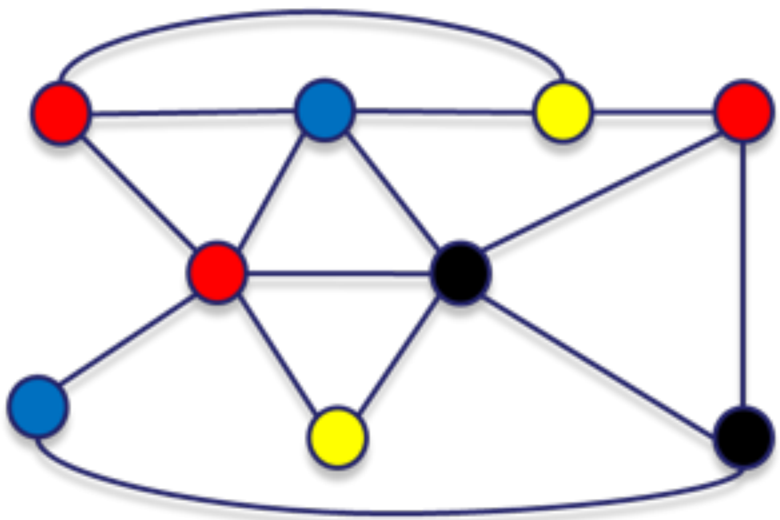
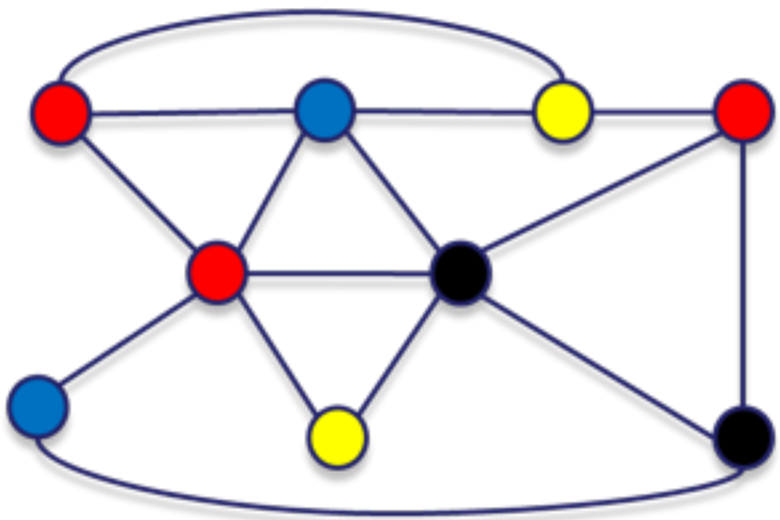
Distance Oracles for Vertex-Colored Graphs

Oren Weimann
Weizmann Institute



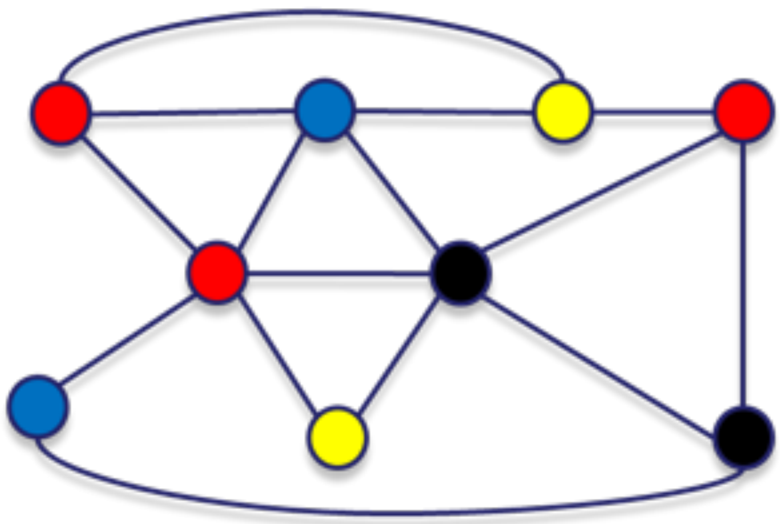
Joint work with:
Danny Hermelin, Avivit Levy, and Raphael Yuster

Vertex-Colored Networks



Vertex-Colored Networks

- A graph G where each vertex v has a color $c(v)$.



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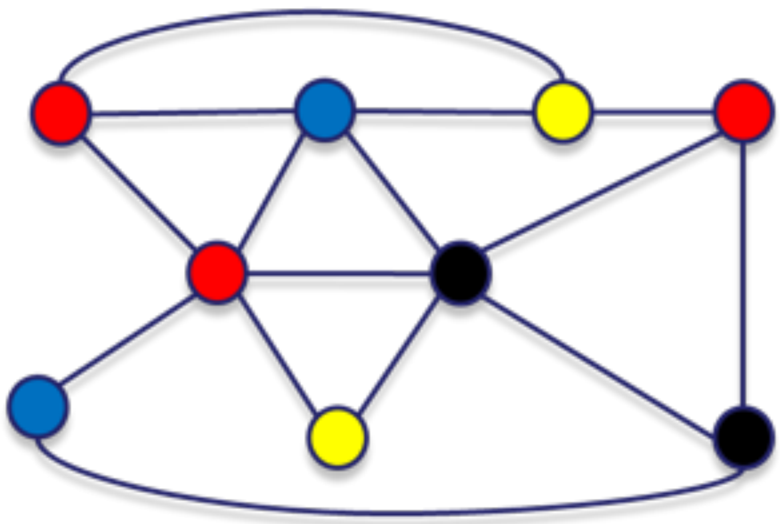

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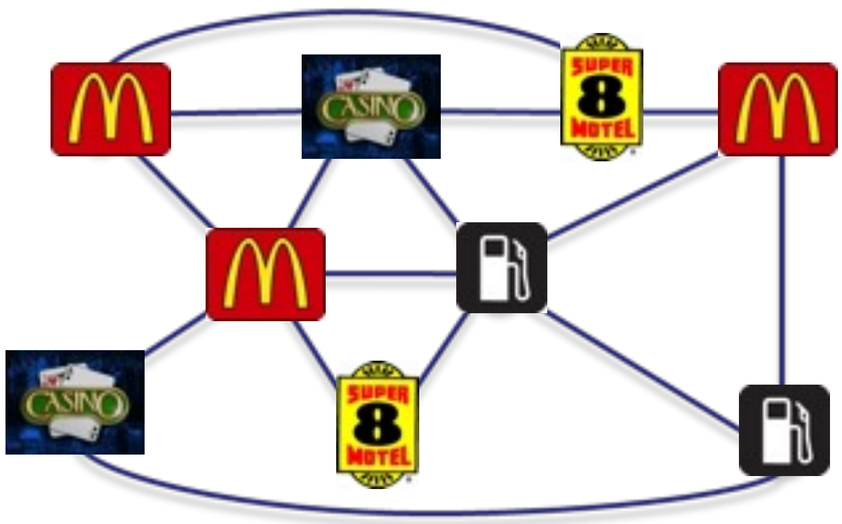
road network



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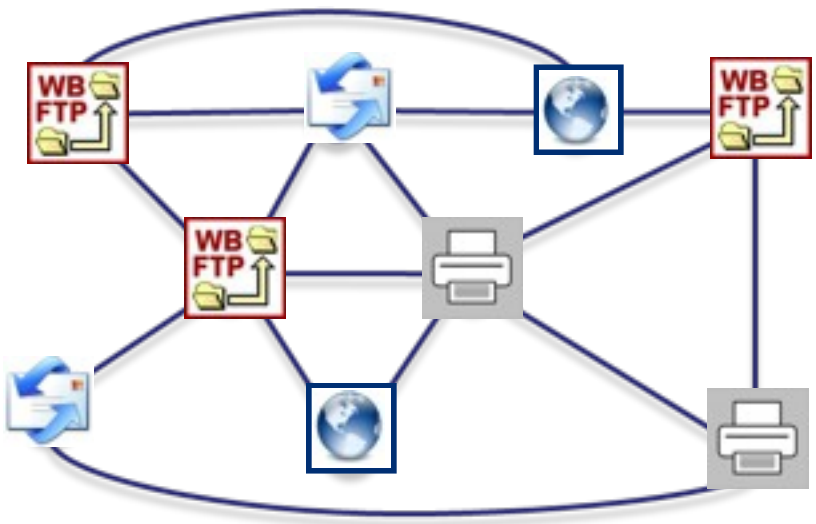
computer
network



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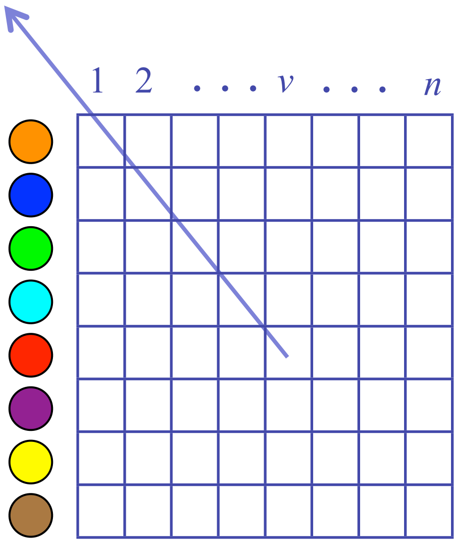

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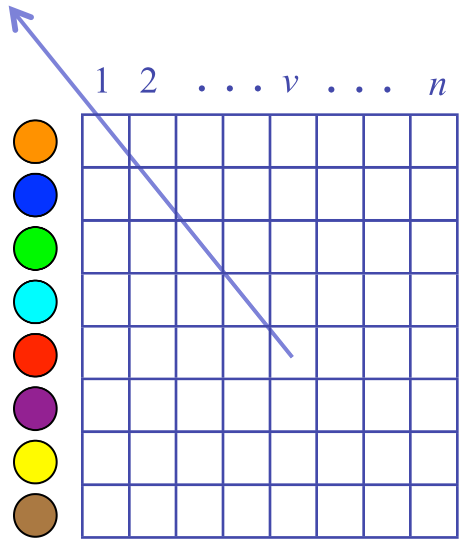

