-f)^*R$, Lemma 4.9.5 implies that there is $e \in \mathrm{E}(R)$ such that $e = e^*$ and $eR = fR$. Finally, Lemma 4.9.5 also implies that ε is the only $*$-invariant idempotent generating $\varepsilon\overline{R}$ and therefore $\overline{e} = \varepsilon$. □

Corollary 4.9.7. *Any $*$-invariant unital decomposition $\{\varepsilon_i\}$ of $\overline{R}$ can be lifted to a $*$-invariant unital decomposition of R. That is, there exists a $*$-invariant unital decomposition $\{e_i\}$ of R such that $\overline{e}_i = \varepsilon_i$ for all i. In particular, if R is semiperfect, then any $*$-invariant unital decomposition can be lifted from $R/\operatorname{Jac}(R)$ to R.*

Proof. Lift ε_1 to $e_1 \in \mathrm{E}(R)$ with $e_1^* = e_1$ using Theorem 4.9.6. Now induct on $(1-e_1)R(1-e_1)$ and $(1-e_1)J(1-e_1)$ (we are allowed do this due to Lemma 4.9.4(ii)). □

Example 4.9.8. If $*$ is not an involution but merely an anti-automorphism, then Theorem 4.9.6 might fail. For example, let $p > 2$ be a prime number, $S = \mathbb{Z}_{\langle p\rangle}$, $R = \mathrm{M}_2(S)$, $J = \operatorname{Jac}(R) = pR = \mathrm{M}_2(pS)$ and let $*$ be the anti-automorphism defined by:

$$A^* = \left[\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right]^{-1} A^T \left[\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right] .$$

Then J is idempotent lifting since R is semiperfect. In addition, $*$ acts as the transpose involution on $R/\operatorname{Jac}(R) \cong \mathrm{M}_2(\mathbb{Z}/p)$, so $R/\operatorname{Jac}(R)$ has plenty of non-trivial $*$-invariant idempotents. On the other hand, the set of $*$-invariant elements in R is contained in the subring $\{a \in R : a^{**} = a\}$ which is the centralizer of:

$$X = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1-p^2 & -p \\ p & 1 \end{bmatrix}$$

in $\mathrm{M}_2(S)$. It is not hard to verify that the centralizer of X in $\mathrm{M}_2(\mathbb{Q})$ is $\mathbb{Q}[X] \cong \mathbb{Q}[x]/\langle x^2 + (p^2-2) + 1\rangle$, which is a field (since $p > 2$). Therefore, $\mathrm{M}_2(S)$ admits no non-trivial $*$-invariant idempotents. In particular, there are $*$-invariant idempotents in $R/\operatorname{Jac}(R)$ that cannot be lifted to R.

4.9.2. Lifting Isotropic Idempotents. We will now consider lifting of idempotents $\varepsilon \in \overline{R}$ satisfying $\varepsilon + \varepsilon^* = 1$ and, more generally, idempotents $\varepsilon \in \overline{R}$ such that ε is orthogonal to ε^*. Such idempotents are called *isotropic* (or $*$-isotropic). In contrast to the previous subsection, the mere assumption that J is idempotent lifting does not guarantee such a lifting, as shown in the following example.

Example 4.9.9. Let $n \in \mathbb{N}$ be such that $p = n^2 + 1$ is prime (e.g. $n = 2$). Let S be the ring $\mathbb{Z}[\sqrt{p}]$ localized at the prime ideal $\langle\sqrt{p}\rangle$ and let $I = \sqrt{p}S = \operatorname{Jac}(S)$. Let $R = M_2(S)$ and define $* : R \to R$ by

$$A^* = \begin{bmatrix} \sqrt{p} & n \\ n & \sqrt{p} \end{bmatrix}^{-1} A^T \begin{bmatrix} \sqrt{p} & n \\ n & \sqrt{p} \end{bmatrix} .$$

Let $J = \operatorname{Jac}(R)$. We have $R/\operatorname{Jac}(R) = M_2(S)/M_2(I) \cong M_2(S/I) \cong M_2(\mathbb{Z}/p)$. The action of $*$ on R/J can be described as:

$$\overline{A}^* = \begin{bmatrix} 0 & \overline{n} \\ \overline{n} & 0 \end{bmatrix}^{-1} \overline{A}^T \begin{bmatrix} 0 & \overline{n} \\ \overline{n} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \overline{1} \\ \overline{1} & 0 \end{bmatrix}^{-1} A^T \begin{bmatrix} 0 & \overline{1} \\ \overline{1} & 0 \end{bmatrix} .$$

It is now easy to check that the matrix unit $\varepsilon := e_{11} \in M_2(\mathbb{Z}/p) \cong R/J$ satisfies $\varepsilon + \varepsilon^* = 1$. However, despite that R is semiperfect and J is idempotent lifting, there is no $e \in \mathrm{E}(R)$ such that $e + e^* = 1$ and $\overline{e} = \varepsilon$.

To see this, consider the bilinear form $b : S^2 \times S^2 \to S$ defined by $b(x,y) = x^T \left[\begin{smallmatrix} \sqrt{p} & n \\ n & \sqrt{p} \end{smallmatrix}\right] y$. Then b is stable and $*$ is its corresponding anti-endomorphism. In addition, $(W_b, \beta(b)) \cong (R, *)$ by Proposition 4.6.9. Thus, e as above exists $\iff$ $\beta(b)$ is hyperbolic $\iff$ b is hyperbolic (Proposition 4.7.8). But the latter is impossible since b is anisotropic, i.e. $b(x,x) \neq 0$ for all $0 \neq x \in S^2$. (Indeed, consider b as a form over $\mathbb{R}$ rather than S. Then the quadratic form corresponding to b is 2-dimensional and has discriminant $n^2 - q = -1$, hence it is anisotropic.[22])

DEFINITION 4.9.10. *Let $I \trianglelefteq R$. The ideal I is called* ($*$-)symmetric *if $I = I^*$. In this case we define:*

$$\mathrm{Sym}(I) = \{x \in I \mid x = x^*\} \qquad \text{and} \qquad \mathrm{Symd}(I) = \{x + x^* \mid x \in I\} \ .$$

These sets are called the $*$-symmetric *elements of I and the* $*$-symmetrized *elements of I, respectively. The ideal I is called* ($*$-)symmetrized *if it is symmetric and* $\mathrm{Sym}(I) = \mathrm{Symd}(I)$.

A ring with involution $(R, *)$ is dyadic *if it admits a non-symmetrized symmetric ideal. Otherwise, it is* non-dyadic.

The results of this section would apply in their full strength when $(R, *)$ is non-dyadic. The dyadic is more complicated and will not be treated here. The following proposition ensures that $(R, *)$ is non-dyadic whenever $2 \in R^\times$ (take $a = 1$), hence justifying the name "non-dyadic".

PROPOSITION 4.9.11. *Let $a \in \mathrm{Cent}(R)$ and let $I \trianglelefteq R$ be symmetric. Then:*

(i) $\mathrm{Sym}((a + a^*)I) \subseteq \mathrm{Symd}(aI + a^*I) + a\,\mathrm{ann}(a + a^*)$.
(ii) *If $a + a^* \in R^\times$, then I is symmetrized. In particular, R is non-dyadic.*
(iii) *If $a\,\mathrm{ann}(a + a^*) = 0$, then* $\mathrm{Symd}((a + a^*)I) \subseteq \mathrm{Sym}(I)$.

PROOF. (ii) and (iii) easily follow from (i). To prove (i), let $y \in \mathrm{Sym}((a+a^*)I)$. Then there exists $x \in I$ such that $y = (a + a^*)x$. Now, $(a + a^*)x = y = y^* = (a + a^*)x^*$ implying $x - x^* \in \mathrm{ann}(a + a^*)$. Therefore,

$$y = (a + a^*)x = (a^*x) + (a^*x)^* + a(x - x^*) \in \mathrm{Symd}(aI + a^*I) + a\,\mathrm{ann}(a + a^*)$$

(note that we used $a \in \mathrm{Cent}(R)$). □

LEMMA 4.9.12. *Assume that there are symmetric ideals $I, J_0, J_1 \trianglelefteq R$ such that $IJ_0 + J_0I \subseteq J_1 \subseteq J_0 \subseteq I$ and $\mathrm{Sym}(J_0) \subseteq \mathrm{Symd}(I) + J_1$. Let $\varepsilon_0 \in \mathrm{E}(R/J_0)$ be an isotropic idempotent. Then there exists an isotropic idempotent $\varepsilon_1 \in \mathrm{E}(R/J_1)$, and ε_0, ε_1 has the same image in R/I.*

PROOF. Let us work in $R' = R/J_1$ and set $I' = I/J_1$, $J_0' = J_0/J_1$. Then $(J_0')^2 = I'J_0' = J_0'I' = 0$. In particular, J_0' is nilpotent, hence there exists $e \in \mathrm{E}(R')$ whose image in R/J_0 is ε_0. Now take arbitrary $x \in R$ with $x + J_1 = e$. Since $ee^* + J_0' = J_0'$ and $(xx^*) = xx^*$, $xx^* \in \mathrm{Sym}(J_0)$ and hence there is $y \in I$ such that $xx^* - (y + y^*) \in J_1$ (because $\mathrm{Sym}(J_0) \subseteq \mathrm{Symd}(I) + J_1$). Let a be the image of y in R' (so $a \in I'$), then $ee^* = a + a^*$. By replacing a with eae^*, we may assume $a = ea = ae^* \in eI'e^*$. Now, $ee^* \in J_0'$ implies

$$aa = aa^* = a^*a = a^*a^* = ae = a^*e = e^*a = e^*a^* = 0$$

(because $I'J_0' = J_0'I' = 0$). Define $g = e - a$ and observe that

$$\begin{aligned} g^2 &= e^2 - ea - ae - a^2 = e - a = g, \\ gg^* &= (e - a)(e^* - a^*) = ee^* - ea^* - ae^* + aa^* = ee^* - (a + a^*) = 0, \\ g^*g &= (e^* - a^*)(e - a) = e^*e - e^*a - a^*e + a^*a = e^*e \ . \end{aligned}$$

Thus, g is an idempotent with $gg^* = 0$, $g^*g \in J_1'$ and $g = e_1$ in R/I_1.

Repeating the previous argument with g^* in place of e would yield an element $b \in I'$ with $g^*g = b + b^*$ satisfying:

$$bb = bb^* = b^*b = b^*b^* = gb = gb^* = bg^* = b^*g^* = 0 \ .$$

[22] The discriminant of a quadratic form $ax^2 + 2bxy + cy^2$ is defined to be $b^2 - ac$. It is well known that (over fields) the quadratic forms is isotropic if and only the discriminant is a square.

We finish by taking $\varepsilon_1 = g - b$ and noting that as before, we have $\varepsilon_1^2 = \varepsilon_1$, $\varepsilon_1^*\varepsilon_2 = 0$ and $\varepsilon_1\varepsilon_1^* = gg^* = 0$. □

REMARK 4.9.13. Under the lemma's assumptions one can also show that for every $\varepsilon_0 \in \mathrm{E}(R/J_0)$ with $\varepsilon_0^*\varepsilon_0 = 0$, there exists $\varepsilon_1 \in \mathrm{E}(R/J_1)$ with $\varepsilon_1^*\varepsilon_1 = 0$, and $\varepsilon_0, \varepsilon_1$ has the same image in R/I.

For the next theorem, recall that a Hausdorff linearly topologized (abbrev.: LT) ring is called *complete* if $R = \varprojlim \{R/I\}_{I\in\mathcal{B}}$ for some local basis of ideals $\mathcal{B}$. In this case, this holds for any local basis consisting of ideals (e.g. for $\mathcal{I}_R$ – the set of all open ideals of R). See section 1.5 for additional details.

THEOREM 4.9.14. *Assume R is a complete Hausdorff LT ring admitting symmetric ideals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \dots$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ such that for all $n \in \mathbb{N}$:*

(a) *$J_n \subseteq I_n$ and $I_nJ_n + J_nI_n \subseteq J_{n+1}$.*
(b) *$\mathrm{Sym}(J_n) \subseteq \mathrm{Symd}(I_n) + J_{n+1}$.*

Also assume that one of the following holds:

(c) *Every open ideal contains I_n for some n (e.g. if $R = \varprojlim R/I_n$ as a topological ring).*
(c′) *Every open ideal contains J_n for some n and R is compact (i.e. R is an inverse limit of finite rings).*

Then for every isotropic $\varepsilon \in \mathrm{E}(R/J_1)$, there exists isotropic $e \in \mathrm{E}(R)$ such that e, ε has the same image in R/I_1.

PROOF. A repeated application of Lemma 4.9.12 (with I_n, J_n, J_{n+1} in place of I', J_0, J_1) yields idempotents $\varepsilon_n \in \mathrm{E}(R/J_n)$ with $\varepsilon_1 = \varepsilon$ such that $\varepsilon_n\varepsilon_n^* = \varepsilon_n^*\varepsilon_n = 0$, and $\varepsilon_{n+1}, \varepsilon_n$ has the same image in R/I_n.

Now, if (c) holds, then for every $U \in \mathcal{I}_R$ there is $n = n(U)$ such that $U \supseteq I_n$. Let e_U be the image of $\varepsilon_n \in R/J_n$ in R/U. Then e_U is independent of n and the elements $\{e_U\}_{U\in\mathcal{I}_R}$ are compatible with the standard maps $R/U \to R/V$ ($U, V \in \mathcal{I}_R$). As R is complete, there is $e \in R$ such that $e_U = e + U$ for all $U \in \mathcal{I}_R$ and it is routine to verify that e satisfies all the requirements (it is enough to check them modulo I_n for all $n \in \mathbb{N}$).

