The Uniformity Space of Hypergraphs and its Applications

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Abstract

Let $H = (V, E)$ be a hypergraph, and let F be a field. A function $f : V \to F$ is called *stable* if for each $e \in E$, the sum of the values of f on the members of e is the same. The linear space consisting of the stable functions, denoted by $U(H, F)$, is called the *uniformity space* of H over F. The dimension of $U(H, F)$, denoted by $udim(H, F)$, is called the *uniformity dimension* of H over F. The concept of uniformity space carries over to several (weighted) (hyper)graph-theoretic problems, in which we require that all the sub(hyper)graphs having a specific property have the same weight or size. This is done by defining an appropriate hypergraph whose edges represent all the sub(hyper)graphs having the property. Two such natural problems are:

- Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs where G_1 is a subgraph of G_2. A function $f : E_2 \to F$ is called stable if all the copies of G_1 in G_2 have the same weight.
- Let $G = (V, E)$ be a graph. A function $f : V \to F$ is called stable if all the maximal (w.r.t. containment) independent sets of G have the same weight.

Clearly, many other problems can be formulated, and their resulting uniformity space can be defined. The purpose of this paper is twofold. The first is to determine (or, alternatively, compute efficiently) the uniformity dimension, and a corresponding basis, of several problems. The other purpose is to show applications of the uniformity space concept to other graph-theoretic problems, such as the determination of the zero-sum mod 2 Ramsey numbers.

1 Introduction

All graphs and hypergraphs considered here are finite, undirected and have no loops or multiple edges. For the standard graph-theoretic notations the reader is referred to [3]. Let $H = (V, E)$ be a hypergraph, and let F be a field. A function $f : V \to F$ is called *stable* if for some $c \in F$, and for each $e \in E$, $\sum_{v \in e} f(v) = c$. In other words, the sum of the values of f on the members of e is the same. Clearly, if f_1 and f_2 are two stable functions, so is every linear combination of

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them. Thus, the set $U(H,F)$ of all the stable functions is a linear space over F. We call this space the *uniformity space* of H over F. Another way to view $U(H,F)$ is through the incidence matrix of H. This zero-one matrix, denoted by $B(H)$, has $|V|$ columns and $|E|$ rows, and $B(e,v)=1$ iff $v \in e$. Thus, we may identify $U(H,F)$ with all the vectors $u \in F^{|V|}$ such that $B(H)u = cJ$, for some $c \in F$, and where J is the all-one vector in $F^{|E|}$. Clearly, $U(H,F)$ has finite dimension, which is at most $|V|$. The dimension of $U(H,F)$, denoted by $udim(H,F)$, is called the *uniformity dimension* of H over F. Note that $udim(H,F)$ can be immediately computed from the rank of $B(H)$ over F. The problem of computing the rank of incidence matrices of hypergraphs has been investigated by several researchers (cf. [2, 14, 25]) and these results may sometimes be helpful in solving combinatorial problems which rely on the characterization of $U(H,F)$. In this paper, however, we are concerned with combinatorial problems whose $U(H,F)$ characterization cannot be determined from the known results on $rank(B(H))$.

The concept of uniformity space provides a linear algebra framework for many graph-theoretic and hypergraph-theoretic problems. In these problems we wish to assign weights, which are scalars of some field F, to the vertices or the edges of the graph (hypergraph), such that the sum of the weights on all subgraphs (subhypergraphs) of a specific type, is the same. We wish to determine the dimension and a basis of the linear space of these weight-assignment functions. Such problems can be converted to the problem of determining a basis for $U(H,F)$, where H is an appropriately defined hypergraph, which we call the *master* hypergraph. H is constructed in the following obvious manner. The vertices of H are the objects of the original graph (hypergraph) to which weight-assignment is applied (usually, these objects are either edges or vertices). Each edge of H corresponds to a subset of objects which comprise a subgraph (subhypergraph) having the required type. Consider, for example, the following uniformity-space problems:

P1. Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs where G_1 is a subgraph of G_2. A function $f: E_2 \rightarrow F$ is called G_1-*stable* if all the copies of G_1 in G_2 have the same weight (the weight of a copy is the sum of the values of f on the edges of the copy). Let $U(G_1,G_2,F)$ be the vector space of all the G_1-stable functions, and let $udim(G_1,G_2,F)$ be its dimension.

P2. Let $S=(V,E)$ be an r-uniform hypergraph on n vertices. For $1 \le k \le n$, a function $f: E \rightarrow F$ is called k-*stable* if all the induced subhypergraphs of S on k vertices have the same weight. Let $U(S,k,F)$ be the vector space of the k-stable functions, and let $udim(S,k,F)$ be its dimension.

The problems P1 and P2, and dual formulations of them, are closely related to central problems in combinatorics such as Null t-designs [16], Block designs [10] (pages 718-740), signed hypergraph designs [26], and G-decomposition of K_n [10]. In all of these problems, in which the rank of the adjacency matrix of the master hypergraph was computed, there always appears the condition $v \geq k+t$ where v is the number of vertices of the graph or hypergraph in question (e.g. in P1 v is the number of vertices of G_2, and in P2, v is the number of vertices of S), k is the number of vertices of the sub-(hyper)graphs from which the corresponding uniformity property is required (e.g. in P1 k is the number of vertices of G_1 while in P2 k is the same k mentioned there), and t is the number of vertices in each edge (e.g. in P1, $t = 2$ and in P2, $t = r$). This condition is *necessary* in all the algebraic methods mentioned in these references [26]. The main theorem of this paper, Theorem 1.1, avoids the restriction $v \geq k+2$ for graphs, and thus can be directly used to compute the rank of the corresponding adjacency matrix of the master hypergraph.

P3. Let $G = (V, E)$ be a graph. A function $f : V \to F$ is called *MIS-stable* (*DOM-stable*) if all maximal independent sets (minimal dominating sets) of G have the same weight (the maximality and minimality are w.r.t. containment). Let $U(MIS : G, F)$ ($U(DOM : G, F)$) be the vector space of all the MIS-stable (DOM-stable) functions, and let $udim(MIS : G, F)$ ($udim(DOM : G, F)$) be its dimension.

A graph is called *well-covered* if all maximal independent sets (w.r.t. containment) have equal size. Such graphs have been defined and extensively studied (see, e.g., [21, 22, 12]). Clearly, a graph G is well-covered iff the all-one function $f : V(G) \to F$ is MIS-stable, where F is any field of characteristic 0. Weighted well-covered graphs are graphs with real-valued weights on the vertices such that all maximal independent sets have the same weight; in other words, the weight function is an MIS-stable function. Such graphs have been studied in [6]. A similar concept is that of *well-dominated* graphs, in which all minimal dominating sets have the same size. These graphs have been studied in [13]. Clearly, a graph G is well-dominated iff the all-one function $f : V(G) \to F$ is DOM-stable, where F has characteristic 0. Other graph families which have natural correspondence with uniformity space are graphs having 2-packings [17] and equimatchable graphs [20].

P4. Let $G = (V, E)$ be a graph. A function $f : E(G) \to F$ is called *neighborhood-stable* if for every vertex $v \in G$, the sum of the values of f on the edges adjacent to v is the same. Thus, given a graph, one may wish to determine the linear space of neighborhood-stable functions, and its dimension.

Problem P4 includes a set of problems concerning *magic graphs*, in which there is an additional requirement that f is one-to-one. There are many papers on this subject [18, 19, 11, 24].