Dijkstra's algorithm
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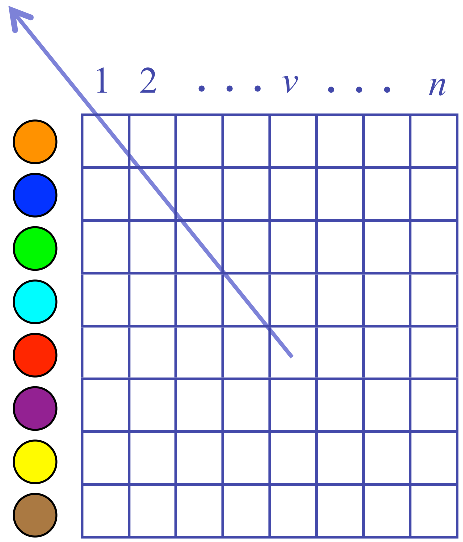

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Theorem: *$O(kn^{1+1/k})$-space $(4k$-$5)$-stretch vertex-color oracles.*

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Theorem: *$O(kn^{1+1/k})$-space $(3^{k-1}\text{-}1)$-stretch vertex-color oracles allowing changing colors in $O(kn^{1/k}\lg n)$ time.*

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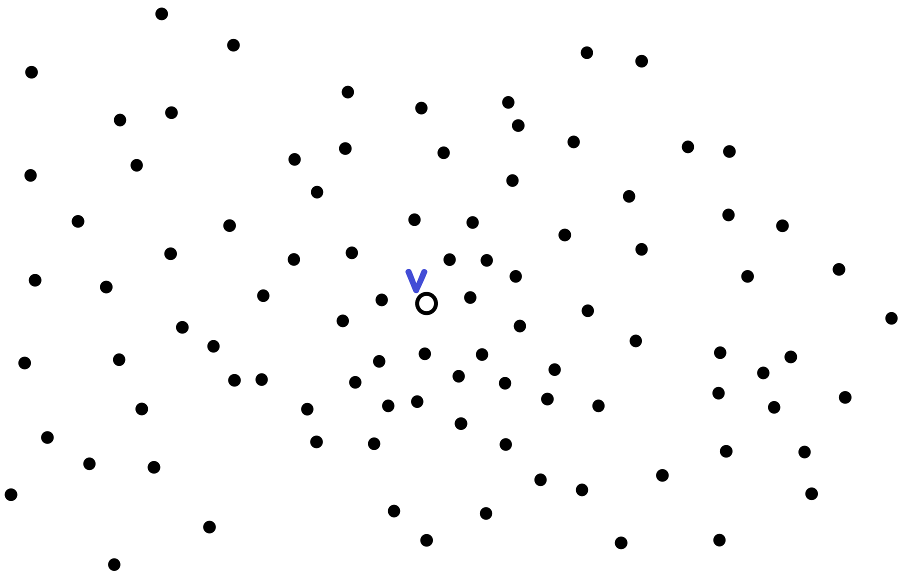

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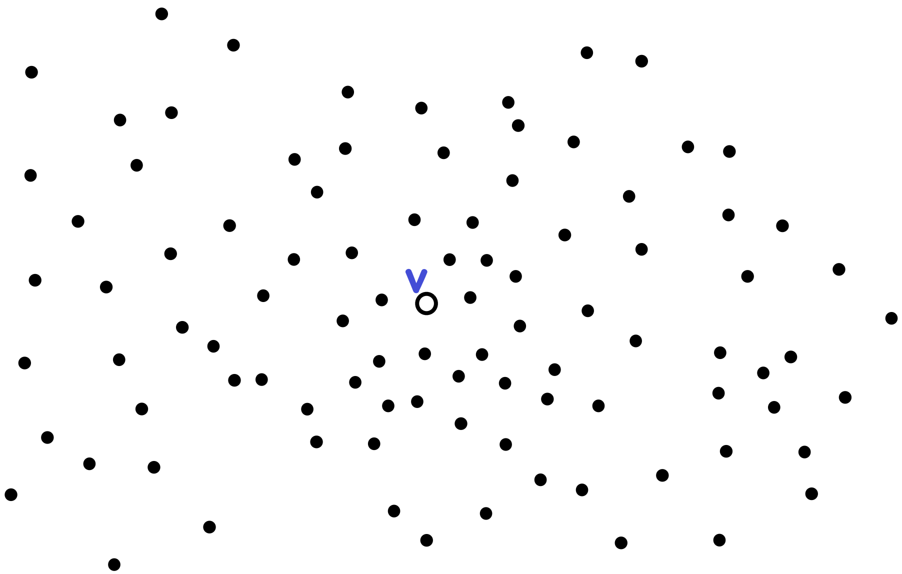

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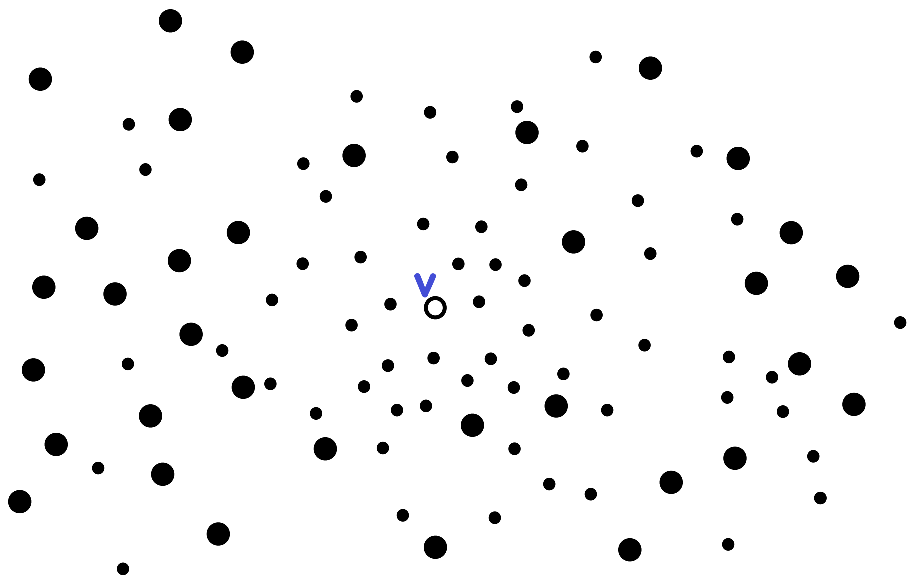

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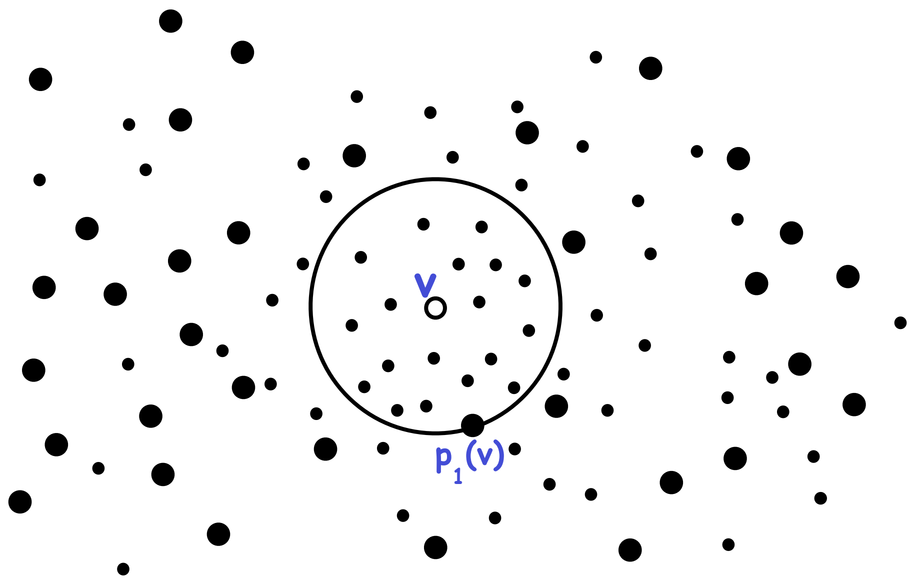

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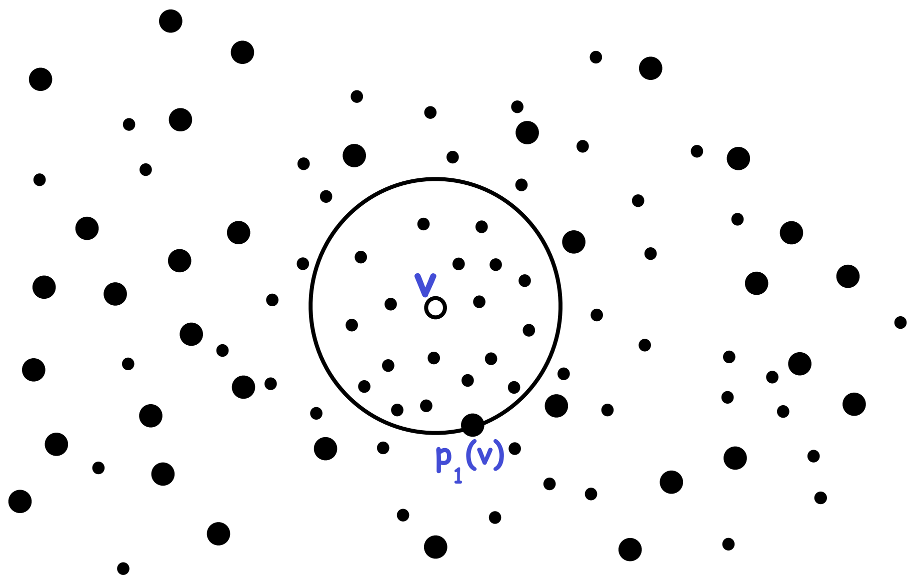

