Workshop permutation patterns 2005

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May 29, 2005

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Workshop permutation patterns

University of Haifa, 29 May - June 3, 2005

Notes:

- All talks are scheduled in one hour blocks. This allows 45 minutes for speaking, 5 minutes for questions, and 10 minutes for a break/changeover period.

Monday May 30

9:45 Opening

10:00 Doron Zeilberger
Computer-generated permutation enumeration.

11:00 Elisheva Kuflik
The mobius function of T_n^A.

12:00 **Lunch**

14:00 Alexander Burstein
Non-intersecting Patience Sorting Shadow Diagrams and Barred Permutation Patterns.

15:00 Eric Egge
Signed permutations counted by the Schröder numbers.

Tuesday May 31

10:00 Gabor Tardos
Extremal problems of forbidden submatrices.

11:00 Ghassan Firro
Pattern avoiding permutations under one roof: Scanning elements method and functional equations.

12:00 **Lunch**

14:00 Simone Severini
Entangling power of permutations.

15:00 Silvia Heubach
Compositions and multisets restricted by patterns of length three.

19:00 Banquet.

Wednesday June 1

10:00 Martin Klazar
Speeds of growth of hereditary classes of structures.

11:00 Vince Vatter
Enumeration schemes for restricted permutations.

12:00 **Lunch**

14:00 Igor Pak
RSK revisited.

15:00 Toufik Mansour
Matchings avoiding partial patterns.

Thursday June 2

10:00 Einar Steingrímsson
Permutation tableaux and permutation patterns.

11:00 Rebecca Smith
Sorting by Stacks in Series.

12:00 **Lunch**

14:00 Dan Bernstein
A Foata bijection for the alternating group and for q-analogues.

15:00 Sergey Kitaev
Introduction to the POPs.

A Foata bijection for the alternating group and for q-analogues

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May 5, 2005

The Foata bijection $\Phi : S_n \to S_n$ is extended to the bijections

$$\Psi : A_{n+1} \to A_{n+1} \text{ and } \Psi_q : S_{n+q-1} \to S_{n+q-1},$$

where S_m, A_m are the symmetric and the alternating groups. These bijections imply bijective proofs for recent equidistribution theorems, by Regev and Roichman, and extend a pattern-avoidance preservation property of the Foata bijection to longer patterns.

Non-intersecting patience sorting shadow diagrams and barred permutation patterns

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[3]May 5, 2005

Abstract

Patience Sorting is a combinatorial algorithm that can be viewed as an iterated, non-recursive form of the Schensted Insertion Algorithm. In recent work the authors have shown that the basic combinatorial properties of Patience Sorting are most naturally expressed in terms of generalized barred permutation pattern avoidance. Moreover, the authors have also recently given a geometric form for Patience Sorting that is in some sense a very natural dual algorithm to G. Viennot's well-known geometric form for RSK. Unlike Geometric RSK, though, the lattice paths coming from Patience Sorting are allowed to intersect.

In this work we characterize the intersections of these lattice paths and relate them to generalized barred permutation pattern avoidance.

Key words: patience sorting, generalized permutation patterns, barred permutation patterns, shadow diagrams, intersecting lattice paths

AMS Mathematical Subject Classifications: Primary: 05A05, 05A18; Secondary: 05E10

1. Introduction

The term *Patience Sorting* was first introduced in the 1960's by C.L. Mallows [11] within the context of studying a certain card sorting algorithm invented by A.S.C. Ross. This algorithm works by first partitioning a shuffled deck of cards (which we take to be a permutation $\sigma \in \mathfrak{S}_n$) into its left-to-right minima subsequences (called *piles* in this context), using what Mallows originally termed a "patience sorting procedure". The formation of these piles under Patience Sorting can be viewed as an iterated, non-recursive form of the Schensted Insertion Algorithm for interposing values into the rows of a Young tableau (see [2] and [4]). For a given $\sigma \in \mathfrak{S}_n$, we call the resulting collection of piles (given as part of the more general Algorithm 1.1 in Section 1.1 below) the *pile configuration* corresponding to σ and denote it by $R(\sigma)$.

In [4] the authors augment the formation of $R(\sigma)$ so that the resulting extension of Patience Sorting essentially becomes a full non-recursive analogue of the celebrated Robinson-Schensted-Knuth (or RSK) Correspondence. As with RSK, this Extended Patience Sorting Algorithm (Algorithm 1.1) takes a simple idea—that of placing cards into piles—and uses it to build a bijection between elements of the symmetric group $\mathfrak{S}_n$ and certain pairs of pile configurations. In the case of RSK, one uses the Schensted Insertion Algorithm to build a bijection with pairs of standard Young tableau having the same shape (see [12]). However, in the case of Patience Sorting, one achieves a bijection between permutations

[3]The work of the second author was supported in part by the U.S. National Science Foundation under Grants DMS-0135345 and DMS-0304414.

and somewhat more restricted pairs of pile configurations. We denote this latter bijection by $\sigma \overset{PS}{\longleftrightarrow} (R(\sigma), S(\sigma))$ and call $R(\sigma)$ (resp. $S(\sigma)$) the *insertion piles* (resp. *recording piles*) corresponding to σ.

Given a pile configuration R, one forms its *reverse patience word* $RPW(R)$ by listing the piles in R "from bottom to top, left to right" (see Example 1.2 below). In [4] these words are characterized as being exactly the elements of $S_n(3\text{-}\bar{1}\text{-}42)$ (this notation is described in Section 2 below; see Bóna [3] for more about permutation patterns in general), and the pairs of pile configurations $(R(\sigma), S(\sigma))$ given by Extended Patience Sorting are characterized by a certain simultaneous pattern avoidance property in their reverse patience words. Moreover, in [5] the authors define a geometric form for Extended Patience Sorting that is naturally dual to G. Viennot's Geometric RSK (originally defined in [13]). This gives, among other things, a geometric interpretation for what we call "stable pairs" of $3\text{-}\bar{1}\text{-}42$-avoiding permutations. (See [4] for details.)

Geometric Patience Sorting is unlike Geometric RSK in that the lattice paths formed under Extended Patience Sorting are allowed to cross. In order to understand these crossings, we first give some basic results for barred permutation pattern avoidance in Section 2 and then in Section 3 look at how such pattern avoidance affects both the resulting pile configurations and the corresponding shadow diagrams. Then in Section 4 we characterize those permutations corresponding to non-crossing shadowlines and discuss how this is related to pattern avoidance.

We close this introduction by describing both Extended and Geometric Patience Sorting.

1.1. **Extended Patience Sorting.** Mallows' original "patience sorting procedure" can be extended to a full bijection between the symmetric group $\mathfrak{S}_n$ and certain pairs of pile configurations using the following algorithm (which was first introduced in [4]):

Algorithm 1.1 (Extended Patience Sorting Algorithm)**.** *Given a shuffled deck of cards $\sigma = c_1 c_2 \cdots c_n$, inductively build* insertion piles $R = R(\sigma) = \{r_1, r_2, \ldots, r_m\}$ *and* recording piles $S = S(\sigma) = \{s_1, s_2, \ldots, s_m\}$ *as follows:*

- *Place the first card c_1 from the deck into a pile r_1 by itself, and set $s_1 = \{1\}$.*
- *For each remaining card c_i ($i = 2, \ldots, n$), consider the cards $d_1, d_2, \ldots, d_k$ atop the piles $r_1, r_2, \ldots, r_k$ that have already been formed.*
  - *If $c_i > \max\{d_1, d_2, \ldots, d_k\}$, then put c_i into a new pile r_{k+1} by itself and set $s_{k+1} = \{i\}$.*
  - *Otherwise, find the left-most card d_j that is larger than c_i and put the card c_i atop pile r_j while simultaneously putting i at the bottom of pile s_j.*

We visual represent the pile configurations $R(\sigma)$ and $S(\sigma)$ by listing their constituent piles vertically as illustrated in the following example. (See [4] for motivation and further discussion of the algorithm itself.)

Example 1.2. *Given $\sigma = 64518723 \in \mathfrak{S}_8$, one forms the following pile configurations under Algorithm 1.1:*

$$R(\sigma) = \begin{matrix} 1 & & 3 \\ 4 & 2 & 7 \\ 6 & 5 & 8 \end{matrix} \quad \text{and} \quad S(\sigma) = \begin{matrix} 1 & & 5 \\ 2 & 3 & 6 \\ 4 & 7 & 8 \end{matrix}$$

Moreover, we define the reverse patience word *for σ to be $RPW(R(64518723)) = 64152873$, which is formed by "reading up the columns, from left to right". Similarly, we can form $RPW(S(64518723)) = 42173865$.*

1.2. **Geometric Patience Sorting.** In order to describe a natural geometric form for the Extended Patience Sorting Algorithm given in Section 1.1 above, we begin with the following fundamental definitions:

Definition 1.3. *Given a lattice point $(m, n) \in \mathbb{Z}^2$, we define the* (southwest) shadow *of (m, n) to be the quarter space $U(m, n) = \{(x, y) \in \mathbb{R}^2 \mid x \leq m,\ y \leq n\}$.*

As with the northeasterly-oriented shadows that Viennot used when building his geometric form for RSK (see [13]), the most important use of these southwesterly-oriented shadows is in building shadowlines:

Definition 1.4. *Given lattice points $(m_1, n_1), (m_2, n_2), \ldots, (m_k, n_k) \in \mathbb{Z}^2$, we define their* (southwest) shadowline *to be the boundary of the union of the shadows $U(m_1, n_1), U(m_2, n_2), \ldots, U(m_k, n_k)$.*

In particular, we wish to associate to each permutation a certain collection of (southwest) shadowlines called its shadow diagram. However, unlike Geometric RSK, these shadowlines can intersect as illustrated in Figure 1. (We characterize those permutations having intersecting shadow diagrams in Theorem 4.2 below.)

