Unavoidable Tournaments

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Abstract

A basic result in Ramsey theory states that any tournament contains a "large" transitive subgraph. Since transitive tournaments contain only transitive subgraphs, it is natural to ask which subgraphs must appear in any large tournament that is "far" from being transitive. One result of this type was obtained by Fox and Sudakov who characterized the tournaments that appear in any tournament that is ϵ-far from being transitive. Another result of this type was obtained by Berger et al. who characterized the tournaments that appear in any tournament that cannot be partitioned into a bounded number of transitive sets.

In this paper we consider the common generalization of the above two results, namely the tournaments that must appear in any tournament that is ϵ-far from being the union of a bounded number of transitive sets. Our main result is a precise characterization of these tournaments.

1 Introduction

A *tournament* $T = (V, E)$ is a digraph such that for every two distinct vertices u, v exactly one of the ordered pairs (u, v) or (v, u) is an edge. A tournament is transitive if it contains no directed cycle, or equivalently, if it is possible to order its vertices so that all edges "point" from left to right. We use T_n to denote the (unique) n-vertex transitive tournament. If T is a tournament, we say that a subset of vertices $X \subseteq V(T)$ is *transitive* if the sub-tournament induced by X is transitive. One of the most basic results in graph theory (sometimes attributed to [19] and [13]) states that any tournament on 2^{k-1} vertices contains a transitive subset of size k (i.e., a copy of T_k). Since T_n contains only transitive subsets, it is clear that transitive tournaments are the only subgraphs that are guaranteed to appear in any tournament. It is thus natural to ask if there are any tournaments that are guaranteed to appear in any tournament that is "far" from being transitive?

Before describing the first result of this type let us introduce some definitions. We say that an n-vertex tournament T is ϵ-far from being transitive if one should change the direction of at least $\epsilon \binom{n}{2}$ of T's edges in order to turn[1] it into a transitive tournament. For a tournament H with

[1]Note that the number of edges whose direction needs to be changed in order to turn T into a transitive tournament is precisely the number of edges that need to be removed from T in order to make it cycle-free.

$V(H) = \{v_1, \ldots, v_h\}$ and for a vector $(a_1, \ldots, a_h)$ of positive integers, the *transitive* $(a_1, \ldots, a_h)$-*blowup* of H is the tournament obtained by replacing each vertex v_i with the transitive tournament T_{a_i}, and connecting all edges between T_{a_i} and T_{a_j} in the same direction as the edge connecting v_i and v_j. We say that H' is a transitive blowup of H if there exists $(a_1, \ldots, a_h)$ such that H' is the transitive $(a_1, \ldots, a_h)$-blowup of H. In the case that $c = a_1 = \cdots = a_h$ we say that H' is a c-blowup. Notice that, trivially, every tournament is a transitive blowup of itself. The directed cycle on three vertices is denoted by C_3.

The first result addressing the above mentioned meta-problem was obtained by Fox and Sudakov [15] who characterized the tournaments that appear in every large enough tournament that is ϵ-far from being transitive. More precisely, let us say that a tournament H is 1-unavoidable[2] if for any $\epsilon > 0$ and $n \geq n_0(\epsilon)$, every n-vertex tournament that is ϵ-far from being transitive contains a copy of H. The result of Fox and Sudakov [15] states that a tournament H is 1-unavoidable if and only if H is either a transitive tournament or a transitive blowup of C_3.

To describe the second result we need some more definitions. For an integer $k \geq 1$, a k-*coloring* of a tournament is a partition of its vertices into k parts, where each part induces a transitive subset. The chromatic number $\chi(T)$ of a tournament T is the minimum k such that T admits a k-coloring. Berger et al. [5] call a graph H a *hero* if there is a constant c_H so that any tournament T satisfying $\chi(T) > c_H$ contains a copy of H. As noted in [8], the heroes are the tournaments that satisfy the extreme case of the well-known Erdős-Hajnal conjecture. The main result of [5] is a precise characterization of heroes (see Theorem 3 for the precise characterization).