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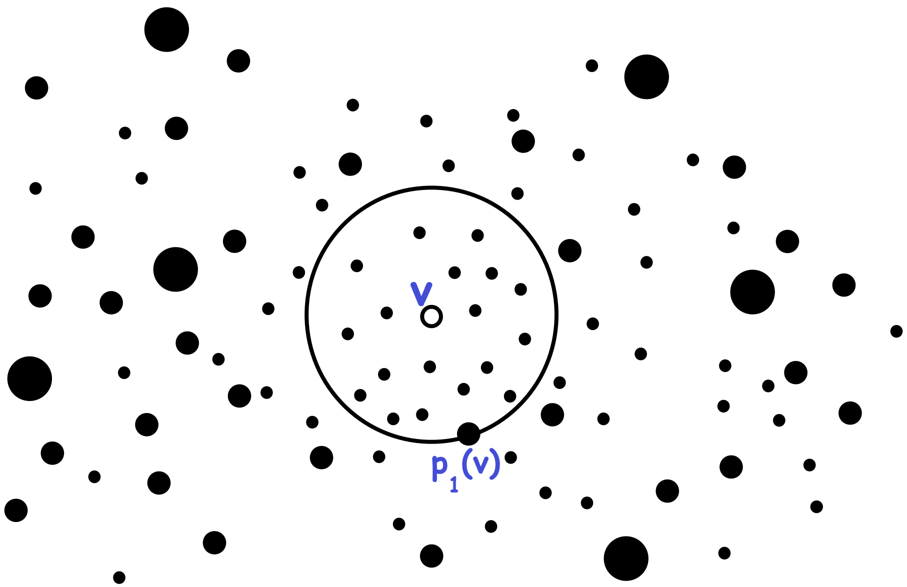

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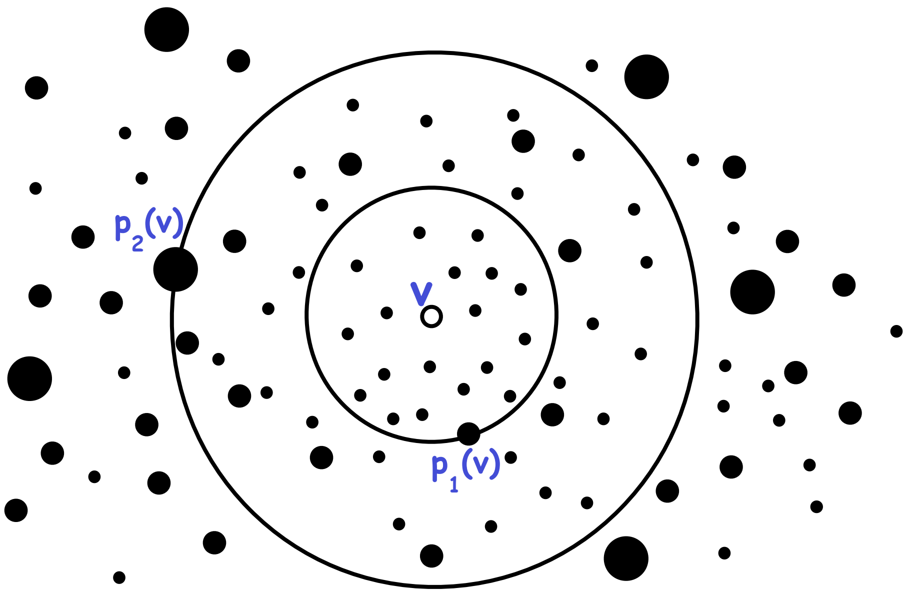

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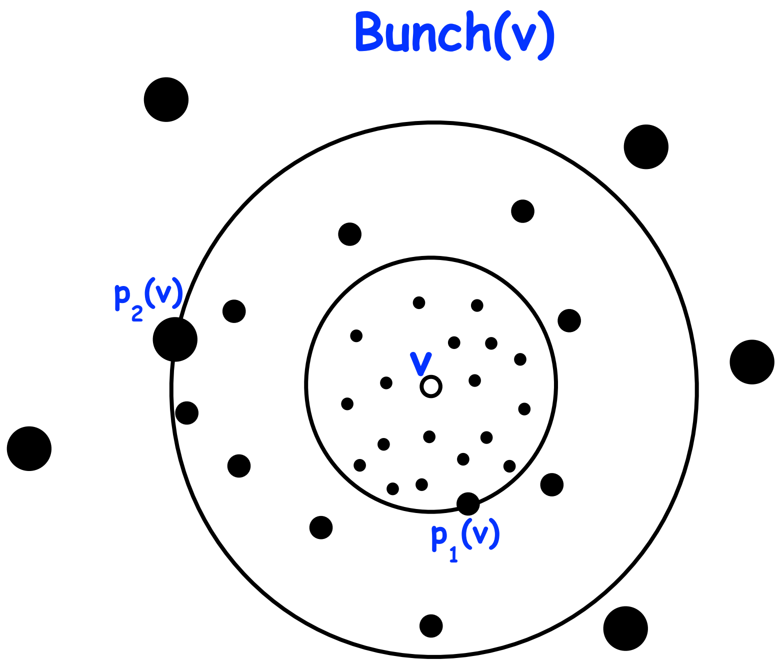

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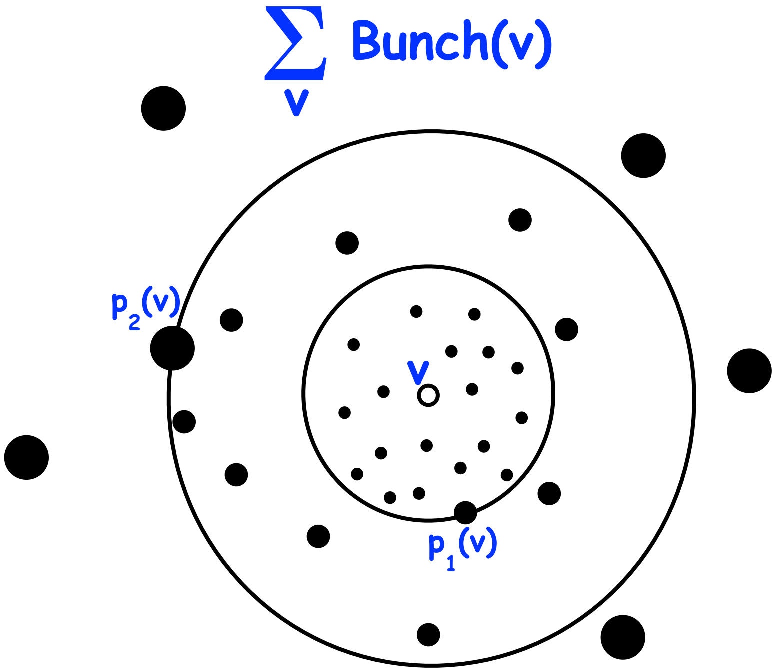

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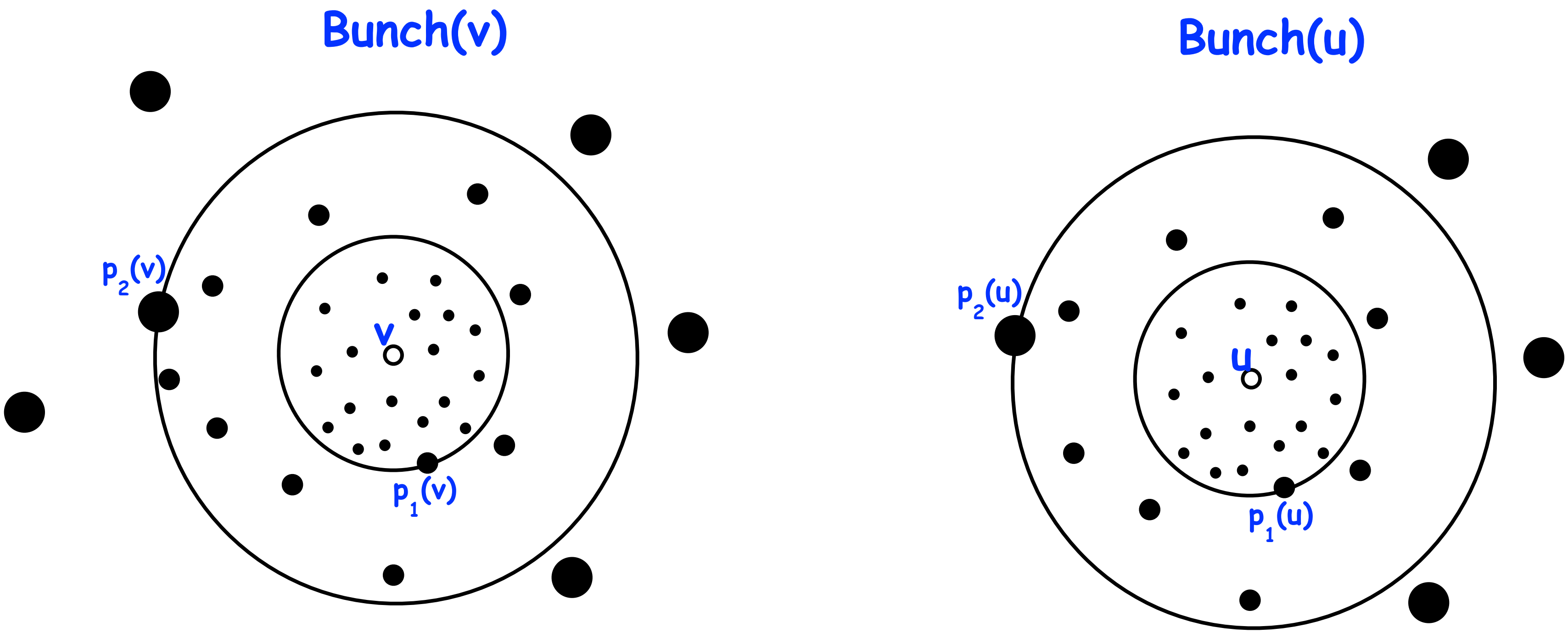

