On zero-sum and almost zero-sum subgraphs over $\mathbb{Z}$

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Abstract

For a graph H with at most n vertices and a weighing of the edges of K_n with integers, we seek a copy of H in K_n whose weight is minimal, possibly even zero. Of a particular interest are the cases where H is a spanning subgraph (or an almost spanning subgraph) and the case where H is a fixed graph. In particular, we show that relatively balanced weighings of K_n with $\{-r,\dots,r\}$ guarantee almost zero-sum copies of spanning graphs with small maximum degree, guarantee zero-sum almost H-factors, and guarantee zero-sum copies of certain fixed graphs.

Keywords: zero-sum, subgraph

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. Graph theory notation follows [4].

For positive reals r, q, an (r,q)-weighting of the edges of the complete graph K_n is a function $f : E(K_n) \to [-r, r]$ such that $|\sum_{e \in E(K_n)} f(e)| \le q$. We call $w(f) = \sum_{e \in E(K_n)} f(e)$ the *total weight* of f. We say that an (r,q)-weighting is *integral* if $f : E(K_n) \to \{-r,\dots,r\}$.

Our main objective in this paper is to study such (r,q)-weightings with the goal of finding nontrivial conditions that guarantee the existence of certain bounded weight subgraphs and even zero weighted subgraphs (also called zero-sum subgraphs). Our main source of motivation is zero-sum Ramsey theory, a well-studied topic in graph theory, as well as some results about balanced colorings of integers. In zero-sum Ramsey theory we have a function $f : E(K_n) \to X$ where X is usually the cyclic group Z_k or (less often) an arbitrary finite abelian group. The goal is to show that under some necessary divisibility conditions imposed on the number of the edges $e(G)$ of a graph G and for sufficiently large n, there is always a zero-sum copy of G. For some results in this direction that are also related to results that shall be proved here see [1, 2, 5, 6, 8, 13].

Our first result has no counterpart in zero-sum Ramsey theory as it states that every (r,q)-weighting of K_n where q and r are relatively small, has an almost zero-sum copy of any *spanning* subgraph with relatively small maximum degree.

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Theorem 1.1 *Let H be a graph with n vertices and maximum degree Δ. Let $f : E(K_n) \to [-r, r]$ be an edge weighing with $|w(f)| \leq 2(n-1)r$. Then, there is a copy of H in K_n with absolute weight at most $2\Delta r$. Furthermore, if H is connected and $|w(f)| \leq 2(n-1)r(1 - \frac{1}{\Delta})$, then there is a copy of H in K_n with absolute weight at most $2(\Delta - 1)r$.*

The dependence on Δ in Theorem 1.1 is essential. For example, it is easy to see that there are integral $(1, 0)$-weighings of K_n such that any spanning star has absolute weight roughly $n/2$. Indeed, say, for simplicity, that n is a multiple of 4. Take two vertex-disjoint cliques A and B on $n/2$ vertices each. Label the edges of A with 1 and the edges of B with -1. Label $n/4$ disjoint perfect matchings between A and B with 1 and label the remaining edges between A and B with -1. The absolute weight of any spanning star in this example is $n/2 - 1$.

We note that one of the corollaries of theorem 1.1, given as Proposition 2.2, is the existence of a zero-sum path on at least $n - 2$ vertices in any integral $(1, n-1)$-weighting (namely, a zero-sum almost Hamilton path).

Our next result is about zero-sum large matchings and zero-sum graphs of the form tH , where H is a fixed graph. The main distinction here is that in zero-sum Ramsey theory we cannot get a zero-sum matching of size $t \approx n/2$ neither $t \approx n/|H|$ for tH, rather a fraction smaller than these magnitudes, see [3]. On the other hand, as the following result shows, this is possible to achieve for integral (r, q)-weighings. Recall that an H-factor of a graph G is a set of pairwise vertex-disjoint copies of H that cover all the vertices of G. For example, a perfect matching is just a K_2-factor. Theorem 1.1 guarantees the existence of an H-factor of K_n (under the assumption that the number of vertices of H divides n) with relatively small total weight (here H is a fixed graph and n is large). However, if we settle for an *almost* H-factor, we can do much better, and obtain nontrivial conditions which do guarantee zero-sum. More formally, define an (H, c)*-factor* of G to be a set of pairwise vertex-disjoint copies of H that cover all but at most c vertices of G. Our second main result concerns zero-sum (H, c)-factors where c is bounded as a function of H alone (independent of n).

Theorem 1.2 *Let H be a graph with h vertices, m edges, and maximum degree Δ. Let $f : E(K_n) \to \{-r, \ldots, r\}$ be an edge weighing with $|w(f)| \leq 2(n-1)r$. Then K_n contains a zero-sum (H, c)-factor where*

$$c < \max\{h(2rm - 1), h(2\Delta r + rm)\} .$$

We next show that as in zero-sum Ramsey theory, zero-sum copies of the complete bipartite graphs $K_{s,t}$ as well as many other bipartite graphs do exist once certain divisibility conditions hold. But on the other hand, quite distinct from Ramsey theory and zero-sum Ramsey theory, no zero-sum copies of a complete graph K_m necessarily exist already for integral $(1, 0)$-weightings, unless $m = 4k^2$. In fact, the only complete graph for which we can show zero-sum existence given any integral $(1, 0)$-weighting and large n is K_4 and the proof of the latter is somewhat involved. Let us state our results more formally.

For a positive integer r, consider the set of integers

$$B_r = \left\{ \frac{a+b}{\gcd(a,b)} \mid 1 \le a \le r \ , \ 1 \le b \le r \right\} \ .$$

Specifically, $B_1 = \{2\}$, $B_2 = \{2,3\}$, $B_3 = \{2,3,4,5\}$, $B_4 = \{2,3,4,5,7\}$. Observe that $B_r \subset B_{r+1}$. We say that the complete bipartite graph $K_{s,t}$ is *r-good* if each element of B_r divides at least one of s or t (in Section 4 we extend the notion of r-goodness to bipartite graphs that are not necessarily complete). For example, $K_{2,2}$ is 1-good, $K_{2,3}$ is 2-good, and $K_{5,12}$ is 3-good. Our next result gives a sufficient condition for the existence of zero-sum complete bipartite graphs.