Note that a transitive tournament T satisfies $\chi(T) = 1$ thus the 1-unavoidable tournaments studied by Fox and Sudakov [15] are those that must appear in any large enough tournament that is ϵ-far from being 1-colorable. Given the result of Berger et al. [5] it is thus natural to combine to two notions studied in [5] and [15] and introduce the following one:

Definition 1.1 (Unavoidable) *A tournament W is c-unavoidable if for every $\epsilon > 0$ and $n \geq n_0(\epsilon, W)$, every n-vertex tournament T that is ϵ-far[3] from satisfying $\chi(T) \leq c$ contains a copy of W. A tournament W is* unavoidable *if it is c_W-unavoidable for some constant c_W.*

Our first result gives a precise characterization of the unavoidable tournaments.

Theorem 1 *A tournament is unavoidable if and only if it is a transitive blowup of a hero.*

It is of course natural to ask for any given $c \geq 1$ which tournaments are c-unavoidable. We discuss this in Section 4. Also, as we note later in Section 4, the proof actually shows that for any unavoidable W there is some $\delta > 0$, so that if one should change the direction of at least $n^{2-\delta}$ edges in order to make an n-vertex T satisfy $\chi(T) \leq c_W$, then T contains a copy of W.

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[2]The reason for using the 1 will become clear soon.

[3]A tournament is ϵ-far from satisfying $\chi(T) \leq c$ if one should change the direction of at least $\epsilon\binom{n}{2}$ edges in order to obtain a tournament T' that satisfies $\chi(T') \leq c$. Also, if T is not ϵ-far from satisfying $\chi(T) \leq c$ then we say that it is ϵ-close to satisfying $\chi(T) \leq c$.

Our second result concerns a weaker notion of being unavoidable. To this end it might be better to consider the contra-positive versions of the notions introduced above. Then H is a hero if every H-free tournament T satisfies $\chi(T) \le c_H$, and W is unavoidable if for every $\epsilon > 0$ and large enough n, if T is an n-vertex W-free tournament, then T is ϵ-close to satisfying $\chi(T) \le c_W$. Note that c_H and c_W are constants that depend only on H and W respectively. It is thus natural to ask what happens if we relax the condition on the chromatic number and allow it to depend on ϵ.

Definition 1.2 (Weakly-Unavoidable) *A tournament W is* weakly-unavoidable *if for every $\epsilon > 0$ there is $c_W = c_W(\epsilon)$ so that for every $n \ge n_0(\epsilon, W)$, every n-vertex tournament T that is ϵ-far from satisfying $\chi(T) \le c_W$ contains a copy of W.*

Our second result gives a precise characterization of the weakly-unavoidable tournaments.

Theorem 2 *Every tournament is weakly-unavoidable.*

As it turns out, the proofs of Theorems 1 and 2 are quite different. While the first one turns out to be a Turán-type problem, the second turns out to be a Ramsey-type problem. In Section 2 we prove Theorem 1, which naturally has two main steps. In the first step we show that a transitive blowup of a hero must be unavoidable. To this end we combine several combinatorial tools that can all be classified as being related to the area of graph/hypergraph property testing (see e.g. [17]). To show that any unavoidable tournament must be a transitive blowup of a hero we need to rely on the characterization of heroes obtained in [5].

Let us mention an interesting aspect of the proof of the second part of Theorem 1. Recall that if H is a hero, then any H-free tournament T can be partitioned into c_H transitive sets. It is then natural to ask what if we only asked T to contain a transitive subset of linear size. It is shown in [5], that this (apparently) weaker condition is equivalent to the notion of a hero. One can thus formulate a similar weaker notion in our setting as well, and only ask that if a tournament T is W-free then T must contain a set of vertices of linear size that is ϵ-close to transitive. As our proof shows, this (seemingly) weaker notion is equivalent to the one in Definition 1.1. In other words, what we show is that if W is not a transitive blowup of a hero, then it does not satisfy even this weaker condition.