If (c′) holds, then take arbitrary elements $\{x_n\}_{n=1}^\infty$ with $\varepsilon_n = x_n + I_n$. Since R is compact, $\{x_n\}_{n=1}^\infty$ has a converging subsequence. Let e denote its limit. Then e is easily seen to satisfy all the requirements. □

REMARK 4.9.15. Assume that conditions (a),(b) of the previous theorem hold for $(R,*)$ and for the ideals $\{I_n\}_{n=1}^\infty$, $\{J_n\}_{n=1}^\infty$. Then for any $e \in \mathrm{E}(R)$ with $e = e^*$, conditions (a) and (b) also hold for eRe and $\{eI_ne\}_{n=1}^\infty$, $\{eJ_ne\}_{n=1}^\infty$. To see this, notice that $\mathrm{Sym}(eIe) = e\,\mathrm{Sym}(I)e$, $\mathrm{Symd}(eIe) = e\,\mathrm{Symd}(I)e$ for every symmetric ideal $I \trianglelefteq R$. The proof is straightforward.

COROLLARY 4.9.16. *Assume R is pro-semiprimary and $(R,*)$ is non-dyadic (e.g. when there exists $a \in \mathrm{Cent}(R)$ such that $a+a^* \in R^\times$). Then for every isotropic idempotent $\varepsilon \in \mathrm{E}(R/\mathrm{Jac}(R))$, there is isotropic $e \in \mathrm{E}(R)$ such that $\varepsilon = e + \mathrm{Jac}(R)$. In particular, $*$ is hyperbolic on $R \iff *$ is hyperbolic on $R/\mathrm{Jac}(R)$.*

PROOF. Take $I_n = J_n = \mathrm{Jac}(R)^{2^{n-1}}$ in the previous theorem. Conditions (a) is clear and condition (c) follows from Proposition 1.5.17. To see (b), note that since $(R,*)$ is non-dyadic, $\mathrm{Sym}(J_n) = \mathrm{Symd}(J_n) = \mathrm{Symd}(I_n) \subseteq \mathrm{Symd}(I_n) + J_{n+1}$. □

4.9.3. Lifting Congruences and Isometries. Recall that two elements $a, b \in R$ are called *($*$-)congruent*, denoted $a \sim_* b$, if there exists $u \in R^\times$ such that $u^*au = b$. Congruence is an equivalence relation. In addition, two $*$-invariant idempotents $e, e' \in \mathrm{E}(R)$ are called *isometric* if there exists $s \in e'Re$ such that

$s^*s = e$ and $ss^* = e'$. In this case, s is called an *isometry* from e to e'. In this subsection, we will prove that under certain assumptions, congruences between elements and isometries between idempotents can be lifted from R/J to R. As in the previous subsection, we begin with a counterexample showing that the assumption that J is idempotent lifting is insufficient for this.

EXAMPLE 4.9.17. Let $S = \mathbb{Z}_{\langle 3\rangle}$ be $\mathbb{Z}$ localized at $\langle 3\rangle$, let $R = \mathrm{M}_2(S)$ and let $J = \mathrm{Jac}(R) = 3R$. Let $*$ be the transpose involution and consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} .$$

Since $\overline{A} = \overline{B}$, $\overline{A} \sim_* \overline{B}$. However, if $A \sim_* B$, then there would be $P \in \mathrm{M}_2(S)$ such that $A = P^TBP$ implying $-1 = \det A = (\det P)^2(\det B) = 8(\det P)^2$. But this implies $-\frac{1}{8} \in (\mathbb{Q}^\times)^2$, a contradiction.

EXAMPLE 4.9.18. Let S, R, J be as in the previous example and let $A = [\begin{smallmatrix} 1 & 0 \\ 0 & 7 \end{smallmatrix}] \in \mathrm{M}_2(S)$. Define $b : S^2 \times S^2 \to S$ by $b(x, y) = x^TAy$. Then b is regular and its corresponding anti-endomorphism $*$ is given by $X^* = A^{-1}X^TA$. Let $e = e_{11}$ and $e' = e_{22}$, where $\{e_{ij}\}$ are the standard matrix units in R. Then $e = e^*$, $e' = e'^*$ and in $\overline{R} = R/J$ one has $\overline{e}_{21}^* = \overline{e}_{12}$ (because $\overline{A} = 1_{\overline{R}}$). Therefore, $\overline{e}_{21}^*\overline{e}_{21} = \overline{e}$ and $\overline{e}_{21}\overline{e}_{21}^* = \overline{e'}$, hence $\overline{e}_{21}$ is an isometry from $\overline{e}$ to $\overline{e'}$. However, e and e' are not isometric. Indeed, by Proposition 4.8.8, it is enough to verify that the restrictions of b to $\mathrm{im}(e) = S \times \{0\}$ and $\mathrm{im}(e') = \{0\} \times S$ are not isometric, which is clear because these forms have discriminants 1 and 7, respectively, and $1 \not\equiv 7 \mod (S^\times)^2$.

LEMMA 4.9.19. *Let $I, J_0, J_1 \subseteq \mathrm{Jac}(R)$ be symmetric ideals such that $I^2 \subseteq J_1 \subseteq J_0 \subseteq I$ and $\mathrm{Sym}(J_0) \subseteq \mathrm{Symd}(I) + J_1$. Let $e, e' \in \mathrm{E}(R)$ be $*$-invariant and let $a \in (eRe)^\times \cap \mathrm{Sym}(eRe, *)$, $b \in (e'Re')^\times \cap \mathrm{Sym}(e'Re', *)$. Assume that there exists $u_0 \in e'Re$ and $v_0 \in eRe'$ such that all the following equations hold modulo J_0:*

$$v_0u_0 = e, \qquad u_0v_0 = e', \qquad u_0au_0^* = b\ .$$

Then there exists $u_1 \in e'Re$ and $v_1 \in eRe'$ such that the following equations hold modulo J_1:

$$v_1u_1 = e, \qquad u_1v_1 = e', \qquad u_1au_1^* = b$$

and $u_0 + I = u_1 + I$, $v_0 + I = v_1 + I$.

PROOF. Let a' denote the inverse of a in eRe and note that $a'^* = a'$ since $a^* = a$. Observe that $u_0au_0^* - b \in \mathrm{Sym}(J_0)$. As $\mathrm{Sym}(J_0) \subseteq \mathrm{Symd}(I) + J_1$, there exists $c \in I$ such that $(c + c^*) + (u_0au_0^* - b) \in J_1$. By replacing c with $e'ce'$, we may assume $c = e'ce'$. Define $u_1 = u_0 + cv_0^*a'$. Calculating modulo J_1, we have:

$$\begin{aligned} u_1au_1^* &= (u_0 + cv_0^*a')a(u_0^* + a'v_0c^*) = u_0au_0^* + cv_0^*a'au_0^* + u_0aa'v_0c^* \\ &= u_0au_0^* + cv_0^*u_0^* + u_0v_0c^* = u_0au_0^* + c(u_0v_0)^* + u_0v_0c^* \\ &= u_0au_0^* + ce' + c^*e'^* = u_0au_0^* + c + c^* = b\ . \end{aligned}$$

Now let $v_1 = 2v_0 - v_0u_1v_0$. Then modulo J_1:

$$\begin{aligned} u_1v_1 &= 2u_1v_0 - u_1v_0u_1v_0 = u_1v_0 + (e' - u_1v_0)u_1v_0 \\ &= u_1v_0 + (e' - u_1v_0)e' = u_1v_0 + e' - u_1v_0 = e' \end{aligned}$$

(the third equality holds since $e' - u_1v_0 \in J_1$, $e' + J_0 = u_1v_0 + J_0$ and $J_0^2 \subseteq I^2 \subseteq J_1$) and in the same way, $v_1u_1 = e \mod J_1$. □

THEOREM 4.9.20. *Assume R is a complete Hausdorff LT ring, let $J_1 \supseteq J_2 \supseteq J_3 \supseteq \dots$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ be symmetric ideals such that:*

(a) *$J_n \subseteq I_n$ and $I_n^2 \subseteq J_{n+1}$,*[23]

[23]Notice that this is stronger than condition (a) of Theorem 4.9.14.

(b) $\mathrm{Sym}(J_n) \subseteq \mathrm{Symd}(I_n) + J_{n+1}$,

and at least one of the following holds:

(c) *every open ideal contains I_n for some n (e.g. if $R = \varprojlim R/I_n$ as a topological ring),*

(c′) *every open ideal contains J_n for some n and R is compact (i.e. R is an inverse limit of finite rings).*

Then the following hold:

(i) *Let $a, b \in \mathrm{Sym}(R) \cap R^\times$. Then $a \sim_* b$ if and only if $a + J_1 \sim_* b + J_1$. Furthermore, if $x + J_1 \in R/J_1^\times$ satisfies $b = x^*ax \mod J_1$, then exists $y \in R^\times$ such that $b = yay^*$ and x, y have the same image in R/I_1.*

(ii) *Let $e, e' \in \mathrm{E}(R)$ be $*$-invariant. Then e is is isometric to e' $\iff$ $e + J_1$ is isometric to $e' + J_1$. Furthermore, if $u + J$ is an isometry from e to e', then there exists an isometry v from e to e' such that u, v has the same image in R/I_1.*

Proof. We will prove (i) and (ii) in the same manner: If the assumptions of (i) hold, let $e = e' = 1$, $u = x^*$ and $v = (x^*)^{-1}$, and if the assumptions of (ii) hold let $a = e$, $b = e'$ and $v = u^*$. Then $u \in e'Re$, $v \in e'Ve$ and modulo J_1, $vu = e$, $uv = e'$ and $uau^* = b$.

Let $u_1 = u$ and $v_1 = v$. A repeated application of Lemma 4.9.19 (with $u_{n+1}, u_n, v_{n+1}, v_n, I_n, J_{n+1}, J_n$ in place of $u_1, u_0, v_1, v_0, I, J_1, J_0$) would yield $u_n, v_n \in R$ such that

$$v_n u_n = e, \qquad u_n v_n = e', \qquad u_n a u_n^* = b$$

modulo J_n and $u_{n+1} + I_n = u_n + I_n$, $v_{n+1} + I_n = v_n + I_n$ for all $n \in \mathbb{N}$. Arguing as in the proof of Theorem 4.9.14, this implies the existence of $\hat{u}, \hat{v} \in R$ such that $\hat{v}\hat{u} = e$, $\hat{u}\hat{v} = e'$, $\hat{u}a\hat{u}^* = b$ and $\hat{u} + I_1 = u + I_1$, $\hat{v} + I_1 = v + I_1$.

If $e = e' = 1$ as in (i), then this clearly implies $a \sim_* b$. If $a = e$ and $b = e'$ as in (ii), then $e' = b = \hat{u}a\hat{u}^* = \hat{u}e\hat{u}^* = \hat{u}\hat{u}^*$. Multiplying by $\hat{v}$ on the left yields $\hat{v} = \hat{u}^*$ and hence, $e = \hat{v}\hat{u} = \hat{u}^*\hat{u}$, implying $\hat{u}$ is an isometry from e to e'. □

Corollary 4.9.21. *Assume R is pro-semiprimary and $(R, *)$ is non-dyadic (e.g. if exists $a \in \mathrm{Cent}(R)$ such that $a + a^* \in R^\times$). Then:*

(i) *Let $a, b \in \mathrm{Sym}(R) \cap R^\times$. Then $a \sim_* b \iff a + \mathrm{Jac}(R) \sim_* b + \mathrm{Jac}(R)$.*

(ii) *Let $e, e \in \mathrm{E}(R)$ be $*$-invariant. Then e is isometric to e' $\iff$ $e + \mathrm{Jac}(R)$ is isometric to $e' + \mathrm{Jac}(R)$.*

Proof. This is similar to the proof of Corollary 4.9.16. □

Remark 4.9.22. The assumption $a, b \in \mathrm{Sym}(R) \cap R^\times$ in Corollary 4.9.21(i) (and also in Theorem 4.9.20(i)) is essential. For example, take any ring R with $2 \in R^\times$ and $\mathrm{Jac}(R)^2 = 0$. Then the conditions of Corollary 4.9.21 hold. However, for any non-congruent $a, b \in \mathrm{Sym}(\mathrm{Jac}(R))$ with $a \nsim_* b$, we have $a + \mathrm{Jac}(R) = 0 + \mathrm{Jac}(R) \sim_* 0 + \mathrm{Jac}(R) = b + \mathrm{Jac}(R)$. In addition, for every $x \in \mathrm{Jac}(R) \setminus \mathrm{Sym}(R)$ and $a \in \mathrm{Sym}(R) \cap R^\times$, the elements $a, a + x$ lie in $R^\times$ and are not congruent (since $a \in \mathrm{Sym}(R)$ and $a + x \notin \mathrm{Sym}(R)$), but a and $a + x$ have the same image in $R/\mathrm{Jac}(R)$, hence $a + \mathrm{Jac}(R) \sim_* (a + x) + \mathrm{Jac}(R)$.

We finish this section with the following open question, which has many consequences if answered in the positive. The motivation for the question is that *quasi-π_∞-regular* rings (see section 1.5) resemble Henselian valuation rings.

Question 3. *Do Theorem 4.9.14 and Theorem 4.9.20 hold under the weaker assumption that R is a quasi-π_∞-regular LT ring?*

4.10. Sufficient Conditions for W_b to Be Semiperfect

In the flavor of conditions (C2), (C2′) and (C2″) above, most of our results would apply to bilinear spaces (M, b, K) such that W_b is semiperfect or semiperfect and pro-semiprimary. This section is therefore devoted to supplying sufficient conditions for this to happen. Some general results of this type were already obtained in section 4.4. However, we will focus here on more explicit conditions.

Throughout, R is a ring and K is a double R-module. Recall that if R is an LT ring, then any right R-module M can be topologized by taking $\{MJ \mid J \in \mathcal{I}_R\}$ as a local basis. By saying that M is Hausdorff we mean it is Hausdorff w.r.t. this topology, which amounts to $\bigcap_{J\in\mathcal{I}_R} MJ = 0$ (see section 1.8 for a detailed discussion).

Proposition 4.10.1. *Let (M, b, K) be a bilinear space. Assume that one the following holds:*

(A0) *R is semiperfect π_∞-regular (e.g. right or left artinian) and M is f.p.*
(A1) *R is a semiperfect quasi-π_∞-regular LT ring, M is Hausdorff f.p. and $\bigcap_{J\in\mathcal{I}_R}(K_0J + K_1J) = 0$.*
(A2) *R is complete semilocal with Jacobson radical f.g. as a right ideal, M is f.p. and b is stable.*
(A3) *K has an anti-isomorphism κ, b is κ-symmetric and stable and M is a finite direct sum of LE-modules.*
(A4) *M is of finite length.*
(A5) End(M) *is right or left noetherian and complete semilocal and M is reflexive (e.g. if there exists a regular bilinear space (M, b', K)).*

Then W_b is semiperfect.

