Permutation Reconstruction

Rebecca Smith
Department of Mathematics,
University of Florida,
Gainesville, FL 32611-8105
smithr@math.ufl.edu

Abstract

A well known problem in graph theory is graph reconstruction. The question is that if in all possible ways, one eliminates a single vertex of a graph containing at least three vertices, can the original graph be reconstructed from this multiset of minors? This problem has also been looked at using elimination of edges instead of vertices. We consider the corresponding problem using permutations and their entries instead of graphs.

Let p be a permutation of length n. In all possible ways, delete k entries of p. Then renumber (retaining the order) the obtained $\binom{n}{k}$ strings as permutations (so their entries are from 1 through $n-k$). Call the multiset of these permutations the $(n-k)$*-minor multiset of* p and denote it $M_{n-k}(p)$.

Let N_k be the smallest number such that if given $M_{n-k}(p)$, where p is a permutation of length $n \geq N_k$, we can figure out what p must have been. That is, each permutation of length $n \geq N_k$ gives a unique multiset of $(n-k)$-minors, but there is a pair of permutations p and q each of length $N_k - 1$ where $M_{n-k}(p) = M_{n-k}(q)$.

For the case $k = 1$, it can be shown that $N_1 = 5$. When permutations have length of at least five, one can first prove that the position of entry 1 must be in the same position in any two permutations with the same $(n-1)$-minor multiset. Then using induction, one can show each entry must be in the same position in both permutations. To complete the proof, one can see that the permutations 2413 and 3142 have the same 3-minor multisets and so N_1 is exactly five.

We show that the $(n-2)$-minor multiset of a permutation of length n where n is at least six is unique to that permutation.

This is done by first showing that if two permutations p and q of length $n \geq 6$ were to have the same $(n-2)$-minor multiset, then the entries 1 and 2 must be in the same respective positions in both permutations. Considering where 1 must be in the minors initially forces the position of the entry 1 in p to be no more than two spaces away from where it is in q. Then examining the cases where the positions are different by one or two places, we force a contradiction. To establish that the same must be true for the entry 2, we consider what happens when 1 is one of the entries eliminated, but 2 is not. Then after renumbering, the position of 2 in the original permutations will determine the position of 1 in $n-1$ of the $(n-2)$-minors for each of p and q. This can once again be used to show that 2 was in fact in the same position both p and q.

Once this is established, one can use induction to show that in fact, each entry is in the exact same position in p as it is in q and thus $p = q$. This is done by contradiction as follows:

Say p_i is the smallest entry of p that occurs in a different position in q. From the previous two lemmas, we know $p_i > 2$.

So $1, 2, ..., p_i - 1$ each occur in the same position in p as in q. Thus, when either zero or one entry less than or equal to $p_i - 2$ is eliminated, the entries $p_i - 2$ and $p_i - 1$ respectively from the original permutations give the exact same contributions to the position of $p_i - 2$ in the $(n-2)$-minors for both p and q.

The final case involves $(n-2)$-minors of p and q for which both eliminated values were less than p_i. In this case, p_i will be reassigned the value of $p_i - 2$.

By considering where the entry $p_i - 2$ is in the $(n-2)$-minors, we are able to show that p_i must have been in the same position in both p and q. Thus p and q are the same permutation.

We complete the proof that N_2 is exactly six, by considering the permutations 13524 and 14253. These permutations have the same 3-minor multisets.

Note that these permutations are the same as those considered in the case when $k = 1$ except with a 1 added to the beginning of each and the remaining entries renumbered accordingly.