Theorem 1.3 *For a positive integer r, an r-good complete bipartite graph $K_{s,t}$ and a real $\epsilon > 0$ the following holds. For all n sufficiently large, any weighing $f : E(K_n) \to \{-r, \dots, r\}$ with $|w(f)| \le n^2(\frac{1}{2} - \epsilon)$ contains a zero-sum copy of $K_{s,t}$.*

Notice that the requirement $|w(f)| \le n^2(\frac{1}{2} - \epsilon)$ is essentially tight as one can label $\binom{n}{2} - ex(n, K_{s,t})$ edges of K_n with 1 and $ex(n, K_{s,t})$ edges with 0 where $ex(n, K_{s,t})$ is the Turán number of $K_{s,t}$ in K_n, and obtain a labeling where any copy of $K_{s,t}$ has nonzero weight.

Theorem 1.4 *For a real $\epsilon > 0$ the following holds. For all n sufficiently large, any weighing $f : E(K_n) \to \{-1, 0, 1\}$ with $|w(f)| \le (1-\epsilon)n^2/6$ contains a zero-sum copy of K_4. On the other hand, for any positive integer k which is not of the form $k = 4d^2$, there are infinitely many n and weighings $f : E(K_n) \to \{-1, 0, 1\}$ with $|w(f)| = 0$ that do not contain a zero-sum copy of K_k.*

Again, notice that requirement $|w(f)| \le (1-\epsilon)n^2/6$ is essentially tight as the Turán number of K_4 is $\lfloor n^2/3 \rfloor$. Hence, one can label $\lfloor n^2/3 \rfloor$ edges with 0 and the remaining edges with 1 and obtain a labeling where any copy of K_4 has nonzero weight.

Our final main result concerns the existence of zero-sum spanning trees in integral $(1, n-2)$-weighings.

Theorem 1.5 *For $n = 1 \bmod 2$, any integral $(1, n-2)$-weighing of K_n has a zero-sum spanning tree.*

The result is tight as one can weigh all $n-1$ edges incident with the same vertex with 1 and the remaining edges with zero, and there is no zero sum tree. The requirement that n is odd is necessary as trivially, any weighing that only uses the weights -1 and 1 has no zero sum tree when n is even. We note that the highly nontrivial problem concerning the existence of zero-sum spanning trees in the context of zero-sum Ramsey theory was completely solved in [8, 13].

The rest of this paper is organized as follows. In Section 2 we consider almost zero-sum spanning graphs and prove Theorem 1.1. Section 3 considers almost H-factors and consists of the proof of Theorem 1.2 preceded by a lemma regarding the existence of a relatively short zero-sum subsequence of a sequence of integers. Section 4 is about zero-sum fixed graphs and contains the proofs of Theorem 1.3 and Theorem 1.4. Section 5 is about zero sum trees and contains the proof of Theorem 1.5. Section 6 contains some concluding remarks and open problems.

2 Almost zero-sum spanning subgraphs

Proof (Theorem 1.1): Consider a labeling of H with $\{1,\ldots,n\}$ and a labeling of K_n with $\{1,\ldots,n\}$. Each copy of H in K_n therefore corresponds to a permutation $\pi \in S_n$. Notice that $|Aut(H)|$ distinct permutations produce the same (non-labeled) copy of H where $Aut(H)$ denotes the automorphism group of H. However, for convenience, we consider all $n!$ labeled copies and denote by H_π the copy of H corresponding to π. Let m denote the number of edges of H. As any copy H_π occupies a fraction of $m/\binom{n}{2}$ of the edges of K_n, we have that each edge of K_n appears in $n!m/\binom{n}{2}$ distinct H_π. Let $f : E(K_n) \to [-r,r]$ be an edge weighing of K_n with total weight $w(f)$. We therefore have:

$$\sum_{\pi \in S_n} w(H_\pi) = \frac{n!m}{\binom{n}{2}} w(f) \ .$$

It follows that the average weight of a copy of H is $\frac{m}{\binom{n}{2}} w(f)$.

For the rest of the proof assume that $w(f) \geq 0$. This may be assumed as otherwise we can multiply each weight by -1 without affecting the statement of the theorem. Let $H_{\max}$ be a copy with maximum weight and let $H_{\min}$ be a copy with minimum weight. We therefore have:

$$w(H_{\max}) \geq \frac{m}{\binom{n}{2}} w(f) \geq 0 \ , \quad w(H_{\min}) \leq \frac{m}{\binom{n}{2}} w(f) \ .$$

Consider first the case $w(H_{\min}) \geq 0$. The theorem follows in this case since we have

$$|w(H_{\min})| = w(H_{\min}) \leq \frac{m}{\binom{n}{2}} w(f) \leq \frac{\Delta}{n-1} w(f) \leq \frac{\Delta}{n-1} 2(n-1)r = 2\Delta r$$

where we have used that $2m \leq n\Delta$ and the stated assumption that $w(f) \leq 2(n-1)r$. Observe that if $w(f) \leq 2(n-1)r(1-\frac{1}{\Delta})$ as assumed in the second part of the theorem, then, in fact, $|w(H_{\min})| \leq 2\Delta r(1-\frac{1}{\Delta}) = 2(\Delta-1)r$ so the second part of the theorem holds as well in this case.

We may now assume that $w(H_{\min}) < 0$. We start by proving the first part of the theorem where H is not assumed to be connected. Let P be the graph whose vertices are all the $n!$ copies of H in K_n. We connect vertex H_π of P with vertex H_σ of P if π and σ differ in a single transposition. Clearly, P is connected as any permutation can be obtained from any other permutation by a sequence of transpositions. Consider some edge (H_π, H_σ) of P and let (uv) be the transposition connecting π and σ. The symmetric difference between the edge set of H_π and the edge set of H_σ consists only of edges that are incident with u in H_π or H_σ or edges that are incident with v in H_π or H_σ. As the number of such edges is at most 4Δ, it follows that $|w(H_\pi) - w(H_\sigma)| \leq 4\Delta r$. Consider a path of P connecting $H_{\max}$ and $H_{\min}$. As $w(H_{\max}) \geq 0$ and $w(H_{\min}) < 0$, there must be some edge (H_π, H_σ) on this path such that $w(H_\pi) \geq 0$ and $w(H_\sigma) \leq 0$. It follows that

$$\min\{w(H_\pi)\,,\, -w(H_\sigma)\} \leq 2\Delta r$$

as required.