Proof. Throughout, let $W = \mathrm{End}(M)$. Note that (A0) is just a special case of (A1) (endow R with the discrete topology).

If (A1) holds, then End(M) is semiperfect quasi-π_∞-regular (w.r.t. τ_1^M) by Theorem 1.8.3(ii). By Proposition 4.4.5, W_b is a T-semi-invariant subring of $W \times W^{\mathrm{op}}$ and hence semiperfect quasi-π_∞-regular by Theorem 1.7.1.

If (A2) holds, then $W = \mathrm{End}(M)$ is semilocal complete by Corollary 1.8.5. Let α be the corresponding anti-*auto*morphism of b. Then $W_b \cong W^{\{\alpha^2\}}$ by Proposition 4.6.9. As any automorphism is continuous w.r.t. the Jacobson topology, α^2 is continuous. Therefore, $W^{\{\alpha^2\}}$ is a T-semi-invariant subring of W, hence semiperfect pro-semiprimary by Theorem 1.7.1.

If (A3) holds, then W is semiperfect. Since b is symmetric, its corresponding anti-*auto*morphism is an involution, hence by Proposition 4.6.9, $W_b = W$.

If (A4) holds, then W is semiprimary,[24] hence $W \times W^{\mathrm{op}}$ is semiprimary. By Proposition 4.4.1, W_b is a semi-invariant subring of the latter and hence semiprimary by Theorem 1.7.1.

If (A5) holds, then Proposition 4.4.3 implies W_b is complete semilocal. □

In the same manner one can prove:

Proposition 4.10.2. *Let (M, b, K) be a bilinear space and assume that one of the following holds:*

(B0) *R is semiprimary and M is f.p.*

[24] Sketch of the proof: Write $M = \bigoplus_{i=1}^t M_i$ with each M_i indecomposable. Let $\{e_i\}_{i=1}^t$ be the unital decomposition of W corresponding to $M = \bigoplus_{i=1}^t M_i$. By Proposition 1.2.3(ii), it is enough to verify that $e_iWe_i \cong \mathrm{End}(M_i)$ is semiprimary, and this follows from Fitting's Lemma (see [**80**, §2.9]).

(B1) *R is a semiperfect first-countable pro-semiprimary LT ring, M is Hausdorff f.p. and $\bigcap_{J\in\mathcal{I}_R}(K_0J+K_1J)=0$.*
(B2) *R is complete semilocal with Jacobson radical f.g. as a right ideal, M is f.p. and b is stable.*
(B3) *K has an anti-isomorphism κ, b is κ-symmetric and stable and* $\mathrm{End}(M)$ *is semiperfect pro-semiprimary (see Theorem 1.8.3 for conditions on M that imply this).*
(B4) *M is of finite length.*
(B5) $\mathrm{End}(M)$ *is right or left noetherian and complete semilocal and M is reflexive (e.g. if there exists a regular bilinear space (M,b',K)).*

Then W_b is semiperfect pro-semiprimary.

The next results present additional sufficient conditions that apply for systems of bilinear forms. Henceforth, I is a set and $K'=\prod_{i\in I}K$. Observe that any system of bilinear forms $\{(M,b_i,K)\}_{i\in I}$ corresponds to a bilinear form $b:M\times M\to K'$ (see section 4.5) which we denote by $\prod_{i\in I}b_i$.

PROPOSITION 4.10.3. *Assume R is an LT ring and M is Hausdorff and endow $W:=\mathrm{End}(M)$ with the topology τ_M defined in Proposition 4.4.5. Let $\{b_i\}_{i\in I}$ be a system of bilinear forms on M taking values in K such that there exists $i_0\in I$ for which b_{i_0} is regular. Let $b=\prod_{i\in I}b_i$ and assume that one of the following holds:*
(1) *W is complete semilocal.*
(2) *K has an anti-isomorphism.*

Then W_b is isomorphic as a ring to a T-semi-invariant subring of W.

PROOF. Let α be the corresponding anti-automorphism of b_{i_0} and let $h_i=(\mathrm{Ad}^r_{b_{i_0}})^{-1}\circ\mathrm{Ad}^r_{b_i}\in W$. Observe that $b_i(x,y)=(\mathrm{Ad}^r_{b_i}y)x=(\mathrm{Ad}^r_{b_{i_0}}h_iy)x=b_{i_0}(x,h_iy)$. As b_{i_0} is regular, b is reduced, hence we may identify W_b with its projection to W (Proposition 4.6.8(iii)). We claim that under this identification $W_b=W^{\{\alpha^2\}}\cap\mathrm{Cent}_W(\{h_i,h_i^{\alpha^{-1}}\,|\,i\in I\})$.

Indeed, if $w\in W^{\{\alpha^2\}}\cap\mathrm{Cent}_W(\{h_i,h_i^{\alpha^{-1}}\,|\,i\in I\})$, then $b_i(wx,y)=b_{i_0}(wx,h_iy)=b_{i_0}b(x,w^\alpha h_iy)=b_{i_0}(x,(h_i^{\alpha^{-1}}w)^\alpha y)=b_{i_0}(x,(wh_i^{\alpha^{-1}})^\alpha y)=b_{i_0}(x,h_iw^\alpha y)=b_i(x,w^\alpha y)$ and $b_i(w^\alpha x,y)=b_{i_0}(w^\alpha x,h_iy)=b_{i_0}(x,w^{\alpha\alpha}h_iy)=b_{i_0}(x,wh_iy)=b_{i_0}(x,h_iwy)=b_i(x,wy)$. Hence $b(wx,y)=b(x,w^\alpha y)$ and $b(x,wy)=b(w^\alpha x,y)$, implying $w\in W_b$. Conversely, if $w\in W_b$, then there exists $u\in W$ such that $b_i(wx,y)=b_i(x,uy)$ and $b_i(x,wy)=b_i(ux,y)$. Taking $i=i_0$ implies $w^\alpha=u$ and $u^\alpha=w$, hence $w^{\alpha\alpha}=w$ and $w\in W^{\{\alpha^2\}}$. Furthermore, we now have $b_{i_0}(x,h_iwy)=b_i(x,wy)=b_i(w^\alpha x,y)=b_{i_0}(w^\alpha x,h_iy)=b_{i_0}(x,wh_iy)$ and $b_{i_0}(h_i^{\alpha^{-1}}wx,y)=b_{i_0}(wx,h_iy)=b_i(wx,y)=b_i(x,w^\alpha y)=b_{i_0}(x,h_iw^\alpha y)=b_{i_0}(wh_i^{\alpha^{-1}}x,y)$, so $h_iw=wh_i$ and $h_i^{\alpha^{-1}}w=wh_i^{\alpha^{-1}}$, as required.

By Proposition 1.5.4(b), $\mathrm{Cent}_W(\{h_i,h_i^{\alpha^{-1}}\,|\,i\in I\})$ is a T-semi-invariant subring of W. As part (e) of that proposition implies that the intersection of two T-semi-invariant subrings of W is T-semi-invariant, we are done if we prove that $W^{\{\alpha^2\}}$ is T-semi-invariant. To show this, it is enough to verify α^2 is continuous. Indeed, if (1) holds, then any *auto*morphism γ of W is continuous since $\gamma(\mathrm{Jac}(W))\subseteq\mathrm{Jac}(W)$, implying $\gamma(\mathrm{Jac}(W)^n)\subseteq\mathrm{Jac}(W)^n$ (and $\{\mathrm{Jac}(W)^n\,|\,n\in\mathbb{N}\}$ is a basis for the topology on W). If (2) holds, then b_{i_0} has an invertible right κ-asymmetry λ, so α^2 is inner by Proposition 2.3.9(i), hence continuous. □

COROLLARY 4.10.4. *Let $\{b_i\}_{i\in I}$ be a system of bilinear forms on M taking values in K and assume that there exists $i_0\in I$ such that b_{i_0} is regular. Let $b=\prod_{i\in I}b_i$. If the following holds:*

(A6) *K has an anti-isomorphism, R is semiperfect quasi-π_∞-regular LT ring and M is Hausdorff f.p.,*

then W_b is semiperfect. If one of the following holds:

(B6) *K has an anti-isomorphism, R is first countable semiperfect and pro-semiprimary LT ring and M is Hausdorff f.p.,*

(B7) *R is complete semilocal with Jacobson radical f.g. as a right ideal and M is f.p.,*

then W_b is semiperfect and pro-semiprimary.

PROOF. In all cases, the previous proposition implies that W_b is isomorphic to a T-semi-invariant of $W := \mathrm{End}(M)$, once endowed with τ_M (or τ_1^M). If (A6) holds, then W is semiperfect quasi-π_∞-regular (by Theorem 1.8.3), hence W_b is semiperfect by Theorem 1.7.1. The same argument implies that when (B6) holds, W_b is semiperfect and pro-semiprimary. If (B7) holds, then W is complete semilocal by Corollary 1.8.5 and again we are though by Theorem 1.7.1. □

REMARK 4.10.5. The following important observation will be used implicitly throughout the following sections: If b is a bilinear form such that W_b is semiperfect (semiperfect and pro-semiprimary), then so is $W_{b'}$ for every summand b' of b. This follows from the fact that being semiperfect (semiperfect and pro-semiprimary) passes from a ring S to eSe for every $e \in \mathrm{E}(S)$ (Proposition 1.2.3).

4.11. Indecomposable Bilinear Forms

It is time to put the infrastructure we have developed into action. We begin with classifying the indecomposable bilinear spaces in various situations. To view this in the right context, note that classical regular indecomposable bilinear forms were classified implicitly in [**76**], [**75**] and explicitly in [**38**], [**93**] (the latter uses a different approach than the others). The degenerate case was treated in [**44**]. In addition, a characterization of indecomposable regular hermitian forms in a *Krull-Schmidt category with duality* appears in [**86**, Ch. 7, Th. 10.8]. In contrast to previous works, our exposition applies to all bilinear forms, regular or non-regular, and shows that both cases can be treated with the same tools.[25]

Let us set some general notation: (M, b, K) is a bilinear space over a ring R and $W = \mathrm{End}_R(M)$. We let $\overline{W}_b := W_b / \mathrm{Jac}(W_b)$ and $\overline{w} = w + \mathrm{Jac}(W_b)$ for all $w \in W_b$. The involution $\beta := \beta(b)$ induces an involution on $\overline{W}_b$ which we keep denoting by β.

Recall that a Kronecker module Z is *of bilinear type* if $Z \cong Z(b)$ for some bilinear form, *self-dual* if $Z \cong Z^*$ and *non-self-dual* otherwise. We let $[Z]$ denote the isomorphism class of Z.

THEOREM 4.11.1. *Keep the previous assumptions and assume (M, b, K) is indecomposable. If W_b is semiperfect (e.g. if one of the conditions (A0)-(A6),(B7) of the previous section holds), then exactly one of the following holds:*

(i) *$\overline{W}_b$ is a division ring. In this case $Z(b)$ is indecomposable.*

(ii) *$\overline{W}_b \cong D \times D^{\mathrm{op}}$ for some division ring D and β exchanges D and D^{op}. In this case $Z(b) \cong Z' \oplus Z'^*$ for a* non-self-dual *indecomposable Kronecker module Z'. The set $\{[Z'], [Z'^*]\}$ is uniquely determined by b.*

[25] The description to follow relates the indecomposable forms with the result of Osborn classifying rings with involution without non-trivial idempotents invariant under the involution (see section 3.8). Based on a conversation the author had with Manfred Knebusch several years ago, this connection also seems to be new.

(iii) $\overline{W}_b \cong \mathrm{M}_2(F)$ *for some field* F *and* β *(on* $\overline{W}_b$*) is a symplectic involution. In this case,* $Z(b) \cong Z' \oplus Z'^*$ *for some* self-dual *indecomposable Kronecker module* Z'*. If* $2 \in R^\times$*, then* Z' *is not of* bilinear type*. The isomorphism class* $[Z']$ *is uniquely determined by* b*.*

The proof requires the following two well-known lemmas.

Lemma 4.11.2. *Let* $\mathscr{A}$ *be an additive category in which all idempotents split. Let* $A \in \mathscr{A}$*,* $W = \mathrm{End}(A)$*,* $J = \mathrm{Jac}(W)$ *and* $e, e' \in \mathrm{E}(W)$*. Let* $B = eA := \mathrm{im}(e)$ *and* $B' = e'A := \mathrm{im}(e')$*. Then the following are equivalent:*

(a) $B \cong B'$.
(b) *There are* $x \in e'We$ *and* $y \in eWe'$ *such that* $yx = e$ *and* $xy = e'$*.*
(c) *The right* W*-modules* eW *and* $e'W$ *are isomorphic.*
(d) *The right* W/J*-modules* eW/eJ *and* $e'W/eJ$ *are isomorphic.*

Proof. The equivalences (a) $\iff$ (b) is routine (take x to be the isomorphism from B to B' and y to be its inverse; x, y can be understood as elements of $e'We, eWe'$, respectively). (b) $\iff$ (c) is just a special case of (a) $\iff$ (b) — take $\mathscr{A} = $ Mod-W and $A = W_W$. To see (b)$\Longrightarrow$(d) note that the equations $yx = e$, $xy = e'$ also hold modulo J. Applying (b)$\Longrightarrow$(c) with W/J in place of W now yields (d). Finally, (d)$\Longrightarrow$(c) follows from the fact that eW is the projective cover of eW/eJ (as W-modules); the uniqueness of the projective cover implies that any isomorphism $eW/eJ \to e'W/e'J$ lifts to an isomorphism $eW \to e'W$. □

Lemma 4.11.3. *Let* F *be a field, let* V *be a two-dimensional vector space, let* $b : V \times V \to F$ *be a regular classical alternating bilinear form and let* α *be its corresponding anti-endomorphism. Then for every non-trivial idempotent* $e \in \mathrm{E}(\mathrm{End}_F(V))$ *and* $w \in e\,\mathrm{End}_F(V)e^\alpha$*, one has* $w^\alpha = -w$*.*

Proof. Clearly e^α is a non-trivial idempotent. As V is 2-dimensional, this implies $\dim e^\alpha V = 1$. Write $e^\alpha V = vF$. Then for all $x, y \in V$, there are $s, t \in F$ such that $e^\alpha x = vs$ and $e^\alpha y = vt$. We now have

$$\begin{aligned} b(x, wy) &= b(x, ewe^\alpha y) = b(e^\alpha x, we^\alpha y) = b(vs, wvt) \\ &= b(v, wv)st = -b(wv, v)st = -b(v, w^\alpha v)st \\ &= -b(vs, w^\alpha vt) = -b(e^\alpha x, w^\alpha e^\alpha y) = -b(x, ew^\alpha e^\alpha y) \\ &= b(x, -w^\alpha y)\ , \end{aligned}$$

so $w^\alpha = -w$ (since b is right stable).[26] □

Proof of Theorem 4.11.1. By Proposition 4.8.1, W_b does not have non-trivial β-invariant idempotents. As W_b is semiperfect, Theorem 4.9.6 implies that $\overline{W}_b$ does not have non-trivial β-invariant idempotents. Therefore, by applying Theorem 3.8.2 to $\overline{W}_b$ (which is semisimple) we get that either (i) $\overline{W}_b$ is a division ring, (ii) $\overline{W}_b \cong D \times D^{\mathrm{op}}$ for some division ring D and β exchanges D and D^{op} or (iii) $\overline{W}_b \cong \mathrm{M}_2(F)$ for some field F and β (on $\overline{W}_b$) is a symplectic involution.