Definition 1.5. *Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$, the* (southwest) shadow diagram *$D^{(0)}(\sigma)$ of σ consists of the (southwest) shadowlines*

$$D^{(0)}(\sigma) = \{L_1^{(0)}(\sigma), L_2^{(0)}(\sigma), \ldots, L_k^{(0)}(\sigma)\}$$

formed as follows:

- *$L_1^{(0)}(\sigma)$ is the shadowline for those lattice points $(x, y) \in \{(1, \sigma_1), \ldots, (n, \sigma_n)\}$ such that the shadow $U(x, y)$ does not contain any other lattice points.*
- *While at least one of the points $(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)$ is not contained in the shadowlines $L_1^{(0)}(\sigma), L_2^{(0)}(\sigma), \ldots, L_j^{(0)}(\sigma)$, define $L_{j+1}^{(0)}(\sigma)$ to be the shadowline for the points*

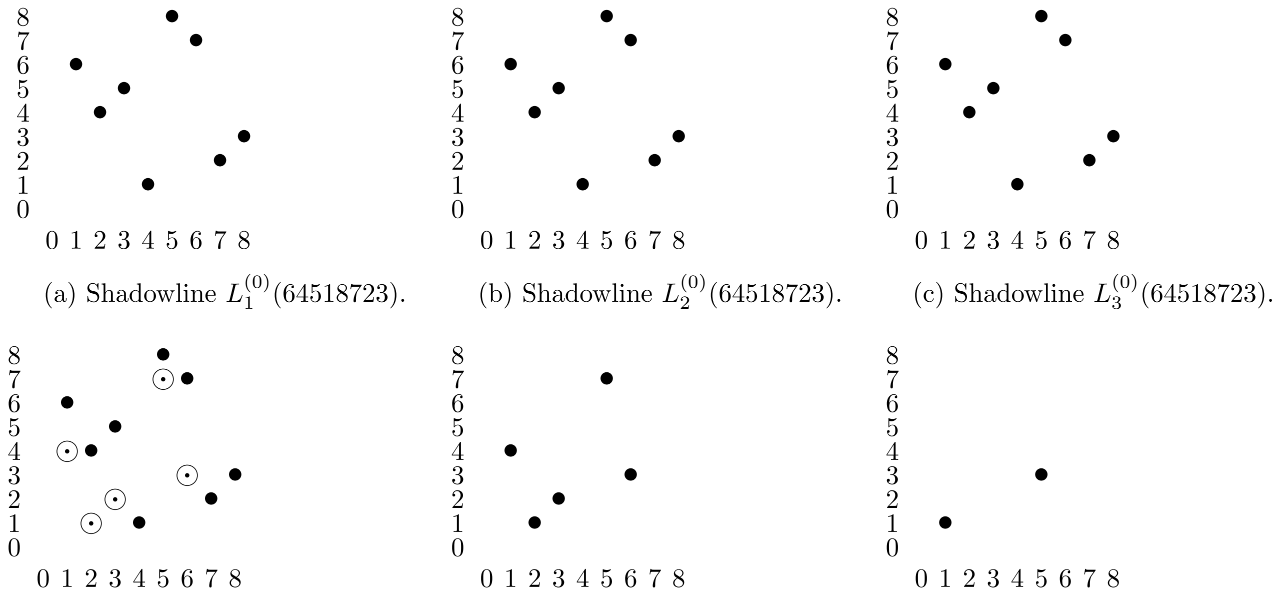
$$(x, y) \in A := \{(i, \sigma_i) \mid (i, \sigma_i) \notin \bigcup_{k=1}^{j} L_k^{(0)}(\sigma)\}$$

  such that the shadow $U(x, y)$ does not contain any other lattice points from the set A.

In other words, we define a shadow diagram by inductively eliminating points in the permutation diagram until every point has been used to define a shadowline (as illustrated in Figure 1).

We can then produce a sequence $D(\sigma) = (D^{(0)}(\sigma),\ D^{(1)}(\sigma),\ D^{(2)}(\sigma),\ \ldots)$, of shadow diagrams for a given permutation $\sigma \in \mathfrak{S}_n$ by recursively applying Definition 1.5 to the southwest corners (called *salient points*) of a given set of shadowlines (as illustrated in Figure 1). The only difference is that, with each iteration, newly formed shadowlines can only connect salient points along the same pre-existing shadowline. One can then uniquely reconstruct the pile configurations $R(\sigma)$ and $S(\sigma)$ from these shadowlines by taking their intersections with the x- and y-axes in the correct order. (See [5].)

Definition 1.6. *We call $D^{(k)}(\sigma)$ the k^{th}* iterate *of the* exhaustive shadow diagram *$D(\sigma)$ for $\sigma \in \mathfrak{S}_n$.*



(a) Shadowline $L_1^{(0)}(64518723)$.

(b) Shadowline $L_2^{(0)}(64518723)$.

(c) Shadowline $L_3^{(0)}(64518723)$.

(d) Salient points for $D^{(1)}(64518723)$.

(e) Shadow Diagram $D^{(2)}(64518723)$.

(f) Shadow Diagram $D^{(3)}(64518723)$.

FIGURE 1. Examples of Shadowline and Shadow Diagram Constructions

2. BARRED AND UNBARRED GENERALIZED PATTERN AVOIDANCE

Definition 2.1. *A* barred pattern β *is a generalized permutation pattern in which overbars are used to indicate that barred values cannot occur at the barred positions. We denote by* $S_n(\beta)$ *the set of all permutations* $\sigma \in \mathfrak{S}_n$ *avoiding* β *(i.e., permutations that do not contain a subsequence that is order-isomorphic to* β*).*

Example 2.2. *An occurrence of the barred pattern* $3\text{-}\bar{5}\text{-}2\text{-}4\text{-}1$ *means that we have an occurrence of the generalized pattern* 3-2-4-1 *(i.e., of the classical pattern 3241) in which no element greater than "4" is allowed to occur between "3" and "2". This is one of the two smallest excluded patterns for the set of 2-stack-sortable permutations* [9, 10]. *(The other pattern is* 2-3-4-1*, i.e., the classical pattern 2341.)*

Theorem 2.3. $S_n(3\text{-}\bar{1}\text{-}42) = S_n(3\text{-}\bar{1}\text{-}4\text{-}2) = S_n(23\text{-}1)$ *and* $|S_n(3\text{-}\bar{1}\text{-}42)| = B_n$ *(the* n^{th} *Bell number).*

Proof. As in [6], we see that each of these set consists of permutations having the form

$$\sigma = \sigma_1 a_1 \sigma_2 a_2 \ldots \sigma_k a_k,$$

where $a_k > a_{k-1} > \cdots > a_2 > a_1$ are the successive right-to-left minima of σ and where each segment $\sigma_i a_i$ is a decreasing subsequence. □

Remark 2.4. *Even though* $S_n(3\text{-}\bar{1}\text{-}42) = S_n(23\text{-}1)$*, it is more natural to use avoidance of the barred pattern* $3\text{-}\bar{1}\text{-}42$ *in studying Patience Sorting. As shown in* [4], $S_n(3\text{-}\bar{1}\text{-}42)$ *is the set of equivalence classes of* $\mathfrak{S}_n$ *modulo* $3\text{-}\bar{1}\text{-}42 \sim 3\text{-}\bar{1}\text{-}24$*, where every permutation in a given equivalence class has the* *same insertion piles R under Patience Sorting. This equivalence relation is much more difficult to describe for occurrences of* 23-1.

Corollary 2.5. *As in Theorem 2.3,*

(1) $S_n(31\text{-}\bar{4}\text{-}2) = S_n(3\text{-}1\text{-}\bar{4}\text{-}2) = S_n(3\text{-}12)$
(2) $S_n(\bar{2}\text{-}41\text{-}3) = S_n(\bar{2}\text{-}4\text{-}1\text{-}3) = S_n(2\text{-}4\text{-}1\text{-}\bar{3}) = S_n(2\text{-}41\text{-}\bar{3})$
(3) $|S_n(\bar{2}\text{-}41\text{-}3)| = |S_n(31\text{-}\bar{4}\text{-}2)| = |S_n(3\text{-}\bar{1}\text{-}42)| = B_n$.

