Zero-Sum Ascending Waves

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Abstract

A sequence of positive integers $a_1 \leq a_2 \leq \ldots \leq a_n$ is called an *ascending monotone wave* of length n, if $a_{i+1} - a_i \geq a_i - a_{i-1}$ for $i = 2, \ldots, n-1$. If $a_{i+1} - a_i > a_i - a_{i-1}$ for all $i = 2, \ldots, n-1$ the sequence is called an *ascending strong monotone wave* of length n. Let Z_k denote the cyclic group of order k. If $k \mid n$, then we define $MW(n, Z_k)$ as the least integer m such that for any coloring $f : \{1, \ldots, m\} \to Z_k$ there exists an ascending monotone wave of length n, where $a_n \leq m$, such that $\sum_{i=1}^{n} f(a_i) \equiv 0 \bmod k$. Similarly, define $SMW(n, Z_k)$, where the ascending monotone wave in $MW(n, Z_k)$ is replaced by an ascending strong monotone wave. The main results of this paper are:

- $\frac{\sqrt{k}}{2} n \leq MW(n, Z_k) \leq c_1(k) n$. Hence, this result is tight up to a constant factor which depends only on k.
- $\binom{n}{2} < SMW(n, Z_k) \leq c_2(k) n^2$. Hence, this result is tight up to a constant factor which depends only on k.
- $MW(n, Z_2) = 3n/2$.
- $\frac{23}{12} n - 7/6 \leq MW(n, Z_3) \leq 2n + 3$.

These results are the zero-sum analogs of theorems proved in [1] and [5].
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1 Introduction

A sequence of positive integers $a_1 \leq a_2 \leq \ldots \leq a_n$ is called an *(ascending) monotone wave* of length n, if $a_{i+1} - a_i \geq a_i - a_{i-1}$ for $i = 2, \ldots, n-1$. If $a_{i+1} - a_i > a_i - a_{i-1}$ for all $i = 2, \ldots, n-1$

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the sequence is called an *(ascending) strong monotone wave* of length n. In particular, every arithmetic progression is a monotone wave. Alon and Spencer [1] and Brown, Erdös and Freedman [5] considered the least positive integer $t(n)$ such that in any coloring of the integers in the interval $[1, \ldots, t(n)]$, using two colors, there is always a monochromatic monotone wave of length n. It is shown in [1] that $c_1 n^3 \le t(n) \le c_2 n^3$ where c_1 and c_2 are some positive constants. Bollobás, Erdös and Jin [4] considered monochromatic pairs of strong monotone waves of length 2, in a k-coloring of the integers.

In the zero-sum direction, a theorem of Alon and Caro [2] states that in any coloring of the integers $1, \ldots, 2n-1$, where n is even, with the colors 0 and 1, there is an arithmetic progression $a_1, \ldots, a_n$, such that $\sum_{i=1}^{n} c(a_i) \equiv 0 \bmod 2$, where $c(a_i)$ is the color of a_i. Another strongly related paper is [8]. Motivated by these results and the recent trend in zero-sum Ramsey Theory (see, e.g. [3, 6, 7]), we shall consider the zero-sum analogs of the above mentioned results.

Define $MW(n, Z_k)$ as the least integer m such that for any coloring $f : \{1, \ldots, m\} \to Z_k$, there exists a monotone wave of length n with $a_n \le m$, such that $\sum_{i=1}^{n} f(a_i) \equiv 0 \bmod k$. Define $SMW(n, Z_k)$ as the least integer m such that for any coloring $f : \{1, \ldots, m\} \to Z_k$, there exists a strong monotone wave of length n with $a_n \le m$, such that $\sum_{i=1}^{n} f(a_i) \equiv 0 \bmod k$. Finally, define $W(n, Z_k)$ as the least integer m such that for any coloring $f : \{1, \ldots, m\} \to Z_k$, there exists an arithmetic progression of length n with $a_n \le m$, such that $\sum_{i=1}^{n} f(a_i) \equiv 0 \bmod k$. The purpose of this paper is to investigate these three functions. As usual in zero-sum theory, we shall assume that $k \ge 2$ and that k divides n. Our main results determine the asymptotic behavior of $MW(n, Z_k)$ and $SMW(n, Z_k)$, for fixed k. It turns out that $MW(n, Z_k)$ is a linear function of n, and $SMW(n, Z_k)$ is a quadratic function of n. In case $k = 2$, $MW(n, Z_2)$ is determined precisely, and in case $k = 3$, tight upper and lower bound are determined. For $W(n, Z_k)$, a quadratic lower bound is determined for fixed $k \ge 3$. We summarize these results in the following theorems:

Theorem 1.1 *$\frac{\sqrt{k}}{2} n \le MW(n, Z_k) \le cn$, where $c = c(k)$ is a constant depending only on k.*

Note that for fixed k, Theorem 1.1 is best possible up to a constant factor.