If (i) holds, then we are clearly through, so suppose (ii) or (iii) hold. In both cases, there are primitive idempotents $\varepsilon, \varepsilon' \in \overline{W}_b$ with $\varepsilon + \varepsilon' = 1$. As W_b is semiperfect, $\varepsilon, \varepsilon'$ can be lifted to primitive idempotents $e, e' \in W_b$ with $e + e' = 1$. Define $Z = eZ(b)$ and $Z' = e'Z(b)$. Then $Z(b) = Z \oplus Z'$. Since e, e' are primitive and $\mathrm{End}(Z(b)) = W_b$ is semiperfect, $Z(b) = Z \oplus Z'$ is Krull-Schmidt decomposition, i.e. it is the only decomposition of $Z(b)$ up to isomorphism of terms and reordering (see Theorem 1.1.1). As $Z(b) = Z(b)^* = Z^* \oplus Z'^*$ is another such decomposition, either $Z' \cong Z^*$ (and then $Z \cong Z'^*$) or $Z \cong Z^*$ and $Z' \cong Z'^*$.

[26] Using similar ideas one can also prove $e + e^\alpha = 1$.

We now apply the equivalence (a) $\iff$ (d) of Lemma 4.11.2 to e, e' of the previous paragraph with $\mathscr{A}$ being the category of Kronecker modules and $A = Z(b)$. This equivalence now reads as $Z = eZ(b) \cong e'Z(b) = Z' \iff \varepsilon\overline{W}_b \cong \varepsilon'\overline{W}_b$ as right $\overline{W}_b$-modules. When (ii) holds, it is well known that $\varepsilon\overline{W}_b \ncong \varepsilon'\overline{W}_b$ (e.g. since these modules have different annihilators), hence $Z \ncong Z'$. On the other hand, in case (iii) holds, $\varepsilon\overline{W}_b \cong \varepsilon'\overline{W}_b$, so $Z \cong Z'$, implying $Z(b) \cong Z' \oplus Z'$ and $Z' \cong Z'^*$ (since either $Z \cong Z'^*$ or $Z' \cong Z'^*$). It is therefore left to verify that $Z' \ncong Z'^*$ when (ii) holds (which would imply $Z \cong Z'^*$ and $Z(b) \cong Z' \oplus Z'^*$) and that Z' is not of bilinear type when (iii) holds and $2 \in R^\times$.

Write $e' = (e_1, e_2^{\mathrm{op}})$, $h_0 = \mathrm{Ad}_b^\ell$, $h_1 = \mathrm{Ad}_b^r$ and identify $M^{[i]}$ with $(e_2M)^{[i]} \oplus ((1-e_2)M)^{[i]}$. Then $Z' = e'Z(b) = (e_1M, h_0|_{e_1M}, h_1|_{e_1M}, e_2M)$ and by Corollary 2.2.5, $Z'^* = (e_2M, h_0|_{e_2M}, h_1|_{e_2M}, e_1M)$. Assume that there is an isomorphism $(\sigma, \tau) : Z' \to Z'^*$. We consider σ, τ as elements of e_2We_1 (where $W = \mathrm{End}(M)$), which we identify with $\mathrm{Hom}(e_1M, e_2M)$. Then $\tau^{[1]} \circ h_1|_{e_1M} = h_1|_{e_2M} \circ \sigma$, hence $b(\tau x, y) = b(x, \sigma y)$ for all $x, y \in e_1M$. Recalling (17) and the fact that $(e_1, e_2^{\mathrm{op}}) \in W_b$, we get that $b(\tau x, y) = b(e_2\tau e_1 x, y) = b(\tau e_1 x, e_1 y) = b(e_1 x, \sigma e_1 y) = b(x, e_2\sigma e_1 y) = b(x, \sigma y)$ and similarly $b(x, \tau y) = b(\sigma x, y)$. Thus, $w := (\sigma, \tau^{\mathrm{op}}) \in W_b$. Now let $u := (\sigma', \tau'^{\mathrm{op}}) : Z'^* \to Z'$ be the inverse of $(\sigma, \tau^{\mathrm{op}})$, i.e. $\sigma', \tau' \in e_1We_2 = \mathrm{Hom}(e_2M, e_1M)$, $\sigma'\sigma = \tau'\tau = e_1$ and $\sigma\sigma' = \tau\tau' = e_2$. Then the same argument would imply that $u = (\sigma', \tau'^{\mathrm{op}}) \in W_b$. Now, $uw = (\sigma'\sigma, (\tau\tau')^{\mathrm{op}}) = (e_1, e_2^{\mathrm{op}}) = e'$ and similarly, $wu = (e_2, e_1^{\mathrm{op}}) = e'^\beta$. Modulo $\mathrm{Jac}(W_b)$ these equations become $\overline{uw} = \varepsilon'$ and $\overline{wu} = \varepsilon'^\beta$.

Assume (ii) holds. Then ε'^β is necessarily ε. Therefore, the previous equations and Lemma 4.11.2 imply that $\varepsilon\overline{W}_b \cong \varepsilon'\overline{W}$, in contradiction to what shown above. Thus, $Z' \ncong Z'^*$.

Finally, assume by contradiction that (iii) holds, $2 \in R^\times$ and Z' is of bilinear type. Then by Proposition 4.3.4, we can take $\sigma = \tau$ in the above computation, thus obtaining $w^\beta = (\sigma, \sigma^{\mathrm{op}}) = w$. However, $w = (\sigma, \tau^{\mathrm{op}}) = (e_1\sigma e_2, (e_1\tau e_2)^{\mathrm{op}}) = e'we'^\beta$, so by Lemma 4.11.3, $w^\beta = -w$. As $2 \in R^\times$, this implies $w = 0$, which is absurd. Thus, Z' is not of bilinear type. □

For brevity, an indecomposable bilinear space whose Kronecker module has a semiperfect endomorphism ring will be called a *block*. The previous theorem asserts that there are essentially three families of blocks. A block satisfying conditions (i), (ii) or (iii) we will be said to be of *type-I, -II or -III*, respectively. In case $2 \in R^\times$, these types can also be characterized as follows: If (M, b, K) is a block, then

- (M, b, K) is of type-I if $Z(b)$ is indecomposable,
- (M, b, K) is of type-II if $Z(b) \cong Z' \oplus Z'^*$ with indecomposable non-self-dual Z',
- (M, b, K) is of type-III if $Z(b) \cong Z' \oplus Z'^*$ with indecomposable self-dual non-bilinear Z'.

EXAMPLE 4.11.4. The previous characterization type-III blocks fails when $2 \notin R^\times$. That is, there are type-III blocks (M, b, K) such that $Z(b) \cong Z' \oplus Z'^*$ and Z' is of bilinear type. For example, let F be a field of characteristic two and let $b : F^2 \times F^2 \to F$ be the classical alternating bilinear form defined by $b(x, y) = x^T \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] y$. It is well known that b is indecomposable and it is straightforward to verify that $W_b \cong \overline{W}_b \cong \mathrm{M}_2(F)$, hence b is of type-III. Observe that $Z(b) = (x_1F, h_0|_{x_1F}, h_1|_{x_1F}, x_2F) \oplus (x_2F, h_0|_{x_2F}, h_1|_{x_2F}, x_1F)$ with $\{x_1, x_2\}$ being the standard basis of F^2 and $h_0 = \mathrm{Ad}_b^\ell$, $h_1 = \mathrm{Ad}_b^r$. Since b is symmetric and $(F^2)^{[0]} = (F^2)^{[1]} = (F^2)^* := \mathrm{Hom}_F(F^2, F)$, we actually have $h_0 = h_1$ and $h_0(x_i) = x_{3-i}^*$ where $\{x_1^*, x_2^*\}$ is the dual basis of $\{x_1, x_2\}$. It is now easy to verify that the specified summands of $Z(b)$ are isomorphic to $Z(b')$ where $b' : F \times F \to F$ is defined by $b'(x,y)=xy$. (For example, the isomorphism from $(x_1F, h_0|_{x_1F}, h_1|_{x_1F}, x_2F)$ to $Z(b')$ is given by $(x_1a\mapsto 1_Fa, (1_Fa\mapsto x_2a)^{\mathrm{op}})$.) Thus, $Z(b)\cong Z(b')\oplus Z(b')^*$ with $Z(b')$ clearly being bilinear.

EXAMPLE 4.11.5. Let F be a field of characteristic not 2 and let (V,b) be a classical regular bilinear space over F. Let λ denote the asymmetry of b and let f_λ and m_λ be its characteristic and minimal polynomial, respectively. Recall from section 4.1 that for every monic polynomial $f\in F[x]$ with $f(0)\neq 0$, we define $f^*=f(0)^{-1}x^{\deg f}f(x^{-1})$. Then in [**38**] (my M.Sc. thesis) I proved that

(i) b is a type-I block $\iff$ there exists a prime power $p(x)\in F[x]$ with $p=p^*$ such that $m_\lambda=f_\lambda=p$ and $p(x)\neq(x-(-1)^n)^n$ for all $n\in\mathbb{N}$.
(ii) b is a type-II block $\iff$ there exists prime power $q(x)\in F[x]$ with $q\neq q^*$ such that $m_\lambda=f_\lambda=qq^*$.
(iii) b is a type-III block $\iff$ there exists $n\in\mathbb{N}$ such that $m_\lambda^2=f_\lambda=(x-(-1)^n)^{2n}$.

In particular, whether b is a block can be determined from the conjugacy class of λ. (This should be expected as Corllary 4.6.5(ii) implies that the asymmetry determines the Kronecker module up to isomorphism.)

We now show that when $2\in R^\times$ and W_b is semiperfect and pro-semiprimary (w.r.t. some linear ring topology), type-II and type-III blocks are hyperbolic and determined up isometry by their Kronecker-modules (compare with Theorem 4.1.3).

THEOREM 4.11.6. *Assume $2\in R^\times$ and let (M,b,K) be a type-II or type-III block such that W_b is semiperfect pro-semiprimary (e.g. if one of the conditions (B0)-(B6) is satisfied). Write $Z(b)\cong Z'\oplus Z'^*$. Then b is hyperbolic and $b\cong b_{Z'}$. In particular, b is determined up to isometry by $[Z(b)]$.*

PROOF. By Theorem 4.11.1, either $\overline{W}_b\cong D\times D^{\mathrm{op}}$ for some division ring D with β exchanging D and D^{op} or $\overline{W}_b\cong \mathrm{M}_2(F)$ for some field F with β being a symplectic involution. In both cases, it is easy to see that β is hyperbolic on $\overline{W}_b$, i.e. there exists $\varepsilon\in\mathrm{E}(\overline{W}_b)$ with $\varepsilon+\varepsilon'=1$,[27] hence by Corollary 4.9.16, β is hyperbolic on W_b. Therefore, by Proposition 4.7.8, b is hyperbolic, i.e. there are totally isotropic $M_1,M_2\leq M$ such that $M=M_1\oplus M_2$. On the other hand, $b_{Z'}$ is also hyperbolic, so we can write $M=M_1'\oplus M_2'$ with $b_{Z'}(M_1',M_1')=b_{Z'}(M_2',M_2')=0$. Let Z_1,Z_1',Z_2,Z_2' be as in Proposition 4.7.5 (with $b_{Z'}$ in place of b'). By part (ii) of that proposition, it is enough to prove $Z_1\cong Z_1'$ or $Z_1\cong Z_2'$ (in the latter case replace Z_1' and Z_2'). However, this follows from the Krull-Schmidt Theorem since $Z_1\oplus Z_2=Z(b)\cong Z'\oplus Z'^*\cong Z(b_{Z'})=Z_1'\oplus Z_2'$. □

We finish with the following observation: Assume $2\in R^\times$. In order to find the Kronecker modules of the indecomposable blocks, it is enough to (1) find all indecomposables Kronecker modules with semiperfect endomorphism ring and (2) determine for each indecomposable whether it is bilinear, self-dual and non-bilinear or non-self-dual. After this is accomplished, the Kronecker modules of the blocks are given (up to isomorphism) by:

- $[Z]$ where Z is indecomposable of bilinear type. (In this case, $Z\cong Z(b)$ for a type-I block b.)
- $[Z\oplus Z^*]$ where Z is a non-self-dual indecomposable. (In this case, $Z\oplus Z^*\cong Z(b)$ for a type-II block b.)
- $[Z\oplus Z^*]$ where Z is a self-dual non-bilinear indecomposable. (In this case, $Z\oplus Z^*\cong Z(b)$ for a type-III block b.)

[27] In fact, this holds for all non-trivial idempotents.

After finding the indecomposables in Kr(Mod-R), the hardest part in this process is usually determining which self-dual indecomposables are bilinear.