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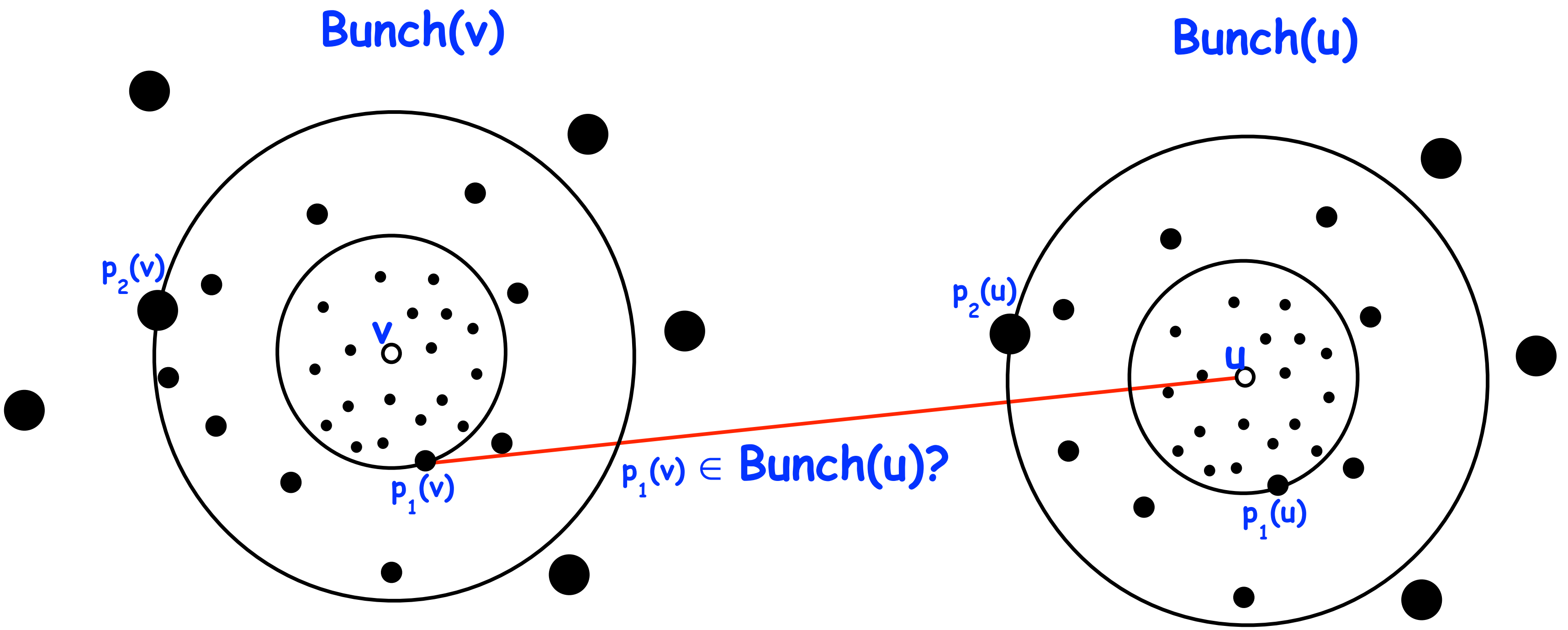

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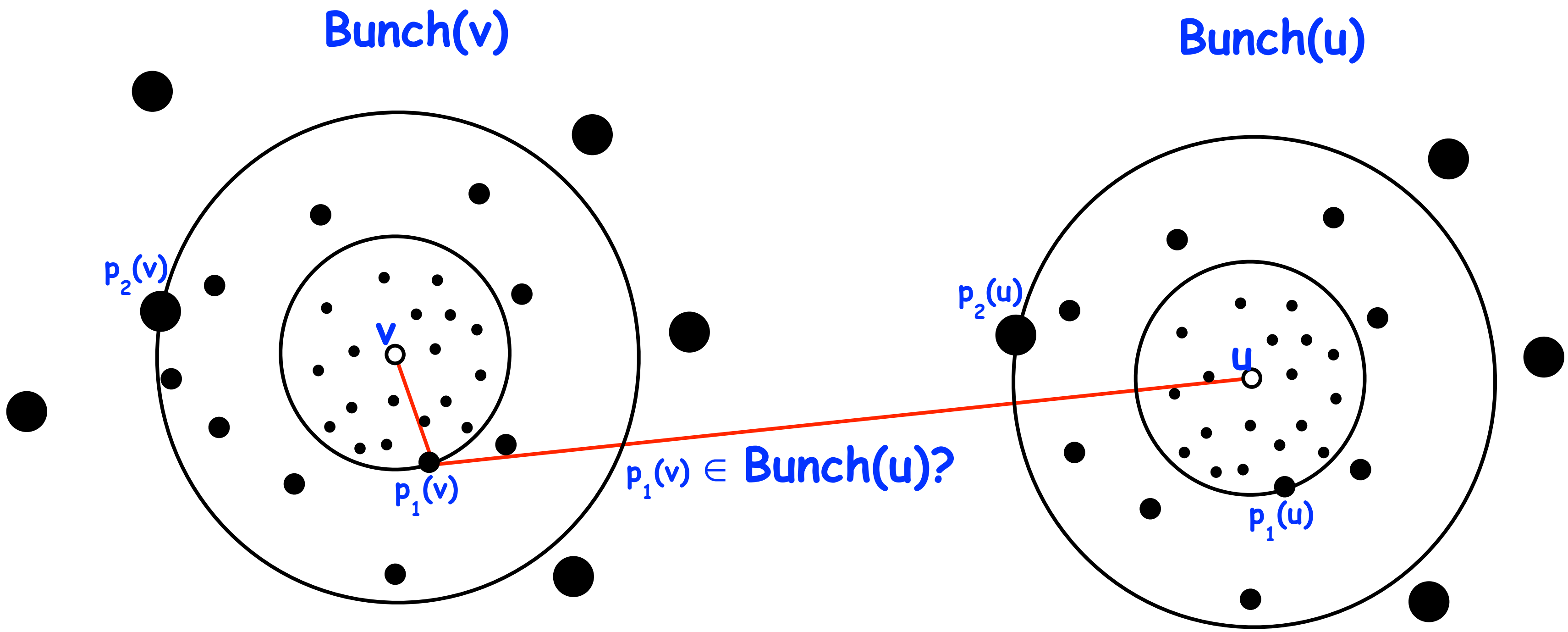

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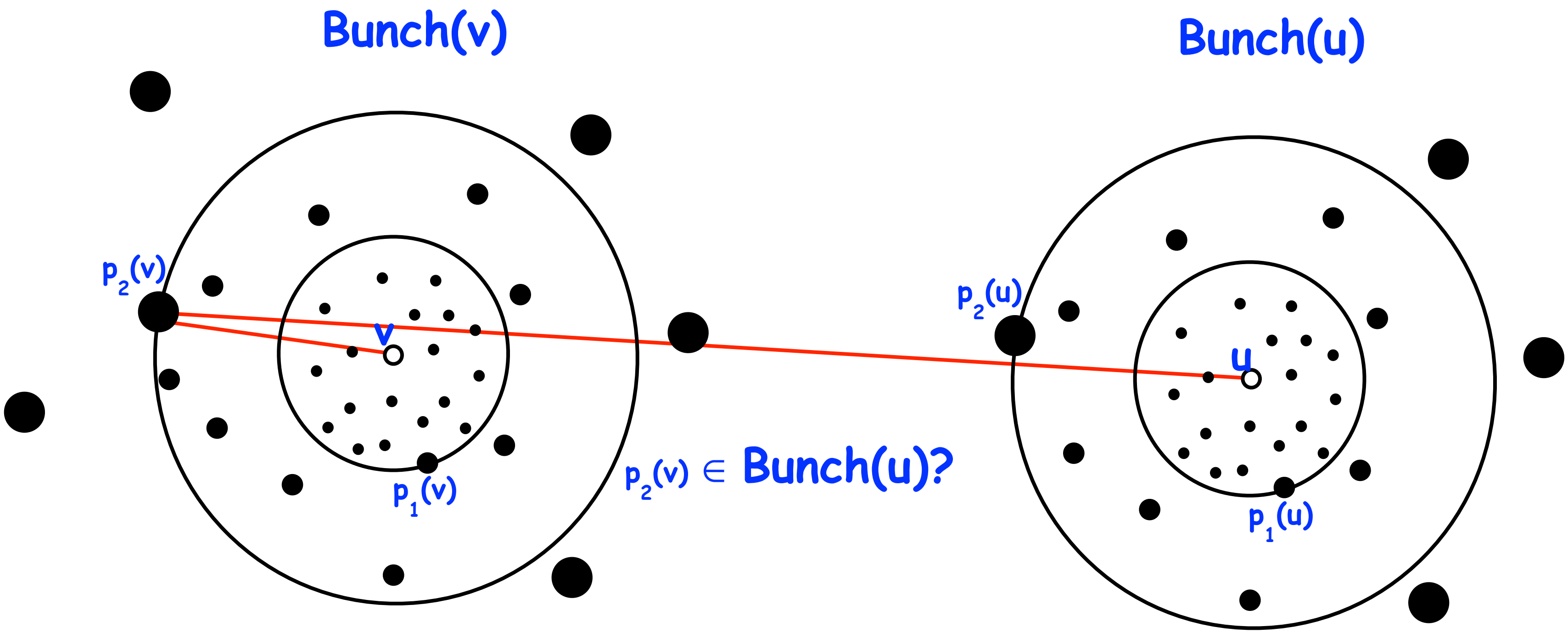