Theorem 1.2 *$SMW(n, Z_k) \le cn^2$, where $c = c(k)$ is a constant depending only on k.*

Note that for fixed k, Theorem 1.2 is best possible up to a constant factor, as any strong monotone wave of length n has $a_n \ge n(n-1)/2 + 1$. Thus, $SMW(n, Z_k) \ge n(n-1)/2 + 1$.

Theorem 1.3 *$MW(n, Z_2) = 3n/2$, $\frac{23}{12} n - 7/6 \le MW(n, Z_3) \le 2n + 3$.*

Theorem 1.4 *$W(n, Z_k) \ge n^2(1 - o(1))$ whenever $k \ge 3$. Furthermore, if $n + 1$ is a prime, $W(n, Z_k) \ge n^2$.*

The last theorem should be compared with the aforementioned theorem of Alon and Caro, showing that $W(n, Z_2) = 2n - 1$.

The rest of this paper is organized as follows. In section 2 we prove the upper and lower bounds for $MW(n, Z_k)$ and $SMW(n, Z_k)$, namely Theorems 1.1 and 1.2. In Section 3 we focus on monotone waves in Z_2 and Z_3 and prove Theorem 1.3. Arithmetic progressions and Theorem 1.4 are dealt with in Section 4. The final section contains some concluding remarks and open problems.

2 Upper bounds for $MW(n, Z_k)$ and $SMW(n, Z_k)$

We begin this section by proving the lower bound in Theorem 1.1. Clearly, $MW(n, Z_k) \geq n$. However, the lower bound of Theorem 1.1, which is $\sqrt{k}n/2$, shows that no absolute multiple of n can bound $MW(n, Z_k)$ for general k:

Proof that $MW(n, Z_k) \geq \sqrt{k}n/2$: Let $m = \lfloor \sqrt{k-1} \rfloor$. Let $x = \lfloor (n-m)/2 \rfloor$. Consider the coloring $f : [1, \ldots, (m+1)x + m^2] \to Z_k$ defined by $f((x+m)i + j) = 0$ for $i = 0, \ldots, m$ and $j = 1, \ldots, x$, and otherwise $f = 1$, That is, f assigns x consecutive zeroes followed by m consecutive ones, followed by x consecutive zeroes, and so forth. We claim that there is no zero-sum monotone wave of length n, in the defined interval. Indeed, if $a_1, \ldots, a_t$ is a zero-sum monotone wave, then we shall show that $t < n$. To see this, we note first that there are exactly m^2 integers in the interval $[1, \ldots, (m+1)x + m^2]$ which are assigned 1 by f. Since $m^2 < k$ it follows that f must be constantly zero on any zero-sum monotone wave. In particular, $f(a_i) = 0$ for $i = 1, \ldots, t$. If $t \leq x$ we are done since $x < n$. Otherwise, the definition of f implies that there must be some $j \leq x$, such that $a_{j+1} - a_j \geq m + 1$. The monotonicity now implies that $a_{j+p} - a_j \geq p(m+1)$ for $p = 1, \ldots, t - j$. Thus, $a_t - a_j \geq (t-j)(m+1)$ and therefore

$$t \leq t - j + x \leq \frac{a_t - a_j}{m+1} + x < \frac{a_t}{m+1} + x \leq \frac{(m+1)x + m^2}{m+1} + x < 2x + m \leq n$$

as required. It follows that $MW(n, Z_k) \geq (m+1)x + m^2 + 1 \geq \sqrt{k}n/2$.

We now prove the upper bounds for both $MW(n, Z_k)$ and $SMW(n, Z_k)$, thereby completing the proofs of Theorems 1.1 and 1.2. Although the claimed upper bounds in both theorems are different, the proofs are similar. Before we proceed with the proof we need two definitions. Let $MW(n, Z_k, s)$ ($SMW(n, Z_k, s)$) be the least integer m, such that for any coloring $f : \{1, \ldots, m\} \to Z_k$, using only s colors from Z_k, there exists a zero-sum (strong) monotone wave of length n. Clearly, $MW(n, Z_k, 1) = n$, $SMW(n, Z_k, 1) = \binom{n}{2} + 1$, $MW(n, Z_k, k) = MW(n, Z_k)$, $SMW(n, Z_k, k) = SMW(n, Z_k)$. The following lemma establishes a relation between $MW(n, Z_k, s)$ and $MW(n, Z_k, s+1)$, and between $SMW(n, Z_k, s)$ and $SMW(n, Z_k, s+1)$.