We will now apply this principle to get an easy proof of Gabriel's classification of classical indecomposable *degenerate* bilinear spaces over a field (see [**44**] or the end of section 4.1). Strictly speaking, Gabriel proved that any such bilinear space is hyperbolic and determined up to isomorphism by its Kronecker module; his proof is based on a careful analysis of the different families of indecomposable Kronekcer modules over a field. We will prove a slightly more accurate statement:

COROLLARY 4.11.7. *Let F be a field of characteristic not 2. Any classical degenerate indecomposable bilinear form over F is a block of type-II. Furthermore, it is hyperbolic and thus determined up to isometry by it by its Kronecker module.*

PROOF. In the proof we will use the classical notion of Kronecker modules over a field, namely, quartets (U, f_0, f_1, V) such that U, V are f.d. vector spaces and $f_0, f_1 \in \mathrm{Hom}(U, V)$. This is allowed by Example 4.3.2.

Let $Z = (U, f_0, f_1, V)$ be an indecomposable Kronecker-module. To see the first assertion, it is enough to prove that if Z is self-dual, then f_0 and f_1 are bijective (as this wouldn't allow degenerate blocks of types I or III). The indecomposable Kronecker modules over F are well known (e.g. see [**44**]) and the ones with $U \cong V$ (which is required for $Z \cong Z^*$) are of the form

$$(F^n, 1, A, F^n) \qquad \text{or} \qquad (F^n, 1, J_n, F^n), (F^n, J_n, 1, F^n)$$

where A is any indecomposable invertible linear transformation and J_n is a 0-diagonal Jordan block. As $(F^n, 1, J_n, F^n) \ncong (F^n, 1, J_n, F^n)^* = (F^n, J_n^T, 1, F^n)$, the Kronecker modules $(F^n, 1, J_n, F^n)$ and $(F^n, J_n, 1, F^n)$ are not self dual. Thus, $Z \cong (F^n, 1, A, F^n)$, implying f_0 and f_1 are invertible. The other claims follow from Theorem 4.11.6 since (B4) holds (i.e. the base module is of finite length). □

QUESTION 4. *Can the previous corollary be generalized to degenerate forms over other rings?*

4.12. Isotypes

In this section we define and study isotypes. We keep the notation $\overline{W}_b := W_b/\operatorname{Jac}(W_b)$ of the previous section.

DEFINITION 4.12.1. *Let Z be an indecomposable Kronecker module and let $\zeta = \{[Z], [Z^*]\}$. A bilinear space (M, b, K) is called a ζ-*isotype *if $Z(b)$ is of* type-ζ*, i.e. $Z(b)$ is isomorphic to a direct sum of copies of Z and Z^*. If moreover (M, b, K) is a block, then it called a ζ-*block.

By Theorem 4.11.1, every block is a ζ-isotype for a uniquely determined ζ. Moreover, a bilinear form b for which W_b is semiperfect is a ζ-isotype if and only if it is a sum of ζ-blocks (as implied by Theorem 4.11.1 and the Krull-Schmidt Theorem).

Henceforth, for every Kronecker module Z, let Σ_Z denote the isomorphism classes of the indecomposable summands of Z and set

$$\overline{\Sigma}_Z = \{\{[Z'], [Z'^*]\} \mid [Z'] \in \Sigma_Z\}$$

(compare with the notation Σ_M, $\overline{\Sigma}_M$ of section 4.2).

THEOREM 4.12.2. *Let (M, b, K) be a bilinear space.*

(i) *If W_b is semiperfect, then $b = \perp_{\zeta \in \overline{\Sigma}_{Z(b)}} b_\zeta$ where each b_ζ is a ζ-isotype.*
(ii) *If moreover W_b is pro-semiprimary and $2 \in W_b^\times$, then the isotypes b_ζ of (i) are determined up to isometry by b and ζ.*

The proof requires the following two lemmas.

Lemma 4.12.3. *Let $\mathscr{A}$ be an additive category and let $A_1, \dots, A_t \in \mathscr{A}$ be pair-wise non-isomorphic objects with local endomorphism ring. Let $W_i = \mathrm{End}(A_i)$, $J_i = \mathrm{Jac}(W_i)$ and $A = \bigoplus_{i=1}^t A_i^{n_i}$ (where $n_1, \dots, n_t \in \mathbb{N}$). Then*

$$\mathrm{Jac}(\mathrm{End}(A)) = \sum_{i=1}^{t} \mathrm{M}_n(J_i) + \sum_{i \neq j} \mathrm{Hom}(A_i^{n_i}, A_j^{n_j})$$

(we identify $\mathrm{M}_{n_i}(W_i)$ and $\mathrm{Hom}(A_i^{n_i}, A_j^{n_j})$ as subsets of $\mathrm{End}(A)$ in the standard way). In particular,

$$\mathrm{End}(A)/\mathrm{Jac}(\mathrm{End}(A)) \cong \prod_{i=1}^{t} \mathrm{M}_{n_i}(W_i/J_i)$$

Proof. This is a well-known argument. Let $W = \mathrm{End}(A)$. Since $\mathrm{Jac}(eWe) = e\,\mathrm{Jac}(W)e = \mathrm{Jac}(W) \cap eWe$ for all $e \in \mathrm{E}(W)$, we can reduce the proof to showing that $\mathrm{Jac}(\mathrm{M}_{n_i}(\mathrm{End}(W_i))) = \mathrm{M}_{n_i}(\mathrm{Jac}(W_i))$, which is well known, and $\mathrm{Hom}(A_i, A_j) \subseteq \mathrm{Jac}(\mathrm{End}(A_i \oplus A_j))$ for any two distinct $1 \le i, j \le t$, which we verify below. This would imply $\mathrm{Jac}(W) \supseteq \sum_{i=1}^t \mathrm{M}_n(J_i) + \sum_{i \neq j} \mathrm{Hom}(A_i^{n_i}, A_j^{n_j})$ and the reverse inclusion follows since a quotient by the r.h.s. (which is an ideal by the argument below), yields a semisimple ring.

It is thus left to verify that $\mathrm{Hom}(A_i, A_j) \subseteq \mathrm{Jac}(\mathrm{End}(A_i \oplus A_j))$ for any two distinct $1 \le i, j \le t$. W.l.o.g. $i = 1$ and $j = 2$. Let $U = \mathrm{End}(A_1 \oplus A_2)$ and let e_1, e_2 be the projections from $A_1 \oplus A_2$ to A_1, A_2, respectively. Then $\mathrm{Hom}(A_1, A_2) = e_2Ue_1$, hence we need to prove that $w \in \mathrm{Jac}(U)$ for all $w \in e_2Ue_1$. Let $u \in U$. It is enough to prove that $1 + uw$ is invertible. Indeed, $e_1uw \in e_1Ue_1$. We claim that $e_1uw \notin (e_1Ue_1)^\times$. Assume by contradiction that $e_1uw \in (e_1Ue_1)^\times$. Then $w \neq 0$ and there exists $u' \in e_1Ue_1$ such that $u'uw = e_1$. This implies $(wu'ue_2)^2 = w(u'ue_2w)u'ue_2 = wu'ue_2$. As $wu'ue_2 \in e_2Ue_2 = \mathrm{End}(A_2)$ and $\mathrm{End}(A_2)$ is local, $wu'ue_2 \in \{e_2, 0\}$. Since $0 \neq w = (wu'ue_2)w$, necessarily $wu'ue_2 = e_2$ and it follows that $u'ue_2$ is an isomorphism from A_2 to A_1 (its inverse is w), a contradiction to the assumption $A_1 \not\cong A_2$. Thus, e_1uw is not a unit in $e_1Ue_1 = \mathrm{End}(A_1)$. As the latter is local, $e_1 + e_1uw$ is invertible in e_1Ue_1. Let a be its inverse. Then $a + e_2 \in U^\times$ and $(a+e_2)(1+uw) = ae_1(1+uw) + e_2(1+uw) = e_1 + e_2 + e_2uw = 1 + e_2uw$. As $(1+e_2uw)^{-1} = (1 - e_2uw)$ (straightforward), it follows that $1 + uw$ is invertible in U, as required. (Notice that this argument also implies that $\mathrm{Hom}(A_1, A_2) \cdot \mathrm{Hom}(A_2, A_1) \subseteq e_2\,\mathrm{Jac}(U)e_2 = \mathrm{Jac}(A_2)$ which justifies the claim in parenthesis at the end of the previous paragraph.) □

Lemma 4.12.4. *Let $(R, *)$ be a ring with involution that does not contain an infinite set of orthogonal idempotents. Then there exists* unique *unital decomposition $\{e_i\}_{i=1}^t$ of R with the following properties: (1) each e_i is $*$-invariant and central in R and (2) e_iRe_i does not contain non-trivial central $*$-invariant idempotents.*

Proof. Let S be the set of all non-zero central $*$-invariant idempotents satisfying (2). It is enough to prove that S is a unital decomposition of R. First, note that $S \neq \phi$ since R does not contain an infinite sum of orthogonal idempotents. Next, let $e, f \in S$ be distinct. We claim that $ef = fe = 0$. Indeed, since e, f are central and $*$-invariant, ef is a central $*$-invariant idempotent in eRe. Thus, $ef = 0$ or $ef = e$. In the latter case, $0 \neq e \in fRf$. As e is central and β-invariant, we must have $e = f$, a contradiction. Therefore, $ef = fe = 0$ which means that S consists of pair-wise orthogonal idempotents. The assumptions on R imply that S is finite and hence $h := 1 - \sum_{e \in S} e$ is a $*$-invariant cental idempotent which is orthogonal to all elements of S. If $h \neq 0$, take central β-invariant $0 \neq h' \in \mathrm{E}(hRh)$ satisfying

(2). Then $h' \in S$ (by definition) and $0 \neq h' = hh' = 0$ (since h is orthogonal to all elements of S), a contradiction. Thus, necessarily $h = 0$ and we are through. □

Proof of Theorem 4.12.2. (i) By Theorem 4.11.1, (M, b, K) is an orthogonal sum of blocks $\{(M_i, b_i, K)\}_{i=1}^t$. Take (M_ζ, b_ζ, K) to be the orthogonal sum of the ζ-blocks in $\{(M_i, b_i, K)\}_{i=1}^t$.

(ii) Since $\mathrm{Jac}(W_b)$ is idempotent lifting, $\overline{W}_b$ does not contain an infinite set of orthogonal idempotents. Therefore, by Lemma 4.12.4(i), there exists a *unique* central β-invariant unital decomposition $\{\varepsilon_i\}_{i=1}^t$ such that $\varepsilon_i \overline{W}_b \varepsilon_i$ does not contain non-trivial central β-invariant idempotents. Now let $\{e_\zeta\}_{\zeta \in \overline{\Sigma}_{Z(b)}}$ be the β-invariant unital decomposition of W_b corresponding to $b = \perp_{\zeta \in \overline{\Sigma}_{Z(b)}} b_\zeta$ (see Proposition 4.8.1). We claim that $\{\overline{e}_\zeta \,|\, \zeta \in \overline{\Sigma}_{Z(b)}\} = \{\varepsilon_i \,|\, 1 \leq i \leq t\}$. By the uniqueness of the set $\{\varepsilon_i \,|\, 1 \leq i \leq t\}$, it is enough to check that for any ζ, $\overline{e}_\zeta$ is β-invariant, central and $\overline{e}_\zeta \overline{W}_b \overline{e}_\zeta$ does not contain non-trivial central β-invariant idempotents.

Let $\zeta = \{[Z], [Z^*]\} \in \overline{\Sigma}_{Z(b)}$. That $\overline{e}_\zeta$ is β-invariant is clear. To see it is central, let $Z' := \sum_{\zeta \neq \zeta' \in \overline{\Sigma}_{Z(b)}} Z(b_{\zeta'})$. By definition, $Z(b) = Z(b_\zeta) \oplus Z'$ and $[Z], [Z^*] \notin \Sigma_{Z'}$. Therefore, by Lemma 4.12.3, e_ζ, which is the projection from $Z(b)$ to $Z(b_\zeta)$ with kernel Z', becomes central in $\overline{W}_b$.

We now show that $\overline{e}_\zeta \overline{W}_b \overline{e}_\zeta$ does not contain non-trivial central β-invariant idempotents. If Z is self-dual, then $Z(b_\zeta) \cong Z^n$ for some $n \in \mathbb{N}$. Thus, $\overline{e}_\zeta \overline{W}_b \overline{e}_\zeta$ is simple artinian by Lemma 4.12.3 and thus has no non-trivial central idempotents, as required. If Z is non-self-dual, then $Z(b_\zeta) \cong Z^n \oplus (Z^*)^m$ for some $n, m \in \mathbb{N}$ and $Z \not\cong Z^*$ (in fact, $n = m$ since $Z(b_\zeta)^* = Z(b_\zeta)$). Let f_1, f_2 be the projections from $Z(b_\zeta)$ to $Z^n, (Z^*)^m$, respectively. Then by Lemma 4.12.3, then only non-trivial central idempotents in $\overline{e}_\zeta \overline{W}_b \overline{e}_\zeta$ are $\overline{f}_1, \overline{f}_2$. It is enough to show that they are not β-invariant. Indeed, assume by contradiction that $\overline{f}_1$ is β-invariant. Then by Theorem 4.9.6, there exists a β-invariant $f' \in \mathrm{E}(e_\zeta W_b e_\zeta)$ such that $\overline{f}' = \overline{f}_1$. Let $b = b_1 \perp b_2$ be the decomposition corresponding the unital decomposition $\{f', 1 - f'\}$ (Proposition 4.8.1). Then $f' Z(b) = Z(b_1)$, hence $f' Z(b)$ is self-dual. However, by Lemma 4.11.2, $\overline{f}' = \overline{f}_1$ implies $f' Z(b) \cong f_1 Z(b) = Z^n$ which is not self-dual by the Krull-Schmidt Theorem (as $Z \not\cong Z^*$), a contradiction. That $\overline{f}_2$ is not β-symmetric follows by symmetry.