Consider next the case where H is connected. Let Q be the spanning subgraph of P where (H_π, H_σ) is an edge if and only if π and σ differ in a single transposition (uv) and, furthermore, uv is an edge in both H_π and H_σ (notice that uv is either an edge in both of them or in none of them). We claim that Q is connected. Since P is connected, it suffices to show that for any two permutations π and σ that differ in a single transposition (uv), there is a path in Q connecting H_π and H_σ. We prove it by induction on the length of a shortest path connecting u and v in H (which is finite as H is connected). For shortest paths of length 1 this is true as H_π and H_σ are adjacent in Q, by its definition. For shortest paths of length $k > 1$, consider a path $u = x_0, x_1, \ldots, x_k = v$ connecting u and v in H. Then H_π is connected to H_φ where φ is obtained from π by the transposition (x_0, x_1). Now, as the length of a shortest path from x_1 to $v = x_k$ is only $k-1$, we have by induction that H_φ and H_σ are connected in Q. Thus H_π and H_σ are connected in Q as well.

Now, for an edge (H_π, H_σ) of Q, the symmetric difference between the edge set of H_π and the edge set of H_σ consists only of edges that are incident with u in H_π or H_σ or edges that are incident with v in H_π or H_σ, but this symmetric difference does not include the edge uv which appears in both H_π and H_σ. The number of such edges is therefore at most $4(\Delta-1)$. It follows that $|w(H_\pi) - w(H_\sigma)| \le 4(\Delta-1)r$. Consider a path of Q connecting $H_{\max}$ and $H_{\min}$. As $w(H_{\max}) \ge 0$ and $w(H_{\min}) < 0$, there must be some edge (H_π, H_σ) on this path such that $w(H_\pi) \ge 0$ and $w(H_\sigma) \le 0$. It follows that

$$\min\{w(H_\pi)\,,\, -w(H_\sigma)\} \le 2(\Delta-1)r$$

as required. ∎

Two graphs H_1 and H_2 with the same vertex set are *k-edge switchable* if H_2 can be obtained from H_1 by replacing at most k edges of H_1 with edges of H_2. Call a family of graphs with the same vertex set k-edge switchable if any graph in the family can be obtained from any other by a sequence of k-edge switches. For example, results of Havel [10] and Hakimi [9] (sometimes attributed to Berge) show, in particular, that the family of spanning k-regular subgraphs of K_n is 2-edge switchable. Also, the family of spanning trees is 1-edge switchable (see also Lemma 5.1). The proof of Theorem 1.1 uses the fact that the family of labeled copies of a given spanning graph H of K_n is 2Δ-edge switchable. A similar proof can thus be obtained for any other family of k-edge switchable graphs, as long as one can guarantee that the average weight of a graph in the family is small. We summarize this in the following corollary.

Corollary 2.1 *Let $\mathcal{H}$ be a family of graphs with n vertices that is k-edge switchable, such that each graph in $\mathcal{H}$ has m edges. Let $f : E(K_n) \to [-r, r]$ be an edge weighing with $|w(f)| \le kr\binom{n}{2}/m$. Then, there is a copy of $\mathcal{H}$ in K_n with absolute weight at most kr.*

One simple consequence of Theorem 1.1 is that, for even n, in any weighing of K_n with weights in $[-1, 1]$ where the total sum of the weights is at most $2(n-1)$, there is a perfect matching whose total absolute weight is at most 2 (apply the case $\Delta = 1$ and $r = 1$ in Theorem 1.1). This is tight for, say, K_8, as we can label seven edges incident with the same vertex with -1 and label the remaining edges with 1, having $w(f) = 21 - 7 = 14 = 2 \cdot (8-1)$, and yet any perfect matching of this weighing of K_8 has weight 2. The same extremal example is true for the weight interval $[-r, r]$ by multiplying each edge weight by r. Observe, however that for the special case of weights in $\{-1, 0, 1\}$ one may delete at most two edges from a perfect matching of absolute total weight at most 2 and obtain a zero-sum matching. A more illustrative application is given in the following proposition.

Proposition 2.2 *Let $f : E(K_n) \to \{-1, 0, 1\}$ be an edge weighing with $|w(f)| \le n-1$. Then, there is a zero-sum path with at least $n-2$ vertices.*

Proof: We use Theorem 1.1 where H is a cycle of length n (hence the connected case where $\Delta = 2$), $r = 1$, and observe that the assumption $|w(f)| \le n-1$ satisfies the stated condition in the theorem. The theorem guarantees that if C is a Hamilton cycle of minimum total absolute weight, then $|w(C)| \le 2$.

If $w(C) = 0$ (a zero-sum Hamilton cycle), then either C contains a zero edge, which, once removed, show that there is a zero-sum Hamilton path, or else n must be even and exactly half of the edges of C have weight 1 and the other half have weight -1. We may remove a vertex incident with one positive and one negative edge and obtain a zero-sum path with $n-1$ vertices.

If $|w(C)| = 1$, then we can remove an edge with weight 1 if $w(C) = 1$ or an edge with weight -1 if $w(C) = -1$ and obtain a zero-sum Hamilton path.

We remain with the case $|w(C)| = 2$. We prove the case $w(C) = 2$ as the negative case is symmetric. The proof of Theorem 1.1 shows that either all Hamilton cycles have weight 2 or else there must be both a positive and a negative weight Hamilton cycle (since the average weight of a Hamilton cycle is at most 2, we cannot have that all Hamilton cycles have nonnegative weight, as otherwise at least one cycle has nonnegative weight smaller than 2 and the proposition holds by one of the previous cases). If all Hamilton cycles have weight 2, then we must have $w(f) = n-1$ and hence there must be two edges with weight 1 incident with the same vertex u. Taking any Hamilton cycle that contains these two edges consecutively and then deleting u, we obtain a zero-sum path with $n-1$ vertices. We remain with the case where there are both a positive weight Hamilton cycle and a negative weight Hamilton cycle. The proof of Theorem 1.1 shows that there are two Hamilton cycles C_π and C_σ with $w(C_\pi) = 2$, $w(C_\sigma) = -2$ and π differs from σ in a single transposition (uv) where uv is an edge of both C_π and C_σ. Hence, if x is the other neighbor of u in C_π and y is the other neighbor of v in C_π (which implies that xv and yu are both edges of C_σ), then we must have that the weights of xu and yv are 1 and the weights of xv and yu are -1. Now, if the weight of uv is zero, we can delete u and v from C_π and obtain a zero-sum path with $n-2$ vertices. If the weight of uv is 1 we can delete u from C_π and obtain a zero-sum path with $n-1$ vertices. If the weight of uv is -1 we can delete u from C_σ and obtain a zero-sum path with $n-1$ vertices. ∎

3 Zero-sum almost H-factors

Theorem 1.2 is based on the following lemma.