Note that in all of the uniformity space problems, once the appropriate master hypergraph H (in fact, its incidence matrix $B(H)$) is constructed, computing a basis for $U(H, F)$ is easy since it is merely a problem of solving a set of linear equations. However, the master hypergraph might be much larger than the size of the original problem. Consider, for example, computing $udim(G_1, G_2, F)$ defined in problem P1 above. The number of rows of the master hypergraph is equal to the number of copies of G_1 in G_2, which may be exponential in the size of G_2. Furthermore, one needs also to detect all copies of G_1 within G_2, which may also be difficult. Thus, computing $udim(G_1, G_2, F)$ through the master hypergraph is impractical. The same arguments hold for problems P2 and P3 described above. On the other hand, problem P4, which is to compute the dimension of the neighborhood-stable functions, can be solved in polynomial time since the master hypergraph can be constructed from the original graph in polynomial time.

The first goal of this paper is to determine, and to compute efficiently, the uniformity dimension, and a corresponding basis, of several graph-theoretic problems. We now describe our main results in this area. Recall that for every natural number p, a graph is called *regular* mod p if the degrees of all the vertices are the same, modulo p. In our applications, p denotes the characteristic of a field, and therefore we shall also allow $p = 0$, and a regular graph is considered regular mod 0. We also assume $r = s$ mod 0, iff $r = s$.

Theorem 1.1 *Let G be a connected graph with $n \geq 3$ vertices, and let F be a field of characteristic $\chi(F) = p$. Then:*

1. *If G is not regular* mod p *and $G \neq K_{1,n-1}$, then $udim(G, K_n, F) = 1$, unless $p = 2$ and G is complete bipartite.*
2. *If G is regular* mod p, *and $G \notin \{K_n, K_{1,n-1}\}$ then $udim(G, K_n, F) = n$, unless $p = 2$ and G is complete bipartite.*
3. *If $G = K_{1,n-1}$ then $udim(G, K_n, F) = \binom{n-1}{2}$, unless $p = 2$ and n is even, in which case $udim(G, K_n, F) = \binom{n-1}{2} + 1$.*
4. *If $G = K_n$ then $udim(G, K_n, F) = \binom{n}{2}$.*
5. *If $p = 2$ and G is complete bipartite $udim(G, K_n, F) = \binom{n-1}{2}$ if n is odd and $udim(G, K_n, F) = \binom{n-1}{2} + 1$ if n is even.*

In all cases, a basis of $U(G, K_n, F)$ can be computed in $O(n^4)$ time. Furthermore, given $f : E(K_n) \to F$, one can decide in $O(n^4)$ time if f is G-stable, and if it is not, two copies of G in K_n having different weights can be produced.

Theorem 1.1 enables us to determine $udim(G, K_n, F)$, and compute a basis of $U(G, K_n, F)$ for *all* connected n-vertex graphs (if $n = 1, 2$ the problem is trivial), and all fields. Now, it is easy to see that if $\overline{G}$ is the complement of G in K_n, then $U(G, K_n, F) = U(\overline{G}, K_n, F)$. This is because f is G-stable iff it is $\overline{G}$-stable. Furthermore, if two copies of G in K_n have different weights, then the complements of these copies are copies of $\overline{G}$, which also have different weights. Since the complement of a non-connected graph is always connected we have that Theorem 1.1 also enables us to determine $udim(G, K_n, F)$ and compute a basis for $U(G, K_n, F)$ in case G is non-connected. If G has m vertices and $m < n$ then, by adding $n - m$ isolated vertices to G, we obtain an n-vertex graph G' where, clearly, $U(G, K_n, F) = U(G', K_n, F)$. Consequently, Theorem 1.1 can be applied to all graphs with $m \leq n$ vertices. We emphasize here that Theorem 1.1 can be easily applied, via standard linear algebra, to compute the p-rank of the incidence matrix of any graph G on m vertices in K_n, $n \geq m$. This goes below the barrier $n \geq m + 2$ mentioned in [26]. In Section 2 we prove Theorem 1.1, and we also show that computing $udim(G_1, G_2, F)$ is, in general, NP-Hard (see [15] for the definition of NP-Hardness).

Our next result shows that in several cases, one can efficiently compute $udim(MIS : G, F)$, and a basis of $U(MIS : G, F)$.

Theorem 1.2 *Let F be a field, and let G be an n-vertex graph. Then, $udim(MIS : G, F)$, and a basis for $U(MIS : G, F)$ can be computed in polynomial time in the following cases:*

1. *G has girth at least 7, and $\chi(F) = 0$.*

2. *The maximum degree of G is $O((\log n)^{1/3})$.*

Note that, in particular, the first part of Theorem 1.2 shows that $udim(MIS : T, F)$ can be computed for any tree T. In fact, we show that $udim(MIS : G, F)$ is equal to the number of degree-one vertices of G, plus the number of C_7 components of G. As a corollary of this result, one can obtain the result of Ravindra [23], which determines the well-covered trees. The second part of Theorem 1.2 is a consequence of a result of Caro et. al. [6]. In Section 3 we prove Theorem 1.2, and its related corollaries.

The second goal of this paper is to exhibit applications of the uniformity space to other graph-theoretic problems. The first one we consider is the *zero-sum* mod 2 *Ramsey numbers*. Let $G = (V, E)$ be an n-vertex graph with $|E| = 0 \bmod k$. Denote by $R(G, Z_k)$ the smallest integer m such that for every $f : E(K_m) \to Z_k$, there exists a zero-sum copy of G in K_m (i.e. the sum of the values of f on the edges of the copy is $0 \bmod k$). The first author, in [4], determined the value of $R(G, Z_2)$ for all possible graphs G (i.e. all the graphs with an even number of edges). However, the proof is involved, and contains a detailed case analysis. In Section 4 we show how $R(G, Z_2)$ can be rather easily determined, as a consequence of Theorem 1.1. Furthermore, our proof also supplies an *algorithm* which, given $f : E(K_m) \to Z_2$, where $m = R(G, Z_2)$, produces a zero-sum copy of G in K_m. This algorithmic aspect is a new result, since the proof in [4] is non-algorithmic.

As another application, consider the following Theorem:

Theorem 1.3 *Let r, k and n be positive integers such that $r \le k \le n - r$. Let H be an r-uniform hypergraph on n vertices having the property that every induced k-vertex subhypergraph has the same number of edges. Then H is either the complete r-uniform hypergraph, or the empty hypergraph.*

We prove this theorem by showing that it is a consequence of a more general result which states, in the language of problem P2, that $udim(S_{r,n}, k, F) = 1$, and $U(S_{r,n}, k, F)$ is spanned by the all-one constant function, where $S_{r,n}$ is the complete r-uniform hypergraph on n vertices, and F is a field with characteristic 0. This proof also appears in Section 4. The final section contains some concluding remarks.

2 Determining $U(G, K_n, F)$ and $udim(G, K_n, F)$

The main goal of this section is to prove Theorem 1.1. Since the proof is rather detailed, we split it into several lemmas. In this section we shall always assume, unless otherwise stated, that $G = (V, E)$ is a connected graph with $n \ge 3$ vertices. The degree of a vertex $v \in G$ is denoted by $d(v)$. F denotes a field, and $p = \chi(F)$ is the characteristic of F. It will be convenient to denote the vertices of K_n by the numbers $1, \ldots, n$. Using this convention, we may identify a copy of G in K_n with a one-to-one mapping $g : V(G) \to \{1, \ldots, n\}$, which defines the obvious isomorphism between G and its copy in K_n. We denote by $g^{-1}(i)$ the vertex of G which maps by g to i. For a weight function $f : E(K_n) \to F$, and for a copy g of G in K_n, let $w(f, g)$ be the sum of the values of f on the edges of the copy g (the summation is performed in the field F). Thus, if f is G-stable, $w(f, g_1) = w(f, g_2)$ for any two copies g_1 and g_2.