Now let $b = \perp_{\zeta \in \overline{\Sigma}_{Z(b)}} b'_\zeta$ be another decomposition of b into isotypes and let $\{e'_\zeta\}_{\zeta \in \overline{\Sigma}_{Z(b)}}$ be the corresponding β-invariant unital decomposition of W_b. By what we have just shown, $\{\overline{e}'_\zeta \,|\, \zeta \in \overline{\Sigma}_{Z(b)}\} = \{\varepsilon_i \,|\, 1 \leq i \leq t\} = \{\overline{e}_\zeta \,|\, \zeta \in \overline{\Sigma}_{Z(b)}\}$. Since $e_\zeta Z(b)$ and $e'_{\zeta'} Z(b)$ cannot be isomorphic for distinct ζ, ζ' (Krull-Schmidt Theorem), $\overline{e}_\zeta \overline{W}_b \not\cong \overline{e}'_{\zeta'} \overline{W}_b$ as right $\overline{W}_b$-modules (Lemma 4.11.2). In particular, $\overline{e}_\zeta$ must be distinct from $\overline{e}'_{\zeta'}$, which forces $\overline{e}_\zeta = \overline{e}'_\zeta$ for all $\zeta \in \overline{\Sigma}_{Z(b)}$. Now, $\overline{e}_\zeta$ is isometric to $\overline{e}'_\zeta$ (as they are equal), hence e_ζ is isometric to e'_ζ (Theorem 4.9.20), so $b_\zeta \cong b'_\zeta$ by Proposition 4.8.8. □

Let (M, b, K) be a bilinear space and let $\zeta = \{[Z], [Z^*]\}$ with Z indecomposable. Assume that $2 \in W_b^\times$ and W_b is semiperfect and pro-semiprimary. We define (M_ζ, b_ζ, K) as in Theorem 4.12.2 in case $\zeta \in \overline{\Sigma}_{Z(b)}$ and $(M_\zeta, b_\zeta, K) = (0, 0, K)$ otherwise. The bilinear space (M_ζ, b_ζ, K) is then uniquely determined up to isometry and the map $b \mapsto b_\zeta$ is additive in sense that

$$(b_1 \perp b_2)_\zeta \cong (b_1)_\zeta \perp (b_2)_\zeta$$

(whenever $W_{b_1 \perp b_2}$ is a semiperfect pro-semiprimary ring in which 2 is a unit).

Remark 4.12.5. One can define isotypes without assuming W_b is semiperfect by defining them to be bilinear spaces (M, b, K) in which $\overline{W}_b$ does not contain non-trivial central β-invariant idempotents. It can then be shown that if $\mathrm{Jac}(W_b)$ is idempotent lifting and W_b does not contain an infinite set of orthogonal idempotents, then b is an orthogonal sum of isotypes $b = \perp_{i=1}^{k} b_i$ and $b_i \perp b_j$ is not an isotype for $i \neq j$. If moreover W_b is complete in the Jacobson topology (e.g. if $\mathrm{Jac}(W_b)$ is nilpotent), then the isotypes $\{b_i\}_{i=1}^{k}$ are determined up to isometry and reordering. However, there is no obvious way to form families of isotypes as we did above. (It is possible to say that two isotypes b and b' are of the same kind if $b \perp b'$ is also an isotype, but the author does not know if this is an equivalence relation in general. In addition, it is not clear what are the equivalence classes. Nevertheless, we suspect that if R is an algebra over $\mathbb{Z}$ which is f.g. as a $\mathbb{Z}$-module, then some positive results can be shown.)

REMARK 4.12.6. Consider regular classical bilinear forms over a field F. Then the Kronecker module of a bilinear forms is determined by the conjugacy class of its asymmetry (Corollary 4.6.5) and the conjugacy class is determined by the Jordan decomposition (or, equivalently, the canonical rational form). Using this, one can see that the isotypes we have defined in this section agree with the definition given in section 4.1 for classical regular bilinear forms. However, now we also have degenerate isotypes.

4.13. Isometry and Cancelation

In this section, we show how to reduce the isometry problem of bilinear forms (M, b, K) for which W_b is semiperfect pro-semiprimary with $2 \in W_b^{\times}$ to isometry of hermitian forms over division rings. This is then used to prove Witt's Cancellation Theorem.

Recall that Theorem 4.12.2 reduces the isometry problem of bilinear forms b for which is W_b is semiperfect pro-semiprimary with $2 \in W_b^{\times}$ to isometry of ζ-isotypes. However, if $\zeta = \{[Z], [Z^*]\}$ for Z which is *not of bilinear type*, then the ζ-blocks are necessarily of type-II or type-III. Therefore, in this case any ζ-isotype (M, b, K) is hyperbolic and determined up to isometry by $[Z(b)]$, as implied by Theorem 4.11.6. We may thus restrict our attention to ζ-isotypes with $\zeta = \{[Z], [Z^*]\}$ and Z of bilinear type (which implies $\zeta = \{[Z]\}$).

Fix an indecomposable Kronecker module Z of bilinear type with $\mathrm{End}(Z)$ being local and pro-semiprimary. W.l.o.g. we may assume $Z = Z(b_0)$ for some bilinear space (M_0, b_0, K). Let $L = \mathrm{End}(Z)$ and let $D = L/\mathrm{Jac}(L)$. Then D is a division ring. For every $n \in \mathbb{N}$, let

$$n \cdot b_0 = \underbrace{b_0 \perp \cdots \perp b_0}_{n}$$

and set $W_n = W_{n \cdot b_0}$ and $\overline{W}_n = \overline{W}_{n \cdot b_0}$. As $Z(n \cdot b_0) = Z(b_0)^n = Z^n$, we may identify W_n with $\mathrm{M}_n(W_1)$ and $\overline{W}_n$ with $\mathrm{M}_n(\overline{W}_1) = \mathrm{M}_n(D)$. We let β_n denote the involution induced by $\beta(n \cdot b_0)$ on $\overline{W}_n$.

PROPOSITION 4.13.1. *Under the identification* $W_n \cong \mathrm{M}_n(W_1)$ *we have*

$$\begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix}^{\beta(n \cdot b_0)} = \begin{bmatrix} w_{11}^{\beta(b_0)} & \dots & w_{n1}^{\beta(b_0)} \\ \vdots & \ddots & \vdots \\ w_{1n}^{\beta(b_0)} & \dots & w_{nn}^{\beta(b_0)} \end{bmatrix}$$

PROOF. Recall that $W_n = \mathrm{End}(Z^n)$ and $\beta(n \cdot b_0)$ is nothing more than the map $* : \mathrm{End}(Z^n) \to \mathrm{End}((Z^*)^n) = \mathrm{End}(Z^n)$. The proposition holds since for any element $(f_{ij})_{i,j}$ of $\mathrm{End}(Z^n)$ (written in matrix form), we have $(f_{ij})_{i,j}^* = (f_{ji}^*)_{i,j}$. (This is a general fact about additive contravariant functors.) □

Let V_n be an n-dimensional right D-vector space. Then $\mathrm{End}_D(V_n) \cong \mathrm{M}_n(D) = \overline{W}_n$. By Theorem 3.5.5, the involution β_n correspond to a *regular* bilinear form $b_n : V_n \times V_n \to K_n$, where K_n is some double D-module. Furthermore, by the discussion of section 3.1, there is an involution κ_n of K_n such that b_n is κ_n-symmetric.

To get an explicit realization of (V_n, b_n, K_n) and κ_n, we invoke Proposition 3.3.9 (with $e = e_{11}$; here $\{e_{ij}\}$ are the standard matrix units in $\mathrm{M}_n(D)$). This proposition implies that once identifying V_n with $\overline{W}_n e_{11} = \mathrm{M}_n(D)e_{11}$ and letting D act on V_n from the right in the standard way, we can take $K_n = e_{11}^{\beta_n}\mathrm{M}_n(D)e_{11}$, with $\odot_0, \odot_1$ defined by

$$(e_{11}^{\beta_n} x e_{11}) \odot_0 a = (e_{11} a e_{11})^{\beta_n}(e_{11}^{\beta_n} x e_{11}), \qquad (e_{11}^{\beta_n} x e_{11}) \odot_1 a = (e_{11}^{\beta_n} x e_{11})a,$$

for all $x \in \overline{W}_n$, $a \in D$ and $\kappa_n = \beta_n|_{K_n}$. The form $h_n : V_n \times V_n \to K_n$ is then given by

$$h_n(x, y) = x^{\beta_n} y \qquad \forall\, x, y \in V_n = \mathrm{M}_n(D)e_{11}\ .$$

By Proposition 4.13.1, $e_{11}^{\beta_n} = e_{11}$, hence $K_n = e_{11}De_{11}$ and there is a set isomorphism $K_n \to D$ given by $e_{11}ae_{11} \mapsto a$. Pulling $\odot_0$ and $\odot_1$ to D along this isomorphism, we get a double D-module structure on D given by

$$d \odot_0 a = a^{\beta_1} d, \qquad d \odot_1 a = da\ .$$

Thus, identifying K_n with D, we get that h_n is nothing but a $(\beta_1, 1)$-hermitian form over D. We henceforth consider h_n as a hermitian form taking values in D. It is given by $h_n(x, y) = d$ where d is the unique element of D satisfying $x^\beta y = e_{11} d e_{11}$. Even more explicitly, taking $\{e_{11}, \dots, e_{n1}\}$ as the standard basis of V_n, one has

$$h_n\Big(\sum_i e_{i1} d_i, \sum_i e_{i1} d'_i\Big) = \sum_i d_i^{\beta_1} d'_i\ .$$

We now apply the following argument: Isometry classes of bilinear forms b over R with $b \sim_{\mathrm{Kr}} n \cdot b_0$ correspond to congruence classes of invertible elements in $\mathrm{Sym}(W_n, \beta(n \cdot b_0))$ (Proposition 4.8.12), which correspond to congruence classes of invertible elements in $\mathrm{Sym}(\mathrm{M}_n(D), \beta_n)$ (Theorem 4.9.20) which in turn correspond to isometry classes of bilinear forms h over D with $h \sim_{\mathrm{Kr}} h_n$ (again by Proposition 4.8.12). As bilinear forms h with $h \sim_{\mathrm{Kr}} h_n$ are just $(\beta_1, 1)$-hermitian forms defined over n-dimensional D-vector spaces (straightforward), it follows that the isometry classes of bilinear forms b over R with $b \sim_{\mathrm{Kr}} n \cdot b_0$ are in one-to-one correspondence with isometry classes of n-dimensional $(\beta_1, 1)$-hermitian forms over D. This is phrased more formally in the theorem below.

To handle extremal cases, we define both W_0 and $\overline{W}_0$ to be the zero ring (or, if one insists, $\mathrm{M}_0(W_1)$ and $\mathrm{M}_0(D)$). We also let V_0 be the zero right module over D and $h_0 : V_0 \times V_0 \to D$ be the zero $(\beta_1, 1)$-hermitian form.

THEOREM 4.13.2. *Keeping the previous notation, let (M, b, K) be a ζ-isotype. Then there exists unique $n \in \mathbb{N} \cup \{0\}$ such that $b \sim_{\mathrm{Kr}} n \cdot b_0$. Let $(\sigma, \tau^{\mathrm{op}}) : Z(b) \to Z(n \cdot b_0) = Z^n$ be any isomorphism. Define $\overline{b} : V_n \times V_n \to D$ by*

$$\overline{b}(x, y) = x^{\beta_n}\overline{\tau\sigma} y \qquad \forall x, y \in V_n = \overline{W}_n e_{11}$$

and let (M', b', K) be another ζ-isotype. Then:

(i) *$\overline{b}$ is well-defined up to isometry. Moreover, $b \cong b' \iff \overline{b} \cong \overline{b'}$.*
(ii) *The map $[b] \mapsto [\overline{b}]$ is additive in sense that $\overline{b \perp b'} \cong \overline{b} \perp \overline{b'}$.*

PROOF. (i) Since b is a ζ-isotype and $\zeta = \{[Z]\}$, we have $Z(b) \cong Z^n = Z(n \cdot b_0)$ for some $n \in \mathbb{N} \cup \{0\}$, which is unique by the Krull-Schmidt Theorem. Tracking along the proofs of Proposition 4.8.12 (and also Theorem 4.9.20), one sees that the $(\beta_1, 1)$-hermitian form corresponding to b in the preceding discussion is precisely $\overline{b}$, hence $[\overline{b}]$ determines $[b]$ and vice versa.

(ii) Let $(\sigma,\tau^{\mathrm{op}}) : Z(b)\to Z^n$ and $(\sigma',\tau'^{\mathrm{op}}) : Z(b)\to Z^{n'}$ be isomorphisms of Kronecker modules. Then $(\sigma\oplus\sigma',(\tau\oplus\tau')^{\mathrm{op}}) : Z(b\perp b')\to Z^{n+n'}$ is an isomorphism and considering all terms involved as matrices over D, we have $\overline{\tau\sigma}\oplus\overline{\tau'\sigma'}=\overline{(\tau\oplus\tau')(\sigma\oplus\sigma')}$. It is now straightforward to check that by identifying $V_n\oplus V_{n'}$ with $V_{n+n'}$ via $(\sum_i e_{i1}a_i,\sum_j e_{j1}b_i)\mapsto\sum_{i=1}^n e_{i1}a_i+\sum_{j=1}^{n'}e_{n+j}b_i$ we have $\overline{b}\perp\overline{b'}=\overline{b\perp b'}$. □

Theorem 4.13.2 implies that, under mild assumptions, the isometry problem of bilinear forms can be reduced to isomorphism of Kronecker modules and isometry of hermitian forms over division rings. We have noted this at the end of section 4.3, but now we have presented an explicit way to see this. However, the categorical approach taken in section 4.3 has the advantage that the reduction is functorial (i.e., rather than an additive correspondence between isometry classes, there is a functor from the category of bilinear forms over R to the product of certain categories of hermitian forms over certain division rings). This can be shown explicitly as well, but as it requires additional notation that does not benefit the text, we have omitted the details. We finish this section by proving Witt's Cancellation Theorem.