Lemma 2.1 *If $n \geq k^2$ is divisible by k then*

$$MW(n, Z_k, s+1) \leq \binom{(k-1)(s+1)+1}{2} MW(n, Z_k, s) + n - (k-1)(s+1).$$

$$SMW(n, Z_k, s+1) \leq \binom{(k-1)(s+1)+1}{2} SMW(n, Z_k, s) + \binom{n-(k-1)(s+1)+1}{2} + 1.$$

Proof: Put $x = MW(n, Z_k, s)$ ($x = SMW(n, Z_k, s)$ for the strong monotone wave case). Put $q = (k-1)(s+1)$ and put $y = \binom{q+1}{2}x + n - q$ ($y = \binom{q+1}{2}x + \binom{n-q+1}{2} + 1$ in the strong case). Consider $f : [1, \ldots, y] \to Z_k$ where $f(j) \in S$, and $S = \{u_0, \ldots, u_s\}$ is an $s+1$-subset of Z_k. We must prove that there is a (strong) monotone wave of length n which is zero-sum. For $i = 0, \ldots s$, let (in the non-strong case) $r_i = |\{j \ : \ f(j) = u_i \ , \ j \leq n - q\}|$ be the number of times f assigns u_i to integers in the interval $[1, n-q]$. In the strong case, let r_i be the number of times f assigns u_i to members of the sequence $\binom{j}{2} + 1$ for $j = 1, \ldots, n-q$. Let $0 \leq t_i \leq k-1$ be selected so that $t_i + r_i \equiv 0 \bmod k$. Now, define $p = q - \sum_{i=0}^{s} t_i$. Clearly, $p \geq 0$. Since $\sum_{i=0}^{s} r_i = n - q$ and since $k \mid n$, it follows that $k \mid p$. Clearly, in the non-strong case,

$$(\sum_{j=1}^{n-q} f(j)) + (\sum_{i=0}^{s} u_i \cdot t_i) + p \cdot u_0 = \sum_{i=0}^{s} u_i(r_i + t_i) + p \cdot u_0 \equiv 0 \bmod k.$$

In the strong case we similarly have

$$(\sum_{j=1}^{n-q} f(\binom{j}{2} + 1)) + (\sum_{i=0}^{s} u_i \cdot t_i) + p \cdot u_0 = \sum_{i=0}^{s} u_i(r_i + t_i) + p \cdot u_0 \equiv 0 \bmod k.$$

Thus, if we can find a strong monotone wave $a_1, \ldots, a_q$ of length $q = p + \sum_{i=0}^{s} t_i$ in the interval $[n-q+1, y]$ (or in the interval $[\binom{n-q+1}{2} + 1, y]$ in the strong case) having t_i elements colored by u_i, for $i = 1, \ldots, s$, and $t_0 + p$ elements colored by u_0, and such that $a_1 - (n-q) < a_2 - a_1$, ($a_1 - \binom{n-q}{2} - 1 < a_2 - a_1$ in the strong case) then extending the interval $[1, \ldots, n-q]$ (or extending the sequence $\binom{j}{2} + 1$ for $j = 1, \ldots, n-q$ in the strong case) with this wave, we obtain a monotone wave (strong monotone wave) of length n which is zero-sum. We will show that such a (strong) monotone wave exists, under the assumption that the interval $[n-q+1, y]$ contains no monotone wave (strong monotone wave) of length n which is zero-sum (and we can assume the latter, since otherwise we are done).