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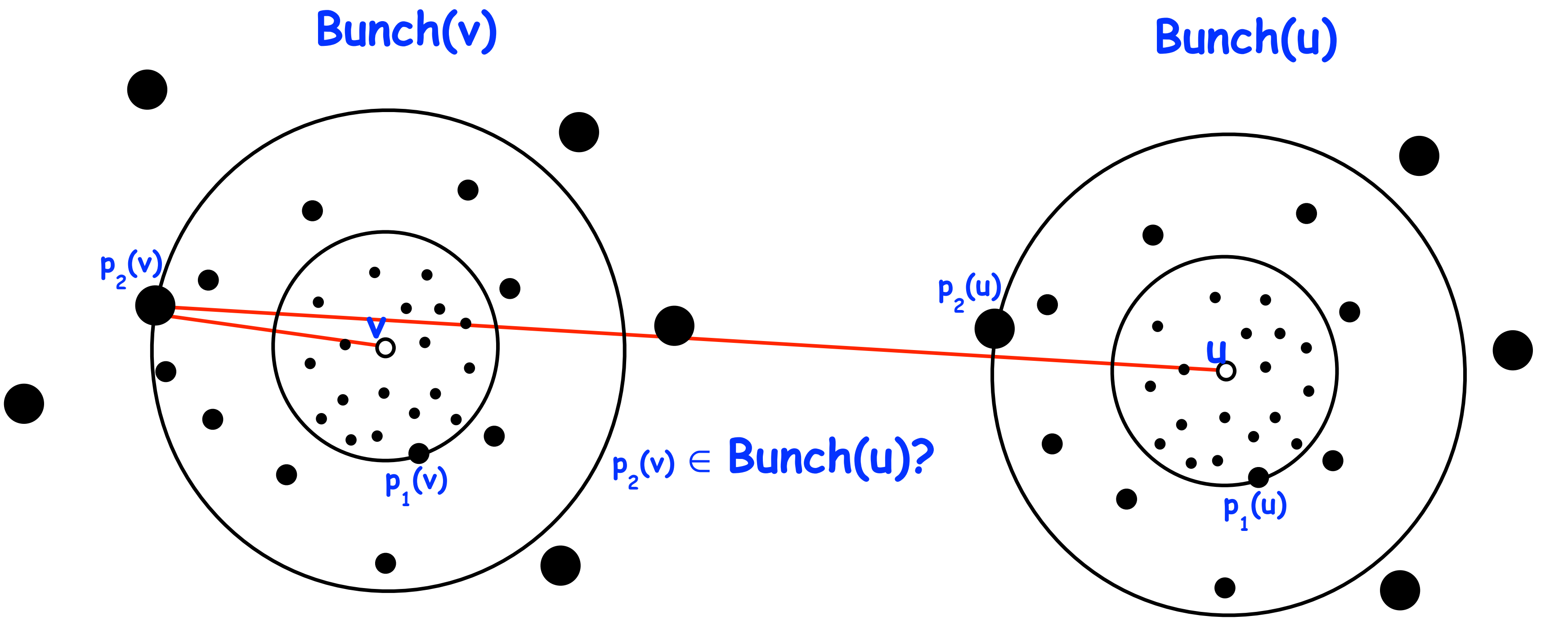

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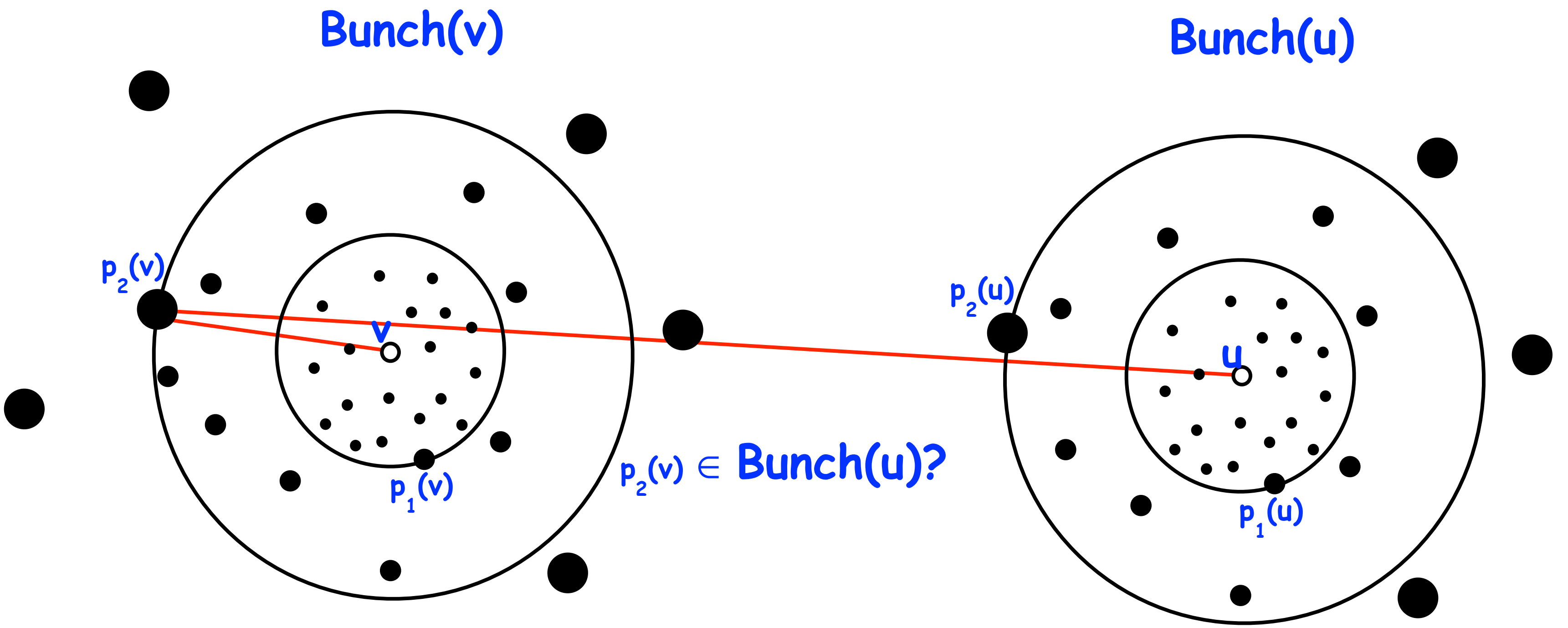

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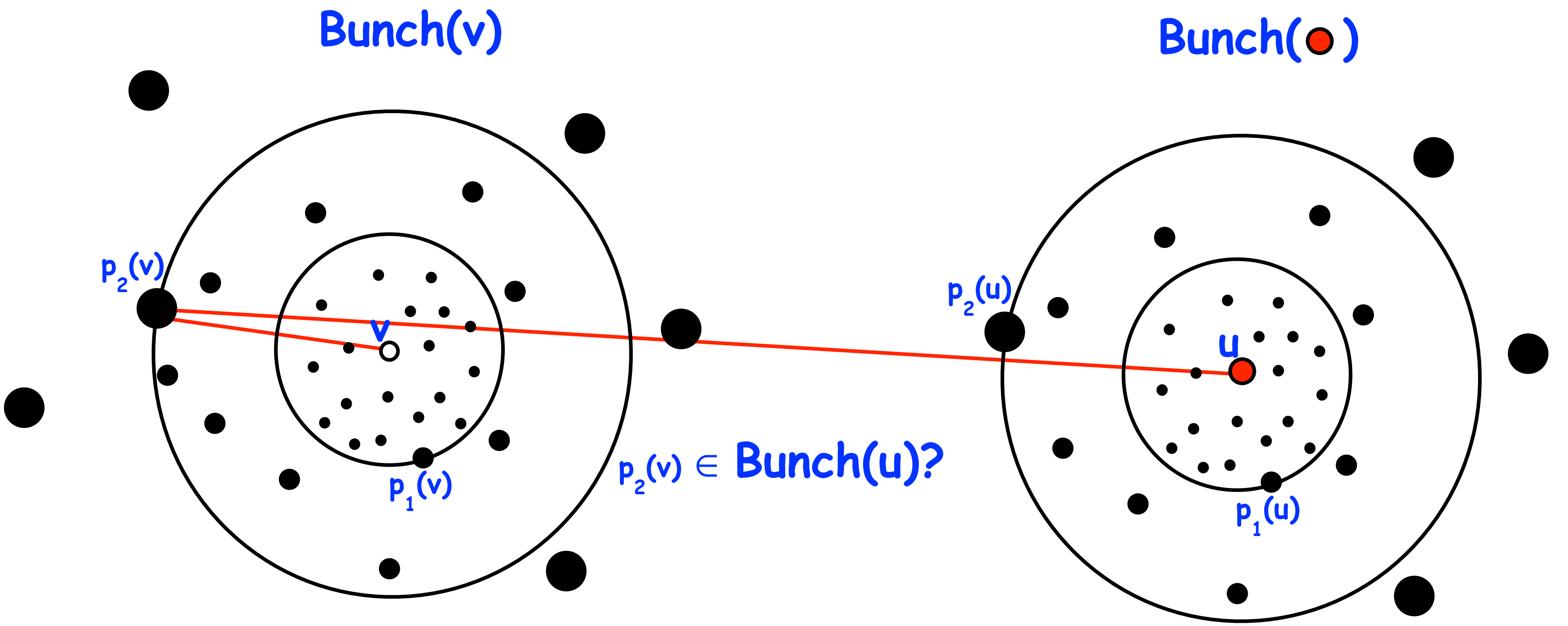

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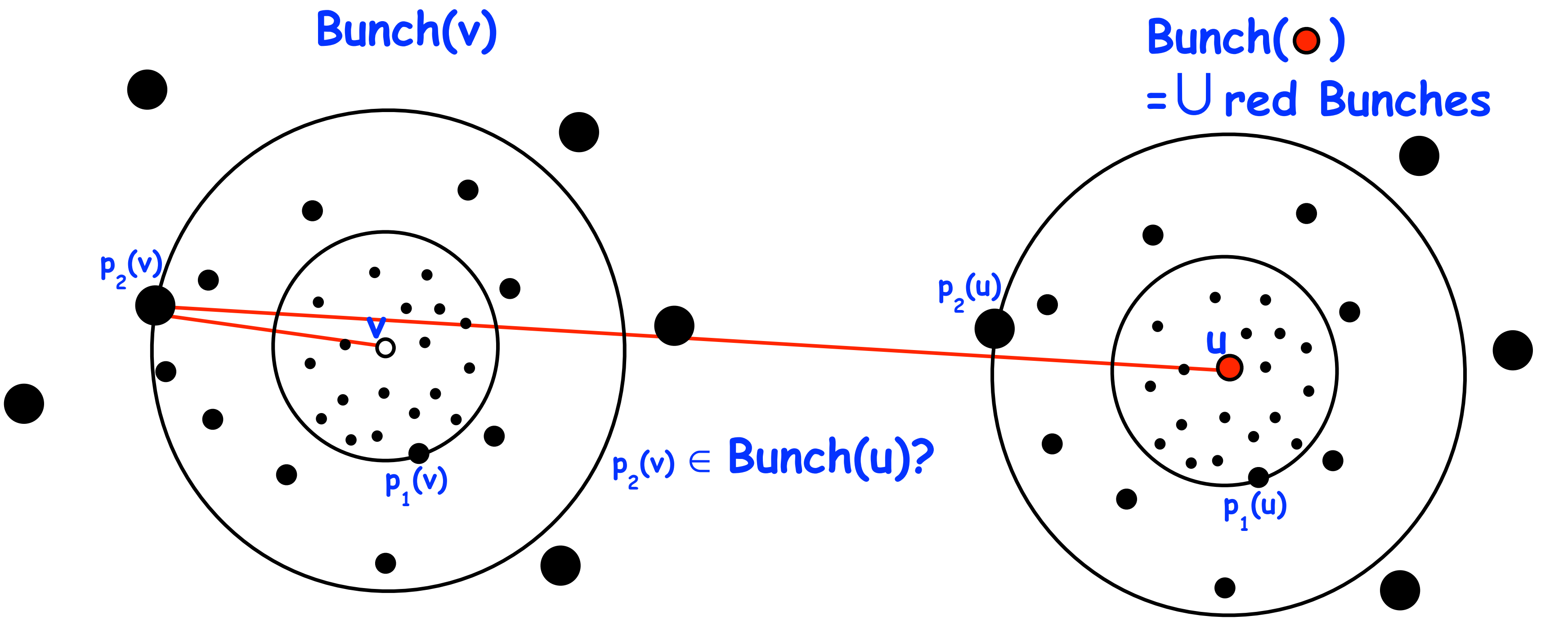