Proof. (Sketches)

(1) Take reverse complement in $S_n(3\text{-}\bar{1}\text{-}42)$ and apply Theorem 2.3.
(2) Similar to (1). (Note that (2) is also proven in [1].)
(3) This follows from the fact that the patterns 3-1-$\bar{4}$-2 and $\bar{2}$-4-1-3 are inverses of each other.

□

3. Patience Sorting of Restricted Permutations

Proposition 3.1. *Let* $\imath_k = 12\cdots k$ *and* $\jmath_k = k\cdots 21$. *Then there is*

(1) *a bijection between* $S_n(\imath_k)$ *and pairs of pile configurations having the same shape with at most* k *piles.*
(2) *a bijection between* $S_n(\jmath_k)$ *and pairs of pile configurations having the same shape but with no pile having more than* k *cards in it.*

In [5] the authors prove

Proposition 3.2. $S_n(3\text{-}\bar{1}\text{-}42) = \{RPW(R(\sigma)) \mid \sigma \in \mathfrak{S}_n\}$. *In particular, given* $\sigma \in S_n(3\text{-}\bar{1}\text{-}42)$, *the entries in each column of the insertion piles* $R(\sigma)$ *(when read from bottom to top) occupy successive positions in* σ.

Using Proposition 3.2 and taking inverses, we obtain

Proposition 3.3. $S_n(\bar{2}\text{-}41\text{-}3) = \{RPW(R(\sigma^{-1})) \mid \sigma \in \mathfrak{S}_n\}$. *In particular, given* $\sigma \in S_n(\bar{2}\text{-}41\text{-}3)$, *the columns of the insertion piles* $R(\sigma)$ *(when read from top to bottom) contain successive values.*

Corollary 3.4. $S_n(3\text{-}\bar{1}\text{-}42, \bar{2}\text{-}41\text{-}3) = S_n(3\text{-}\bar{1}\text{-}42) \cap S(\bar{2}\text{-}41\text{-}3)$ *is the set of layered permutations in* $\mathfrak{S}_n$.

Proof. Apply Propositions 3.1–3.3 noting that $S_n(3\text{-}\bar{1}\text{-}42, \bar{2}\text{-}41\text{-}3) = S_n(23\text{-}1, 31\text{-}2)$ (as considered in [7]). □

Theorem 3.5. *The set* $S_n(3\text{-}\bar{1}\text{-}42, 31\text{-}\bar{4}\text{-}2)$ *consists of all reverse patience words having non-intersecting shadow diagrams. (I.e., no shadowlines cross in the* 0^{th} *iterate shadow diagram.) Moreover, given a permutation* $\sigma \in S_n(3\text{-}\bar{1}\text{-}42, 31\text{-}\bar{4}\text{-}2)$, *the values in the bottom rows of* $R(\sigma)$ *and* $S(\sigma)$ *increase from left to right.*

Proof. From Theorem 2.3 and Corollary 2.5, $S_n(3\text{-}\bar{1}\text{-}42, 31\text{-}\bar{4}\text{-}2) = S_n(23\text{-}1, 3\text{-}12)$ consists exactly of set partitions of $[n] = \{1, 2, \ldots, n\}$ whose components can be ordered so that both the minimal and maximal elements of the components simultaneously increase. (These are called *strongly monotone partitions* in [8]).

Let $\sigma \in S_n(3\text{-}\bar{1}\text{-}42, 31\text{-}\bar{4}\text{-}2)$. Since σ avoids 3-$\bar{1}$-42, we have that $\sigma = RPW(R(\sigma))$ by Proposition 3.2. Thus, the i^{th} shadowline $L_i^{(0)}(\sigma)$ of σ is the boundary of the union of shadows with generating points in decreasing segments $\sigma_i a_i$, $i \in [k]$, where $\sigma_i a_i$ are as in the proof of Theorem 2.3. Let b_i be the i^{th} left-to-right maximum of σ. Then b_i is the left-most (i.e. maximal) entry of $\sigma_i a_i$, so $\sigma_i a_i = b_i \sigma_i' a_i$ for some decreasing subsequence σ_i'. Note that σ_i' may be empty so that $b_i = a_i$.

Since b_i is the i^{th} left-to-right maximum of σ, it must be at the bottom of the i^{th} column of $R(\sigma)$ (similarly, a_i is at the top of the i^{th} column). So the bottom rows of both $R(\sigma)$ and $S(\sigma)$ must be in increasing order.

Now consider the i^{th} and j^{th} shadowlines $L_i^{(0)}(\sigma)$ and $L_j^{(0)}(\sigma)$ of σ, respectively, where $i < j$. We have that $b_i < b_j$ from which the initial horizontal segment of the i^{th} shadowline is lower than that of the j^{th} shadowline. Moreover, a_i is to the left of b_j, so the remaining segment of the i^{th} shadowline is completely to the left of the remaining segment of the j^{th} shadowline. Thus, $L_i^{(0)}(\sigma)$ and $L_j^{(0)}(\sigma)$ do not intersect. □

4. Non-intersecting shadow diagrams

Definition 4.1. *Given two shadowlines, $L_i^{(m)}(\sigma), L_j^{(m)}(\sigma) \in D^{(m)}(\sigma)$ with $i < j$, we call $L_i^{(m)}(\sigma)$ the* lower *shadowline and $L_j^{(m)}(\sigma)$, the* upper *shadowline. Moreover, if $L_i^{(m)}(\sigma)$ and $L_j^{(m)}(\sigma)$ intersect, then we call this a* vertical crossing *(resp.* horizontal crossing*) if it involves a vertical (resp. horizontal) segment of $L_j^{(m)}(\sigma)$.*

Each shadowline $L_i^{(m)}(\sigma) \in D^{(m)}(\sigma)$ corresponds to the pair of segments of the i^{th} columns of $R(\sigma)$ and $S(\sigma)$ that are above the m^{th} row (or are the i^{th} columns if $m = 0$). We number rows from bottom to top.

Theorem 4.2. *Each iterate $D^{(m)}(\sigma)$ $(m \geq 0)$ of $\sigma \in \mathfrak{S}_n$ is free from crossings if and only if every row in both $R(\sigma)$ and $S(\sigma)$ is monotone increasing from left to right.*

Proof. Since each $L_i^{(m)} = L_i^{(m)}(\sigma)$ depends only on the i^{th} columns of $R = R(\sigma)$ and $S = S(\sigma)$ above row m, we may assume without loss of generality that R and S have the same shape with exactly two columns.

Let $m + 1$ be the highest row where a descent occurs in either R or S. If this descent occurs in R, then $L_2^{(m)}$ is the upper shadowline in a horizontal crossing since $L_2^{(m)}$ has y-intercept below that of $L_1^{(m)}$, which is the lower shadowline in this crossing (as in 312). If this descent occurs in S, then $L_2^{(m)}$ is the upper shadowline in a vertical crossing since $L_2^{(m)}$ has x-intercept to the left of $L_1^{(m)}$, which is the lower shadowline in this crossing (as in 231). Note that both descents may occur simultaneously (as in 4231 or 45312).

Conversely, suppose m is the last iterate at which a crossing occurs in $D(\sigma)$ (i.e., $D^{(\ell)}(\sigma)$ has no crossings for $\ell > m$). We will prove that $L_2^{(m)}$ may have a crossing only at the first or last segment. This, in turn, implies that row m in R or S is decreasing. A crossing occurs when there is a vertex of $L_1^{(m)}$ not in the shadow of any point of $L_2^{(m)}$. We will prove that it can only be the first or last vertex. Let $\{(s_1, r_1), (s_2, r_2), \dots\}$ and $\{(u_1, t_1), (u_2, t_2), \dots\}$ be the vertices that define $L_1^{(m)}$ and $L_2^{(m)}$, respectively. Then $\{r_i\}_{i \geq 1}$ and $\{t_i\}_{i \geq 1}$ are decreasing while $\{s_i\}_{i \geq 1}$ and $\{u_i\}_{i \geq 1}$ are increasing. Write $(a, b) \leq (c, d)$ if (a, b) is in the shadow of (c, d) (i.e. if $a \leq b$ and $c \leq d$), and consider $L_1^{(m+1)}$ and $L_2^{(m+1)}$. They are non-crossing and defined by points $\{(s_1, r_2), (s_2, r_3), \dots\}$ and $\{(u_1, t_2), (u_2, t_3), \dots\}$, respectively. Then, for any i, $(s_i, r_{i+1}) \leq (u_j, t_{j+1})$ for some j. Suppose $(s_i, r_{i+1}) \leq (u_j, t_{j+1})$

and $(s_{i+1}, r_{i+2}) \leq (u_k, t_{k+1})$ for some $j < k$. Each upper shadowline vertex must contain some lower shadowline vertex in its shadow, so for all $\ell \in [j, k]$, $(s_i, r_{i+1}) \leq (u_\ell, t_{\ell+1})$ or $(s_{i+1}, r_{i+2}) \leq (u_\ell, t_{\ell+1})$. Choose the least $\ell \in [j, k]$ such that $(s_{i+1}, r_{i+2}) \leq (u_\ell, t_{\ell+1})$. If $(s_i, r_{i+1}) \leq (u_\ell, t_{\ell+1})$, then $(s_{i+1}, r_{i+1}) \leq (u_\ell, t_{\ell+1}) \leq (u_\ell, t_\ell)$. If $(s_i, r_{i+1}) \not\leq (u_\ell, t_{\ell+1})$, then $(s_i, r_{i+1}) \leq (u_{\ell-1}, t_\ell)$, so $(s_{i+1}, r_{i+1}) \leq (u_\ell, t_\ell)$. Thus, in both cases, $(s_{i+1}, r_{i+1}) \leq (u_\ell, t_\ell)$, and the desired conclusion follows. □