As we mentioned above, Theorem 2 turns out to be a Ramsey-type problem. The proof of this theorem, which appears in Section 3, uses the approach of Graham, Rödl and Ruciński [16] in their study of the Ramsey numbers of bounded degree graphs. As pointed to us by one of the referees, the main technical part of the proof of Theorem 2 can also be deduced from Lemma 2.8 in [4]. However, this gives a tower-type dependence for the function $c_W(\epsilon)$, while our proof gives a single exponential dependence. Section 4 contains some concluding remarks and open problems.

2 Unavoidable Tournaments

We start with the positive side of Theorem 1 showing that a transitive blowup of a hero is unavoidable. We start with the following lemma which proves a "removal lemma" (in the sense of [18]) for

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triangles[4] in tournaments with a polynomial bound. Such a lemma appears in [15] with a better bound, but for completeness we give a shorter and simpler proof.

Lemma 2.1 *If an n-vertex tournament T is ϵ-far from transitive, then it contains at least $\frac{\epsilon^6}{128}n^3$ copies of C_3.*

Proof. First observe that T has a subtournament with minimum out-degree at least $\epsilon n/2$. One can construct such a subtournament as follows. As long as there is a vertex with out-degree less than $\epsilon n/2$, we remove it from T, and repeat. If this process exhausts all vertices, then T has an ordering of vertices $v_1, \ldots, v_n$ such that each v_i has fewer than $\epsilon n/2$ edges pointing from it to vertices with higher index. But this means that the set of forward edges under this ordering is of size less than $(n-1)\epsilon n/2$, so T can be made transitive by changing the direction of fewer than $\epsilon \binom{n}{2}$ edges, contradicting the assumption. Thus, let G^* denote a subtournament of T with minimum out-degree at least $\epsilon n/2$.

Let αn denote the number of vertices of G^* and observe that $\alpha \geq \epsilon$ as in every tournament the number of vertices is at least twice as large as the minimum out-degree.

Suppose we sample a set Q of $q = (2\alpha/\epsilon)^2$ vertices of G^*. What is the probability that Q induces a transitive tournament? For this to happen, we must have at least one vertex v of Q such that all other $q - 1$ vertices of Q are in-neighbors of v. The probability of this occurring for a particular $v \in Q$ is at most $((\alpha n - 1 - d_v)/(\alpha n - 1))^{q-1}$ where d_v is the number of out-neighbors of v in G^*. Since $d_v \geq \epsilon n/2$ we have

$$\left(\frac{\alpha n - 1 - d_v}{\alpha n - 1}\right)^{q-1} \leq \left(\frac{\alpha n - d_v}{\alpha n}\right)^{q-1} \leq \left(1 - \frac{\epsilon}{2\alpha}\right)^{(2\alpha/\epsilon)^2 - 1} < \frac{\epsilon^2}{8\alpha^2} .$$

Hence, by the union bound, the probability that Q is transitive is less than $1/2$. In particular, with probability at least $1/2$, Q contains a triangle. As there are $\binom{\alpha n}{q}$ sets of q vertices in G^* and each triangle of G^* appears in $\binom{\alpha n - 3}{q - 3}$ such sets we have, by double counting, that the number of triangles in G^* (and therefore in T) is at least

$$\frac{1}{2} \cdot \frac{(\alpha n)^3}{q^3} \geq \frac{1}{2}\left(\frac{\alpha n \epsilon^2}{4\alpha^2}\right)^3 \geq \frac{\epsilon^6}{128}n^3 .$$

∎

We recall that an h-uniform hypergraph (h-graph for short) on vertex set V is a collection of edges, where each edge is a subset of h distinct vertices from V. An h-graph is k-colorable if we can color its vertices with k colors so that no edge is monochromatic. The following result is proved in [10] (see also [20] for an improved bound).

