RESTRICTED 132-AVOIDING k-ARY WORDS, CHEBYSHEV POLYNOMIALS, AND CONTINUED FRACTIONS

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Abstract

We study generating functions for the number of n-long k-ary words that avoid both 132 and an arbitrary ℓ-ary pattern. In several interesting cases the generating function depends only on ℓ and is expressed via Chebyshev polynomials of the second kind and continued fractions.

1. Extended abstract

1.1. **Permutations.** Let $\mathfrak{S}_n$ denote the set of permutations of $[n] = \{1, 2, \dots, n\}$, written in one-line notation, and suppose $\pi \in \mathfrak{S}_n$. We write π_i to denote the ith element of π, for $i = 1, 2, \dots, n$. Let $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$ be two permutations. We say that π *contains* τ if there exists a subsequence $\pi_{i_1}, \dots, \pi_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, such that it is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that π *avoids* τ, or is τ-*avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in $\mathfrak{S}_n$ is denoted by $\mathfrak{S}_n(\tau)$. For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids any $\tau \in T$; the corresponding subset of $\mathfrak{S}_n$ is denoted by $\mathfrak{S}_n(T)$. For example, the permutation 562314 avoids 132, but it has 634 as a subsequence so it does not avoid 312.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ^1, τ^2. Several recent papers [5, 8, 7, 9, 10, 11] deal with the case $\tau^1 \in \mathfrak{S}_3$, $\tau^2 \in \mathfrak{S}_k$ for various pairs τ^1, τ^2. The tools involved in these papers include continued fractions, Chebyshev polynomials of the second kind, and Dyck words. For example, Chow and West [5] have show that

$$\sum_{n\geq 0} |\mathfrak{S}_n(132, 12\dots k)| x^n = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)},$$

where $U_n(t)$ is the nth Chebyshev polynomial of the second kind (in what follows just Chebyshev polynomials), which may be defined by $U_n(\cos t) = \sin(n+1)t/\sin t$. Clearly, $U_n(t)$ is a polynomial of degree n in t with integer coefficients, and the following recurrence holds:

$$U_0(t) = 1,\ U_1(t) = 1,\ \text{and}\ U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)\ \text{for all}\ n \geq 2.$$

The same recurrence is used to define $U_n(t)$ for $n < 0$ (for example, $U_{-1}(t) = 0$ and $U_{-2}(t) = -1$). Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [13]). Apparently, the relation between restricted permutations and Chebyshev polynomials was discovered for the first time by Chow and West in [5], and later by Mansour and Vainshtein [8, 9, 10, 11] and Krattenthaler [7]. For example, Mansour and Vainshtein [9] have shown a recursive formula for the generating function of the number of permutations in $\mathfrak{S}_n(132,\tau)$ for any $\tau \in \mathfrak{S}_k(132)$. In particular, they proved that

$$\sum_{n\geq 0} |\mathfrak{S}_n(132,\tau)|x^n = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)}$$

for any wedge pattern $\tau \in \mathfrak{S}_k(132)$ as defined in [9]. For other results involving 132-avoiding permutations and continued fractions or Chebyshev polynomials, see [8] and the references therein.

1.2. **Words.** Let $[k]$ be a (totally ordered) alphabet on k letters. We call the elements of $[k]^n$ either *n-long k-ary words*, or *words on the letters* $1, 2, \ldots, k$, or just *words*. Consider two words, $\sigma \in [k]^n$ and $\tau \in [\ell]^m$. In other words, σ is an n-long k-ary word and τ is an m-long ℓ-ary word. Assume additionally that τ contains all letters 1 through ℓ. We say that σ contains an *occurrence* of τ, or simply that σ *contains* τ, if σ has a subsequence *order-isomorphic* to τ, i.e. if there exist $1 \leq i_1 < \ldots < i_m \leq n$ such that, for any relation $\phi \in \{<, =, >\}$ and indices $1 \leq a, b \leq m$, $\sigma(i_a)\phi\sigma(i_b)$ if and only if $\tau(a)\phi\tau(b)$. In this situation, the word τ is called a *pattern*. If σ contains no occurrences of τ, we say that σ *avoids* τ. For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids any $\tau \in T$; the corresponding subset of $[k]^n$ is denoted $[k]^n(T)$.