We conclude by characterizing intersecting shadowlines beyond the 0^{th} iterate of $\sigma \in \mathfrak{S}_n$ in terms of sub-pile patterns for the entries in $R(\sigma)$ and $S(\sigma)$. We state this result only for horizontal crossings, but vertical crossings can then be characterized by inverting σ (i.e., by transposing within these pairs of patterns via a Schützenberger-type symmetry result proven in [4]). Moreover, it is not difficult to show that avoiding both horizontal and vertical crossings in every iterate is equivalent to avoiding all crossings.

Theorem 4.3. *If $R(\sigma)$ and $S(\sigma)$ contain either of the following two simultaneous sub-pile patterns, then the permutation $\sigma \in \mathfrak{S}_n$ has a horizontal crossing in $D^{(m)}(\sigma)$ (here $\{x_s\}_{s\geq 1}$ and $\{y_r\}_{r\geq 1}$ are monotone increasing; $m \leq k, l$; and the numbers in the boxes indicate the number of elements in respective sub-piles):*

$$
\begin{array}{cc} \boxed{i} & \\ y_1 & \boxed{j} \\ y_3 & y_2 \\ \boxed{k} & \boxed{m} \\ \hline \end{array} \subset R
\qquad
\begin{array}{cc} \boxed{k-m} & \\ x_1 & \boxed{0} \\ x_2 & x_3 \\ \boxed{i+m} & \boxed{j+m} \\ \hline \end{array} \subset S
\qquad \text{or} \qquad
\begin{array}{cc} \boxed{i} & \\ y_1 & \boxed{j} \\ y_3 & y_2 \\ \boxed{k} & \boxed{l} \\ \hline \end{array} \subset R
\qquad
\begin{array}{cc} \boxed{k-m} & \boxed{l-m} \\ x_2 & x_1 \\ x_3 & x_4 \\ \boxed{i+m} & \boxed{j+m} \\ \hline \end{array} \subset S
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SIGNED PERMUTATIONS COUNTED BY THE SCHRÖDER NUMBERS

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A signed permutation of $1, 2, \ldots, n$ is a permutation of $1, 2, \ldots, n$ in which each entry may or may not have a bar over it. The type of a subsequence of a signed permutation is the type of the underlying sequence, with bars over the elements corresponding to barred elements of the subsequence. For instance, the subsequence $51\overline{3}$ of the signed permutation $51\overline{4}2\overline{3}$ has type $31\overline{2}$. It is well-known that the Catalan numbers count permutations which avoid a single pattern of length 3, and that the Schröder number count permutations which avoid certain pairs of patterns of length 4. In addition, Mansour and West have shown that the Catalan numbers count signed permutations which avoid $12, 1\overline{2}$, and $\overline{12}$. In this talk I will discuss several sets of signed permutations which are counted by the Schröder numbers.

Pattern avoiding permutations under one roof: Scanning elements method and functional equations

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Many families of pattern-avoiding permutations can be described by a generating tree in which each node carries one integer label, computed recursively via a rewriting rule. In this article, we will describe how to be an enumerator of number of permutations of length n that satisfy a certain set of conditions. More precisely, we will present an algorithm for finding a system of recurrence relations for the number of permutations of length n that satisfy a certain set of conditions. The rewriting of these relations automatically gives a system of functional equation satisfied by the multivariate generating function that counts the permutations by their length and the indexes of the corresponding recurrence relations. We propose an approach to describing such equations. We thus recover and refine, in a unified way, some results on τ-avoiding permutations, permutations containing τ exactly once, 1234-avoiding permutations, and 1243-avoiding permutations, where τ any classical (generalized, distanced) pattern of length three.

Compositions and multisets restricted by patterns of length three

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Abstract

We find generating functions for the number of compositions avoiding a single pattern or a pair of patterns of length three on the alphabet $\{1, 2\}$ and determine which of them are Wilf-equivalent on compositions. We also derive the number of permutations of a multiset which avoid these same patterns and determine the Wilf-equivalence of these patterns on permutations of multisets.

2000 Mathematics Subject Classification: 05A05, 05A15

1. Introduction

Pattern avoidance was first studied for $\mathfrak{S}_n$, the set of permutations of $[n] = \{1, 2, \ldots, n\}$, avoiding a pattern $\tau \in \mathfrak{S}_3$. Knuth [Kn] found that, for any $\tau \in \mathfrak{S}_3$, the number of permutations of $[n]$ avoiding τ is givien by the nth Catalan number. Later, Simion and Schmidt [SS] determined $|\mathfrak{S}_n(T)|$, the number of permutations of $[n]$ simultaneously avoiding any given set of patterns $T \subseteq \mathfrak{S}_3$. Burstein [Bu] extended this to words of length n on the alphabet $[k] = \{1, \ldots, k\}$, determining the number of words that avoid a set of patterns $T \subseteq \mathfrak{S}_3$. Burstein and Mansour [BM] considered forbidden patterns with repeated letters and we will use techniques similar to the ones used in their paper. Recently, pattern avoidance has been studied for compositions. Heubach and Mansour [HM2] counted the number of times a pattern τ of length 2 occurs in compositions, and determined the number of compositions avoiding such a pattern. Most recently, Savage and Wilf [SW] considered pattern avoidance in compositions for a single pattern $\tau \in \mathfrak{S}_3$, and showed that the number of compositions of n with parts in $\mathbb{N}$ avoiding $\tau \in \mathfrak{S}_3$ is independent of τ.

Savage and Wilf posed some open questions, one of which asked about pattern avoidance in compositions where the patterns are not themselves permutations, i.e., the pattern has repeated letters. We will answer this questions for all such patterns of length 3, and also consider pattern avoidance for pairs of such patterns. We will derive generating functions and determine which patterns or sets of patterns are avoided equally often.

2. Preliminaries

Let $\mathbb{N}$ be the set of all positive integers, and let A be any ordered finite (or infinite) set of positive integers, say $A = \{a_1, a_2, \ldots, a_d\}$, where $a_1 < a_2 < a_3 < \cdots < a_d$. For ease of notation, "ordered set" will always refer to a set whose elements are listed in increasing order.

A *composition* $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$ of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of *summands* or *letters*, namely m, is called the number of *parts* of the composition. For any ordered set $A = \{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{N}$, we denote the set of all compositions of n with parts in A (with m parts in A) by C_n^A ($C_{n;m}^A$).

To define a pattern, we utilize the notion of words. Let $[k]^n$ denote the set of all words of length n over the (totally ordered) alphabet $[k] = \{1, 2, \ldots, k\}$. We call these words *$k$-ary words of length n.* A *pattern* τ is a word in $[\ell]^m$ that contains each letter from $[\ell]$, possibly with repetitions. We say that the composition $\sigma \in C_n^A$ (resp., $\sigma \in C_{n;m}^A$) *contains* a pattern τ, if σ contains a subsequence isomorphic to τ. Otherwise, we say that σ *avoids* τ and write $\sigma \in C_n^A(\tau)$ (resp., $\sigma \in C_{n;m}^A(\tau)$). Moreover, if T is a set of patterns on $[k]^n$, then $C_n^A(T)$ (resp., $C_{n;m}^A(T)$) denotes the set all compositions in C_n^A (resp., $\sigma \in C_{n;m}^A$) that avoid all patterns from T simultaneously.

We say that two sets of patterns T_1 and T_2 belong to the same *cardinality class*, or *Wilf class*, or are *Wilf-equivalent*, if for all values of A, m and n, we have $|C_{n;m}^A(T_1)| = |C_{n;m}^A(T_2)|$. It is easy to see that for each $\tau \in [\ell]^k$, the *reversal map* defined by $r : \tau_i \mapsto \tau_{k+1-i}$ produces a pattern that is Wilf-equivalent to τ. For example, if $\tau = 1232$, then $r(\tau) = 2321$. We call $\{\tau, r(\tau)\}$ the *symmetry class* of τ. Hence, to determine cardinality classes of patterns it is enough to consider only one representative from each symmetry class.