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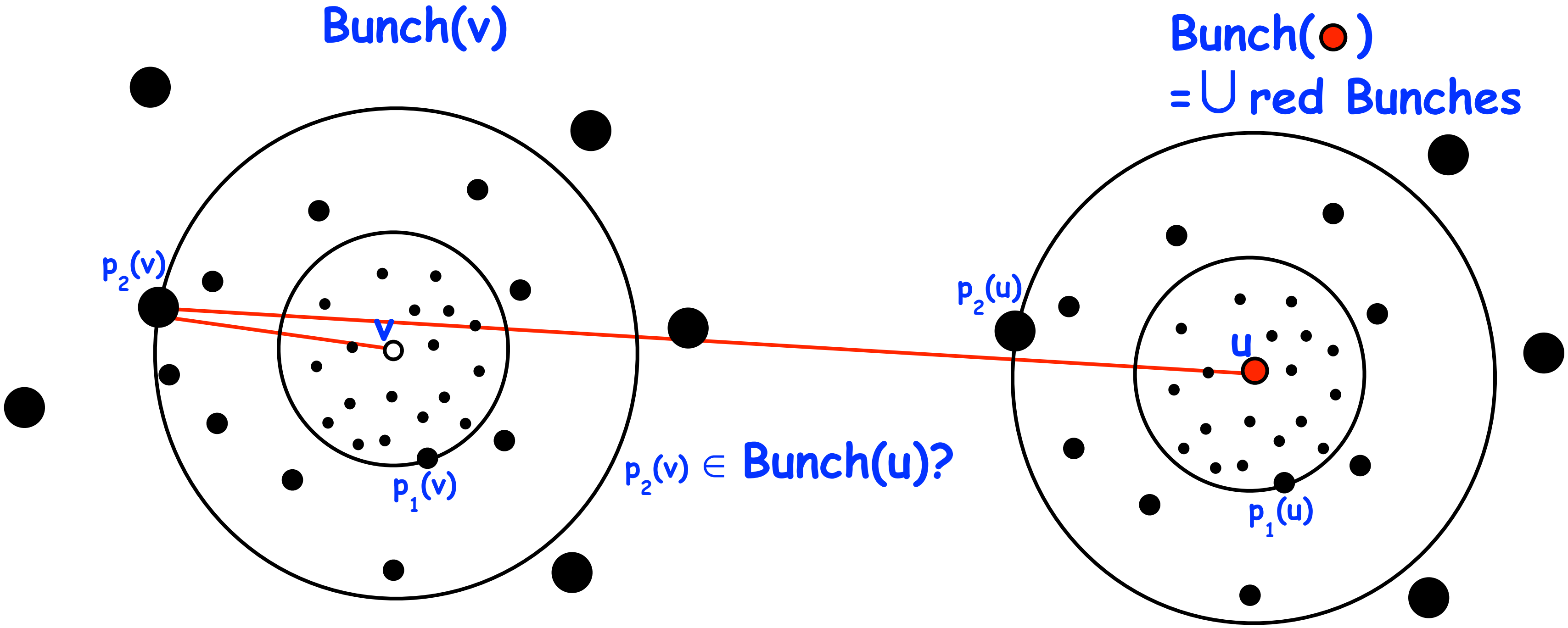
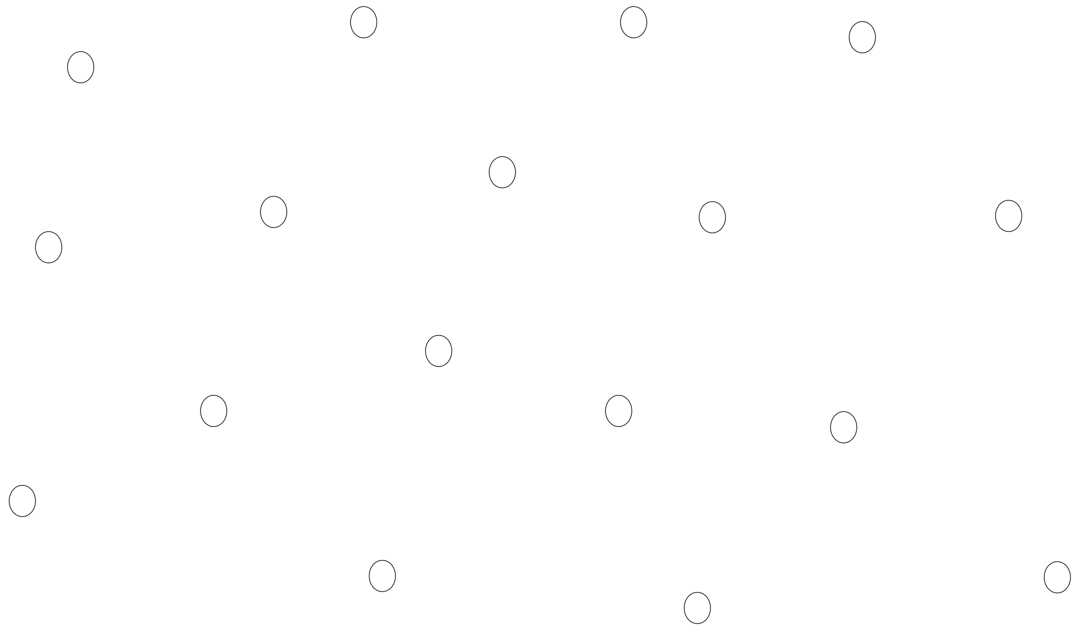
3. check all $p_i(v)$



An $O(nc^{1/2})$-space 3-stretch Oracle

- Select *routers* uniformly at random with prob. $c^{-1/2}$.



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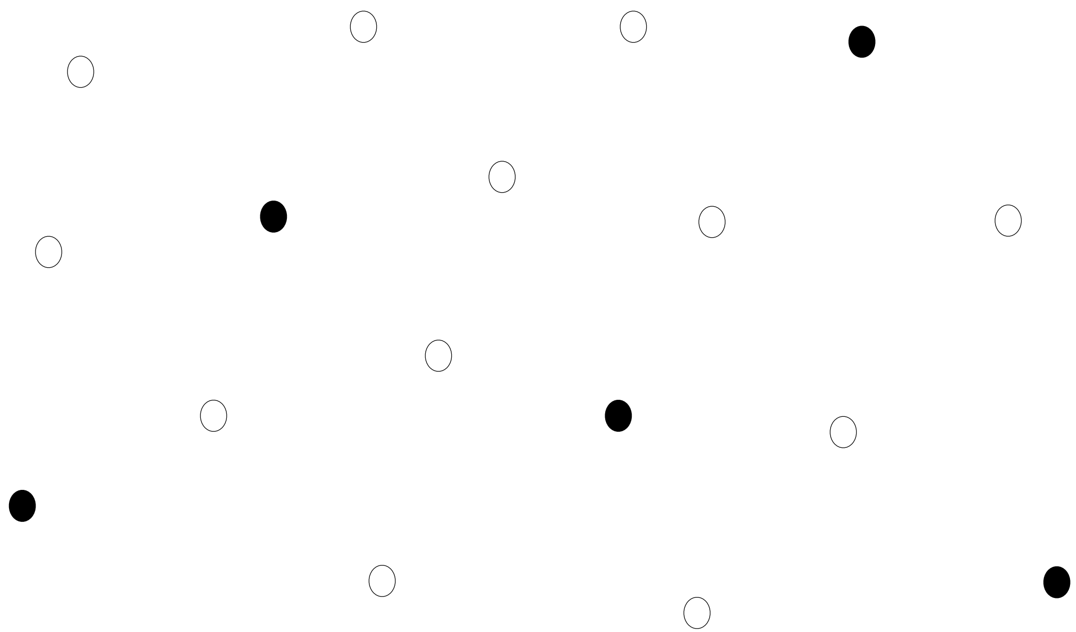
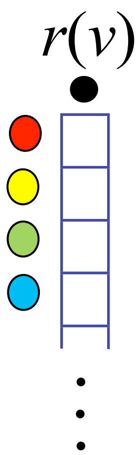
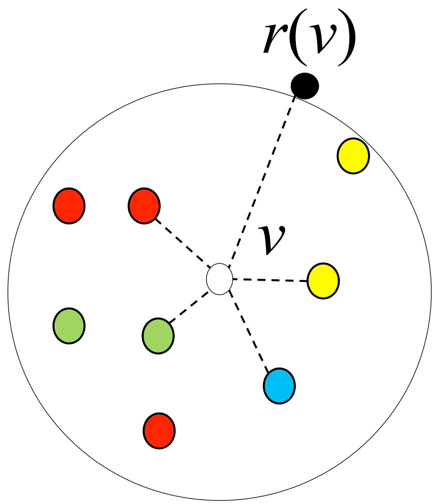


An $O(nc^{1/2})$-space 3-stretch Oracle

- Select *routers* uniformly at random with prob. $c^{-1/2}$.
- Store all distances
  - $\delta(r,\lambda)$ for every router r and color λ.
  - $\delta(v,r(v))$ from every vertex v to its closest router $r(v)$.
  - $\delta(v,\lambda)$ from every vertex v to every color λ with $\delta(v,\lambda) < \delta(v,r(v))$.