Lemma 2.1 *If G is regular* mod p*, then $udim(G, K_n, F) \ge n$. Furthermore, a set Q of n linearly-independent G-stable functions can be constructed in $O(n^3)$ time.*

Proof: Let r be the degree of every vertex of G, modulo p (recall that if $p = 0$, then G is regular, and r denotes the degree of all vertices). We define a set $Q = \{f_1, \ldots, f_n\}$ of n distinct linearly independent G-stable functions, where $f_i : E(K_n) \to F$. For all $i = 1, \ldots, n - 1$, the value of f_i is 1 on every edge which is adjacent to the vertex i of K_n. The value of f_n is 1 on all the edges of K_n. Clearly, each f_i can be constructed in $O(|E(K_n)|) = O(n^2)$ time, and Q is therefore constructed in $O(n^3)$ time. Note that for all $i = 1, \ldots, n - 1$, $w(f_i, g) = d(g^{-1}(i)) \bmod p = r$. Also,

$w(f_n, g) = |E(G)| \bmod p$. Thus, in any case, f_i is G-stable for all $i = 1, \ldots, n$. It remains to show that the f_i's are linearly independent. Indeed, assume that $c_1 f_1 + \ldots + c_n f_n = 0$. Let $1 \le i \le n-1$, and let $j \notin \{i, n\}$ (such a j exists since $n \ge 3$). Consider the edge (j, n). $f_k((j, n)) = 1$ iff $k = j$ or $k = n$. Thus, $c_j + c_n = 0$. Now consider the edge (i, j). $f_k((i, j)) = 1$ iff $k \in \{i, j, n\}$. Thus, $c_i + c_j + c_n = 0$. These two equalities imply $c_i = 0$. Thus, for all $i = 1, \ldots, n-1$, $c_i = 0$. Hence, also, $c_n = 0$. $\Box$

Lemma 2.2 *If G is not the complete graph and not a star then G has four vertices x, y, z, w such that $(x, z) \in E$, $(y, z) \notin E$ and $(y, w) \in E$. Furthermore, if G is not complete bipartite then one may choose w such that $(x, w) \in E$. These vertices can be detected in $O(n^2)$ time.*

Proof: The assumptions in the lemma imply that G has at least $n \ge 4$ vertices, since otherwise $n = 3$ and G would have been a K_3 or a $K_{1,2}$. If G is complete bipartite the result is obvious. Assume, therefore, that G is not complete bipartite. The fact that G is a connected graph which is not a star and not the complete graph implies that G has a vertex y with $2 \le d(y) \le n-2$. Let $N(y)$ denote the neighbor-set of y, and let $N^2(y)$ denote the vertices at distance 2 from y. Since $d(y) \le n-2$, we have that $N^2(y)$ is non-empty. If there exists a vertex $x \in N^2(y)$ which is connected to some vertex $z \notin N(y)$ then let $w \in N(y)$ be any neighbor of x. The four vertices x, y, z, w satisfy the conditions of the lemma. If every vertex of $N^2(y)$ is only connected to vertices of $N(y)$ then the fact that G is connected implies that $V = \{y\} \cup N(y) \cup N^2(y)$, and now the fact that G is not complete-bipartite implies that there is a vertex $x \in N^2(y)$ and a vertex $z \in N(y)$ such that $(x, z) \notin E$. Let $w \in N(y)$ be a neighbor of x. By replacing the roles of x and y we have that the four vertices x, y, z, w satisfy the conditions of the lemma. Note that the operations we have performed only involve degree counting and Breadth-First Search, and these can be performed in $O(n^2)$ time using the adjacency matrix of G $\Box$.

Lemma 2.3 *Assume that $G \notin \{K_{1,n-1}, K_n\}$ and that if $p = 2$ G is also not complete bipartite. Let $f : E(K_n) \to F$. If f is G-stable then for every four vertices a, b, c, d of K_n, $f(a, b) + f(c, d) = f(b, c) + f(d, a)$ holds. If f is not G-stable, and there exist four vertices a, b, c, d for which $f(a, b) + f(c, d) \neq f(b, c) + f(d, a)$ then two copies of G in K_n with different weights can be produced in $O(n^2)$ time.*

Proof: If $n = 3$ there is nothing to prove, so assume $n \ge 4$. Fix four vertices x, y, z, w as in Lemma 2.2. If $p = 2$, G is not complete bipartite, and thus we may also assume by Lemma 2.2 that $(x, w) \in E$. Put $N(y) \setminus \{x\} = \{y_1, \ldots, y_r\}$ where $w = y_1$. Put $N(x) \setminus (N(y) \cup \{y\}) = \{x_1, \ldots, x_s\}$ where $x_1 = z$. We may assume that $\{y_1, \ldots, y_t\}$ are also neighbors of x for some $0 \le t \le r$, and if G is not complete bipartite we know that $t > 0$. Fix any four vertices a, b, c, d of K_n. Consider a copy g_1 of G in K_n for which $g_1(x) = a$, $g_1(y) = c$, $g_1(z) = b$, $g_1(w) = d$. g_1 maps the $n-4$ remaining vertices of G to the remaining $n-4$ vertices of K_n in some arbitrary way. Now consider a copy g_2 of G which coincides with g_1 on all vertices except x and y, which are permuted with respect to g_1. Thus, $g_2(x) = c$ and $g_2(y) = a$. If f is G-stable we must have

$$0 = w(f, g_1) - w(f, g_2) = (\sum_{i=1}^{s} f(a, g_1(x_i)) + \sum_{i=t+1}^{r} f(c, g_1(y_i))) - (\sum_{i=1}^{s} f(c, g_1(x_i)) + \sum_{i=t+1}^{r} f(a, g_1(y_i))). \tag{1}$$

We now define two additional copies, g_3 and g_4, of G in K_n. g_3 coincides with g_1 on all vertices except w and z, which are permuted. Thus, $g_3(z) = d$ and $g_3(w) = b$. g_4 coincides with g_3 on all vertices except x and y, which are permuted. Thus $g_4(x) = c$ and $g_4(y) = a$. Once again, if f is G-stable,

$$0 = w(f, g_3) - w(f, g_4) = (\sum_{i=1}^{s} f(a, g_3(x_i)) + \sum_{i=t+1}^{r} f(c, g_3(y_i))) - (\sum_{i=1}^{s} f(c, g_3(x_i)) + \sum_{i=t+1}^{r} f(a, g_3(y_i))). \tag{2}$$

We now subtract (2) from (1). However, we must distinguish between the case $t = 0$ and the case $t > 0$. If $t > 0$, we obtain

$$0 = (w(f, g_1) - w(f, g_2)) - (w(f, g_3) - w(f, g_4)) = f(a, b) - f(c, b) - f(a, d) + f(c, d)$$

which implies $f(a, b) + f(c, d) = f(b, c) + f(d, a)$, as required. If $t = 0$ (recall that this only happens if G is complete bipartite) we obtain

$$0 = (w(f, g_1) - w(f, g_2)) - (w(f, g_3) - w(f, g_4)) =$$

$$f(a, b) + f(c, d) - f(c, b) - f(a, d) - f(a, d) - f(c, b) + f(c, d) + f(a, b)$$

which implies $f(a, b) + f(c, d) = f(b, c) + f(d, a)$, in case $p \neq 2$. If f is not G-stable, and $f(a, b) + f(c, d) \neq f(b, c) + f(d, a)$ for some four vertices of K_n, then one can create the two copy pairs (g_1, g_2) and (g_3, g_4) as before, and compute their weights, in $O(n^2)$ time. By the above equalities, we must have that either $w(f, g_1) \neq w(f, g_2)$ or $w(f, g_3) \neq w(f, g_4)$. $\square$

Lemma 2.4 *Let $f : E(K_n) \to F$ be such that for any four vertices a, b, c, d of K_n, $f(a, b) + f(c, d) = f(b, c) + f(d, a)$ holds. Let*

$$S = \{(1, 2), \ldots, (1, n), (2, 3)\} \subset E(K_n).$$

Then for any two vertices a, b of K_n, $f(a, b)$ is a linear combination of the values of f on the members of S.