We also look at pattern avoidance on $\mathfrak{S}_{m_1 m_2 \ldots m_k}$, the set of permutations of the multiset $S = 1^{m_1} 2^{m_2} \ldots k^{m_k}$ with $m_i > 0$. Thus, $\mathfrak{S}_{m_1 m_2 \ldots m_k}$ is the set of all words of length $m = m_1 + \cdots + m_k$ that contain the letter i exactly m_i times. For a given set of patterns T, we denote the set of permutations of the multiset S which avoid T by $\mathfrak{S}_{m_1 m_2 \ldots m_k}(T)$.

3. Single patterns of length 3

For single patterns of length 3, there are eight symmetry classes for which we will use the following class representatives: 111, 112, 121, 221, 212, 123, 132, and 213. We derive results for the first five symmetry classes, those where the pattern τ is not a permutation on [3] and give the generating functions for both the number of compositions with parts from a set A as well as for the number of permutations of the multiset $S = 1^{m_1} 2^{m_2} \ldots k^{m_k}$ that avoid a given pattern. In addition, we show that 111 is not Wilf-equivalent to any representative of the other symmetry classes, that 112 and 121 are Wilf-equivalent, and that 221 and 212 are Wilf-equivalent. To show the Wilf-equivalence of the first two patterns we exhibit a bijection ρ which can be adapted to show Wilf-equivalence of 221 and 212. Furthermore, for $A = \{1, s\}$ we give explicit formulas for the number of compositions avoiding 112 (221, respectively) using combinatorial arguments and obtain connections to known sequences listed in [S]. (The remaining three symmetry classes were considered by Savage and Wilf [SW], who showed that they are Wilf-equivalent and gave a generating function.)

4. Pairs of patterns of length three

We classify pairs of patterns of length three on [2] and determine their equivalence classes and generating functions. Note that if τ_1 and τ_2 are two patterns, then any composition that avoids $\{\tau_1, \tau_2\}$ will also avoid $\{r(\tau_1), r(\tau_2)\}$, where r is the reversal map defined in Section 2. Using this argument, the 21 possible pairs formed from the seven patterns of length 3 on [2] are reduced to 13 symmetry classes:

$$\{111, 112\}, \{111, 121\}, \{111, 212\}, \{111, 221\}, \{112, 121\}, \{112, 122\}, \{112, 211\},$$
$$\{112, 212\}, \{112, 221\}, \{121, 212\}, \{122, 212\}, \{122, 221\}, \{122, 121\}.$$

We show that the first and second symmetry classes of this list are Wilf-equivalent, and likewise, the third and fourth symmetry classes. We establish the Wilf-equivalence of the representative pairs by restricting the set of compositions to those that avoid 111, and then apply the bijection ρ used for single patterns to obtain a bijection for the pairs. None of the other symmetry classes are in the same equivalence class. We give generating functions for the number of compositions with parts from a set A and as well as for the number of permutations of the multiset $S = 1^{m_1}2^{m_2}\ldots k^{m_k}$ that avoid a given pair of patterns, with the exception of the pair $\{112, 122\}$. (This pair of patterns has also proved difficult for pattern avoidance on words.) As before, we use the set $A = \{1, s\}$ as an example and give explicit formulas derived with combinatorial arguments.

5. Some patterns of arbitrary length

Finally, we consider two types of patterns of arbitrary length for which we give generating functions for the number of compositions with parts from a set A and as well as for the number of permutations of the multiset $S = 1^{m_1}2^{m_2}\ldots k^{m_k}$ that avoid one of these types of patterns. The first pattern generalizes the pattern 111, and the second one generalizes 121. We denote the pattern consisting of s 1's (respectively, t 1's) to the left (respectively, right) of the single 2 by $v_{s,t}$ and show that all patterns $v_{s,t}$ of the same length $(= s + t + 1)$ are Wilf-equivalent.

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Introduction to Partially Ordered Patterns

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Abstract

We review selected known results on partially ordered patterns (POPs) and provide some new results on a class of POPs.

1. Introduction and background

An occurrence of a *pattern* τ in a permutation π is defined as a subsequence in π (of the same length as τ) whose letters are in the same relative order as those in τ. For example, the permutation 31425 has three occurrences of the pattern 1-2-3, namely the subsequences 345, 145, and 125. *Generalized permutation patterns* (*GPs*) being introduced in [1] allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. We indicate this requirement by removing a dash in the corresponding place. If we write, say 2-31, then we mean that if this pattern occurs in a permutation π, then the letters in π that correspond to 3 and 1 are adjacent. For example, the permutation 516423 has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563.

Further generalization of the GPs is *partially ordered patterns* (*POPs*) when the letters of a pattern form a partially ordered set (poset), and an occurrence of such a pattern in a permutation is a linear extension of the corresponding poset in the order suggested by the pattern (we also pay attention to eventual dashes). For instance, if we have a poset on three elements labelled by $1'$, 1, and 2, in which the only relation is $1 < 2$, then in an occurrence of $p = 1'$-12 in a permutation π the letter corresponding to the $1'$ in p can be either larger or smaller than the letters corresponding to 12. Thus, the permutation 31254 has three occurrences of p, namely 3-12, 3-25, and 1-25.

Let $\mathfrak{S}_n(p_1, \ldots, p_k)$ denote the set of n-permutations avoiding simultaneously each of the patterns $p_1, \ldots, p_k$.

The POPs were introduced in [5][3] as an auxiliary tool to study the maximum number of non-overlapping occurrences of *segmented* GPs (*SGPs*), that is, the GPs, occurrences of which in permutations form contiguous subwords (there are no dashes). However, the most useful property of POPs known so far is their ability to "encode" certain sets of GPs which provides a convenient notation for those sets and often gives an idea how to treat them. For example, the original proof of the fact that $|\mathfrak{S}_n(123, 132, 213)| = \binom{n}{\lfloor n/2 \rfloor}$ was on 3 pages ([4]); on the other hand, if one notices that $|\mathfrak{S}_n(123, 132, 213)| = |\mathfrak{S}_n(11'2)|$, where $11'2$ is as above, then the result is easy to see. Indeed, we

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[3]The POPs in this paper is the same as the POGPs in [5], which is abbreviation for Partially Ordered Generalized Patterns

may use the property that the letters in odd and even positions of a "good" permutation do not affect each other because of the form of $11'2$. Thus we choose the letters in odd positions in $\binom{n}{\lfloor n/2 \rfloor}$ ways, and we must arrange them in decreasing order. We then must arrange the letters in even positions in decreasing order too.

In what follows we need the following notations. Let σ and τ be two POPs of length greater than 0. We write $\sigma < \tau$ to indicate that any letter of σ is less than any letter of τ. We write $\sigma <> \tau$ when no letter in σ is comparable with any letter in τ. Also, *SPOP* abbreviates Segmented POP.

Section 2 reviews selected results in the literature, and Section 3 provides some new results, on POPs.

2. Review of selected results on POPs

2.1. **Co-unimodal patterns.** For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, the *inversion index*, $\mathrm{inv}(\pi)$, is the number of ordered pairs (i,j) such that $1 \le i < j \le n$ and $\pi_i > \pi_j$. The major index, $\mathrm{maj}(\pi)$, is the sum of all i such that $\pi_i > \pi_{i+1}$. Suppose σ is a SPOP and

$$\mathrm{place}_\sigma(\pi) = \{i \mid \pi \text{ has an occurrence of } \sigma \text{ starting at } \pi_i\}.$$

Let $\mathrm{maj}_\sigma(\pi)$ be the sum of the elements of $\mathrm{place}_\sigma(\pi)$.

If σ is *co-unimodal*, meaning that $k = \sigma_1 > \sigma_2 > \cdots > \sigma_j < \cdots < \sigma_k$ for some $2 \le j \le k$, then the following formula holds [2]:

$$\sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{maj}_\sigma(\pi^{-1})} q^{\mathrm{maj}_\sigma(\pi)} = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{maj}_\sigma(\pi^{-1})} q^{\mathrm{inv}(\pi)}.$$

2.2. **A pattern of the form σ-m-τ.** Let σ and τ be two SGPs (the results below work for SPOPs as well). We consider the POP $\alpha = \sigma\text{-}m\text{-}\tau$ with $m > \sigma$, $m > \tau$, and $\sigma <> \tau$, that is, each letter of σ is incomparable with any letter of τ and m is the largest letter in α. The POP α is an instance of so called *shuffle patterns* (see [5, Sec 4]).