We shall construct $a_1, \ldots, a_q$ such that the first $p + t_0$ elements are colored u_0, the next t_1 elements are colored u_1, and so on, the last t_s elements are colored u_s. Indeed, a_1 can be found in the interval $[n-q+1, n-q+x]$, (or in the interval $[\binom{n-q+1}{2} + 1, \binom{n-q+1}{2} + x]$ in the strong case) since all $s+1$ colors appear in this interval, otherwise, by the definition of x, there would have been a zero-sum (strong) monotone wave of length n. Having determined a_1, we can similarly find a_2 in the interval $[2a_1-(n-q)+1, 2a_1-(n-q)+x]$ (or $[2a_1-\binom{n-q}{2}+2, 2a_1-\binom{n-q}{2}+x+1]$ in the strong case). Note that this guarantees $a_2-a_1 > a_1-(n-q)$ (or $a_2-a_1 > a_1-\binom{n-q}{2}-1$ in the strong case), maintaining (strong) monotonicity. Now a_3 can be found in the interval $[2a_2-a_1+1, 2a_2-a_1+x]$, and so forth, where in the end a_q can be found in the interval $[2a_{q-1}-a_{q-2}+1, 2a_{q-1}-a_{q-2}+x]$. Note that by this construction we get that $a_j \le x\cdot\binom{j+1}{2}+(n-q)$, (or $a_j \le x\cdot\binom{j+1}{2}+\binom{n-q+1}{2}+1$) as can be seen by induction from the fact that $a_{j+1}-a_j \le a_j-a_{j-1}+x$. Since $y = x\binom{q+1}{2}+(n-q)$, (and $y = x\binom{q+1}{2}+\binom{n-q+1}{2}+1$ in the strong case) our construction can be completed. □

The following corollary is immediate from Lemma 2.1, since

$$MW(n,Z_k,s+1) \le MW(n,Z_k,s)\binom{(k-1)(s+1)+1}{2}+n-(k-1)(s+1) \le k^4 MW(n,Z_k,s).$$

Similarly, $SMW(n,Z_k,s+1) \le k^4 SMW(n,Z_k,s)$.

Corollary 2.2 *If $n \ge k^2$ then $MW(n,Z_k) < k^{4k}n$ and $SMW(n,Z_k) < k^{4k}n^2$.*

In case $n \le k^2$ we trivially have $MW(n,Z_k) \le c(k)$ and $SMW(n,Z_k) \le c(k)$ where $c(k)$ is an appropriately chosen constant since in this case, $MW(n,Z_k)$ and $SMW(n,Z_k)$ are only functions of k, bounded by the Van der Waerden constant. This completes the proof of Theorems 1.1 and 1.2. □

The constant $c(k) = k^{4k}$ which appears in the upper bounds for $MW(n,Z_k)$ and $SMW(n,Z_k)$ can be significantly improved when k is prime. We will show how to obtain $MW(n,Z_k) \le k^2 n$ (a similar proof shows that $SMW(n,Z_k) \le k^2n^2$). Consider a coloring $f : [1,\ldots,k^2n] \to Z_k$. Let $0 \le t \le k-1$ be chosen such that $\sum_{i=k}^{n-1} \equiv t \bmod k$. For $j = 1,\ldots,k$, let $I_j = [(j^2-j+1)n, (j^2-j+2)n-1]$. These k intervals are disjoint and if we select an element from each interval we obtain a monotone wave of length k. If f is constant on some I_j, then I_j is a zero-sum monotone wave. Assume, therefore, that $A_j = \{f(a_j), f(b_j)\}$ where $\{a_j, b_j\} \subset I_j$ and $f(a_j) \ne f(b_j)$. Applying the Cauchy-Davenport theorem (see, e.g., [9, 10]) to the sets A_j $j = 1,\ldots,k$ (here we use the fact that k is prime), we can obtain k integers $m_1,\ldots,m_k$ such that $m_j \in \{a_j,b_j\}$ and $\sum_{i=1}^{k} f(m_i) \equiv -t \bmod k$. Thus, $k, k+1, \ldots, n-1, m_1, \ldots, m_k$ is a zero-sum monotone wave of length n.

3 Zero-sum waves in Z_2 and Z_3

In this section we prove Theorem 1.3, which determines $MW(n,Z_2)$, and provides a very tight bound for $MW(n,Z_3)$. We shall begin with the easier case:

Determining $MW(n,Z_2)$: Since $Z_k = Z_2$ in this case, we are interested only in the case where n is even. A lower bound of $3n/2$ is established as follows. Put $n = 2m$, and consider the interval