Burstein [3] found the complete answer for the cardinalities $|[k]^n(T)|$ where $T \subseteq \mathfrak{S}_3$. For example, he has shown that (see also Brändén and Mansour [2])

$$\sum_{n,k\geq 0} |[k]^n(132)|x^n y^k = 1 + \frac{y}{1-x} + \frac{2y^2}{(1-2x)(1-y) + \sqrt{((1-2x)^2-y)(1-y)}}. \tag{1.1}$$

Recently, Burstein and Mansour [4] gave a complete answer for the cardinalities $|[k]^n(\tau)|$ where $\tau \in [3]^3$, for example they showed that

$$\sum_{n,k\geq 0} |[k]^n(112)|x^n y^k = \frac{1}{1-y}\left(\frac{1-y}{1-y-xy}\right)^{1/y}. \tag{1.2}$$

1.3. **Main results.** Motivated by the parallels among restricted 132-avoiding permutations (see [8, 9]) and restricted n-long k-ary words (see [3]), we present in this paper a general approach to the study of n-long k-ary words that avoid both 132 and an arbitrary ℓ-ary pattern. As a consequence, we derive all the previously known results for these kinds of problems (see [3]), and we give analogies and generalization for some of the results in [5, 8, 7, 9, 10, 11]), as well as many new results. For example, we prove the following result.

Theorem 1.1. *Let $\mathcal{F}_\ell(x,y)$ be the generating function for the number of n-long k-ary words that avoids $12\ldots\ell$, that is, $\mathcal{F}_\ell(x,y) = \sum_{n,k\geq 0}\sum_{\pi\in[k]^n(12\ldots\ell)} x^n y^k$. Then for any $\ell \geq 3$ we have the* *recurrence*

$$\mathcal{F}_\ell(x,y)=1+\cfrac{y}{(1-x)(1-y)+\cfrac{xy(1-x)(1-y)}{x(1-y)+\cfrac{y}{1-\mathcal{F}_{\ell-1}(x,y)}}},$$

with $F_2(x,y)=\frac{1-x}{1-x-y}$. *Thus,* $\mathcal{F}_\ell(x,y)$ *can be expressed as*

$$\mathcal{F}_\ell(x,y)=1+\cfrac{y}{(1-x)(1-y)-\cfrac{xy(1-x)(1-y)}{(1-2x)(1-y)-\cfrac{xy(1-x)(1-y)}{\cfrac{\ddots}{(1-2x)(1-y)-\cfrac{}{(1-2x)(1-y)-xy}}}}},$$

where the fraction has ℓ *levels, or in terms of Chebyshev polynomials of the second kind, as*

$$\mathcal{F}_\ell(x,y)=1+\sqrt{xy^3(1-x)(1-y)}\frac{(1-2x-y)U_{\ell-2}(t)+\frac{xy^2(1-x-y)}{\sqrt{xy^3(1-x)(1-y)}}U_{\ell-3}(t)}{(1-2x-y)U_{\ell-3}(t)+\frac{xy^2(1-x-y)}{\sqrt{xy^3(1-x)(1-y)}}U_{\ell-4}(t)},$$

where $t=-\frac{1-2x}{2}\sqrt{\frac{1-y}{xy(1-x)}}$.

In particular, the above theorem gives that the number of n-long k-ary words that avoids 132 is given by 1, 2^n, $(n^2+3n+8)2^{n-2}$, and $\frac{1}{3}(n^4+10n^3+59n^2+122n+192)2^{n-6}$, where $k=1,2,3,4$, respectively. Also, we remark that the above theorem with ℓ increase to infinite gives (1.1).

Burstein [3, Section 4] has shown analytically that the number of n-long k-ary words that avoids 123 is the same as the number of n-long k-ary words that avoids 312, and the generating function is given by $\frac{(1-x)(1-2x)-y(1-2x+2x^2)}{(1-x)(1-y)(1-2x-y)}$. However, as stated in [3], a challenging question is to prove this fact bijectively. In the following theorem we give a complete answer for this question.

Theorem 1.2. *Let* $\ell\geq 1$. *There exists a bijection between the set of all* n*-long* k*-ary words that avoid both* 132 *and* $12\ldots\ell$, *and the set of all* n*-long* k*-ary words that avoid both* 132 *and* $\ell 12\ldots(\ell-1)$).

For a further generalization of Theorem 1.1, consider the pattern $\tau=\tau'\ell(\ell+1)$ where τ' is a $(\ell-1)$-ary pattern.

Theorem 1.3. *Let* $\tau=\tau'\ell(\ell+1)\in[\ell+1]^m$ *such that* $\tau'\in[\ell-1]^{m-2}$. *Then the generating function for the number of* n*-long* k*-ary words that avoids* τ *satisfies that*

$$\mathcal{F}_\tau(x,y)=1+\cfrac{y}{(1-x)(1-y)+\cfrac{xy(1-x)(1-y)}{x(1-y)+\cfrac{y}{1-\mathcal{F}_{(\tau',\ell)}(x,y)}}},$$

where

$$\mathcal{F}_{213}(x,y)=\frac{(1-x)^2(1-2x-(1-x)y)}{(1-x)^2(1-2x)-(1-x)(2-4x+x^2)y+(1-2x)y^2}.$$

Also, we study the generating functions for the number of n-long k-ary words that avoid both 132 and an arbitrary pattern τ with repeated letters, that is, $\tau\in[\ell]^m$ with $\ell<m$. For example, in this paper we prove the following result.