Theorem 2.1. ([5, Thm 16]) *Let $A(x)$, $B(x)$ and $C(x)$ be the EGF for the number of permutations that avoid σ, τ and α respectively. Then $C(x)$ is the solution to the following differential equation with $C(0) = 1$:*

$$C'(x) = (A(x) + B(x))C(x) - A(x)B(x).$$

Corollary 2.2. ([5, Thm 13]) *Let $\alpha = \sigma\text{-}m$, where σ is a SGP on $[k-1]$. Let $A(x)$ (resp. $C(x)$) be the EGF for the number of permutations that avoid σ (resp. α). Then $C(x) = e^{F(x,A(y))}$, where $F(x, A(y)) = \int_0^x A(y)\, dy$.*

2.3. **Multi-patterns.** Suppose $\{\sigma_1, \ldots, \sigma_k\}$ is a set of SGPs and $p = \sigma_1\text{-}\cdots\text{-}\sigma_k$ where each letter of σ_i is incomparable with any letter of σ_j whenever $i \ne j$ ($\sigma_i <> \sigma_j$). We call such POPs *multi-patterns*. Clearly, the Hasse diagram for such patterns is k disjoint chains. The following theorem is the basis for calculating the number of permutations that avoid a multi-pattern.

Theorem 2.3. ([5, Thm 28]) *Let $p = \sigma_1\text{-}\cdots\text{-}\sigma_k$ be a multi-pattern and let $A_i(x)$ be the EGF for the number of permutations that avoid σ_i. Then the EGF $A(x)$ for the number of permutations that avoid p is*

$$A(x) = \sum_{i=1}^{k} A_i(x) \prod_{j=1}^{i-1} ((x-1)A_j(x) + 1).$$

Remark 2.4. *Although the result in theorem 2.3 is stated in* [5] *for σ_i's which are SGPs, one can see that the same arguments work for σ_i's which are SPOPs. Thus we have a generalization of this theorem.*

2.4. **Non-overlapping patterns – an application of POPs.** Theorem 2.3 and its counterpart in the case of words [7, Thm 4.3] and [7, Cor 4.4], as well as Remark 2.4 applied for these results, give Theorem 2.5 generalizing [5, Thm 32] and [7, Thm 5.1].

Theorem 2.5. ([6, Thm 16]) *Let p be a SPOP and $B(x)$ (resp. $B(x;k)$) is the EGF (resp. GF) for the number of permutations (resp. words over $[k]$) avoiding p. Let $D(x,y) = \sum_\pi y^{N(\pi)} \frac{x^{|\pi|}}{|\pi|!}$ and $D(x,y;k) = \sum_{n\geq 0}\sum_{w\in[k]^n} y^{N(w)}x^n$ where $N(s)$ is the maximum number of non-overlapping occurrences of p in s. Then $D(x,y)$ and $D(x,y;k)$ are given by*

$$\frac{B(x)}{1-y(1+(x-1)B(x))} \quad \textit{and} \quad \frac{B(x;k)}{1-y(1+(kx-1)B(x;k))}.$$

2.5. **Segmented patterns of length four.** In this subsection we state two results on SPOPs of length four. Several other patterns are considered in [6]. Moreover, corollaries 3.3 and 3.5 in subsection 3.1 give extra results in this direction.

Theorem 2.6. ([5, Thm 30]) *The EGF for $122'1'$-avoiding permutations is*

$$\frac{1}{2}+\frac{1}{4}\tan x(1+e^{2x}+2e^x\sin x)+\frac{1}{2}e^x\cos x.$$

Proposition 2.7. ([6, Prop 8,9]) *There are $\binom{n-1}{\lfloor (n-1)/2\rfloor}\binom{n}{\lfloor n/2\rfloor}$ permutations in $\mathfrak{S}_n$ that avoid the SPOP $12'21'$. The $(n+1)$-permutations avoiding $12'21'$ are in one-to-one correspondence with different walks of n steps between lattice points, each in a direction N, S, E or W, starting from the origin and remaining in the nonnegative quadrant.*

3. Patterns built on the letters $a, a_1, \dots, a_k$ with the only relations $a < a_i$ for all i.

3.1. **Avoidance and distribution of the patterns.** The following proposition generalizes [3, Prop. 6].

Proposition 3.1. *The permutations in $\mathfrak{S}_n$ having cycles of length at most k are in one-to-one correspondence with permutations in $\mathfrak{S}_n$ that avoid a-$a_1\cdots a_k$. Thus, the EGF for the number of permutations avoiding a-$a_1\cdots a_k$ is given by $\exp(\sum_{i=1}^k x^i/i)$.*

Proposition 3.2. *One has $|\mathfrak{S}_n(a\text{-}a_1\cdots a_k)| = |\mathfrak{S}_n(aa_1\cdots a_k)|$.*

Corollary 3.3. *The EGF for the number of permutations avoiding $aa_1a_2a_3$ is given by $\exp(x+x^2/2+x^3/3)$.*

Theorem 3.4. (Distribution of $a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$) *Let*

$$P := P(x,y) = \sum_{n\geq 0}\sum_{\pi\in\mathfrak{S}_n} y^{e(\pi)}x^n/n!$$

be the BGF for permutations where $e(\pi)$ is the number of occurrences of the SPOP $a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$ in π. Then P is the solution to

$$\frac{\partial P}{\partial x} = y\left(P-\frac{1-x^k}{1-x}\right)\left(P-\frac{1-x^\ell}{1-x}\right)+\frac{2-x^k-x^\ell}{1-x}P-\frac{1-x^k-x^\ell+x^{k+\ell}}{(1-x)^2}. \tag{3.1}$$

with the initial condition $P(0,y)=1$.

Corollary 3.5. *The EGF for the number of permutations avoiding* $a_1aa_2a_3$ *is*

$$1+\sqrt{\frac{\pi}{2}}(\mathit{erf}(\frac{1}{\sqrt{2}}x+\sqrt{2})-\mathit{erf}(\sqrt{2}))e^{\frac{1}{2}x(x+4)+2}$$

where $\mathit{erf}(x)=\frac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt$ *is the error function.*

Corollary 3.6. *The BGF for the distribution of peaks (valleys) in permutations is given by*

$$1-\frac{1}{y}+\frac{1}{y}\sqrt{y-1}\cdot\tan\left(x\sqrt{y-1}+\arctan\left(\frac{1}{\sqrt{y-1}}\right)\right).$$

3.2. **Distribution of the patterns with additional restrictions.** Let $P_k=\sum_{n=0}^{k-1}\frac{1}{n+1}\binom{2n}{n}x^n$. That is, P_k is the k initial terms in the expansion of the generating function $\frac{1-\sqrt{1-4x}}{2x}$ for the *Catalan numbers.*

Theorem 3.7. (Distribution of $a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$ on $\mathfrak{S}_n$(2-1-3)) *Let*

$$P:=P(x,y)=\sum_{n\geq 0}\sum_{\pi\in\mathfrak{S}_n(2\text{-}1\text{-}3)}y^{e(\pi)}x^n$$

be the BGF for 2-1-3*-avoiding permutations where* $e(\pi)$ *is the number of occurrences of the SPOP* $a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$ *in* π. *Then* P *is given by*

$$\frac{1-x(1-y)(P_k+P_\ell)-\sqrt{(x(1-y)(P_k+P_\ell)-1)^2-4xy(x(y-1)P_kP_\ell+1)}}{2xy}.$$

For certain choices of k, ℓ, and y in theorem 3.7 one gets Catalan numbers, *Pell numbers*, and the *triangle of Narayna numbers.*

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On growth rates of closed sets of permutations, set partitions, ordered graphs and other objects

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If X is a closed set of permutations (i.e., a lower ideal in $(\mathcal{S}, \preceq)$ where $\mathcal{S}$ is the set of all finite permutations and $\preceq$ is the standard containment ordering) and $X_n \subset X$ denotes the set of all n-permutations in X, then it is known that the counting function $n \mapsto |X_n|$ is subject to various dichotomies and restrictions (T. Kaiser and M. Klazar, *Electronic J. of Combinatorics* **9(2)** (2002/3), R10). For example, either $|X_n|$ is eventually constant or $|X_n| \geq n$ for all $n \geq 1$, or—another dichotomy—either $|X_n| \leq n^c$ for all $n \geq 1$ with a constant $c > 0$ or $|X_n| \geq F_n$ for all $n \geq 1$, where $F_n = 1, 2, 3, 5, 8, 13, \ldots$ are the Fibonacci numbers.

In my talk I will discuss a general approach to extend these results from permutations to other classes of objects, like those mentioned in the title, and to prove them uniformly as instances of a general (meta) result.

THE MOBIUS FUNCTION OF T_n^A

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Noncrossing partitions form a relatively new subject in algebraic combinatorics. An investigation of structural and enumerative properties of the lattice of noncrossing partitions of type A was initiated by Kreweras and continued by many other researchers to achieve many interesting and important results.

Bóna and Simion defined a bijection between NC_n^A (the lattice of non-crosssing partitions) and $S_n(132)$ (the set of permutations avoiding the pattern 132). We define a simpler bijection which is more natural. The bijection between NC_n^A and $S_n(132)$ proves that $S_n(132)$ is a lattice with respect to the partial order induced by the bijection, but this order is not simply defined in terms of the elements of $S_n(132)$. It impossible to define a slightly different order which is more natural on the elements of $S_n(132)$. This was first done by Bóna and Simion, who defined a partial order on the elements of $S_n(132)$, and turned it into a poset P_n^A.