$[1,\ldots,3m-1]$ with the coloring f which is zero everywhere except for $f(m+1)=1$. Clearly, any monotone wave with length t satisfying $\sum_{i=1}^{t} f(a_i) \equiv 0 \bmod 2$ cannot include the element $m+1$. Thus, if $a_1 > m+1$ we must have $t \leq 2m-2$, and if $a_1 < m+1$ we must have $t \leq m+(2m-2)/2 = 2m-1$. In any case $t \leq 2m-1 = n-1$, showing $MW(n,Z_2) \geq 3m = 3n/2$.
We now prove the upper bound. Let $n=2m$ and consider a fixed Z_2-coloring f of the interval $[1,\ldots,3m]$. Let $b = \sum_{i=1}^{2m-1} f(a_i) \bmod 2$. Consider first the case $b=1$. If $f(i)=1$ for some $2m \leq i \leq 3m$ we are done, since the sequence $\{1,\ldots,2m-1,i\}$ has even sum, and is a monotone wave of length n. Thus, we assume $f(i)=0$ for all $2m \leq i \leq 3m$. Let j be the largest index having $f(j)=1$. If $j \leq m$ we are done since the sequence $\{m+1,\ldots,3m\}$ has zero sum, and is a monotone wave of length n. We may therefore assume $j \geq m+1$. Clearly, $\sum_{i=1}^{j-1} f(a_i) \bmod 2 = b-1 = 0$. Thus, the sequence $\{1,\ldots,j-1,j+1,\ldots,4m-j+1\}$ has even sum, and is a monotone wave of length n. Note that the sequence is within the interval bounds since $4m-j+1 \leq 3m$.
Now consider the case $b=0$. By defining the coloring $g(i)=1-f(i)$ we now have a coloring satisfying $\sum_{i=1}^{2m-1} g(a_i) \bmod 2 \equiv b+(2m-1) \bmod 2 = 1$. According to the previous arguments, there is a monotone wave of length n with zero sum, with respect to g. The same sequence has zero sum with respect to f, since n is even.
A tight bound for $MW(n,Z_3)$ We start with a lower bound for $MW(n,Z_3)$. We wish to prove that $MW(n,Z_3) \geq \frac{23}{12}n - 7/6$. The case $n=3$ is therefore trivial, so we may assume $n \geq 6$ is divisible by 3. Let x and y be two positive integers satisfying $1 < y < x < n$, and having different parity. Put

$$z = \min\{n+x-1, 2y-x+2n-1, -x/2-3y/2+3n-1/2\}.$$

Consider the coloring $f:[1,\ldots,z] \rightarrow Z_3$ which satisfies $f(x)=f(y)=1$ and $f(i)=0$ for $i \notin \{x,y\}$. We now show that there is no monotone wave of length n whose sum (w.r.t f) is divisible by 3. Consider any monotone wave $T=\{a_1,\ldots,a_t\}$ satisfying $\sum_{i=1}^{t} f(a_i) \equiv 0 \bmod 3$. Clearly, x and y do not belong to T. We distinguish three cases:

1. If $a_1 > x$ then $t < n$ since $z < x+n$.

2. If $x > a_1 > y$, (possible only if $x-y>1$), then there are at most $x-a_1$ members of T smaller than x, and the remaining members of T are greater than x, and have difference at least 2 between each other. Thus, either $t \leq x-a_1 \leq x < n$, or $a_t \geq 2(t-(x-a_1))+x-1$. Since $a_t \leq z \leq 2y-x+2n-1$ it follows that $2y-x+2n-1 \geq 2t-x+2a_1-1 \geq 2t-x+2y+1$ which implies $t \leq n-1$.

3. If $a_1 < y$ there are at most $y-a_1$ members of T smaller than y. Since $x-y$ is odd, there are at most $(x-y-1)/2$ members of T between x and y, since the difference between these

elements is at least 2. Furthermore, since $x-y$ is odd, the gap between any two elements of T which are larger than x (if there are any) is at least 3. Thus, either $t<n$ or

$$a_t \geq 3(t-(y-a_1)-(x-y-1)/2)+x-2 = 3t-3y/2+3a_1-x/2-1/2.$$

Since $a_t \leq z \leq -x/2-3y/2+3n-1/2$ it follows that

$$-x/2-3y/2+3n-1/2 \geq 3t-3y/2+3a_1-x/2-1/2 \geq 3t-3y/2+5/2-x/2$$

which implies $t \leq n-1$.

In all cases, $t \leq n-1$. Thus, there is no monotone wave of length n whose sum (w.r.t f) is divisible by 3, which means $MW(n, Z_3) \geq z+1$. It now remains to choose x and y in order to maximize z, under the constraint that $1<y<x<n$ and $x-y$ is odd. If we did not insist that x and y be integers, we can take $x=(11n-2)/12$ and $y=(5n+4)/12$, giving

$$z = \min\{23n/12-7/6,\ 23n/12-1/6,\ 23n/12-11/12\} = 23n/12-7/6$$

Since y must be an integer, we can round $(5n+4)/12$ to the closest integer, and if this number is odd, round $(11n-2)/12$ to the closest even integer, or otherwise to the closest odd integer. Note that these adjustments change y by at most 1/2 and x by at most 1, thus the three expressions whose minimum defines z are reduced to give

$$z = \min\{23n/12-13/6,\ 23n/12-13/6,\ 23n/12-13/6\} = 23n/12-13/6.$$

Hence, $MW(n, Z_3) \geq 23n/12-7/6$, as required.