Corollary 4.13.3 (Witt's Cancellation Theorem). *Let b_1,b_2,b_3 be bilinear forms over R such that $b_1\perp b_2\cong b_1\perp b_3$ (no assumption or regularity or symmetry is needed). If $W_{b_1\perp b_2}$ is semiperfect pro-semiprimary and $2\in W_{b_1\perp b_2}^{\times}$, then $b_2\cong b_3$.*

Proof. By Theorem 4.12.2 and the preceding discussion, $(b_1)_\zeta\perp(b_2)_\zeta\cong(b_1)_\zeta\perp(b_3)_\zeta$ for every $\zeta\in\overline{\Sigma}_{Z(b_1\perp b_2)}$. Therefore, we may assume $b_i=(b_i)_\zeta$ for every $i\in\{1,2,3\}$. Write $\zeta=\{[Z],[Z^*]\}$. If Z is not of bilinear type, then all ζ-blocks are hyperbolic and isomorphic to each other (Theorem 4.11.6), hence b_1,b_2,b_3 are determined up to isometry by their Kronecker module. As $Z(b_1)\oplus Z(b_2)\cong Z(b_1)\oplus Z(b_3)$, the Krull-Schmidt Theorem implies $Z(b_2)\cong Z(b_3)$, and hence $b_2\cong b_3$. If Z is of bilinear type, then by Theorem 4.13.2, it is enough to prove $\overline{b}_2\cong\overline{b}_3$. As $\overline{b}_1\perp\overline{b}_2=\overline{b}_1\perp\overline{b}_3$, we are through by Witt's Cancelation Theorem for hermitian forms over division rings of characteristic not 2, e.g. see [**86**, Ch. 7, §9] or [**73**]. □

Remark 4.13.4. Witt's Cancelation Theorem was proved in various scenarios including hermitian categories satisfying (C2) (see section 4.4), *systems* of classical *regular symmetric* bilinear forms over a field ([**86**, Ch. 7, Ex. 11.8]) and also for classical *non-symmetric regular* bilinear forms over a field ([**76**] and related papers). Non-regular symmetric bilinear forms were treated in [**16**]. Corollary 4.13.3 generalizes all of these results. Even in the classical symmetric case, it seems to be the only result that applies to systems of *non-regular* bilinear forms (this is "almost" obtained in [**86**, Ch. 7, Ex. 11.8]; the assumptions require at least one of the forms to be regular).

4.14. Structure of the Isometry Group

Throughout, F is an *algebraically closed* field of characteristic not 2 and R is a f.d. F-algebra. In this last section, we will use the results of the previous sections to deduce some strong structural results about isometry groups of *F-linear* bilinear of forms over R. (The case when F is not algebraically closed is briefly discussed at the end.) Note that all the results of this section apply to systems of bilinear forms and no regularity or symmetry assumption is needed. The results of this section extend [**14**] and related works.

Definition 4.14.1. *A double R-module K is* F-linear *if $\odot_0|_{K\times F}=\odot_1|_{K\times F}$ or, more explicitly, $k\odot_0 a=k\odot_1 a$ for all $k\in K$ and $a\in F$. A bilinear form taking values in an F-linear double R-module is called* F-linear.

Let (M, b, K) be an F-linear bilinear space over R. Then $\mathrm{End}_R(M)$ is naturally an F-algebra. In addition, the F-linearity of K implies that $b(xa, y) = b(x, ya)$ for all $a \in F$, hence F embeds in W_b via $a \mapsto (a, a^{\mathrm{op}})$, which makes W_b into an F-algebra as well (and W_b is a sub-F-algebra of $\mathrm{End}_R(M) \times \mathrm{End}_R(M)^{\mathrm{op}}$). Also note that $\beta(b)$ is clearly F-linear. These facts will be used freely below.

We begin by showing that F-linear bilinear forms (M, b, K) over R with M f.g. are determined up to isometry by $Z(b)$ (this was already noted at the end of section 4.3). Observe that condition (A4) of section 4.10 is satisfied (i.e. M is of finite length), hence W_b is semiprimary (and in particular, semiperfect and pro-semiprimary).

Proposition 4.14.2. *Let (M, b, K) be an F-linear bilinear form over R with M finitely generated. Then b is determined up to isometry by $[Z(b)]$.*

Proof. By Theorem 4.12.2, we may restrict to isotypes. Let $Z = (A, f_0, f_1, B)$ be an indecomposable Kronecker module such that $\dim_F A + \dim_F B < \infty$ and let $\zeta = \{[Z], [Z^*]\}$. We need to prove that for every ζ-isotype, b, the isometry class $[b]$ is determined by $[Z(b)]$. This follows from Theorem 4.11.6 in case Z is not of bilinear type. If Z is of bilinear type, say $Z = Z(b_0)$, then by Theorem 4.13.2, the problem reduces to showing that, up to isometry, there exists exactly one n-dimensional $(\beta_1, 1)$-hermitian form over D, where $D = \mathrm{End}(Z)/\mathrm{Jac}(\mathrm{End}(Z))$ and β_1 is the involution induced by b_0. However, the previous discussion implies that D is a f.d. F-algebra and β_1 is F-linear (D is f.d. since $\mathrm{End}(Z)$ is f.d. and β_1 is F-linear since b_0 is F-linear). As F is algebraically closed, necessarily $D = F$ and $\beta_1 = \mathrm{id}_F$, implying that there exists exactly one n-dimensional $(\beta_1, 1)$-hermitian form over D, as required. □

We now turn our attention to isometry groups of F-linear bilinear forms. Let (M, b, K) be such a form with M finitely generated. Then the isometry group of b, denoted $O(b)$, is an affine algebraic group over F (since it is a closed algebraic set in $\mathrm{End}_F(M)^\times \cong \mathrm{GL}_n(F)$ for some $n \in \mathbb{N}$).

Example 4.14.3. Assume $R = F$ and b is regular. Write $n = \dim_F M$. If b is symmetric, then $O(b)$ is $O_n(F)$, namely the standard orthogonal group (type B/D). However, when b is alternating (n is necessarily even), $O(b)$ is $\mathrm{Sp}_n(F)$, the symplectic group (type C).

We will show that for general R and b the situation is not very different from the last example; roughly speaking, after removing the unipotent radical, $O(b)$ is a product of copies of $O_n(F)$, $\mathrm{Sp}_n(F)$ and $\mathrm{GL}_n(F)$.[28]

To simplify phrasing, we will say that a ζ-isotype is of type-I (-II, -III) if $\zeta = \{[Z], [Z^*]\}$ and Z is of bilinear type (non-self-dual, self-dual but not bilinear). Clearly the type-I (-II, -III) isotypes are precisely those which are sums of type-I (-II, -III) blocks. In addition, for every ζ-isotype b, we let $\dim_\zeta b$ be the unique integer n with $Z(b) \cong Z^n$ in case Z is bilinear and $Z(b) \cong (Z \oplus Z^*)^n$ in case Z is not bilinear.

Theorem 4.14.4. *Let (M, b, K) be an F-linear bilinear form over R with M finitely generated. Then there exists an exact sequence of affine algebraic groups (over F):*

$$1 \to U \to O(b) \to G \to 1$$

[28] These families correspond to the three "typical" kinds of isometry groups: of symmetric forms (a form of $O_n(F)$), of alternating forms (a form of $\mathrm{Sp}_n(F)$) and of (α, λ)-hermitian forms with $\alpha \neq \mathrm{id}$ (a form of $\mathrm{GL}_n(F)$).

such that U is unipotent and G is a product of copies of $O_n(F)$, $\mathrm{GL}_m(F)$, $\mathrm{Sp}_k(F)$ (n, m, k may vary between the copies). If $b = \perp_{\zeta\in\overline{\Sigma}_{Z(b)}} b_\zeta$ is the decomposition of b into isotypes, then the $O_n(F)$ (resp.: $\mathrm{GL}_n(F)$, $\mathrm{Sp}_{2n}(F)$) components in G correspond to the type-I (resp.: type-II, type-III) ζ-isotypes with $\dim_\zeta b_\zeta = n$.

Proof. Recall that by Remark 4.8.9, the group $O(b)$ is isomorphic to the unitary group of $(W_b, \beta(b))$, i.e. $\{w \in W_b \mid w^{\beta(b)}w = 1\}$, via $\sigma \mapsto (\sigma, (\sigma^{-1})^{\mathrm{op}})$. As W_b is a f.d. F-algebra and $\beta(b)$ is F-linear, this is easily seen to be an isomorphism of algebraic groups. We may thus identify $O(b)$ with its copy in W_b.

Observe that $1 + \mathrm{Jac}(W_b)$ is a closed unipotent subgroup of $W_b^\times$ (that $\mathrm{Jac}(W_b)$ is nilpotent follows from Proposition 4.10.1(A4)). Let $U = (1 + \mathrm{Jac}(W_b)) \cap O(b)$ and $G = \{\overline{w} \in \overline{W}_b \mid \overline{w}^{\beta(b)}\overline{w} = 1\}$. By Theorem 4.9.20(ii), the map $w \mapsto \overline{w}$ from $O(b)$ to G is onto (take $e = e' = 1$ and choose the rest of the parameters as in the proof of Corollary 4.9.21). Therefore, there is an exact sequence of algebraic groups $1 \to U \to O(b) \to G \to 1$ and U is unipotent. It is thus left to determine the structure of G.

Let $b = \perp_{\zeta\in\overline{\Sigma}_{Z(b)}} b_\zeta$ be the decomposition of b into isotypes. The proof of Theorem 4.12.2 implies that $\overline{W}_b \cong \prod_\zeta \overline{W}_{b_\zeta}$ with $\beta(b)$ acting as $\beta(b_\zeta)$ on $\overline{W}_{b_\zeta}$. This isomorphism is easily seen to be an isomorphism of F algebras, hence $G \cong \prod_\zeta \{\overline{u} \in \overline{W}_{b_\zeta} \mid \overline{u}^\beta\overline{u} = 1\}$ as algebraic groups. We will now determine the structure of $G_\zeta := \{\overline{u} \in \overline{W}_{b_\zeta} \mid \overline{u}^\beta\overline{u} = 1\}$.

Write $\zeta = \{[Z], [Z^*]\}$ and $n = \dim_\zeta b_\zeta$. Let b_0 be a ζ-block. Then $Z(b_\zeta) \cong Z(b_0)^n$, hence by Proposition 4.14.2, $b_\zeta \cong n \cdot b_0$, so w.l.o.g. we may assume $b_\zeta = n \cdot b_0$. Let $W_1 = W_{b_0}$. Then $W_{b_\zeta} \cong \mathrm{M}_n(W_1)$ and the proof of Proposition 4.13.1 implies that under this isomorphism $\beta(b_\zeta)$ acts by transposing and applying $\beta(b_0)$ component-wise. That is, $(W_{b_\zeta}, \beta(b_\zeta)) \cong (\mathrm{M}_n(F), T) \otimes_F (W_1, \beta(b_0))$ as F-algebras with involution (here T denotes the transpose involution). This implies

$$(\overline{W}_{b_\zeta}, \beta(b_\zeta)) \cong (\mathrm{M}_n(F), T) \otimes_F (\overline{W}_1, \beta(b_0)) \ .$$

As b_0 is a block, the structure of $(\overline{W}_1, \beta(b_0))$ is determined in Theorem 4.11.1.

If b_0 is of type-I, then $(\overline{W}_1, \beta(b_0))$ is a division ring with involution, implying $\overline{W}_1 = F$ and $\beta(b_0) = \mathrm{id}_F$. Thus, $(W_{b_\zeta}, \beta(b_\zeta)) \cong (\mathrm{M}_n(F), T)$ and $G_\zeta \cong O_n(F)$.

If b_0 is of type-II, then $\overline{W}_1 \cong D \times D^{\mathrm{op}}$ and $\beta(b_0)$ exchanges D and D^{op}. We must have $D = F$ and hence, $\overline{W}_1 \cong F \times F$ with $\beta(b_0)$ exchanging the components. Thus, $(\mathrm{M}_n(F), T) \otimes_F (W_1, \beta(b_0)) \cong (\mathrm{M}_n(F) \times \mathrm{M}_n(F), \gamma)$ where γ is given by $(A, B)^\gamma = (B^T, A^T)$. It is easy to check that the group $\{x \in \mathrm{M}_n(F) \times \mathrm{M}_n(F) \mid x^\gamma x = 1\}$ is isomorphic to $\mathrm{GL}_n(F)$, hence $G_\zeta \cong \mathrm{GL}_n(F)$ in this case.

If b_0 is of type-III, then $\overline{W}_1 \cong \mathrm{M}_2(K)$ for some field K containing F and $\beta(b_0)$ is a symplectic involution. Again, we must have $K = F$ and hence $(\mathrm{M}_n(F), T) \otimes_F (W_1, \beta(b_0)) \cong (\mathrm{M}_{2n}(F), S)$ with S a symplectic involution. Thus, $G_\zeta \cong \mathrm{Sp}_{2n}(F)$. □

Remark 4.14.5. If F is not assumed to be algebraically closed, then there is still an exact sequence of algebraic groups as in Theorem 4.14.4 (which induces an exact sequence on the group of rational points). However, the group G is not a product of $O_n(F)$, $\mathrm{GL}_m(F)$ and $\mathrm{Sp}_k(F)$, but merely a *form* of such a product (i.e. G becomes isomorphic to such a product over the algebraic closure of F).

Bibliography

[1] A. Adrian Albert. Symmetric and alternate matrices in an arbitrary field. I. *Trans. Amer. Math. Soc.*, 43(3):386–436, 1938.

[2] A. Adrian Albert. *Structure of algebras*. Revised printing. American Mathematical Society Colloquium Publications, Vol. XXIV. American Mathematical Society, Providence, R.I., 1961.

[3] Afshin Amini, Babak Amini, and Alberto Facchini. Weak Krull-Schmidt for infinite direct sums of cyclically presented modules over local rings. *Rend. Semin. Mat. Univ. Padova*, 122:39–54, 2009.

[4] P. Ara, K. R. Goodearl, and E. Pardo. K_0 of purely infinite simple regular rings. *K-Theory*, 26(1):69–100, 2002.

[5] Gorô Azumaya. On maximally central algebras. *Nagoya Math. J.*, 2:119–150, 1951.

[6] Anthony Bak. On modules with quadratic forms. In *Algebraic K-Theory and its Geometric Applications (Conf., Hull, 1969)*, pages 55–66. Springer, Berlin, 1969.