Lemma 3.1 *For a positive integer $r \geq 2$ and an integer q, any sequence of $n \geq \max\{2r-1, |q|+r\}$ elements from $\{-r, \ldots, r\}$ whose sum is q, has a nonempty subsequence of size at most $2r-1$ whose sum is zero.*

Proof: Denote the sequence by S. We may assume that S has no zero element, as in this case we are done. We further assume that S does not contain an element and its negation, otherwise we are done since $2r-1 > 2$. In particular we may assume that $-r$ and r are not both in S. As we can multiply the whole sequence by -1, without affecting the statement or the result, we may assume that $-r$ is not in S.

A prefix sum c_k is the sum of the first k elements of S. As the lemma is oblivious to reordering the elements, we can consider the ordering of the elements that maximizes k for which $-r \leq c_j \leq r-1$ for all $1 \leq j \leq k$. Observe that $k \geq 1$, as otherwise all elements of S are equal to r and thus $q = nr$ and $n = q/r < q + r$ contradicting the assumption.

Assume first that $k \geq 2r-1$. If some $c_j = 0$ for $j \leq 2r-1$, we are done. Else $c_1, \ldots, c_{2r-1}$ are non-zero. Suppose that $c_{2r-1} > 0$ and let $m \leq 2r-2$ be the largest integer such that $c_m < 0$. If no such m exists, then all the $c_1, \ldots, c_{2r-1}$ are each in the range $\{1, \ldots, r-1\}$ so already among $c_1, \ldots, c_r$ there are $i < j$ such that $c_i = c_j$ and we are done as the subsequence between locations $i+1$ and j has zero-sum. So we may assume $2r-2 \geq m \geq 1$. Write $b = c_{m+1} - c_m$ and observe that $b > 0$ and b is the element of S in location $m+1 \leq 2r-1$. This forces that for $j \leq m$ no $c_j = -b$ for otherwise $c_j + b = 0$. Also for $2r-1 \geq j \geq m+1$ no $c_j = -b$ as all these c_j are positive. So none of $c_1, \ldots, c_{2r-1}$ are equal to $-b$, none are equal to 0 and none are equal to r (because of the definition of k). So, there are only at most $2r-2$ possible values for the $c_1, \ldots, c_{2r-1}$, and hence there are $1 \leq i < j \leq 2r-1$ such that $c_i = c_j$ and we are done as the subsequence between locations $i+1$ and j has zero-sum.

Suppose now that $c_{2r-1} < 0$ and let $m \leq 2r-2$ be the largest integer such that $c_m > 0$. If no such m exists, then all the $c_1, \ldots, c_{2r-1}$ are each in the range $\{-r, \ldots, -1\}$ so already among $c_1, \ldots, c_{r+1}$ there are $i < j$ such that $c_i = c_j$ and we are done as the subsequence between locations $i+1$ and j has zero-sum. So we may assume $2r-2 \geq m \geq 1$. Write $b = c_{m+1} - c_m$ and observe that $b < 0$ and b is the element of S in location $m+1 \leq 2r-1$. This forces that for $j \leq m$ no $c_j = -b$ for otherwise $c_j + b = 0$. Also for $2r-1 \geq j \geq m+1$ no $c_j = -b$ as all these c_j are negative. So none of $c_1, \ldots, c_{2r-1}$ are equal to $-b$, none are equal to 0, and none are equal to r. Observe that $-b \neq r$ since we assumed that $-r$ is not in S. So, there are only at most $2r-2$ possible values for the $c_1, \ldots, c_{2r-1}$, and hence there are $1 \leq i < j \leq 2r-1$ such that $c_i = c_j$ and we are done as the subsequence between locations $i+1$ and j has zero-sum.

Suppose next that $k \leq 2r-2$. If some $c_j = 0$ for $j \leq k$ we are done. Otherwise we are left with at least $\max\{2r-1, |q|+r\} - (2r-2) = max\{1, |q|-r+2\}$ elements at location $k+1 \ldots, n$. If $c_k < 0$, then by the maximality of k all remaining elements of S are negative of order at most $-r-c_k-1$. So the sum of the whole sequence S is at most $c_k+(-r-c_k-1)\max\{1,|q|-r+2\}$. If $|q| \ge r-1$, then we have $c_k+(-r-c_k-1)(|q|-r+2) = -c_k(|q|-r+1)-(r+1)(|q|-r+2)$ which gets maximum at $c_k=-r$, as $c_k \ge -r$. Thus, we get a sum of at most $r(|q|-r+1)-(r+1)(|q|-r+2) = -|q|-2 < q$, which is impossible. If $|q| \le r-2$, then we have a sum of at most $c_k+(-r-c_k-1) = -r-1 < q$ which is impossible.

If $c_k > 0$, then by the maximality of k all remaining elements are positive and of order at least $r-c_k$. So the sum of the whole sequence S is at least $c_k+(r-c_k)\max\{1,|q|-r+2\}$. If $|q| \ge r-1$, then we have $c_k+(r-c_k)(|q|-r+2) = -c_k(|q|-r+1)+r(|q|-r+2)$ which gets minimum at $c_k = r-1$, as $c_k \le r-1$. Thus, we get a sum of at least $-(r-1)(|q|-r+1)+r(|q|-r+2) = |q|+1 > q$, which is impossible. If $|q| \le r-2$, then we have a sum of at least $c_k+(r-c_k) = r > q$ which is impossible. ■

Proof (Theorem 1.2): Let H be a given graph with h vertices, m edges, and maximum degree Δ. We may assume that $m > 1$ or $r > 1$ as the case of matchings with a $\{-1,0,1\}$-weighing trivially satisfies the theorem by the comment following the proof of Theorem 1.1. For a positive integer n, let H_n be the n-vertex graph consisting of $t = \lfloor n/h \rfloor$ vertex-disjoint copies of H and $n-ht < h$ isolated vertices. Let $f : E(K_n) \to \{-r,\ldots,r\}$ be an edge weighing with $|w(f)| \le 2(n-1)r$. By Theorem 1.1, K_n with this weighing has a copy of H_n with $|w(H_n)| \le 2\Delta r$. Let $X_1, X_2, \ldots, X_t$ be the copies of H comprising such a copy of H_n. Thus,