We now prove the claimed upper bound for $MW(n, Z_3)$. Fix a coloring $f : [1, \ldots, 2n+3] \rightarrow Z_3$. We must show that there exists a monotone wave of length n whose sum is divisible by 3. Let $x \equiv \sum_{i=1}^{n-1} f(i) \bmod 3$, where $1 \leq x \leq 3$. Put $c = 3-x$. If $f(t)=c$ for some $n \leq t \leq 2n+3$ we are done since in this case $1, 2, \ldots, n-1, t$ is the desired monotone wave. We may therefore assume that $f(j) \neq c$ for $j=n, \ldots, 2n+3$. We may also assume that f is not constant on the interval $[n, 2n+3]$ since in this case we have, e.g. that $n, \ldots, 2n-1$ is a zero-sum monotone wave. In particular, we may assume $f(n) \neq f(t)$, where $t>n$ is maximum possible.

Put $b \equiv \sum_{i=1}^{n-2} f(i) \bmod 3$, where $0 \leq b \leq 2$. If $b=c$ we are done since we can take the sequence $1, \ldots, n-2, n, t$ which has sum $c+f(n)+f(t) \bmod 3$ and this is the sum of three distinct values mod 3, and hence divisible by 3. The only problem is when $t=n+1$, but the maximality of t implies that in this case $f(n+1) \neq f(n+4) = f(n)$, and then the sequence $1, \ldots, n-2, n+1, n+4$ is monotone, and its sum is divisible by 3. We may now assume $b \neq c$.

Let $0 \leq a \leq 2$, where $a \neq b$ and $a \neq c$. This uniquely determines a. Let $r \geq n$ be the minimal integer satisfying $f(r) = b$, and let $s \geq n$ be the maximal integer satisfying $f(s) = b$. r and s exist since f is not constant on $[n, \ldots, 2n+3]$ and does not get the value c in this interval. Clearly, $r \leq s$, and we can also assume that $r \leq 2n-1$ since otherwise f has the constant value a on the interval $n, \ldots, 2n-1$, and this is a zero-sum monotone wave. Similarly, we can assume $s \geq n+4$, since otherwise f has constant value a on the interval $n+4, \ldots, 2n+3$. We distinguish the following cases, where each case assumes that the cases above it do not hold:

1. $r = n$. In this case the sequence $1, \ldots, n-2, n, s$ is monotone, and has sum $b + f(n) + f(s) = b + b + b \equiv 0 \bmod 3$.

2. $r = s$. Since the previous case does not hold we must have $f(n) = a$. If f is constantly a on $[1, \ldots, n]$ we are done. Otherwise, let $t < n$ be maximal having $f(t) \neq a$. We distinguish the following subcases:

   (a) *$f(t) = c$ and $r - t \leq n-1$.* In this case we take the monotone wave $t, \ldots, r-1, r, \ldots, t+n-1$ in which the value of f is a on all elements except $f(t) = c$ and $f(r) = b$. Thus, the sum is $a(n-2) + b + c \equiv 0 \bmod 3$.

   (b) *$f(t) = c$ and $r - t > n-1$.* In this case we take the monotone wave $t, \ldots, t+n-2, r$ in which the value of f is a on all elements except $f(t) = c$ and $f(r) = b$. Thus, the sum is $a(n-2) + b + c \equiv 0 \bmod 3$.

   (c) *$f(t) = b$, $t < n/2$, and $r - t > n$.* In this case we can take the monotone wave $t+1, \ldots, t+n$ in which the value of f is constantly a.

   (d) *$f(t) = b$, $t < n/2$, and $r - t \leq n$.* In this case we can take the monotone wave $t+1, \ldots, r-1, r+1, \ldots, 2n-r+2t+1$ in which the value of f is constantly a. Note that $2n - r + 2t + 1 \leq 2n+3$, so the wave is within bounds.