[7] Paul Balmer. Witt groups. In *Handbook of K-theory. Vol. 1, 2*, pages 539–576. Springer, Berlin, 2005.

[8] Francesco Barioli, Alberto Facchini, Francisco Raggi, and José Ríos. Krull-Schmidt theorem and homogeneous semilocal rings. *Comm. Algebra*, 29(4):1649–1658, 2001.

[9] Hyman Bass. Finitistic dimension and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.*, 95:466–488, 1960.

[10] Hyman Bass. Unitary algebraic K-theory. In *Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 57–265. Lecture Notes in Math., Vol. 343. Springer, Berlin, 1973.

[11] Eva Bayer-Flickiger, Uriya A. First, and Daniel A. Moldovan. Hermitian categories, extension of scalars and systems of sesquilinear forms. Submitted, 2013.

[12] E. Bayer-Fluckiger. Weak Hasse principle for systems of bilinear forms. In *Number theory, Vol. II (Budapest, 1987)*, volume 51 of *Colloq. Math. Soc. János Bolyai*, pages 581–593. North-Holland, Amsterdam, 1990.

[13] E. Bayer-Fluckiger, Daniel B. Shapiro, and J.-P. Tignol. Hyperbolic involutions. *Math. Z.*, 214(3):461–476, 1993.

[14] Eva Bayer-Fluckiger. Intersections de groupes orthogonaux et principe de Hasse faible pour les systèmes de formes quadratiques sur un corps global. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(20):911–914, 1985.

[15] Eva Bayer-Fluckiger. Systèmes de formes quadratiques d'algèbre donnée. *C. R. Acad. Sci. Paris Sér. I Math.*, 321(10):1291–1293, 1995.

[16] Eva Bayer-Fluckiger and Laura Fainsilber. Non-unimodular Hermitian forms. *Invent. Math.*, 123(2):233–240, 1996.

[17] Eva Bayer-Fluckiger and Daniel A. Moldovan. Sesquilinear forms over rings with involution. Preprint, 2012.

[18] George M. Bergman. The diamond lemma for ring theory. *Adv. in Math.*, 29(2):178–218, 1978.

[19] J.-E. Björk. Conditions which imply that subrings of semiprimary rings are semiprimary. *J. Algebra*, 19:384–395, 1971.

[20] Jan-Erik Björk. Conditions which imply that subrings of artinian rings are artinian. *J. Reine Angew. Math.*, 247:123–138, 1971.

[21] Nicolas Bourbaki. *Elements of mathematics. Theory of sets*. Translated from the French. Hermann, Publishers in Arts and Science, Paris, 1968.

[22] Nicolas Bourbaki. *General topology. Chapters 5–10*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989. Translated from the French, Reprint of the 1966 edition.

[23] Rosa Camps and Warren Dicks. On semilocal rings. *Israel J. Math.*, 81(1-2):203–211, 1993.

[24] Chen-Lian Chuang and Yuan-Tsung Tsai. Symmetric Utumi quotient rings of Ore extensions by skew derivations. *Proc. Amer. Math. Soc.*, 138(9):3125–3133, 2010.

[25] B. Corbas and G. D. Williams. Bilinear forms over an algebraically closed field. *J. Pure Appl. Algebra*, 165(3):255–266, 2001.

[26] Roberta Corisello and Alberto Facchini. Homogeneous semilocal rings. *Comm. Algebra*, 29(4):1807–1819, 2001.

[27] Friedrich Dischinger. Sur les anneaux fortement π-réguliers. *C. R. Acad. Sci. Paris Sér. A-B*, 283(8):Aii, A571–A573, 1976.

[28] Dragomir Ž. Đoković, Konstanze Rietsch, and Kaiming Zhao. Normal forms for orthogonal similarity classes of skew-symmetric matrices. *J. Algebra*, 308(2):686–703, 2007.

[29] Dragomir Ž. Đoković and Fernando Szechtman. Characterization of bilinear spaces with unimodular isometry group. *Proc. Amer. Math. Soc.*, 133(10):2853–2863 (electronic), 2005.

[30] Dragomir Ž. Đoković and Kaiming Zhao. Tridiagonal normal forms for orthogonal similarity classes of symmetric matrices. *Linear Algebra Appl.*, 384:77–84, 2004.

[31] Dragomir Ž. Đoković and Kaiming Zhao. Jordan decomposition of bilinear forms. *Math. Res. Lett.*, 12(2-3):341–356, 2005.

[32] Alberto Facchini. The Krull-Schmidt theorem. In *Handbook of algebra, Vol. 3*, pages 357–397. North-Holland, Amsterdam, 2003.

[33] Alberto Facchini, Şule Ecevit, and M. Tamer Koşan. Kernels of morphisms between indecomposable injective modules. *Glasg. Math. J.*, 52(A):69–82, 2010.

[34] Alberto Facchini and Dolors Herbera. Local morphisms and modules with a semilocal endomorphism ring. *Algebr. Represent. Theory*, 9(4):403–422, 2006.

[35] Alberto Facchini, Dolors Herbera, Lawrence S. Levy, and Peter Vámos. Krull-Schmidt fails for Artinian modules. *Proc. Amer. Math. Soc.*, 123(12):3587–3592, 1995.

[36] Carl Faith. Factor rings of pseudo-Frobenius rings. *J. Algebra Appl.*, 5(6):847–854, 2006.

[37] Carl Faith and Dinh Van Huynh. When self-injective rings are QF: a report on a problem. *J. Algebra Appl.*, 1(1):75–105, 2002.

[38] Uriya A. First. *Extending The Witt Ring*. 2006. Thesis (M.Sc.)– Bar-Ilan University - Israel.

[39] Uriya A. First. General bilinear forms. Submitted, 2013.

[40] Uriya A. First. Rings that are morita equivalent to their opposites. Submitted (currently avaiable at http://arxiv.org/abs/1305.5139), 2013.

[41] Uriya A. First. Semi-invariant subrings. *J. Algebra*, 378:103–132, 2013.

[42] Peter Freyd. *Abelian categories. An introduction to the theory of functors*. Harper's Series in Modern Mathematics. Harper & Row Publishers, New York, 1964.

[43] Vyacheslav Futorny, Roger A. Horn, and Vladimir V. Sergeichuk. Tridiagonal canonical matrices of bilinear or sesquilinear forms and of pairs of symmetric, skew-symmetric, or Hermitian forms. *J. Algebra*, 319(6):2351–2371, 2008.

[44] Peter Gabriel. Appendix: degenerate bilinear forms. *J. Algebra*, 31:67–72, 1974.

[45] A. W. Goldie. Semi-prime rings with maximum condition. *Proc. London Math. Soc. (3)*, 10:201–220, 1960.

[46] Roderick Gow. The equivalence of an invertible matrix to its transpose. *Linear and Multilinear Algebra*, 8(4):329–336, 1979/80.

[47] E. Gregorio. Generalized Morita equivalence for linearly topologized rings. *Rend. Sem. Mat. Univ. Padova*, 79:221–246, 1988.

[48] Yukitoshi Hinohara. Note on non-commutative semi-local rings. *Nagoya Math. J.*, 17:161–166, 1960.

[49] Roger A. Horn and Vladimir V. Sergeichuk. Congruences of a square matrix and its transpose. *Linear Algebra Appl.*, 389:347–353, 2004.

[50] Roger A. Horn and Vladimir V. Sergeichuk. Canonical forms for complex matrix congruence and *congruence. *Linear Algebra Appl.*, 416(2-3):1010–1032, 2006.

[51] Roger A. Horn and Vladimir V. Sergeichuk. A regularization algorithm for matrices of bilinear and sesquilinear forms. *Linear Algebra Appl.*, 412(2-3):380–395, 2006.

[52] Roger A. Horn and Vladimir V. Sergeichuk. Canonical matrices of bilinear and sesquilinear forms. *Linear Algebra Appl.*, 428(1):193–223, 2008.

[53] Daniel M. Kan. Adjoint functors. *Trans. Amer. Math. Soc.*, 87:294–329, 1958.

[54] F. Kasch. *Modules and rings*, volume 17 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982. Translated from the German and with a preface by D. A. R. Wallace.

[55] Manfred Knebusch. Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen. *S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl.*, 1969/70:93–157, 1969/1970.

[56] Max-Albert Knus. *Quadratic and Hermitian forms over rings*, volume 294 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991. With a foreword by I. Bertuccioni.

[57] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[58] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[59] Scott Lanning. The maximal symmetric ring of quotients. *J. Algebra*, 179(1):47–91, 1996.

[60] Daniel Lazard. Sur les modules plats. *C. R. Acad. Sci. Paris*, 258:6313–6316, 1964.

[61] David W. Lewis. Involutions and anti-automorphisms of central simple algebras. *J. Pure Appl. Algebra*, 182(2-3):253–261, 2003.

[62] M. S. Li and J. M. Zelmanowitz. Row rank and the structure of matrix subrings. *J. Algebra*, 136(1):117–131, 1991.

[63] Eben Matlis. Injective modules over Noetherian rings. *Pacific J. Math.*, 8:511–528, 1958.

[64] Daniel A. Moldovan. *Hermitian forms over algebras with involution and hermitian categories.* 2012. Dissertation (Ph.D.)– University of Lausanne, Switzerland.

[65] O. T. O'Meara. *Introduction to quadratic forms.* Die Grundlehren der mathematischen Wissenschaften, Bd. 117. Academic Press Inc., Publishers, New York, 1963.

[66] J. Marshall Osborn. Jordan and associative rings with nilpotent and invertible elements. *J. Algebra*, 15:301–308, 1970.

[67] B. L. Osofsky. A generalization of quasi-Frobenius rings. *J. Algebra*, 4:373–387, 1966.

[68] Donald S. Passman. *A course in ring theory.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991.

[69] Albrecht Pfister. *Quadratic forms with applications to algebraic geometry and topology*, volume 217 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995.

[70] Nicolae Popesco and Pierre Gabriel. Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes. *C. R. Acad. Sci. Paris*, 258:4188–4190, 1964.

[71] H.-G. Quebbemann, W. Scharlau, and M. Schulte. Quadratic and Hermitian forms in additive and abelian categories. *J. Algebra*, 59(2):264–289, 1979.

[72] I. Reiner. *Maximal orders*, volume 28 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, Oxford, 2003. Corrected reprint of the 1975 original, With a foreword by M. J. Taylor.

[73] H. Reiter. Witt's theorem for noncommutative semilocal rings. *J. Algebra*, 35:483–499, 1975.

[74] G. Renault. Sur les anneaux de groupes. In *Rings, modules and radicals (Proc. Colloq., Keszthely, 1971)*, pages 391–396. Colloq. Math. Soc. János Bolyai, Vol. 6. North-Holland, Amsterdam, 1973.

[75] C. Riehm and M. A. Shrader-Frechette. The equivalence of sesquilinear forms. *J. Algebra*, 42(2):495–530, 1976.

[76] Carl Riehm. The equivalence of bilinear forms. *J. Algebra*, 31:45–66, 1974.

[77] Alex Rosenberg and Daniel Zelinsky. Annihilators. *Portugal. Math.*, 20:53–65, 1961.

[78] Louis H. Rowen. Finitely presented modules over semiperfect rings. *Proc. Amer. Math. Soc.*, 97(1):1–7, 1986.

[79] Louis H. Rowen. Finitely presented modules over semiperfect rings satisfying ACC-∞. *J. Algebra*, 107(1):284–291, 1987.

[80] Louis H. Rowen. *Ring theory. Vol. I*, volume 127 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[81] Louis Halle Rowen. Examples of semiperfect rings. *Israel J. Math.*, 65(3):273–283, 1989.

[82] David J. Saltman. Azumaya algebras with involution. *J. Algebra*, 52(2):526–539, 1978.

[83] David J. Saltman. *Lectures on division algebras*, volume 94 of *CBMS Regional Conference Series in Mathematics*. Published by American Mathematical Society, Providence, RI, 1999.

[84] Rudolf Scharlau. Zur Klassifikation von Bilinearformen und von Isometrien über Körpern. *Math. Z.*, 178(3):359–373, 1981.

[85] Winfried Scharlau. Automorphisms and involutions of incidence algebras. In *Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974), Paper No. 24*, pages 11 pp. Carleton Math. Lecture Notes, No. 9, Ottawa, Ont., 1974. Carleton Univ.

[86] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[87] William Schelter. Two-sided rings of quotients. *Arch. Math. (Basel)*, 24:274–277, 1973.

[88] V. V. Sergeĭchuk. Classification problems for systems of forms and linear mappings. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(6):1170–1190, 1358, 1987.

[89] Robert C. Shock. Injectivity, annihilators and orders. *J. Algebra*, 19:96–103, 1971.

[90] Robert C. Shock. Orders in self-injective cogenerator rings. *Proc. Amer. Math. Soc.*, 35:393–398, 1972.

[91] Robert C. Shock. Classical quotient rings. *Trans. Amer. Math. Soc.*, 190:43–48, 1974.

[92] Richard G. Swan. Induced representations and projective modules. *Ann. of Math. (2)*, 71:552–578, 1960.

[93] Alexandru Tupan. On sesquilinear forms over fields with involution. *Linear Algebra Appl.*, 428(1):254–258, 2008.

[94] Yuzo Utumi. Self-injective rings. *J. Algebra*, 6:56–64, 1967.

[95] P. Vámos. The dual of the notion of "finitely generated". *J. London Math. Soc.*, 43:643–646, 1968.

[96] Peter Vámos. Decomposition problems for modules over valuation domains. *J. London Math. Soc. (2)*, 41(1):10–26, 1990.

[97] Ronald L. Wagoner. Cogenerator endomorphism rings. *Proc. Amer. Math. Soc.*, 28:347–351, 1971.

[98] G. E. Wall. On the conjugacy classes in the unitary, symplectic and orthogonal groups. *J. Austral. Math. Soc.*, 3:1–62, 1963.

[99] Seth Warner. *Topological rings*, volume 178 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1993.

[100] G. P. Wene. On a result of Osborn. *Proc. Amer. Math. Soc.*, 69(1):11–15, 1978.

[101] Graham D. Williams. The congruence of a matrix with its transpose. *Results Math.*, 47(1-2):147–154, 2005.