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[4]As is well known, a tournament is not transitive if and only if it contains a copy of C_3. Thus Lemma 2.1 is equivalent to the statement that if a tournament is ϵ-far from being C_3-free then it contains $\frac{\epsilon^6}{128}n^3$ copies of C_3.

Lemma 2.2 (Czumaj-Sohler [10]) *If one must remove from a 3-graph G on n vertices at least ϵn^3 edges in order to make it k-colorable, then a random subset $S \subseteq V(G)$ of $(10k^2/\epsilon)^2$ vertices spans a non k-colorable 3-graph with probability at least $1/2$.*

An h-graph is h-partite if its vertex set can be partitioned into h parts such that each edge contains precisely one vertex from each part. An h-partite h-graph is *complete* if every subset of size h with one vertex in each part is an edge. We need the following classical result of Erdős [12].

Lemma 2.3 (Erdős [12]) *There is a constant $n_{2.3}(\epsilon, h, t)$ such that for all $n > n_{2.3}(\epsilon, h, t)$, every n-vertex h-graph with at least ϵn^h edges contains a complete h-partite subgraph with t vertices in each part.*

The following lemma proves the positive part of Theorem 1.

Lemma 2.4 *Suppose W is obtained from an h-vertex hero H by a transitive blowup where each vertex of H is replaced with a transitive tournament on t vertices. Let k be the constant such that every H-free tournament is k-colorable. Then for any $\epsilon > 0$ and $n > n_{2.4}(\epsilon, t, H)$, every n-vertex tournament that is ϵ-far from being k-colorable contains a copy of W.*

Proof. Fix an arbitrary $\epsilon > 0$ and define $n_{2.4}(\epsilon, t, H) = n_{2.3}((\frac{\epsilon^{18}}{2^{26}hk^{22}})^h, h, 2^{t-1})$. Suppose T is a tournament on $n > n_{2.4}(\epsilon, t, H)$ vertices that is ϵ-far from being k-colorable. We need to prove that T contains W. Consider any partition of $V(T)$ into k parts $U_1, \ldots, U_k$. By assumption we should change the direction of at least $\epsilon\binom{n}{2}$ edges in order to turn this partition into a collection of k transitive tournaments. We claim that there must be at least one part U_i such that $|U_i| \geq \frac{\epsilon}{k}n$ and U_i is $\frac{\epsilon}{k}$-far from transitive. Indeed, there are at most ϵn vertices in parts of size less than $\frac{\epsilon}{k}n$ and hence less than $\epsilon^2 n^2 < (\epsilon/2)\binom{n}{2}$ edges inside such parts. If each larger part was $\frac{\epsilon}{k}$-close to transitive we could have changed less than $\binom{n}{2}\epsilon/k < (\epsilon/2)\binom{n}{2}$ edges in these parts and make them transitive. Thus, one could have made T transitive by changing the direction of less than $\epsilon\binom{n}{2}$ edges, contradicting the assumption. Let therefore U_i be such that $|U_i| \geq \frac{\epsilon}{k}n$ and be $\frac{\epsilon}{k}$-far from transitive. Hence, by Lemma 2.1, the set U_i contains $\frac{(\epsilon/k)^6}{128}(\frac{\epsilon n}{k})^3 \geq (\epsilon/2k)^9 n^3$ copies of C_3. So we see that in every partition of T into k sets there are $(\epsilon/2k)^9 n^3$ copies of C_3 that are fully contained in one of the k sets.