   (e) *$f(t) = b$ and $t \geq n/2$.* Let us denote $u = 2b + (n-t-1)a \bmod 3$, $v = b + (n-t)a \bmod 3$, $w = (n-t+1)a \bmod 3$. Clearly, u, v, w are all distinct mod 3. Assume first that $\sum_{i=1}^{t-1} f(i) + u \equiv 0 \bmod 3$. In this case the monotone wave $1, \ldots, n-1, r$ has sum $\sum_{i=1}^{t-1} f(i) + b + (n-t-1)a + b \equiv 0 \bmod 3$. If $\sum_{i=1}^{t-1} f(i) + v \equiv 0 \bmod 3$, the monotone wave $1, \ldots, n$ has sum $\sum_{i=1}^{t-1} f(i) + b + (n-t)a \equiv 0 \bmod 3$. Finally, if $\sum_{i=1}^{t-1} f(i) + w \equiv 0 \bmod 3$, we must again consider two subcases. If r and t have the same parity, or if $2n - t + 1 < r$ we can take the monotone wave $1, \ldots, t-1, t+1, t+3, \ldots, 2n-t+1$. This sequence does not include r and has sum $\sum_{i=1}^{t-1} f(i) + (n-t+1)a \equiv 0 \bmod 3$. If r and t have different parity and $2n - t + 1 \leq r$ we can take the monotone wave

$1, \ldots, t-1, t+1, \ldots, r-2, r+1, r+4, \ldots, 3n - r/2 - 3t/2 + 5/2$. This sequence does not include r and has sum $\sum_{i=1}^{t-1} f(i) + (n-t+1)a \equiv 0 \bmod 3$. However, we must show that $3n - r/2 - 3t/2 + 5/2 \le 2n+3$, which is equivalent to $n \le (r+3t+1)/2$, and this holds since $t \ge n/2$ and $r \ge n$.

3. $r + 1 = s$. Once again, there are three subcases:

   (a) $\sum_{i=r-n+2}^{r-1} f(i) \equiv b \bmod 3$. In this case the sequence $r-n+2, \ldots, r-1, r, r+1$ has sum $b + b + b \equiv 0 \bmod 3$.

   (b) $\sum_{i=r-n+2}^{r-1} f(i) \equiv a \bmod 3$, and $r \le 2n-2$. In this case the sequence $r-n+2, \ldots, r-1, r+2, r+5$ has sum $a + a + a \equiv 0 \bmod 3$.

   (c) $\sum_{i=r-n+2}^{r-1} f(i) \equiv a \bmod 3$, and $r > 2n-2$. Note that since we always assume $r \le 2n-1$ we must have $r = 2n-1$. In this case f has constant value a on the monotone wave $n, \ldots, 2n-2, 2n+1$.

   (d) $\sum_{i=r-n+2}^{r-1} f(i) \equiv c \bmod 3$. In this case the sequence $r-n+2, \ldots, r-1, r, r+2$ has sum $c + b + a \bmod 3$.

4. $r + 1 < s$. The subcases are:

   (a) $\sum_{i=r-n+1}^{r-2} f(i) \equiv c \bmod 3$. In this case the sequence $r-n+1, \ldots, r-2, r-1, r$ has sum $c + f(r-1) + f(r) = c + a + b \equiv 0 \bmod 3$. (We have used here the fact that Case 1 does not hold, and thus $r > n$, which means $r - 1 \ge n$, and hence $f(r-1) = a$).

   (b) $\sum_{i=r-n+1}^{r-2} f(i) \equiv b \bmod 3$. In this case the sequence $r-n+1, \ldots, r-2, r, s$ has sum $b + b + b \equiv 0 \bmod 3$.

   (c) $\sum_{i=r-n+1}^{r-2} f(i) \equiv a \bmod 3$. If there exists $j > r$ with $f(j) = a$, the monotone wave $r-n+1, \ldots, r-2, r-1, j$ has sum $a + f(r-1) + f(j) = a + a + a \equiv 0 \bmod 3$. Otherwise, we have that $f(j) = b$ for all $j \ge r$. If $r \le n+4$ we have the monotone wave $r, \ldots, r+n-1$ in which f is constantly b, and hence it is zero-sum. If $r > n+4$ we know that $f(r-2) = f(r-1) = a$. This means that the pairs $(r-2, r-1)$, $(r-2, r)$ and $(r, r+3)$ have sums $a + a$, $a + b$ and $b + b$ respectively, and these sums are distinct mod 3. Thus, the monotone wave $r-n, \ldots, r-3$ of length $n-2$ can be continued by one of the three pairs to obtain a monotone wave of length n which is zero-sum.

This proves $MW(n, Z_3) \le 2n+3$, and completes the proof of Theorem 1.3. $\square$

4 Zero-sum arithmetic progressions

Arithmetic progressions are a special case of monotone waves. Although $W(n, Z_2) = 2n - 1$ is completely determined, almost nothing is known about $W(n, Z_k)$ where $k \geq 3$. Theorem 1.4 provides a quadratic ($\Omega(n^2)$) lower bound of for all $k \geq 3$:

Proof of Theorem 1.4: Let p be the largest prime not exceeding n. We will show that $W(n, Z_k) \geq p(n-1)+1$. Since for $n \geq 2$ there is always a prime between $(n+1)/2$ and n, and if n is sufficiently large there is a prime between $n - n^{3/5}$ and n [11], it follows that $W(n, Z_k) \geq n^2(1 - o(1))$.