Proof: If $(a,b)\in S$ the claim is obvious. We may therefore assume that $2\le a<b\le n$. If $a=2$ we have $f(2,b)=f(2,3)+f(1,b)-f(1,3)$ (note that this equality trivially holds when $b=3$). If $a=3$ then $f(3,b)=f(2,3)+f(1,b)-f(1,2)$. If $a>3$ then $f(a,b)=f(1,a)+f(2,b)-f(1,2)=f(1,a)+f(2,3)+f(1,b)-f(1,3)-f(1,2)$. $\square$

Note that the set S in Lemma 2.4 has n members, and thus the following corollary is a consequence of Lemmas 2.3 and 2.4.

Corollary 2.5 *If $G\notin\{K_n,K_{1,n-1}\}$, then $udim(G,K_n,F)\le n$, unless $p=2$ and G is complete bipartite.*

Lemma 2.6 *Assume $G\ne K_{1,n-1}$, and G is not regular* mod p *and that if $p=2$ then G is not complete bipartite. Let $f:E(K_n)\to F$. Then, f is G-stable iff f is constant. If f is not constant, one can find two copies of G in K_n, with different weights, in $O(n^4)$ time.*

Proof: Clearly, a constant function is always G-stable. Assume, therefore, that f is G-stable. According to Lemmas 2.3 and 2.4 we know that f is determined by its values on the set S defined in Lemma 2.4. Furthermore, according to the proof of Lemma 2.4 it suffices to show that f is constant on S. Since G is not regular modulo p, there exist two vertices x and y such that $d(x)\ne d(y)$ mod p. Put $N(x)\backslash(N(y)\cup\{y\})=\{x_1,\dots,x_s\}$, and $N(y)\backslash(N(x)\cup\{x\})=\{y_1,\dots,y_r\}$. Hence, $r\ne s$ mod p. Consider two copies of G in K_n, that differ only in their values on x and y. One of the copies, say g_1, has $g_1(x)=1$ and $g_1(y)=2$ while the other copy, g_2, has $g_2(x)=2$ and $g_2(y)=1$. For any other vertex z, we have $g_1(z)=g_2(z)\ge 3$. Since f is stable it follows that

$$0=w(f,g_1)-w(f,g_2)=(\sum_{i=1}^{s}f(1,g_1(x_i))+\sum_{i=1}^{r}f(2,g_1(y_i)))-(\sum_{i=1}^{s}f(2,g_1(x_i))+\sum_{i=1}^{r}f(1,g_1(y_i))).\quad(3)$$

According to Lemma 2.4, and using the fact that $g_1(x_i)\ge 3$ we know that $f(2,g_1(x_i))=f(2,3)+f(1,g_1(x_i))-f(1,3)$. Similarly, $f(2,g_1(y_i))=f(2,3)+f(1,g_1(y_i))-f(1,3)$. Plugging these two equalities into (3) we get:

$$(s-r)(f(1,3)-f(2,3))=0.$$

This implies that $f(1,3)=f(2,3)$. By symmetric arguments we also have $f(1,2)=f(2,3)$. Using these equalities and the equalities in Lemma 2.4 we obtain that $f(1,b)=f(2,b)=f(3,b)$ for all $b\ge 4$, and $f(a,b)=f(1,a)+f(1,b)-f(1,2)$, for all $2\le a,b$.

It remains to show that $f(1,2)=f(1,b)$ for $b\ge 4$. For this purpose, we define two copies of G in K_n, namely g_3 and g_4. Like before, g_3 and g_4 coincide on all vertices except x and y. Thus, $g_3(x)=2$, $g_3(y)=b$. $g_4(x)=b$, $g_4(y)=2$. Using the stability of f we obtain:

$$0=w(f,g_3)-w(f,g_4)=(\sum_{i=1}^{s}f(2,g_3(x_i))+\sum_{i=1}^{r}f(b,g_3(y_i)))-(\sum_{i=1}^{s}f(b,g_3(x_i))+\sum_{i=1}^{r}f(2,g_3(y_i))).\quad(4)$$

We now show that $f(2, g_3(x_i)) - f(b, g_3(x_i)) = f(1,2) - f(1,b)$. This is clearly true if $g_3(x_i) = 1$. If $g_3(x_i) = 3$ we may use the fact that $f(2,3) = f(1,2)$ and the fact that $f(1,b) = f(3,b)$. If $g_3(x_i) \geq 4$ we may put $g_3(x_i) = a$ and use the fact that $f(a,b) = f(1,a) + f(1,b) - f(1,2) = f(2,a) + f(1,b) - f(1,2)$. Similar arguments show that $f(2, g_3(y_i)) - f(b, g_3(y_i)) = f(1,2) - f(1,b)$. Plugging these two equalities into (4) we get:

$$(s-r)(f(1,2) - f(1,b)) = 0$$

which implies $f(1,2) = f(1,b)$.

Now, if f is not constant, then f is not G-stable. If there are four vertices a, b, c, d in K_n with $f(a,b) + f(c,d) \neq f(b,c) + f(d,a)$ (this can be checked in $O(n^4)$ time by considering all subsets of four vertices), then one can generate two copies with different weights according to Lemma 2.3. Otherwise, we know by Lemma 2.4 that f cannot be constant on S. We may assume w.l.o.g. that $f(1,3) \neq f(2,3)$ (otherwise we may rename the vertices of K_n such that this holds). Hence, according to the first part of the proof of our lemma, we must have that the copies g_1 and g_2 have different weights. These copies are easily created in $O(n^2)$ time. □

The next lemma determines $udim(K_{1,n-1}, K_n, F)$.

Lemma 2.7 *$udim(K_{1,n-1}, K_n, F) = \binom{n-1}{2}$, unless $p = 2$ and n is even, where in this case we have $udim(K_{1,n-1}, K_n, F) = \binom{n-1}{2} + 1$. In both cases, a basis for $U(K_{1,n-1}, K_n, F)$ can be computed in $O(n^4)$ time. Furthermore, given $f : E(K_n) \to F$, one can decide in $O(n^2)$ time if f is $K_{1,n-1}$ stable, and if not, produce two copies with different weights.*

Proof: Let f be $K_{1,n-1}$-stable. The copies of $K_{1,n-1}$ in K_n determine that for all $i = 1, \ldots, n$, $\sum_{j=1, j\neq i}^{n} f(i,j) = c$ where $c \in F$. If $p \neq 2$, these requirements form n linearly independent equations with $\binom{n}{2}$ variables. Thus, for $c = 0$, there are $\binom{n}{2} - n$ linearly-independent solutions, $f_1, \ldots, f_{\binom{n}{2}-n}$. Let f^* be a solution for $c = 2 = 2*1_F$. f^* exists since one may take any Hamiltonian circuit in G and assign the value 1 on the edges of the circuit, and 0 on the non-edges. Note that f^* is not a linear combination of the f_i's, and if f' is any other solution for $c \neq 0$, then $2f' - cf^*$ is a solution for $c = 0$, and thus f' is linearly dependent on f^* and the f_i's. Thus, $udim(K_{1,n-1}, K_n, F) = \binom{n}{2} - n + 1 = \binom{n-1}{2}$. Now consider the case $p = 2$. In this case, the dimension of the linear equations is only $n-1$. Thus, for $c = 0$ there are $\binom{n}{2} - n + 1$ linearly-independent solutions. If n is even, the all-one function f^* is a solution for $c = 1$, and hence $udim(K_{1,n-1}, K_n, F) = \binom{n}{2} - n + 2 = \binom{n-1}{2} + 1$. If n is odd, there is no solution for $c = 1$, and thus $udim(K_{1,n-1}, K_n, F) = \binom{n}{2} - n + 1 = \binom{n-1}{2}$.