Theorem 1.4. *Let $\tau = 122\dots2 \in [2]^{m+1}$. Then*

$$(1-y)\mathcal{F}_\tau(x,y) = 1 + \frac{xy}{1-x} + \sum_{j=1}^{m-1} \frac{x^j(1-y)^j}{y^{j-1}}(\mathcal{F}_\tau(x,y)-1)^{j+1} + \frac{x^m(1-y)^{m-1}}{(1-x)y^{m-2}}(\mathcal{F}_\tau(x,y)-1)^m.$$

The above theorem together with using the Lagrange inversion formula we get the following corollary.

Corollary 1.5. *Let $k \geq 1$. The generating function for the number of n-long k-ary words that avoid $122\dots2 \in [2]^{m+1}$ is given by*

$$\frac{1}{1-x} + \sum_{j\geq 0} \binom{k-1}{j} \frac{M_j(x)}{x},$$

where

$$M_t(x) = \sum_{a=0}^{t}\sum_{b=0}^{t-a}\sum_{j_1=0}^{t}\sum_{j_2=0}^{mb}\sum_{j_3=0}^{(m-1)(t-a-b)} \frac{(-1)^{t+a}(t-1)!}{a!b!(t-a-b)!}\binom{t}{j_1}\binom{mb}{j_2}\binom{(m-1)(t-a-b)}{j_3}\cdot \binom{2t-2-j_1-j_2-j_3}{t-1}\frac{x^{mt-j_1-j_2-j_3-(m-1)a+2b}}{(1-x)^{(m-1)(t-a)+1+b-a}(1-2x)^{t-j_1-j_2-j_3+a+b-1}}.$$

In particular, Theorem 1.4 for $m = 2$ yields

$$\mathcal{F}_{122}(x,y) = \frac{1+x-\sqrt{(x-1)^2 - \frac{4xy}{1-y}}}{2x}.$$

Moreover, Corollary 1.5 obtains that the generating function for the number of n-long k-ary words that avoid both 132 and 122 is given by

$$\frac{1}{1-x} + x\sum_{i=0}^{k-2}\left(\sum_{j=0}^{i}\frac{1}{i+1}\binom{i+1}{j}\binom{i+1}{j+1}\frac{x^{2j}}{(1-x)^{2i-1}}\right).$$

Also, we study the number of n-long k-ary words that avoid 132, 212, and an arbitrary pattern τ. For example, we prove the following result.

Theorem 1.6. *Let $l \geq 3$. Then the generating function for the number of n-long k-ary words that avoid 132, 212 and $12\dots\ell$ is given by*

$$\cfrac{1}{1-\cfrac{(1-2x)y}{(1-x)^2}-\cfrac{xy(1-x)^{-2}(1-y)}{\ddots-\cfrac{\ddots}{1-\cfrac{(1-2x)y}{(1-x)^2}-xy(1-x)^{-2}(1-y)\cfrac{(1-x)}{(1-x-y)}}}},$$

where the fraction has $\ell-2$ levels.

In particular, Theorem 1.6 gives the generating function for the number of n-long k-ary words that avoid both 132 and 212 is given by

$$\frac{(1-x)^2-(1-2x)y-\sqrt{(1-x)^4-2(1-x)^2y+(1-4x^2+4x^3)y^2}}{2xy(1-y)}.$$

Also, we study the number of n-long k-ary words that avoid 132, 121, and an arbitrary pattern τ. For example, we prove the following theorem.

Theorem 1.7. *Let $\ell \geq 2$. Then the generating function for the number of n-long k-ary words that avoid* 132, 121 *and* $12\ldots\ell$ *can be expressed in terms of Chebyshev polynomials as*

$$\frac{U_{\ell-1}\left(\frac{1}{2}\sqrt{\frac{1-y}{x}}\right)}{\sqrt{x(1-y)}U_\ell\left(\frac{1}{2}\sqrt{\frac{1-y}{x}}\right)}.$$

In particular, Theorem 1.7 gives that the generating function for the number of n-long k-ary words that avoids 132 and 121 is given by

$$\frac{1+x+\sqrt{(1-x)^2-\frac{4xy}{1-y}}}{2x}.$$

Moreover, the generating function for the number of n-long k-ary words that avoid both 132 and 121 is given by

$$\frac{1}{1-x}+x\sum_{i=0}^{k-2}\left(\sum_{j=0}^{i}\frac{1}{i+1}\binom{i+1}{j}\binom{i+1}{j+1}\frac{x^{2j}}{(1-x)^{2i-1}}\right),$$

for any given $k \geq 2$.

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