We also give a formula for the Möbius function of the poset P_n^A, thus settling an open problem posed by Bóna and Simion. This is done by converting the enumeration of chains in P_n^A into an enumeration of shapes (Young diagrams, or number partitions) and using a determinantal formula due to MacMahon. In certain special cases we can also use a multiplicative formula due to Proctor.

Matchings avoiding partial patterns

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We show that certain matchings avoiding certain partial partial patterns are counted by the number of ternary trees. We give a characterization of 12312-avoiding matchings in terms of restrictions on the corresponding oscillating tableaux. We also find a bijection between Schröder paths without peaks at level 1 and matchings avoiding the patterns 12312 and 121323. Such objects are counted by the super-Catalan numbers or the little Schröder numbers. We further obtain a refinement of the super-Catalan numbers. In the sense of Wilf-equivalence, we find that the patterns 12132, 12123, 12321, 12231, 12213 are equivalent to 12312.

RSK revisited

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The powerful and partly mysterious Robinson-Schensted-Knuth (RSK) correspondence has been central to many classical and recent developments in Enumerative and Algebraic Combinatorics, and in permutation patterns in particular. We will discuss the structure of RSK, the underlying piecewise linear geometry and elaborate on connections to other combinatorial bijections. Finally, we present an extension of RSK to infinite permutations, which open a new venue for future research.

Quantum Entanglement and Permutations

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Quantum entanglement. I would like to describe a "physical property" of permutations. In Quantum Mechanics there is a fundamental quantity called *entanglement*. The role of entanglement is important and often necessary in many contexts like quantum algorithms, quantum cryptography, *etc.*. First of all, let me fix some terminology. In the Hilbert space formulation of quantum mechanics, the state of a quantum mechanical system (completely isolated from the environment) is a unit vector of the n-dimensional Hilbert space $\mathcal{H} \cong \mathbb{C}^n$ (where n depends on the classical degrees of freedom of the system). In Dirac notation, a unit vector of $\mathcal{H}$ is denoted by $|\psi\rangle$, where ψ is simply a label; given the vectors $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$, the linear functional sending $|\psi\rangle$ to the inner product $\langle\varphi|\psi\rangle$ is denoted by $\langle\varphi|$. Let S_A and S_B be two quantum mechanical systems, associated to the p-dimensional and q-dimensional Hilbert spaces $\mathcal{H}_A \cong \mathbb{C}_A^p$ and $\mathcal{H}_B \cong \mathbb{C}_B^q$, respectively. The state of the composite system S_{AB}, consisting of the subsystems S_A and S_B, is a unit vector in $\mathcal{H}_{AB} \cong \mathbb{C}_A^p \otimes \mathbb{C}_B^q$. We say that $|\chi\rangle \in S_{AB}$ is *entangled* if there are no $|\psi\rangle \in S_A$ and $|\varphi\rangle \in S_B$ such that $|\chi\rangle = |\psi\rangle \otimes |\varphi\rangle$; we say that $|\chi\rangle$ is *separable*, otherwise. Fixed $p = q = d$, an approximate measure of the entanglement in $|\chi\rangle$ is given by (the linear entropy)

$$S_L(|\chi\rangle) := \tfrac{d}{d-1}(1-\mathrm{tr}\rho^2), \qquad \text{where } \rho = \mathrm{Tr}^B|\chi\rangle\langle\chi|,$$

and Tr^B denotes transposition with respect to $\mathcal{H}_B$. The evolution in time of S_{AB} from a state $|\chi\rangle$ to a state $|\chi'\rangle$ is formalized by $U|\chi\rangle = |\chi'\rangle$, where U is a given unitary matrix and $|\chi'\rangle$ is the state after the evolution. The amount of entanglement in $|\chi\rangle$ may, of course, change under the action of U.

Entangling power of permutations. The *entangling power* of a unitary matrix $U \in \mathcal{U}(\mathcal{H}) \cong U(d^2)$ is the average amount of entanglement produced by U acting on a given (uncorrelated) distribution of separable states, that is

$$\epsilon(U) := \int_{\langle\psi_1|\psi_1\rangle=1} \int_{\langle\psi_2|\psi_2\rangle=1} S_L(U|\psi_1\rangle|\psi_2\rangle) d\psi_1 d\psi_2,$$

where $d\psi_1$ and $d\psi_2$ are normalized probability measures on unit spheres. Now, a *permutation* of $[n] = \{1, 2, ..., n\}$ is a bijection from $[n]$ to itself. Every permutation p of $[n]$ induces an $n \times n$ matrix $P = (p_{ij})$, called a *permutation matrix*, such that $p_{ij} = 1$ if $p(i) = j$ and $p_{ij} = 0$, otherwise. Equivalently a permutation on $[n]$ induces a linear map of an n-dimensional Hilbert space which permutes a given basis of the space (a *permutation operator* on $\mathcal{H}$). If $n = d^2$ we can replace $[n]$ by $[d] \times [d]$ and write $p(i, j) = (k_{ij}, l_{ij})$; thus a permutation of $[d^2]$ is represented by a pair of $d \times d$ matrices $K = (k_{ij})$ and $L = (l_{ij})$. The corresponding permutation operator permutes the elements $|i\rangle|j\rangle$ of a basis of $\mathcal{H}$: $P(|i\rangle|j\rangle) = |k_{ij}\rangle|l_{ij}\rangle$. I will describe some results about entangling power of permutation matrices. The main tool for quantifying the entangling power of a permutation matrix is a combinatorial formula given in the following theorem:

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[1]This talk is based on "Lieven Clarisse, Sibasish Ghosh, Simone Severini, Anthony Sudbery, Entangling Power of Permutations, to appear in *Phys. Rev. A. quant-ph/0502040*".

Theorem 1.8. *Let* $P = \sum_{ij} |k_{ij}l_{ij}\rangle\langle ij|$ *be a permutation matrix in* $\mathcal{U}(\mathcal{H})$*. The entangling power of* P *is given by*

$$\epsilon(P) = \tfrac{d^4+d^2-Q_P-Q_{PS}}{d(d-1)(d+1)^2}, \qquad \text{with } Q_P = \sum_{i,j,m,n=1}^{d} a_{ijm}a_{ijn}b_{imn}b_{jmn},$$

where

$$a_{ijm} = \langle l_{im}|l_{jm}\rangle = a_{jim} \quad \text{and} \quad b_{imn} = \langle k_{im}|k_{in}\rangle = b_{inm}.$$

The quantity Q_{PS} *is the corresponding expression for the permutation matrix* PS*, where* S *is the permutation of* $[d] \times [d]$ *such that* $S_{ij,kl} = 1$ *if* $i = l$ *and* $j = k$*;* $S_{ij,kl} = 0$*, otherwise.*

Permutations with zero entangling power. Two permutation matrices $P, Q \in \mathcal{U}(\mathcal{H})$ are said to be *locally unitarily connected* (for short, *LU-connected*) if there are unitaries V acting on $\mathcal{H}_A$ and W on $\mathcal{H}_B$ such that $(V \otimes W)P = Q$. Then V and W are actually permutation operators. Note that if two permutations are LU-connected then they have the same entangling power. The set of non-entangling permutation matrices is denoted by E^0.

Theorem 1.9. *Let* $P \in \mathcal{U}(\mathcal{H})$ *be a permutation matrix. Then* $P \in E^0$ *if and only if one of the following two conditions is satisfied:*

(1) *P is LU-connected to I (where I is the identity);*
(2) *P is LU-connected to S.*

Permutations with maximum entangling power. It is simple to construct the permutations with the maximum entangling power $d/(d+1)$ that can be attained by any unitary matrix in $\mathcal{U}(\mathcal{H})$. The construction makes use of latin squares. Recall that a *latin square* of *side* d is a $d \times d$ matrix with entries from the set $[d] = \{1, \ldots, d\}$ such that every row and column is a permutation of $\{1, \ldots, d\}$, and two $d \times d$ latin squares (k_{ij}) and (l_{ij}) are *orthogonal* if (k_{ij}, l_{ij}) is a permutation of $[d] \times [d]$.

Theorem 1.10. *Let* $P \in \mathcal{U}(\mathcal{H})$ *be a permutation matrix defined by* $P(|i\rangle|j\rangle) = |k_{ij}\rangle|l_{ij}\rangle$*. Then the entangling power of* P *equals the maximum value* $\epsilon(P) = d/(d+1)$ *over* $\mathcal{U}(\mathcal{H})$ *if and only if the matrices* (k_{ij}) *and* (l_{ij}) *are orthogonal latin squares.*

Corollary 1.11. *For every* $d \neq 2, 6$ *there is a permutation matrix* $P \in U(H)$ *such that* $\epsilon(P)$ *is maximum over* $U(H)$*.*

By looking at $d^2 \times d^2$ permutation matrices as made up of d^2 blocks, we can state an alternative version of Theorem 1.10.