By the arguments above, we see that computing a basis for $U(K_{1,n-1}, K_n, F)$ reduces to the problem of solving a set of $O(n)$ linear equations in $\binom{n}{2}$ variables. This can be done in $O(n^4)$ time using

Gaussian elimination.

Given a function $f : E(K_n) \to F$ one can compute, for all $i \in K_n$, the sum of weights of the edges adjacent to i in $O(n^2)$ time, and thus decide whether f is G stable or not. If it is not stable, there are two vertices i and j which are the roots of two copies of $K_{1,n-1}$ with different weights. $\Box$

The final lemma of this section determines $U(G, K_n, F)$ and $udim(G, K_n, F)$ in case G is complete bipartite, and $p = 2$.

Lemma 2.8 *If G is a complete bipartite graph and $p = 2$, $udim(G, K_n, F) = \binom{n-1}{2}$ if n is odd, and $udim(G, K_n, F) = \binom{n-1}{2} + 1$ if n is even. In any case, a basis for $U(G, K_n, F)$ can be generated in $O(n^4)$ time. Furthermore, Given $f : E(K_n) \to F$ one can decide in $O(n^2)$ time if f is G-stable, and if not, produce two copies with different weights.*

Proof: Let x and y be two vertices of G which belong to different vertex classes. Consider two copies g_1 and g_2, where $g_1(x) = i$, $g_1(y) = j$, $g_2(x) = j$, $g_2(y) = i$. $g_1(z) = g_2(z)$ for all $z \notin \{x, y\}$. Let f be G-stable. Since $p = 2$ we have:

$$0 = w(f, g_1) - w(f, g_2) = w(f, g_1) + w(f, g_2) = \sum_{k=1,k\neq i}^{n} f(i,k) + \sum_{k=1,k\neq j}^{n} f(j,k) =$$

$$= \sum_{k=1,k\neq i}^{n} f(i,k) - \sum_{k=1,k\neq j}^{n} f(j,k).$$

It follows that f is also $K_{1,n-1}$-stable, since the sum of the weights of the edges adjacent to each vertex is the same. Thus we have, $U(G, K_n, F) \subset U(K_{1,n-1}, K_n, F)$. According to Lemma 2.7, it remains to show that $udim(G, K_n, F) \geq \binom{n-1}{2}$ when n is odd, and $udim(G, K_n, F) \geq \binom{n-1}{2} + 1$ when n is even. Let $f_{(i,j,k)}$ denote the function which assigns the value 1 to the edges of the triangle (i, j, k), and 0 to all the other edges of K_n. Note that f is G-stable since $w(f, g) = 0$ for every copy g. Now consider the set $T = \{f_{(i,i+1,j)} \mid 1 < i + 1 < j \leq n\}$. T has $\binom{n-1}{2}$ members. We now prove that T is a linearly independent set. Assume, to the contrary, that $f_{(i,i+1,j)}$ is a linear combination of $T' \subset T$. If $i = 1$ the contradiction follows from the fact that the edge $(1, j)$ is assigned 1 only in $f_{(1,2,j)}$. Now consider the case $i > 1$. The edge (i, j) is assigned 1 only in $f_{(i,i+1,j)}$ and in $f_{(i-1,i,j)}$, thus $f_{(i-1,i,j)} \in T'$. The edge $(i - 1, j)$ is assigned 1 in $f_{(i-1,i,j)}$ and 0 in $f_{(i,i+1,j)}$ and therefore $f_{(i-2,i-1,j)} \in T'$. Continuing in the same manner we obtain that $f_{(1,2,j)} \in T'$, which is a contradiction to the fact that the edge $(1, j)$ is assigned 0 in $f_{(i,i+1,j)}$. We have shown that $udim(G, K_n, F) \geq |T| = \binom{n-1}{2}$. If n is even we have that the all-one function f^* is a G-stable function which is linearly independent from T. To see this, note that if f^* were a linear combination of some $T' \subset T$, the fact that $(1, j)$ is assigned 1 only in f^* and $f_{(1,2,j)}$ means that $f_{(1,2,j)} \in T'$, for

all $j = 3, \dots, n$. But now the edge $(1,2)$ is assigned 1 in f^* and $n-2 = 0 \bmod 2$ in T', which is impossible. Thus, $udim(G, K_n, F) \geq |T| + 1 = \binom{n-1}{2} + 1$.
Note that we have shown that $U(G, K_n, F) = U(K_{1,n-1}, K_n, F)$. Hence, as in the previous lemma, a basis for $U(G, K_n, F)$ can be constructed in $O(n^4)$ time, but we can, alternatively, also take the set T in case n is odd, or $T \cup \{f^*\}$ in case n is even, as a basis for $U(G, K_n, F)$.
Given $f : E(K_n) \to F$ one can compute, for all $i \in K_n$, the sum of weights of the edges adjacent to i in $O(n^2)$ time, and thus decide whether f is $K_{1,n-1}$-stable or not, which happens iff f is G-stable. If it is not stable, there are two vertices i and j with different sums of weights on their adjacent edges. We use i and j to construct, in $O(n^2)$ time, the two copies g_1 and g_2 described in the beginning of the proof, which must have different weights. □
We are now ready to prove the main result of this section.
Proof of Theorem 1.1:

1. G is not regular mod p and $G \neq K_{1,n-1}$, and if $p = 2$ then G is not complete bipartite. It follows from Lemma 2.6 that $udim(G, K_n, F) = 1$, and the all-one constant function is a basis for $U(G, K_n, F)$. By the same lemma, if f is not constant, then one can find two copies with different weights in $O(n^4)$ time.

2. G is regular mod p, and $G \notin \{K_n, K_{1,n-1}\}$, and if $p = 2$ then G is not complete bipartite. It follows from Lemma 2.1 and Corollary 2.5 that $udim(G, K_n, F) = n$. Furthermore, by Lemma 2.1, the set of functions Q defined in Lemma 2.1 is a basis of $U(G, K_n, F)$, and Q can be constructed in $O(n^3)$ time. Given $f : E(K_n) \to F$, we can determine if f is a linear combination of Q in $O(n^4)$ by solving the corresponding set of $\binom{n}{2}$ linear equalities in $n + 1$ variables, in $O(n^4)$ time. If f is not G-stable, there is only the trivial solution. In this case we know by Lemmas 2.3 and 2.4 that there *must* be four vertices a, b, c, d with $f(a,b) + f(c,d) \neq f(b,c) + f(d,a)$. We can locate such a foursome in $O(n^4)$ time, and then by Lemma 2.3 we can produce two copies of G with different weights in $O(n^2)$ time.