$$|\sum_{i=1}^{t} w(X_i)| \le 2\Delta r \ .$$

Consider the sequence of integers $w(X_1), w(X_2), \ldots, w(X_t)$. It is a sequence of length t whose sum, call it q, satisfies $|q| \le 2\Delta r$. Furthermore, each element in this sequence is an integer in $\{-rm,\ldots,rm\}$ as H has m edges each having weight in $\{-r,\ldots,r\}$. Hence, by Lemma 3.1, as long as $t \ge \max\{2rm-1 \ , \ 2\Delta r + rm\}$, we can find a nonempty subsequence of $\{X_1,\ldots,X_t\}$ of length at most $2rm-1$, whose sum of weights is zero (here we use the assumption that $rm > 1$). Removing such a subsequence we again obtain a leftover sequence whose sum is still q, but now with less than t elements. We can repeat this process as long as we remain with a leftover sequence containing at least $\max\{2rm-1 \ , \ 2\Delta r+rm\}$ elements. Once we arrive at a smaller leftover sequence we halt. We have constructed a set of vertex-disjoint copies of H that cover all vertices of K_n but at most

$$\begin{aligned} & h\cdot(\max\{2rm-1 \ , \ 2\Delta r+rm\}-1)+(n-ht) \\ \le \ & h\cdot(\max\{2rm-1 \ , \ 2\Delta r+rm\}-1)+h-1 \\ < \ & \max\{h(2rm-1) \ , \ h(2\Delta r+rm)\} \end{aligned}$$

vertices. ■

The case where $H = K_k$ is of particular interest. Plugging in the parameters $m = \binom{k}{2}$, $h = k$, and $\Delta = k-1$ we obtain the following corollary.

Corollary 3.2 *Let $f : E(K_n) \to \{-r, \ldots, r\}$ be an edge weighing with $|w(f)| \le 2(n-1)r$. Then K_n contains a zero-sum (K_k, c)-factor where, for $k \ge 5$ we have $c \le k^2(k-1)r - k - 1$, for $k = 4$ we have $c \le 48r - 1$ and for $k = 3$ we have $c \le 21r - 1$.*

4 Zero-sum fixed graphs

We recall first the theorem of Kövári-Sós-Turán [11] regarding the value of $Z_{a,b}(m, n)$, which is the smallest integer k such that any bipartite graph with vertex classes A and B where $|A| = m$ and $|B| = n$ and k edges has $K_{s,t}$ as a subgraph with the s vertices being in A and the t vertices being in B.

Lemma 4.1 *[Kövári-Sós-Turán [11]]*

$$Z_{a,b}(m, n) < (a-1)^{1/b}(n-b+1)m^{1-1/b} + (b-1)m \ .$$

Proof (Theorem 1.3): Fix $\epsilon > 0$, a positive integer r and a complete bipartite graph $K_{s,t}$ which is r-good. Throughout this proof we assume, wherever necessary, that n is sufficiently large as a function of r, s, t, ϵ. Consider a weighing $f : E(K_n) \to \{-r, \ldots, r\}$ with $|w(f)| \le n^2(\frac{1}{2} - \epsilon)$. We assume that $w(f) \ge 0$ as the proof of the negative case is symmetric.

We may assume that the number of edges having zero weight is at most, say, $(\epsilon^2/4r)n^2$. Otherwise, since the Turán number of $K_{s,t}$ is $o(n^2)$, we would have a copy of $K_{s,t}$ all of whose edges are zero, hence zero-sum, and we are done.

Since $w(f) \le n^2(\frac{1}{2} - \epsilon)$, there must be at least $(\epsilon/4r)n^2$ negative weight edges. Indeed, otherwise, we would have at least $\binom{n}{2} - (\epsilon/4r)n^2 - (\epsilon^2/4r)n^2$ positive weights and thus

$$w(f) \ge 1 \cdot \left(\binom{n}{2} - \frac{\epsilon}{4r}n^2 - \frac{\epsilon^2}{4r}n^2\right) - r \cdot \frac{\epsilon}{4r}n^2 > n^2\left(\frac{1}{2} - \epsilon\right)$$

contradicting the assumption. As $w(f) \ge 0$ we infer that there must be (much more than) $(\epsilon/4r)n^2$ positive weight edges.

Consider the degree sequence $d_1 \ge d_2 \cdots \ge d_n$ of any graph with n vertices and with δn^2 edges (say, $\delta < 1/16$). By a theorem of Erdős and Gallai [7] we infer that for any $3\delta n \le x \le 4\delta n$,

$$n - \delta n \ge d_x \ge \frac{\delta}{2}n \ .$$

As a direct argument, notice that if $d_x > n - \delta n$, then the sum of the degrees of the graph is at least $x(n - \delta n) \ge 3\delta n(n - \delta n)$ which contradicts the fact that the graph only has δn^2 edges. If, on the other hand, $d_x < (\delta/2)n$, then the number of edges is only at most $(\delta/2)n^2 + \binom{x}{2} < \delta n^2$ for $x \le 4\delta n$, again, a contradiction.

It follows from the above discussion that our weighing of K_n has a set of vertices X with, say, $|X| \ge (\epsilon/100r)n$ that are each incident with at least $(\epsilon/10r)n$ positive weight edges and at least $(\epsilon/10r)n$ negative weight edges.

We say that a vertex $u \in X$ is of type $(a, -b)$ if a is the most common positive weight of an edge incident with u in $V(K_n) \setminus X$ and $-b$ is the most common negative weight of an edge incident with u in $V(K_n) \setminus X$. As there are at most r^2 possible types, we have that there is a set $X^* \subset X$ with $|X^*| \geq (\epsilon/100r^3)n$ such that all vertices of X^* have same type $(a, -b)$ and every $u \in X^*$ is incident with at least $(\epsilon/20r^2)n$ edges of weight a in $V(K_n) \setminus X^*$ and at least $(\epsilon/20r^2)n$ edges of weight $-b$ in $V(K_n) \setminus X^*$.

As $(a+b)/gcd(a,b) \in B_r$, and since $K_{s,t}$ is r-good we have, without loss of generality, that $s = q(a+b)/gcd(a,b)$ for some positive integer q. Let $s_a = qa/gcd(a,b)$ and $s_b = qb/gcd(a,b)$. We will prove that there are three vertex-disjoint subsets $T \subset X^*$, S_a and S_b with $|T| = t$, $|S_a| = s_a$, $|S_b| = s_b$ such that all edges between T and S_a have weight $-b$ and all edges between T and S_b have weight a. Hence, the complete bipartite graph with one side consisting of T and the other side $S = S_a \cup S_b$ is a copy of $K_{s,t}$ whose total weight is $ats_b - bts_a = 0$.