Theorem 1.12. *Let* $P \in \mathcal{U}(\mathcal{H})$ *be a permutation matrix. Then* $\epsilon(P)$ *is maximum over* $\mathcal{U}(\mathcal{H})$ *if and only if* P *satisfies the following conditions:*

(1) *Every block contains one and only one nonzero element;*
(2) *All blocks are different;*
(3) *Nonzero elements in the same block-row are in different sub-columns;*
(4) *Nonzero elements in the same block-column are in different sub-rows.*

Theorem 1.12 states that in a $d^2 \times d^2$ permutation matrix P attaining the maximum value $d/(d+1)$, every block contains one and only one nonzero entry. It is then possible to represent P by a $d \times d$

array $\widetilde{P} = (\widetilde{p}_{ij})$, whose cell $\widetilde{p}_{ij}$ specifies the coordinates of the nonzero entry in the ij-th block of P. For the above permutation matrix

$$R = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

we have $\widetilde{R} = \begin{array}{|ccc|} \hline 11 & 23 & 32 \\ 22 & 31 & 13 \\ 33 & 12 & 21 \\ \hline \end{array}$. Note that the ij-th cell of $\widetilde{R}$ is of the form (k_{ij}, l_{ij}) where $K = (k_{ij})$ and $L = (l_{ij})$ are the orthogonal latin squares $K = \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ \hline \end{array}$ and $L = \begin{array}{|ccc|} \hline 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \\ \hline \end{array}$. It follows from Theorem 1.12 that a permutation matrix P has maximal entangling power if and only if $\widetilde{P}$ is obtained by *superimposing* two orthogonal latin squares.

Direct calculations give the following result.

Theorem 1.13. *The following statements are true:*

(1) *For $d = 2$ the matrix $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ attains the value $\epsilon(P) = \frac{4}{9}$ which is maximum over all unitaries in $\mathcal{U}(\mathcal{H})$.*

(2) *For $d = 6$ the value $\epsilon(P) = \frac{628}{735}$ is maximum over all permutations $P \in \mathcal{U}(\mathcal{H})$ and the maximizing P is associated to*

$$\widetilde{P} = \begin{array}{|cccccc|} \hline 11 & 22 & 33 & 44 & 55 & 66 \\ 23 & 14 & 45 & 36 & 61 & 52 \\ 32 & 41 & 64 & 53 & 16 & 25 \\ 46 & 35 & 51 & 62 & 24 & 13 \\ 54 & 63 & 26 & 15 & 42 & 31 \\ 65 & 56 & 12 & 21 & 33 & 44 \\ \hline \end{array}.$$

Permutations with minimum entangling power. The following theorem constructs the permutations with the minimum nonzero entangling power that can be attained by permutation matrices in $\mathcal{U}(\mathcal{H})$.

Theorem 1.14. *Let $P \in \mathcal{U}(\mathcal{H})$ be a permutation matrix. Then $\epsilon(P)$ is nonzero but minimum over all permutations in $\mathcal{U}(\mathcal{H})$ if*

$$\widehat{P} = \begin{array}{|lllll|} 11 & 12 & \ldots & \ldots & 1d \\ 21 & 22 & \ldots & \ldots & 2d \\ \vdots & \vdots & & & \vdots \\ (d-1)1 & (d-1)2 & \ldots & \ldots & (d-1)d \\ d1 & d2 & \ldots & dd & d(d-1) \end{array}.$$

In such a case $\epsilon(P) = \frac{8(d-1)}{d(d+1)^2}$. (The diagram $\widehat{P}$ represents the action of P on the set $[d] \times [d]$. The ij-th cell of $\widehat{P}$ is kl if in P the contribution of the term $|kl\rangle\langle ij|$ is nonzero.)

Two permutation matrices $P, Q \in \mathcal{U}(\mathcal{H})$ are said to be in the same *entangling class* if $\epsilon(P) = \epsilon(Q)$.

Corollary 1.15. *An upper bound to the number of different entangling classes of permutations is given by $B = 2 + \frac{1}{2}(d^4 - d^2 - 8(d-1)^2)$.*

Counting the entangling classes and their respective members.

- Classes of permutations with different entangling power and the number of elements in each class for $d = 2$:

Entangling Power $\epsilon(P)$	Number of elements in entangling class
0	8
4/9	16

- Classes of permutations with different entangling power and the number of elements in each class for $d = 3$:

Entangling Power $\epsilon(P)$	Number of elements in entangling class
0	72
1/3	2592
3/8	864
5/12	1296
182/375	10368
23/48	20736
1/2	27432
25/48	36288
13/24	44064
9/16	101376
7/12	44712
29/48	46656
5/8	22464
2/3	3888
3/4	72

- Number of classes of permutations with different entangling power and the average entangling power as a function of the dimension d.

Dimension d	Number of classes	Average entangling power
2	2	$\frac{8}{27} \approx 0.29$
3	15	$\frac{31}{56} \approx 0.55$
4	≥ 65	0.67 ± 0.01
5	≥ 190	0.74 ± 0.01

Open problems. We conclude with a list of open problems:

- Describe a general classification of the permutation matrices according to their entangling power (that is, give formulas to count the members in each entangling class and the number of classes).
- Give a formula for the average entangling power over all permutation matrices of a given dimension.
- For $d = 6$, does there exist $U \in \mathcal{U}(\mathcal{H})$ such that $\epsilon(U) > \frac{628}{735}$?
- Study the entangling power of permutation matrices in relation to multipartite systems. In this context, it is conceivable that the permutation matrices with maximum entangling power are related to sets of mutually orthogonal latin squares.

Sorting by Stacks in Series

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We will look at two standard algorithms used to sort permutations by t (increasing) stacks in series. The first is the right-greedy algorithm which works on t stacks in series the same way as t-stack sorting as introduced by West. The second is the left-greedy algorithm introduced by Atkinson, Murphy, and Ruškuc. They showed that the left-greedy algorithm is optimal for $t = 2$.

By considering where entries of the permutation are at key points in the process, it can be shown that the use of the left-greedy algorithm on t stacks in series will result in sorting any permutation that could be sorted using the right-greedy algorithm on t stacks in series. For $t \geq 2$, one can construct permutations which can be sorted by just two stacks in series using the left-greedy algorithm, but cannot be sorted by t stacks in series using the right-greedy algorithm. In addition, there are ways to construct permutations that are sortable by t stacks in series using the left-greedy algorithm (or with no restriction to any one given algorithm) from other sortable permutations that does not work when restricting the sorting process to the right-greedy algorithm. It can also be shown that when applied to t stacks in series where $t > 2$, the use of the left-greedy algorithm is not optimal nor is the class of permutations sorted by the left-greedy algorithm closed.

We also consider the addition of a type of movement on the stacks where entries from the last stack can return to the first stack instead of going to the output. It can be shown that this allowance does nothing to improve the sorting ability of two stacks in series. However, when this move is introduced to three or more stacks in series, we are then able to sort any permutation.

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Extremal problems of forbidden submatrices

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A 0-1 matrix A is said to contain another 0-1 matrix (a pattern) P if P is a submatrix of A or it can be obtained from a submatrix of A by changing a few 1 entries to 0. The talk surveys the results on the following extremal problem: How many 1 can be in a n by n 0-1 matrix not containing the pattern P?

This problem can be considered a generalization of the Turan-type extremal graph theory where the graph and the forbidden pattern have a specified vertex-order.

As classical extremal graph theory, the extremal theory of 0-1 matrices was also largely motivated by implications and applications in combinatorial geometry. Later Martin Klazar showed an exciting connection between this theory and the theory of forbidden permutation patterns that lead to the solution of the Stanley-Wilf conjecture. I believe that the two theories are deeply connected and exploiting this connection will lead to further developments in both theories.

Enumeration schemes for restricted permutations

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Almost a decade ago, Zeilberger introduced "enumeration schemes," a systematic method to produce formulas for restricted permutations. Enumeration schemes bear some resemblence to other techniques, such as generating trees (popularized by West) and the insertion encoding (recently introduced by Albert, Linton, and Ruskuc). However, the important distinction between enumeration schemes and other methods is that every step in the derivation of enumeration schemes can be completely automated. This automatation is performed in Zeilberger's Maple package WILF. Unfortunately, as Zeilberger put it, "the success rate of the present method, in its present state, is somewhat disappointing." I will discuss a new extension of enumeration schemes, implemented in my WILFPLUS package, which boasts a less disappointing success rate.

Computer-generated permutation enumeration

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Computers can do, all by themselves, much more than compute the number of n-permutations avoiding such and such, for n less than 10. They can do it, in many cases, for n less than a 100, and in many cases even for all n.

These claims are currently vindicated in the dynamic webbook "Systematic Studies in Pattern Avoidance" (http://www.math.rutgers.edu/ lpudwell/webbook/bookmain.html) written by my brilliant students Lara Pudwell and Vince Vatter, and by my beloved electronic colleague and disciple, Shalosh B. Ekhad.