3. $G = K_{1,n-1}$. This case is completely determined in Lemma 2.7.

4. $G = K_n$. This is a trivial case, since every function is G-stable. Thus, $udim(G, K_n, F) = \binom{n}{2}$, and the standard basis is a basis for $U(G, K_n, F)$.

5. G is complete bipartite and $p = 2$. This case is completely determined in Lemma 2.8.

□

We conclude this section by showing that, in general, computing $udim(G_1, G_2, F)$ is NP-Hard. To prove this, we present a special case of it, in the form of a decision problem:

INSTANCE: two graphs G_1 and G_2 and a homomorphism $h : V(G_1) \to V(G_2)$ which shows that G_1 is, indeed, a subgraph of G_2.
QUESTION: is $udim(G_1, G_2, F) \neq |e(G_2)|$.
We show that this problem, denoted by $\mathcal{P}$, is NP-Complete. $\mathcal{P}$ belongs to NP due to the fact that $udim(G_1, G_2, F) = |e(G_2)|$ iff there is *exactly* one copy of G_1 in G_2. Thus, one proves that the answer to an input is "yes" by supplying another homomorphism $h' : V(G_1) \to V(G_2)$. We perform a polynomial transformation from the CONNECTED SUBGRAPH ISOMORPHISM problem [15] to $\mathcal{P}$. Let H and G be two connected graphs which are input to SUBGRAPH ISOMORPHISM. Construct an input to $\mathcal{P}$ by putting $G_1 = H$ and $G_2 = G \cup H$ (i.e. G_2 is the vertex-disjoint union of G and H). Clearly, H is not a subgraph of G iff there is exactly one copy of G_1 in G_2.

3 The algorithmic aspect of $U(MIS : G, F)$

In this section we consider $U(MIS : G, F)$, the space of all MIS-stable functions of G. It is not difficult to show that computing a basis for $U(MIS : G, F)$ is NP-hard, in general. This follows from the fact that the constant function belongs to $U(MIS : G, F)$ (when $\chi(F) = 0$) iff the graph G is well-covered. However, it is shown in [8] that deciding whether a graph is not well-covered is NP-Complete, even when G is $K_{1,4}$-free. Theorem 1.2 states, however, that there are some large families of graphs for which a basis for $U(MIS : G, F)$ can be computed in polynomial time. In order to prove the first part of Theorem 1.2 we first need several lemmas and definitions.

Lemma 3.1 *If the connected components of $G = (V, E)$ are $G_1, \ldots, G_k$ then $udim(MIS : G, F) = \sum_{i=1}^{k} udim(MIS : G_i, F)$. Furthermore, a basis for $U(MIS : G, F)$ can be constructed in $O(|V| \cdot udim(MIS : G, F))$ time from bases of the spaces $U(MIS : G_i, F)$.*

Proof: Let f_i be an MIS-stable function for G_i. The extension of f_i to G which is defined by $f_i(v) = 0$ for $v \notin G_i$, is MIS-stable for G. Note that we have that f_i is linearly independent from any linear combination of f_j's where $j \neq i$. Clearly, the extended f_i is constructed from the original f_i in $O(|V|)$ time. The Lemma now follows by taking the extensions in the union of bases of all the $U(MIS : G_i, F)$ for $i = 1, \ldots, k$. $\Box$

By lemma 3.1 we only need to prove the first part of Theorem 1.1 for connected graphs. Fix a connected graph $G = (V, E)$, and let $g(G)$ denote the girth of G. For $v \in V$, let $N^i(v)$ be the set of vertices at distance i from v. We partition V into four classes. V_1 contains all the degree-one vertices of G. $V_2 = N^1(V_1) \setminus V_1$. $V_3 = N^1(V_2) \setminus (V_1 \cup V_2)$. Finally, $V_4 = V \setminus (V_1 \cup V_2 \cup V_3)$. If f is MIS-stable, and $I \subset V$ is a maximal independent set of G, we put $f(I) = \sum_{v \in I} f(v)$.

Lemma 3.2 *Let $f : V(G) \to F$ be MIS-stable. If $g(G) \geq 6$ then for all $v \in V_2$ we must have $f(v) = \sum_{u \in V_1 \cap N^1(v)} f(u)$.*

Proof: Since $g(G) \geq 6$, we have that $N^2(v)$ is an independent set. Consider a maximal independent set I_1 which contains $N^2(v) \cup \{v\}$ and possibly some other vertices. Now consider $I_2 = I_1 \cup (V_1 \cap N^1(v)) \setminus \{v\}$. Clearly, I_2 is also a maximal independent set. Since f is MIS-stable, we have $f(I_1) = f(I_2)$, and the lemma follows. $\square$

Lemma 3.3 *Let $f : V(G) \to F$ be MIS-stable. If $g(G) \geq 7$ then for all $z \in V_3$ we must have $f(z) = 0$.*

Proof: Let $v \in V_2$ be a neighbor of z. Let $S = (N^2(z) \cap N^3(v)) \cup (N^2(v) \cap N^3(z))$. Since $g(G) \geq 7$, we have that S is an independent set. Let I_1 be a maximal independent set containing S and v. Let $I_2 = I_1 \cup \{z\} \cup (V_1 \cap N^1(v)) \setminus \{v\}$. I_2 is also a maximal independent set. Since $f(I_1) = f(I_2)$ we have $f(z) + \sum_{u \in V_1 \cap N^1(v)} f(u) = f(v)$. By Lemma 3.2 we know that $\sum_{u \in V_1 \cap N^1(v)} f(u) = f(v)$. Thus, $f(z) = 0$. $\square$

Lemma 3.4 *Let $f : V(G) \to F$ be MIS-stable. If $g(G) \geq 7$ then for all $x, y \in V_3 \cup V_4$ we must have $f(x) = f(y)$. In particular, if $V_1 \neq \emptyset$ then $f(x) = 0$ for all $x \in V_3 \cup V_4$.*

Proof: Assume, to the contrary, that the claim is false. Let $x, y \in V_3 \cup V_4$ be two vertices with $f(x) \neq f(y)$, and which are closest. We may assume that $f(y) \neq 0$ and hence by Lemma 3.3 we must have $y \in V_4$. Let z be any neighbor of y on the shortest path connecting y and x. Let $A_1 = N^2(z) \cap N^3(y)$ and $A_2 = N^2(y) \cap N^3(z)$. Since $y \in V_4$ we have that $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. Furthermore, since $g(G) \geq 7$ we have that $S = A_1 \cup A_2$ is an independent set. Let I_1 be a maximal independent set containing S and y. Let $I_2 = I_1 \cup \{z\} \setminus \{y\}$. Clearly, I_2 is also a maximal independent set. Thus, $f(z) = f(y) \neq 0$. From Lemma 3.3 we have that $z \notin V_3$, and hence $z \in V_4$. But the distance from z to x is shorter than the distance from y to x, a contradiction. Now, if $V_1 \neq \emptyset$ then if $V_4 \neq \emptyset$ then, necessarily, $V_3 \neq \emptyset$, and thus $f(x) = 0$ for all $x \in V_3 \cup V_4$. $\square$

Theorem 3.5 *If $g(G) \geq 7$ and $\chi(F) = 0$, then $udim(MIS : G, F) = |V_1|$, unless $G = C_7$, in which case $udim(MIS : C_7, F) = 1$. Furthermore, a basis of $udim(MIS : G, F)$ can be constructed in $O(|V||V_1|)$ time.*