Consider the bipartite graph B_a whose vertex classes are X^* and $V(K_n) \setminus X^*$ consisting of all the edges between X^* and $V(K_n) \setminus X^*$ having weight a. As $|X^*| = \Theta(n)$, $|V(K_n) \setminus X^*| = \Theta(n)$ and $|E(B_a)| = \Theta(n^2)$ we have, by Lemma 4.1, that B_a contains a copy of $K_{\Theta(\log n), s_b}$ with vertex classes $T^* \subset X^*$ having $|T^*| = \Theta(\log n)$ vertices and $S_b \subset V(K_n) \setminus X^*$ having $|S_b| = s_b$ vertices. Next, consider the bipartite graph B_b whose vertex classes are T^* and $V(K_n) \setminus (X^* \cup S_b)$ consisting of all the edges between T^* and $V(K_n) \setminus (X^* \cup S_b)$ having weight $-b$. As any vertex of T^* has at least $(\epsilon/20r^2)n - O(1)$ neighbors in $V(K_n) \setminus (X^* \cup S_b)$ we have, again by Lemma 4.1, that B_b contains a copy of K_{t,s_a} with vertex classes $T \subset T^*$ having $|T| = t$ vertices and $S_a \subset V(K_n) \setminus (X^* \cup S_b)$ having $|S_a| = s_a$ vertices. We have therefore found a zero-sum copy of $K_{s,t}$. ■

One can extend the notion of r-goodness to bipartite graphs that are not necessarily complete bipartite as follows. We say that a bipartite graph H with vertex classes X and Y and m edges is r-good if for every $1 \leq a \leq r$ and $1 \leq b \leq r$, there is $X' \subset X$ such that the sum of the degrees of the vertices of X' is $ma/(a+b)$. Thus, for example, any k-regular bipartite graph with sides of even cardinality is 1-good. The proof of Theorem 1.3 stays intact for this more general definition of r-good graphs.

Corollary 4.2 *For a positive integer r, an r-good bipartite graph H and a real $\epsilon > 0$ the following holds. For all n sufficiently large, any weighing $f : E(K_n) \to \{-r, \ldots, r\}$ with $|w(f)| \leq n^2(\frac{1}{2} - \epsilon)$ contains a zero-sum copy of H.*

Proof (Theorem 1.4): The proof of the first part of the theorem resembles the proof of Theorem 1.3, but with some necessary nontrivial modifications. Throughout this part of the proof we assume, wherever necessary, that n is sufficiently large as a function of ϵ. Consider a weighing $f : E(K_n) \to \{-1, 0, 1\}$ with $|w(f)| \leq (1-\epsilon)n^2/6$. We assume that $w(f) \geq 0$ as the proof of the negative case is symmetric.

We may assume that the number of edges having zero weight is at most $n^2/3$. Otherwise, since the Turán number of K_4 is at most $n^2/3$, we would have a copy of K_4 all of whose edges are zero, hence zero-sum, and we are done. Thus, there are at least $\binom{n}{2} - n^2/3 \geq (1-\epsilon/2)n^2/6$ edges with nonzero weight. As $w(f) \le (1-\epsilon)n^2/6$ we must therefore have at least $\epsilon n^2/24$ edges with weight -1. As $w(f) \ge 0$ we have at least $(1-\epsilon/2)n^2/12 \ge n^2/24$ edges with weight 1. We call a vertex *positive* if it is incident with at least $(\epsilon/1000)n$ positive edges and *negative* if it is incident with least $(\epsilon/1000)n$ negative edges. Let P denote the set of positive vertices and N denote the set of negative vertices.

Consider first the case $|P \cap N| \ge \epsilon n/1000$. As in the proof of Theorem 1.3 (namely, applying the Kövári-Sós-Turán Theorem twice) we can obtain a copy of $K_{2,31}$ whose vertex classes are A and B with $A = \{u, v\}$, $B = \{x_1, \dots, x_{31}\}$ and all the 31 edges incident with u have weight 1 while all the 31 edges incident with v have weight -1. Consider the complete graph induced by B. As the Ramsey number $R(4,3,3) = 31$ (see [12]) we have that B either induces a copy of K_4 all of whose edges are zero, in which case we are done, or else it induces are triangle $\{x_i, x_j, x_k\}$ all of whose edges have weight 1 or all of whose edges have weight -1. Hence, either $\{u, x_i, x_j, x_k\}$ is a zero-sum copy of K_4 or else $\{v, x_i, x_j, x_k\}$ is a zero-sum copy of K_4.

Consider next the case $|P \cap N| \le \epsilon n/1000$. Observe that by the definition of P and the fact that there are at least $n^2/24$ positive weight edges, we have that $|P| \ge n/24$. Likewise, by the definition of N and the fact that there are at least $\epsilon n^2/24$ negative weight edges we have that $|N| \ge \epsilon n/24$. Let $X = N \setminus P$ and $Y = P \setminus N$ and observe that $|X| \ge \epsilon n/50$ and $|Y| \ge n/50$. We claim that most edges between X and Y have weight 0. Indeed, the number of edges between X and Y with positive weight cannot be more than $|X|(\epsilon/1000)n$ as otherwise some vertex of X would belong to P, contradicting its definition. Likewise, the number of edges between X and Y with negative weight cannot be more than $|Y|(\epsilon/1000)n$ as otherwise some vertex of Y would belong to N, contradicting its definition. Since $(|X| + |Y|)(\epsilon/1000)n \le |X||Y|/10$ we have that at least 90 percent of the edges between X and Y have weight 0 and hence, at least half of the vertices of X are each incident to at least $(2/3)|Y|$ neighbors of Y via zero weight edges. Let X' denote such a set of vertices of X and notice that $|X'| \ge \epsilon n/100$. Consider the complete graph induced by X'. We may assume that X' contains less than $|X'|^2/3$ edges with weight 0 otherwise, by Turán's Theorem, we would have a K_4 in X' all of whose edges are zero and we are done. Hence X' has at least $\binom{|X'|}{2} - |X'|^2/3$ edges with nonzero weight. They cannot all be positive as otherwise X' would contain vertices of P contradicting its definition. Hence, there is an edge (u, v) with weight -1 such that $u, v \in X'$. Let $Y' \subset Y$ be the set of vertices such that each $y \in Y'$ is connected to both u and v via zero weight edges. By the definition of X' we have that $|Y'| \ge |Y|/3 \ge n/200$. Consider the complete graph induced by Y'. We may assume that Y' contains less than $|Y'|^2/3$ edges with weight 0 otherwise, by Turán's Theorem, we would have a K_4 in Y' all of whose edges are zero and we are done. Hence Y' has at least $\binom{|Y'|}{2} - |Y'|^2/3$ edges with nonzero weight. They cannot all be negative as otherwise Y' would contain vertices of N contradicting the definition of Y. Hence, there is an edge (y, w) with weight 1 such that $y, w \in Y'$. Now, $\{u, v, y, w\}$ induce a zero-sum copy of K_4.