Consider the coloring $f : [1, \ldots, p(n-1)] \to Z_k$ which is defined by $f(j) = 1$ if $j \equiv 0 \bmod p$, and $f(j) = 0$ otherwise. We will show that there is no arithmetic progression of length n whose sum is divisible by k. Let $a_1, \ldots, a_n$ be an arithmetic progression. Thus, $a_i = a_1 + (i-1)d$, for some $1 \leq d \leq p - 1$. (Note that if $d \geq p$, then a_n would be out of bounds). We now show that if $1 \leq i < j \leq n$ and $j - i \neq p$, then $a_i \neq a_j \bmod p$. To see this, note that $a_j - a_i = (j-i)d$ and $(j-i)d$ is not a multiple of p as p is prime and $d \leq p-1$ and $j - i \neq p$, and $j - i < 2p$, since $n \leq 2p$. It follows that for every $t = 1, \ldots, p$, there exists an i such that $a_i \equiv t \bmod p$, and if $i + p \leq n$, then also $a_{i+j} \equiv t \bmod p$. Thus, $\sum_{i=1}^{n} f(a_i) = 1$ or $\sum_{i=1}^{n} f(a_i) = 2$. In any case, since $k \geq 3$, f is not zero-sum.

In case $n + 1$ is a prime, we can strengthen Theorem 1.3. by selecting $p = n + 1$, and defining $f : [1, \ldots, n^2 - 1] \to Z_k$ as follows: $f(i) = 1$ if $i \equiv 0 \bmod p$ or $i \equiv -1 \bmod p$ and $f(i) = 0$ otherwise. Now one obtains that for every arithmetic progression $a_1, \ldots, a_n$, any two elements are distinct mod p, thus the sum of the sequence is either 1 or 2, and thus, not zero-sum. Hence $W(n, Z_k) \geq n^2$ if $n + 1$ is a prime. $\square$

5 Concluding remarks and open problems

1. Although Theorem 1.1 shows that, for fixed k, $MW(n, Z_k)$ is a linear function of n, it is still interesting to determine the correct constant, which depends on k. Theorem 1.3 states that for $k = 2$ the constant is $3/2$, and for $k = 3$ the constant is at least $2 - \frac{1}{12}$ and at most 2. For $k > 3$ we only have the lower bound $\sqrt{k}/2$ and the upper bound k^{4k}, unless when k is a prime in which case the upper bound is improved to k^2. We conjecture that the constant for the upper bound is less than k^2 for all k.

2. Similar questions to the ones asked in the preceding paragraph are also valid for $SMW(n, Z_k)$. Theorem 1.2 shows that, for fixed k, $SMW(n, Z_k)$ is a quadratic function of n. A proof that $SMW(n, Z_2) = \binom{n+1}{2}$ is known, although it uses different tools, and will be presented

elsewhere. However, unlike $MW(n, Z_3)$, we do not know of similar tight upper and lower bounds for $SMW(n, Z_3)$. Furthermore, Theorem 1.1 provides a $\sqrt{k}/2$ lower bound for the constant which multiplies n in $MW(n, Z_k)$. We do not know how to prove a lower bound with a similar constant multiplying n^2 in $SMW(n, Z_k)$.

3. Using a computer, we know that $MW(3, Z_3) = 7$, $MW(6, Z_3) = 13$, $MW(9, Z_3) = 18$, $MW(12, Z_3) = 23$ and $MW(15, Z_3) = 29$. We conjecture that $MW(n, Z_3) \leq 23n/12 + C$ where C is some absolute (small) number. We have also computed $MW(4, Z_4) = 13$, $MW(8, Z_4) = 21$, $MW(5, Z_5) = 21$, $SMW(3, Z_3) = 9$ and $SMW(6, Z_3) = 27$.

4. The proof of Theorem 1.4 provides a quadratic lower bound for $W(n, Z_k)$, in case $k \geq 3$. We do not know of any polynomial upper bound for this function. In particular, is it true that $W(n, Z_k) = O(n^k)$? By using a computer we have $W(3, Z_3) = 9$ and $W(6, Z_3) = 36$. In view of Theorem 1.4 it is plausible to conjecture that $W(n, Z_3) = n^2$.

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