Proof: Let $f : V \to F$. Assume first that $V_1 \neq \emptyset$. According to Lemma 3.4, the value of f on V_3 and V_4 is 0. According to Lemma 3.2, the value of f on $v \in V_2$ is a linear combination of the value of f on V_1. Thus, f is determined by its values on V_1. For $v \in V_1$ let $f_v(u) = 1$ if $u = v$ or if

$u \in V_2$ is a neighbor of v. Otherwise, $f_v(u) = 0$. Hence, the set $\{f_v \mid v \in V_1\}$ spans $U(MIS : G, F)$ and, trivially, it is also linearly independent. Clearly, one may construct f_v in $O(|V|)$ time. Now consider the case where $V_1 = \emptyset$. In this case $V_4 = V$. By Lemma 3.4, f must be constant. Since $\chi(F) = 0$, this means that G is well-covered. It is known by [22] Corollary 4.3 that the only (connected) well-covered graph with no vertex of degree 1 and with girth at least 7, is C_7. $\Box$

In Lemmas 3.2, 3.3 and 3.4 we did not require that $\chi(F) = 0$. Thus, if $|V_1| > 0$, we can extend Theorem 3.5 to all fields. Note that the first part of Theorem 1.2 follows from Theorem 3.5 and Lemma 3.1. It is also interesting to note the following alternative of corollary 4.3 of [22].

Corollary 3.6 *Let $G = (V, E)$ be a connected graph with $g(G) \geq 7$, and with $|V_1| > 0$. Then G is well-covered iff $|V_2| = |V_1|$ and $|V_3| = |V_4| = 0$.*

Proof: Let $f : V \to F$ be the all-one function. Recall that G is well-covered iff f is MIS-stable. By Lemma 3.4, if f is MIS-stable, $|V_3| = |V_4| = 0$. By Lemma 3.2, if f is MIS-stable, a vertex of V_2 must have exactly one neighbor in V_1, which implies $|V_2| = |V_1|$. The other direction is trivial. $\Box$

If G is a tree, Corollary 3.6 applies, and we obtain the result of Ravindra [23], which states that a tree is well-covered iff there is a perfect matching between the leaves and the non-leaves of the tree.

The second part of Theorem 1.2 follows from the result in [6]. One of the consequences of their paper is that given a graph $G = (V, E)$ with $|V| = n$, and $\Delta(G) = O((\log n)^{1/3})$, one can determine a basis of $U(MIS : G, F)$ is polynomial time. There is no restriction on the characteristic of F in this case.

4 Applications of the uniformity space

In the first part of this section we show how to use Theorem 1.1 in order to compute the zero-sum mod 2 Ramsey numbers $R(G, Z_k)$. These numbers are computed in [4], where the following theorem in proved:

Theorem 4.1 (The zero-sum characterization theorem [4]) *Let G be a graph on n vertices, with no isolated vertices and an even number of edges. Then:*

1. $R(G, Z_2) = n + 2$ *if* $G = K_n$ (*i.e.* $n = 0, 1 \bmod 4$).
2. $R(G, Z_2) = n + 1$ *if* $G = K_p \cup K_q$ (*i.e.* $\binom{p}{2} + \binom{q}{2} = 0 \bmod 2$).
3. $R(G, Z_2) = n + 1$ *if all the degrees in* G *are odd and* $G \neq K_n$.

4. $R(G, Z_2) = n$ *otherwise.*

Furthermore, given $f : E(K_m) \to Z_2$ *where* $m = R(G, Z_2)$*, one can find a zero-sum copy of* G *in* K_m *in* $O(m^4) = O(n^4)$ *time.*

As mentioned in the introduction, the non-algorithmic part of Theorem 4.1 is proved in [4] directly, and the proof is rather detailed. The algorithmic part of Theorem 4.1 is new, and does not appear in [4]. We now present a rather short proof of Theorem 4.1 which uses the uniformity space results of Theorem 1.1.

Proof of Theorem 4.1: We use the same notation used in Section 2. Note that if $n \le m < R(G, Z_2)$, then there exists $f : E(K_m) \to Z_2$ such that for every copy g of G in K_m, $w(f, g) \neq 0$. But in Z_2 this implies that $w(f, g) = 1$, and thus f is G-stable. Since f is not identically zero, and not identically one, this implies the following two observations:

OB1. $udim(G, K_m, Z_2) \ge 2$ (since the all-one function is also G-stable, and is linearly-independent with f).

OB2. If S is a basis for $udim(G, K_m, Z_2)$ there exists $f' \in S$ and a copy g of G in K_m such that $w(f', g) = 1$.

We now analyze the different cases in Theorem 4.1. We demonstrate the algorithmic part only in the first case. The reader may verify the algorithmic part in the other cases in an analogous way.

1. $G = K_n$. If $R(K_n, Z_2) > n + 2$ then, by Observation 1, we get $udim(K_n, K_{n+2}, Z_2) \ge 2$. Now let G^* be the connected graph on $n + 2$ vertices obtained by adding two isolated vertices to K_n, and taking the complement. As noted in the introduction, $U(K_n, K_{n+2}, Z_2) = U(G^*, K_{n+2}, Z_2)$. However, according to case 1 in Theorem 1.1, $udim(G^*, K_{n+2}, Z_2) = 1$ if n is even, a contradiction. If n is odd we know, by case 2 in Theorem 1.1, that $udim(G^*, K_{n+2}, Z_2) = n + 2$ where a basis to the linear space are the functions $f_1, \ldots, f_{n+2}$ defined in Lemma 2.1. (Recall that f_i assigns 1 to the edges adjacent to vertex i of K_{n+2}, for $i = 1, \ldots, n + 1$, whereas f_{n+2} is the all-one function). In any case, $w(f_i, g) = 0$ for every copy g of K_n in K_{n+2}, which contradicts Observation 2. Thus, $R(K_n, Z_2) \le n + 2$.

   We now prove the algorithmic part. According to Theorem 1.1, given an assignment $f : E(K_{n+2}) \to Z_2$ we can find in $O((n + 2)^4) = O(n^4)$ time whether f is G-stable or not, and if it is not, we can produce two copies with different weights in $O(n^4)$ time. One of these copies has weight 0. If f is stable, then in case n is even, f must be constant, and hence *every* copy of K_n has weight 0. If n is odd, f is a linear combination of $f_1, \ldots, f_{n+2}$, and thus, once again, every copy of K_n has weight 0.

Finally, to see that $R(K_n, Z_2) > n+1$, consider $f : E(K_{n+1}) \to Z_2$ which assigns 1 to the edges of the triangle $(1,2,3)$, and 0 to all other edges. Clearly, every copy g of K_n in K_{n+1} contains one or three edges of the triangle, and hence $w(f,g) = 1$.

2. $G = K_p \cup K_q$. If G in not regular mod 2 or if all the vertices of G have odd degree then according to Theorem 1.1, $udim(G, K_{n+1}, Z_2) = 1$. If all the vertices of G have even degree then $udim(G, K_{n+1}, Z_2) = n+1$, and the functions $f_1, \ldots, f_{n+1}$ in Lemma 2.1 are a basis to the linear space. Note, however, that $w(f_i, g) = 0$ for every copy g of G in K_{n+1}. In any case, we see by Observations 1 and 2 that $R(G, Z_2) \le n+1$.

   To see that $R(G, Z_2) > n$, consider $f : E(K_n) \to Z_2$ which assigns 1 to the edges of some triangle of K_n, and 0 to the other edges. A copy of G must include one or three edges of this triangle.