We now proceed to the second part of the theorem. When $k \equiv 2, 3 \bmod 4$, the number of edges of K_k is odd, hence any coloring of K_n with weights -1 and 1 only does not have a zero-sum copy of K_n. If $k \equiv 1 \bmod 4$, then consider any weighing of K_n obtained by taking two vertex-disjoint cliques of size $n/2$ (assume n is even), labeling the edges of the first with 1, the edges of the second with -1 and the edges between them with zero. The total weight of the coloring is 0 yet is does not contain any zero-sum copy of K_k. Finally, if $k \equiv 2 \bmod 4$, then let $n = 16t^2$ and consider the following weighing of K_n. Take two vertex-disjoint cliques on sizes $n/2 - 2t$ and $n/2 + 2t$. Label the edges inside both cliques with 1 and the edges between them with -1. It is easy to verify that the total weight of this coloring is 0. Now, any copy of K_k has $k/2 + d$ vertices in the first clique and $k/2 - d$ vertices in the second clique for some integer d so its weight is

$$\binom{k/2+d}{2} + \binom{k/2-d}{2} - (k/2+d)(k/2-d) = 2d^2 - k/2 \ .$$

The only way that $2d^2 - k/2 = 0$ is if $k = 4d^2$. ■

5 Zero-sum spanning trees

Consider the graph S whose vertices are all the n^{n-2} labeled spanning trees of K_n. Two trees T_1 and T_2 are connected in S if they differ in a single edge, namely they are 1-edge switchable.

Lemma 5.1 *The family of labeled spanning trees of K_n is 1-edge switchable. Namely, S is connected.*

Proof: The lemma follows from the well-known fact that the spanning trees form a matroid. For completeness, we prove the lemma directly. We need to show that any two trees T_1 and T_2 are connected in S via a path. The proof is by downwards induction on $|E(T_1) \cap E(T_2)|$ where the base case $n-2$ follows from the definition of S. Assume that $|E(T_1) \cap E(T_2)| < n-2$ and consider the forest F induced by $E(T_1) \cap E(T_2)$ which has more than two components. F can thus be completed into a tree T_3 by adding to F at least one edge of $E(T_1) \setminus E(T_2)$ and at least one edge of $E(T_2) \setminus E(T_1)$. As $|E(T_1) \cap E(T_3)| > |E(T_1) \cap E(T_2)|$ and $|E(T_2) \cap E(T_3)| > |E(T_1) \cap E(T_2)|$ we have by the induction hypothesis that T_3 and T_1 are connected in S via a path and T_3 and T_2 are connected in S via a path, thus T_1 and T_2 are connected in S via a path. ■

Proof (Theorem 1.5): Let n be odd and let $f : \{-1, 0, 1\} \to E(K_n)$ with $|w(f)| \le n-2$. We may assume $w(f) \ge 0$ as the negative case is symmetric. We first observe that there is some spanning tree T with $w(T) \le 0$. Consider a partition of K_n into $(n-1)/2$ pairwise edge-disjoint Hamilton cycles. If all of these Hamilton cycles have positive weight, then since $w(f) \le n-2$, at least one of them has weight 1. By deleting some edge with weight 1 on this Hamilton we obtain a zero-sum Hamilton path.

Let therefore T_{max} and T_{min} be two spanning trees with maximum total weight and minimum total weight respectively. As $w(T_{max}) \ge 0$ and $w(T_{min}) \le 0$ we have by Lemma 5.1 that there are two spanning trees T_1 and T_2 with $w(T_1) \ge 0$, $w(T_2) \le 0$ and $w(T_1) - w(T_2) \le 2$. Hence, we can assume that $w(T_1) = 1$ and $w(T_2) = -1$ otherwise we are done.

Consider some edge $e = (x, y)$ of T_1 with weight 0 (notice that there must be such an edge since n is odd and $w(T_1) = 1$) and consider the two disconnected components of $T_1 - e$. Denote them by X and Y where $x \in X$ and $y \in Y$.

If there is some edge f with weight -1 connecting X and Y, then we are done since the spanning tree whose edge set is $(E(T_1) - e) \cup f$ has weight zero. Also notice that not all $|X| \cdot |Y|$ edges between X and Y have weight 0 otherwise there is a spanning tree all of whose edges are zero. Hence there is some edge f between X and Y having weight 1.

Recall that $E(T_2) = (E(T_1) - r) \cup s$ where r is an edge with weight 1 and s is an edge with weight -1. Without loss of generality, the two endpoints of r are both in X (otherwise the two endpoints are both in Y). Now, either the two endpoints of s are both in X and then the tree whose edge set is $(E(T_2) - e) \cup f$ has weight zero. Else, one endpoint of s is in X and the other is in Y, but this case (of an edge with weight -1 between X and Y) was already ruled out. ∎

The following proposition shows that in any balanced $\{-1, 0, 1\}$-weighing of a tree one can still find a relatively large zero-sum subtree, although far from spanning.

Proposition 5.2 *Let T be a tree and $f : E(T) \to \{-1, 0, 1\}$ such that $|w(f)| \leq q$. Then, there is a subtree T^* with $e(T^*) \geq (e(T) - q)/(q + 1)$ which is zero-sum, and this is best possible.*

Proof: We prove the proposition by induction of q. If $q = 0$ the statement is trivially true. Assume we proved it for q and let's prove it for $q + 1$. We may assume without loss of generality that $w(f) = q+1$ as the negative case is symmetric. A branch in T rooted at a vertex z is obtained by taking an edge (z, u) and all the subtree rooted at u. For a vertex $v \in T$ and a branch B with v as its root, let $e(B)$ denote the number of edges in B. Let B^* be a branch with root z with the property that $e(B^*)$ is minimal among all branches B of T with $w(B) > 0$.

If B^* consists of a single edge (z, v), then v is a leaf and by the definition of B^* the weight of (u, v) is 1. Thus, $T^* = T - v$ is a tree with $w(T^*) = q$. Hence by induction, there is a zero-sum subtree T^{**} of T^* with $e(T^{**}) \geq (e(T^*)-q)/(q+1) = (e(T)-(q+1))/(q+1) > (e(T)-(q+1))/(q+2)$.