The ball $B(v)$ of v





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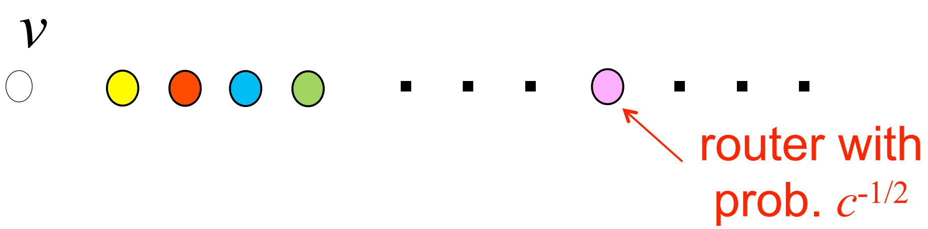

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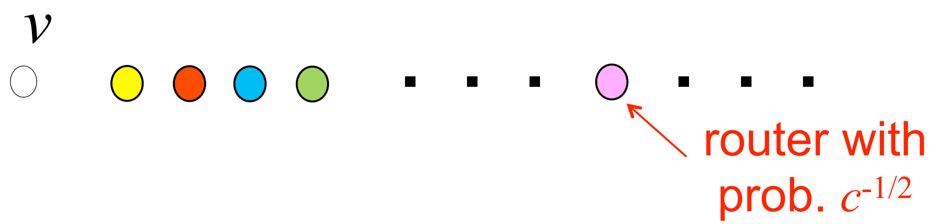

    - $|B(v)|$ bounded by the location of the first router on this list.

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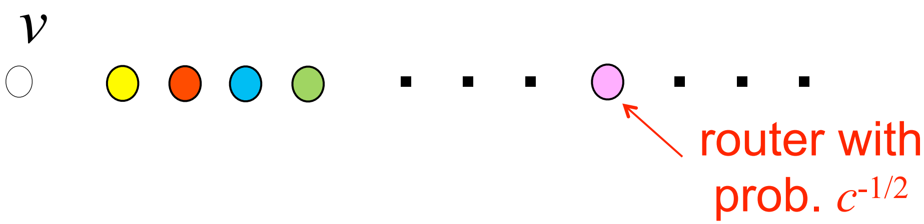

    - $|B(v)|$ bounded by the location of the first router on this list.
  - total size of all ball tables $\le nc^{1/2}$

An $O(nc^{1/2})$-space <u>3-stretch</u> Oracle

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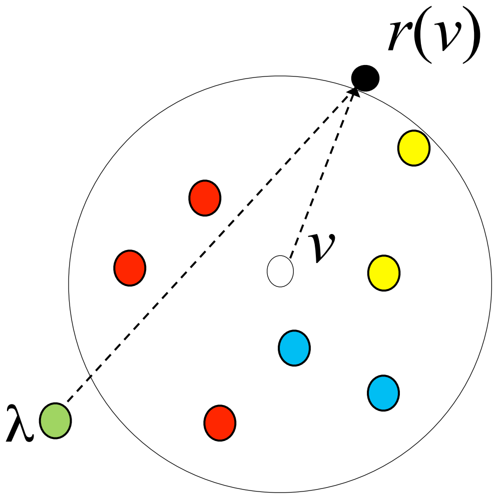

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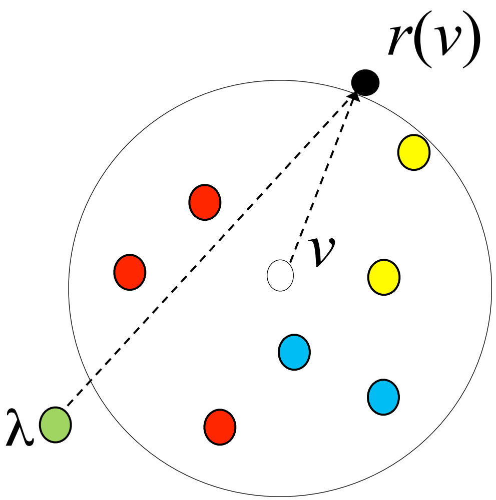

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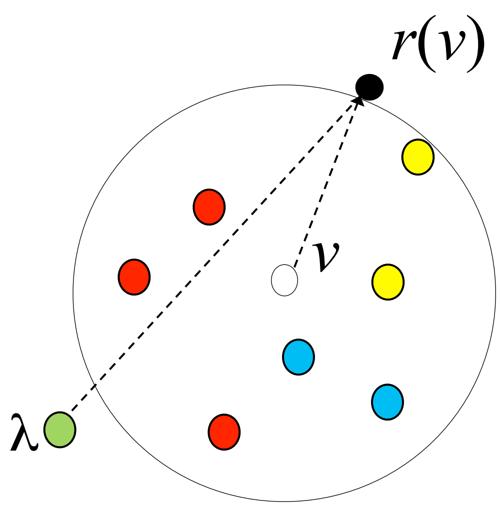

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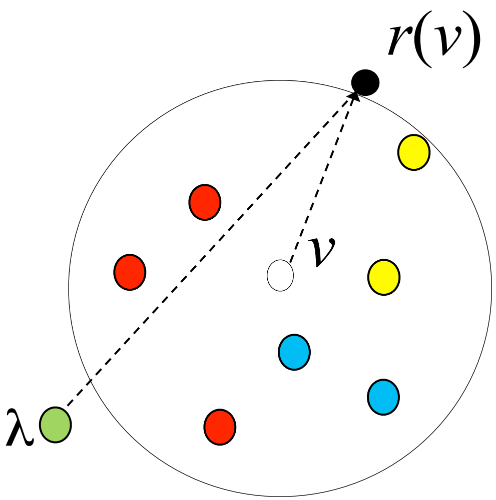

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⇓

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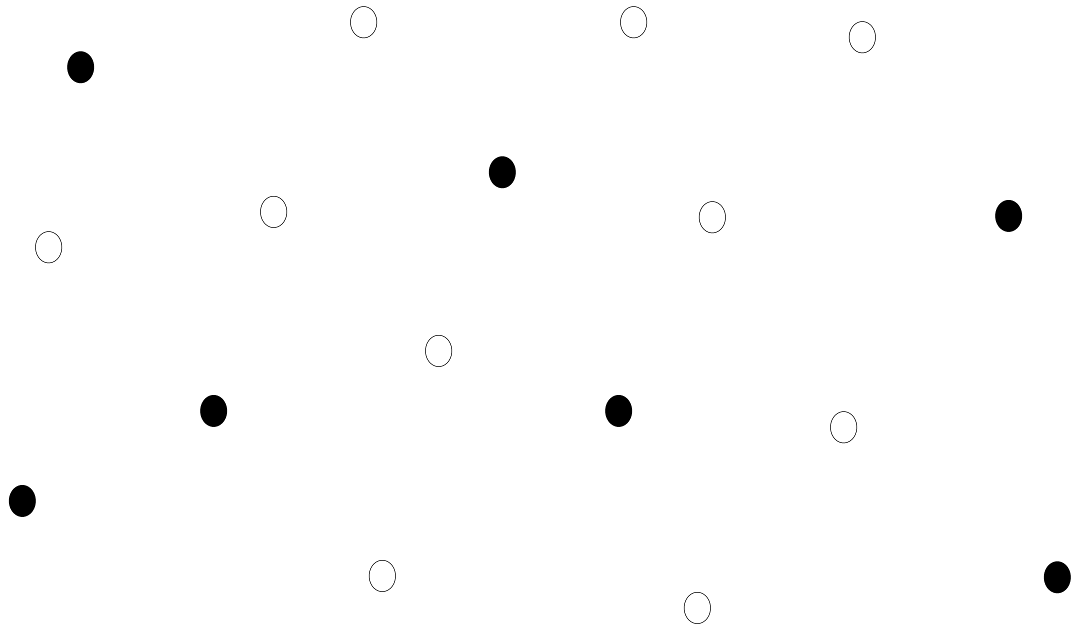
$O(knc^{1/k})$-space (2^k-1)-stretch Oracles



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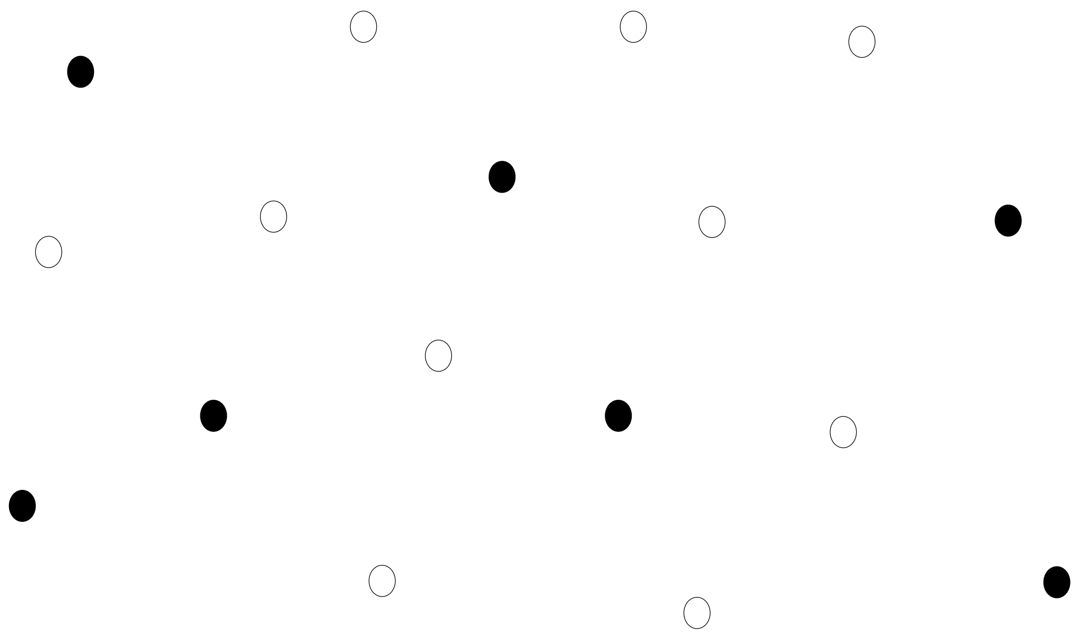