3. All the vertices of G have odd degree and $G \ne K_n$. Note that $G \ne K_{1,n-1}$ since the number of edges of G is required to be even, and thus n is odd, but the root has even degree. Hence, according to Theorem 1.1 $udim(G, K_{n+1}, Z_2) = 1$ and thus $R(G, Z_2) \le n+1$.

   To see that $R(G, Z_2) > n$ consider $f : E(K_n) \to Z_2$ which assigns 1 to the edges adjacent to vertex 1 of K_n, and 0 to all other edges. Every copy of G in K_n includes an odd number of edges assigned 1.

4. G is not one of the graphs mentioned above. Trivially, $R(G, Z_2) \ge n$ for any graph G on n vertices. It thus suffices to show that $R(G, Z_2) \le n$. Consider first the case $G = K_{1,n-1}$. In this case n must be odd. Let $f : E(K_n) \to Z_2$. If every copy g of G in K_n had $w(f,g) = 1$, this means that the subgraph of G on the edges assigned 1 by f has all its degrees odd. This is impossible, since n is odd. Thus, $R(G, Z_2) \le n$. We may now assume $G \notin \{K_n, K_{1,n-1}\}$. Now consider the case where G is not complete bipartite. If G is not regular mod 2, $udim(G, K_n, Z_2) = 1$. If G is regular mod 2 (and thus all the degrees are even), then $udim(G, K_n, Z_2) = n$. A basis for the linear space are the functions $f_1, \ldots, f_n$ of Lemma 2.1. But, $w(f_i, g) = 0$ for every copy g of G in K_n. In any case, we have shown $R(G, Z_2) \le n$. The only remaining case is when G is complete bipartite. In this case $udim(G, K_n, Z_2) = \binom{n-1}{2}$ if n is odd and $udim(G, K_n, Z_2) = \binom{n-1}{2} + 1$ if n is even. The set $T = \{f_{(i,i+1,j)} \mid 1 < i+1 < j \le n\}$ defined in Lemma 2.8 forms a basis of the linear space when n is odd. If n is even we show in Lemma 2.8 that one may add to T the all-one function f^*, in order to form a basis. In any case, $w(f_{(i,i+1,j)}, g) = 0$ for every copy g of G in K_n, since a complete bipartite graph must capture two or zero edges of a triangle. Trivially, we also have $w(f^*, g) = |e(G)| \bmod 2 = 0$. Hence, according to Observation 2, $R(G, Z_2) \le n$. $\Box$

In the second part of this section we wish to see how a pure hypergraph-theoretic theorem which involves no weight functions, namely Theorem 1.3 mentioned in the introduction, can be deduced as a special case of a theorem involving uniformity space. Recall that $S_{r,n}$ denotes the complete r-uniform hypergraph with n vertices.

Theorem 4.2 *Let r, n, k be positive integers where $r \le k \le n - r$. Let F be a field with characteristic 0. Then $udim(S_{r,n}, k, F) = 1$, and any non-zero constant function is a basis for $U(S_{r,n}, k, F)$.*

Proof: We begin the proof with the case $k = n - r$. Let $V = \{1, \ldots, n\}$ denote the vertex-set of $S_{r,n}$. Let E denote the edge-set of $S_{r,n}$. Each $e \in E$ is an r-subset of V. Let $f : E \to F$ be k-stable. This means that every induced subhypergraph of $S_{r,n}$ on $n - r$ vertices has the same weight. Denote this common weight by w^*. For $R \subset V$, $1 \le |R| \le r$, put $w(f, R) = \sum_{R \subset e, e \in E} f(e)$, and put $x(f, R) = \sum_{R \cap e = \emptyset} f(e)$ (the sum of weights on edges which do not contain any vertex of R). We first show that $x(f, R)$ depends only on $|R| = i$. There are $\binom{n-i}{n-r}$ induced subhypergraphs of $S_{r,n}$ on $n-r$ vertices which do not contain any vertex of R. All these subhypergraphs have the same weight, w^*. On the other hand, every edge e which is disjoint from R appears in exactly $\binom{n-i-r}{n-2r}$ of these subhypergraphs. Thus, by counting the sum of the weights of all these subhypergraphs in two ways we have:

$$\binom{n-i}{n-r} w^* = x(f, R) \binom{n-i-r}{n-2r}.$$

Thus, $x(f, R)$ depends only on $|R| = i$. We now show that $w(f, R)$ depends only on $|R| = i$. This is done by induction on i. For $i = 1$ this is true since $w(f, \{v\}) = w(f) - x(f, \{v\})$ where $w(f) = \sum_{e \in E} w(e)$. Assume this is true for all $j < i$. By the inclusion-exclusion principle:

$$w(f) - x(f, R) = \sum_{R' \subset R} (-1)^{|R'|-1} w(f, R').$$

Thus, $w(f, R)$ depends only on $w(f) - x(f, R)$ (which depends only on $|R|$) and on weights of proper subsets R' of R, where these weights, by the induction hypothesis, only depend on $|R'|$. Note that when $|R| = r$, $w(f, R)$ is simply the weight of the edge R, and therefore we have proved that all weights are the same and this means that f is constant and $udim(S_{r,n}, k, F) = 1$.

When $r \le k < n - r$ we can use the fact that every subhypergraph S' on $n - r$ vertices contains exactly $\binom{n-r}{k}$ induced subhypergraphs on k vertices. By the assumption, the sum of weights of each of these k-vertex subhypergraphs is the same, say w^*. Every edge of S' appears in exactly $\binom{n-2r}{k-r}$ of these subhypergraphs. Thus,

$$w^* \binom{n-r}{k} = w(S') \binom{n-2r}{k-r}.$$

This means that $w(S')$ is the same for all subhypergraphs S' on $n-r$ vertices. Thus, by the previous case, f is constant and $udim(S_{r,n}, k, F) = 1$. $\Box$

Theorem 1.3 is a consequence of Theorem 4.2:

Proof of Theorem 1.3 Let S be an n-vertex r-uniform hypergraph, and let $r \le k \le n-r$. Assume that every induced subhypergraph of S on k vertices has the same number of edges. Define the following function $f : E(S_{r,n}) \to F$ (F is any field of characteristic 0). $f(e) = 1$ iff $e \in S$, otherwise $f(e) = 0$. By our assumption, f is k-stable for $S_{r,n}$. By Theorem 4.2, f must be constant. Thus, either f is identically 1 and $S = S_{r,n}$, or f is identically 0, and S has no edges. $\Box$

5 Concluding remarks

1. Theorem 1.1 shows how to compute a basis for $U(G, K_n, F)$ for all possible graphs G with n or less vertices. On the other hand, computing $udim(G_1, G_2, F)$ in NP-Complete in general. It is plausible that there are other non-trivial infinite families of graphs $\mathcal{F}$ for which one can compute $udim(G_1, G_2, F)$, in polynomial time, for all $G_1 \subset G_2 \in \mathcal{F}$.

2. We have shown that the linear algebraic concept of uniformity space can sometimes enable us to solve (hyper)graph-theoretic problems which do not involve linear algebra. Recently [9], using the uniformity space method, we were able to solve completely the determination of the zero-sum bipartite Ramsey numbers, raised by Bialostocki and Dierker [1], and partially solved in [5]. Another recent application of this method is the characterization of the Z_m-well-covered graphs of girth at least 6 [7].

3. Lemma 3.3 cannot be strengthened to graphs with girth 6. To see this, consider a $C_6 = (a, b, c, d, e, f)$ to which a new vertex g of degree one has been added, and connected to a. Assign $+1$ to b, c assign -1 to e, f and assign 0 to g, a, d. This assignment is MIS-stable (for characteristic 0), but $b \in V_3$.

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