So we may assume that B^* is not a single edge. Let $e = (z, w)$ be the unique edge of B^* incident with w and notice that since w is not a leaf, each additional neighbor of w other than z defines a sub-branch of B^*. Denote them by $B_1, \ldots, B_r$. Now, $w(B_i)$ cannot be positive as this contradicts the minimality of B^*. If $w(e) \leq 0$, then since $w(B^*) > 0$, this implies that there is some positive branch B_i again contradicting the minimality of B^*. Hence, it follows that $w(e) = 1$. If there is a negative branch B_i, then again as $w(B^*) > 0$ and $w(e) = 1$, the sum of weights of all branches $B_1, \ldots, B_r$ is non-negative, so there must be another branch B_j which is positive, contradicting the minimality of B^*. So, $w(B_i) \leq 0$ for $i = 1, \ldots, r$. Hence we conclude that $\sum_{i=1}^{r} w(B_i) = 0$ as there is no negative and no positive branches at w. So write B^{**} for the tree consisting of the union of $B_1, \ldots, B_r$ and observe that B^{**} is a zero-sum subtree. If $e(B^{**}) \geq (e(T) - (q+1))/(q+2)$ we are done. Else set $T^* = T \setminus V(B^{**})$. From the above we infer that sum $w(T^*) = q$ (we lost just 1 for the edge (z, w)), and that $e(T^*) = e(T) - e(B^{**}) - 1$. By induction there is a zero-sum tree T^{**}

with

$$\begin{aligned}
e(T^{**}) &\geq (e(T^*) - q)/(q+1) \\
&= (e(T) - e(B^{**}) - q - 1)/(q+1) \\
&> (e(T) - (e(T) - (q+1))/(q+2) - (q+1))/(q+1) \\
&= (e(T) - (q+1))/(q+2)\ .
\end{aligned}$$

To see that the result is best possible consider the following example. Let P_n denote the path on n edges. Label the edges of P_n as follows. First take n such that $n-q = 0 \bmod q+1$. Label the first $(n-q)/(q+1)$ edges on the path with zero and then take q blocks of the form 1 following $(n-q)/(q+1)$ zeros. The sum of the weights of the edges is q. the longest zero-sum path is $(n-q)/(q+1)$ and the total number of elements edges is $(n-q)/(q+1)+q((n-q)/(q+1)+1) = (q+1)(n-q)/(q+1)+q = n$. ■

6 Concluding remarks and open problems

The proof of Theorem 1.4 almost settles the question of existence or nonexistence of zero-sum copies of K_k for weighings with $\{-1,0,1\}$ with total weight zero. The only remaining cases left open are values of k of the form $k = 4d^2$ where $d \geq 2$ (so K_{16} is the first open case).

Problem 6.1 *For $k = 4d^2$, is it true that for all n sufficiently large, any weighing $f : E(K_n) \to \{-1,0,1\}$ with $w(f) = 0$ contains a zero-sum copy of K_k.*

Another open problem is to determine precisely all bipartite graphs that satisfy Corollary 4.2 for a given r. Many bipartite graphs fail to do so for simple divisibility reasons. For example, any bipartite graph with an odd number of edges fails already for $r = 1$. However, the set of bipartite graphs that potentially satisfy Corollary 4.2 may be strictly larger than the set of r-good bipartite graphs.

In this paper we considered weighings of K_n which are close to balanced, trying to find subgraphs that are zero-sum or close to zero-sum. It would be interesting to consider weighing of other combinatorial structures that guarantee the existence of zero-sum or close to zero-sum substructures. For weighings of complete k-uniform hypergraphs, the results of Section 2 and 3 remain almost intact. In fact, the exact same proof of Theorem 1.1 can be reused to prove the following statement which generalizes it.

Proposition 6.2 *Let H be a k-uniform hypergraph with n vertices and maximum degree Δ. Let $f : E(K_n^k) \to [-r,r]$ be an edge weighing with $|w(f)| \leq 2r\binom{n-1}{k-1}$. Then, there is a copy of H in K_n^k with absolute weight at most $2\Delta r$. Furthermore, if H is connected and $|w(f)| \leq 2r\binom{n-1}{k-1}(1-\frac{1}{\Delta})$, then there is a copy of H in K_n^k with absolute weight at most $2(\Delta-1)r$.*

As Lemma 3.1 is only about integers, the proof of Theorem 1.2 can be analogously phrased for almost H-factors of complete hypergraphs, with some adjustments to the constants. We omit the obvious details. Conversely, one may also wish to look at total balanced weighing of sparse structures.

References

[1] P. Balister, Y. Caro, C. Rousseau, and R. Yuster. Zero-sum square matrices. *European Journal of Combinatorics*, 23(5):489–497, 2002.

[2] A. Bialostocki and P. Dierker. Zero sum ramsey theorems. *Congressus Numerantium*, 70:119–130, 1990.

[3] A. Bialostocki and P. Dierker. On zero sum ramsey numbers: multiple copies of a graph. *Journal of Graph Theory*, 18(2):143–151, 1994.

[4] B. Bollobás. *Extremal Graph Theory.* Academic Press, 1978.

[5] Y. Caro. A complete characterization of the zero-sum (mod 2) ramsey numbers. *Journal of Combinatorial Theory, Series A*, 68(1):205–211, 1994.

[6] Y. Caro. Zero-sum problems - a survey. *Discrete Mathematics*, 152(1):93–113, 1996.

[7] P. Erdős and T. Gallai. Graphs with given degrees of vertices. *Mat. Lapok.*, 11:264–274, 1960.

[8] Z. Füredi and D. Kleitman. On zero-trees. *Journal of Graph Theory*, 16(2):107–120, 1992.

[9] S.L. Hakimi. On realizability of a set of integers as degrees of the vertices of a linear graph. *SIAM Journal on Applied Mathematics*, 10(3):496–506, 1962.

[10] V. Havel. A remark on the existence of finite graphs. *Casopis Pest. Mat.*, 80:477–480, 1955.

[11] T. Kövari, V. Sós, and P. Turán. On a problem of k. zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.

[12] P. Radziszowski. Small ramsey numbers. *The Electronic Journal of Combinatorics*, DS1:1–50, 2011.

[13] A. Schrijver and P.D. Seymour. A simpler proof and a generalization of the zero-trees theorem. *Journal of Combinatorial Theory, Series A*, 58(2):301–305, 1991.