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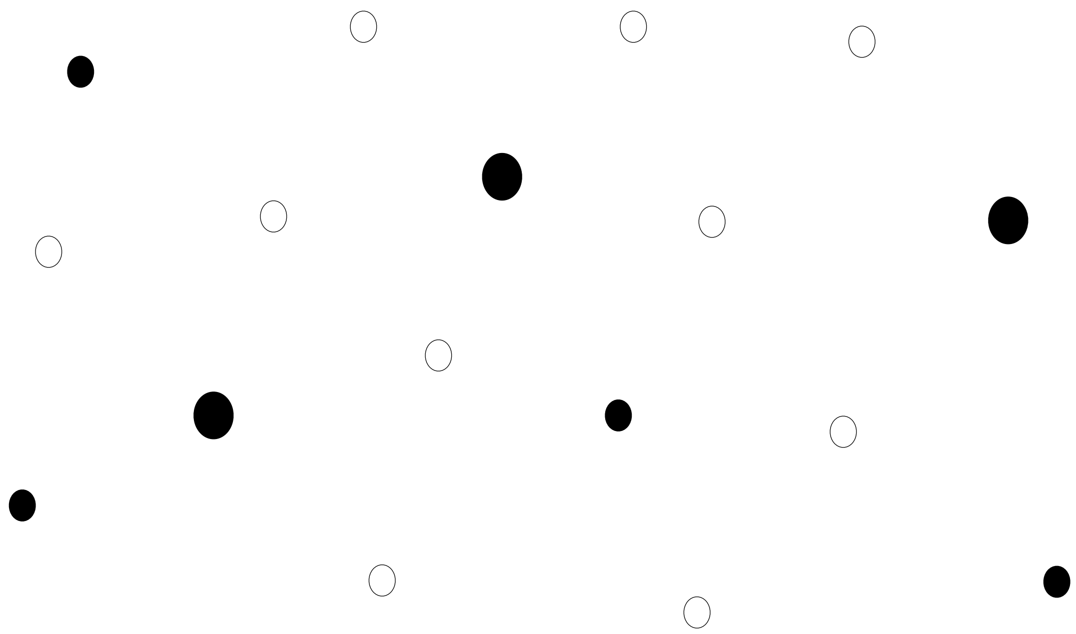

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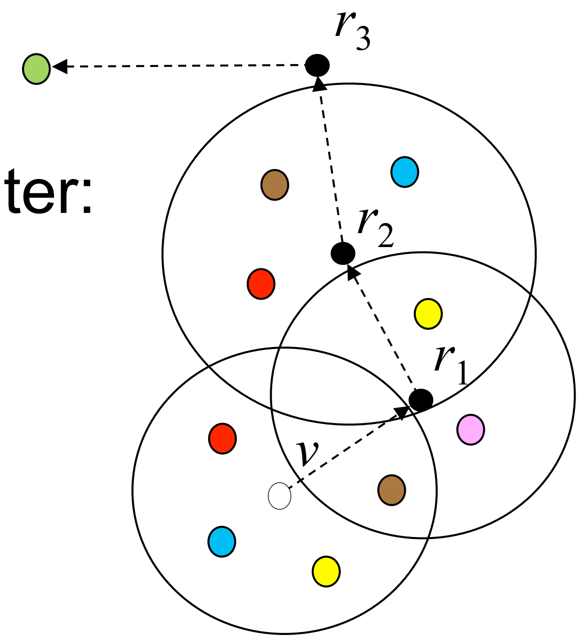

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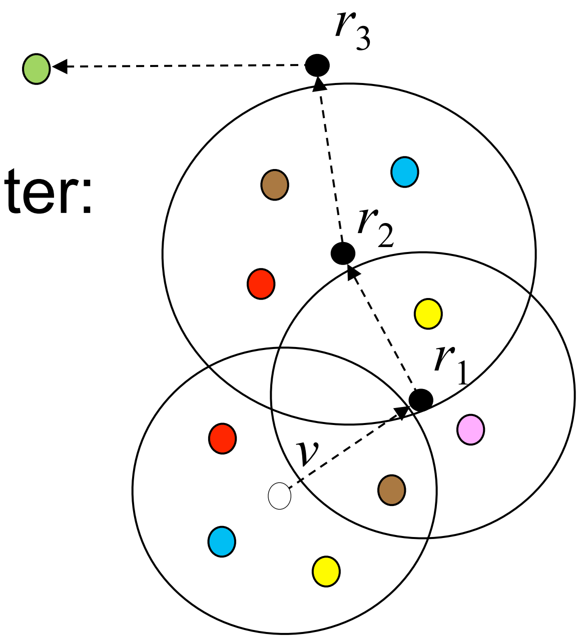

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- On color change of v:
  - Update two heaps in each ball that contains v.

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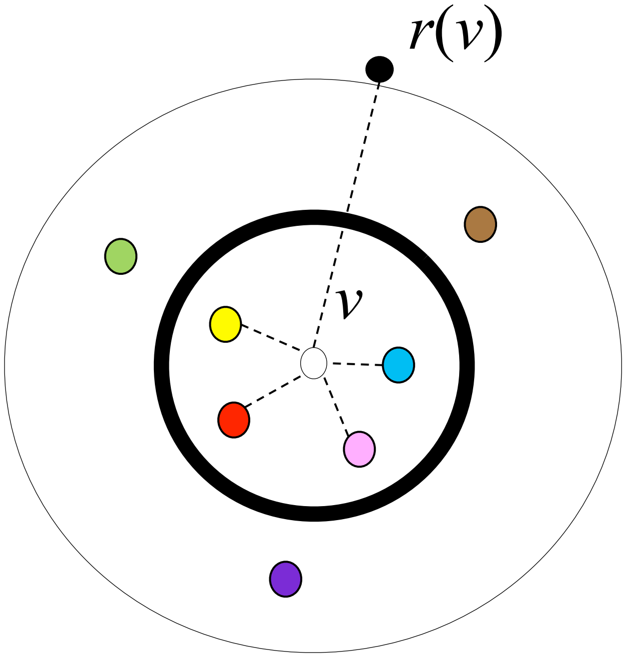

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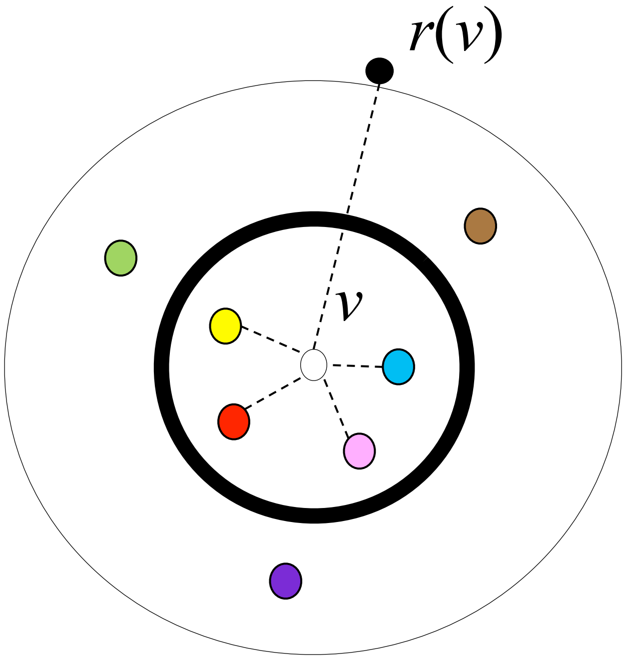

this causes the stretch of the queries to increase

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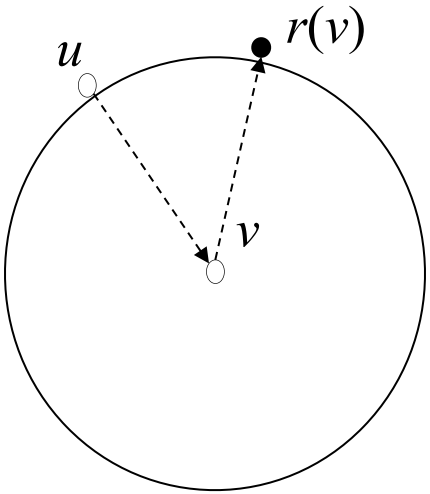

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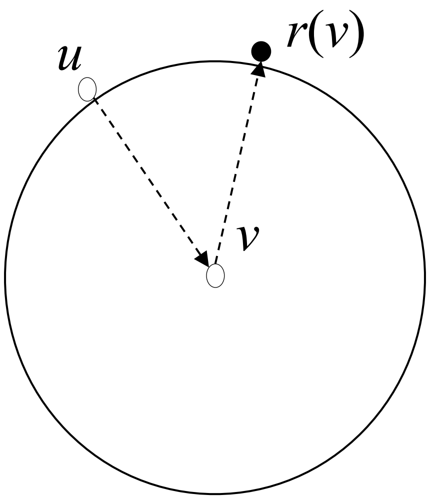
- v is in at most $kn^{1/k}$ balls.



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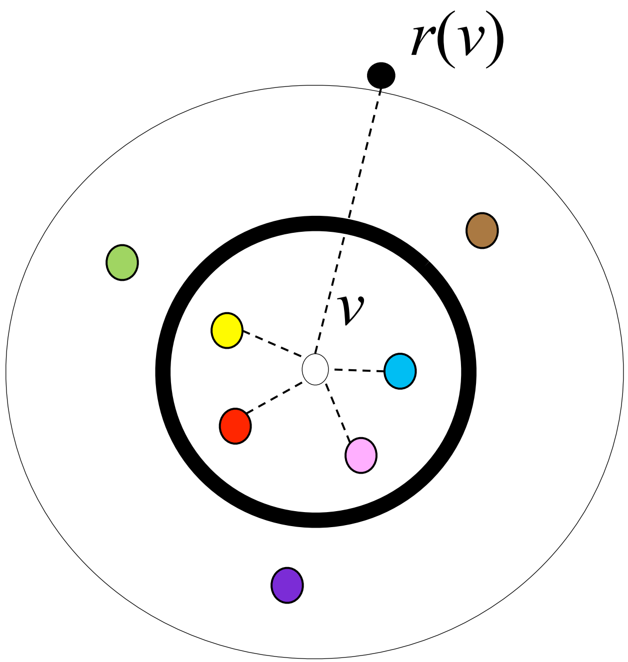
2. $O(knc^{1/k})$-space poly(k)-stretch ?

3. $O(knc^{1/k})$-space with changing colors ?

Thank you!



Thank you!