Define a 3-graph G_3 on $V(T)$ with $\{x, y, z\}$ being an edge of G_3 if and only if they form a C_3 in T. Then the above property of T implies that in every k-partition of $V(G_3)$ there are at least $(\epsilon/2k)^9 n^3$ edges that belong to one of the k parts. This means that one should remove at least $(\epsilon/2k)^9 n^3$ edges from G_3 in order to make it k-colorable. Hence, by Lemma 2.2, a random subset of $2^{25}k^{22}/\epsilon^{18}$ vertices from G_3 spans a non k-colorable 3-graph with probability at least $1/2$. Hence, a random subset of $2^{25}k^{22}/\epsilon^{18}$ vertices from T spans a non k-colorable tournament with probability at least $1/2$. Therefore, each such subset contains a copy of H. This means that T contains at least

$$\frac{\frac{1}{2}\binom{n}{2^{25}k^{22}/\epsilon^{18}}}{\binom{n-h}{2^{25}k^{22}/\epsilon^{18}-h}} \geq \frac{1}{2}\left(\frac{\epsilon^{18}}{2^{25}k^{22}}\right)^h n^h \tag{1}$$

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copies of H.

Consider now a random partition of $V(T)$ into h sets $V_1, \ldots, V_h$ where we randomly, uniformly and independently place each vertex $v \in V(T)$ into one of the sets V_i. Suppose the vertices of H are $x_1, \ldots, x_h$. Now define a (random) h-partite h-graph G_h on $V(T)$, with an h-tuple $v_1, \ldots, v_h$ of vertices being an edge if and only if $v_1 \in V_1, \ldots, v_h \in V_h$ and $v_1, \ldots, v_h$ span a copy of H in T where for every $1 \leq i \leq h$ vertex v_i plays the role of x_i. Note that if $v_1, \ldots, v_h$ span a copy of H in T, then the probability that $v_1, \ldots, v_h$ will be an edge in G_h is at least $1/h^h$. From (1) we deduce that the expected number of edges in G_h is at least

$$\frac{1}{h^h} \cdot \frac{1}{2} \left(\frac{\epsilon^{18}}{2^{25} k^{22}} \right)^h n^h \geq \left(\frac{\epsilon^{18}}{2^{26} h k^{22}} \right)^h n^h \ .$$

Fix an h-graph with at least this many edges. From Lemma 2.3, under assumption that $n > n_{2.3}((\frac{\epsilon^{18}}{2^{26} h k^{22}})^h, h, 2^{t-1}) = n_{2.4}(\epsilon, t, H)$, we can conclude that this h-graph contains a complete 2^{t-1}-partite h-graph on vertex sets $S_1, \ldots, S_h$. Going back to T, it is clear that $S_1, \ldots, S_h$ have the property that for any choice of $v_1 \in S_1, \ldots, v_h \in S_h$ we get a copy of H with v_i playing the role of x_i. By the observation from [19] mentioned in the introduction, every tournament on 2^{t-1} vertices contains T_t. We can now pick from each of the sets $S_1, \ldots, S_h$ a copy of T_t, and thus get a copy of W. ∎

We now turn to prove the other side of Theorem 1. As opposed to the first direction of the proof, in which we didn't use any structural property of heroes, this part of the proof will crucially rely on the characterization of heroes obtained by Berger et al. [5] (see also [9] for a simpler proof). To state their result we need some more definitions. For three tournaments P, Q, R, let $\Delta(P, Q, R)$ be the tournament obtained by taking C_3 and replacing one of its vertices with P, one with Q, and one with R. Notice that trivially, $C_3 = \Delta(T_1, T_1, T_1)$. Also observe that $\Delta(P, Q, R)$ is always a *strong* tournament, where a tournament T is strong if for any ordered pair of vertices u, v, there is a path in T from u to v. The main result of [5] is the following.

Theorem 3 (Berger et al. [5]) *A tournament is a hero if and only if each of its strong components is a hero. A strong tournament is a hero if and only if it is isomorphic to $\Delta(P, T_q, T_1)$ for some positive integer q and for some hero P.*

As we mentioned in Section 1, we will actually show that for any tournament W that is not a transitive blowup of a hero, there are tournaments that are W-free and do not even contain a set of vertices of linear size that is close to being transitive. To make this approach precise let us introduce the following refined version of the notion of being ϵ-far from transitive.

Definition 2.5 *An n-vertex tournament is (c, ϵ)-far from transitive if the induced subgraph on every set of at least cn vertices is ϵ-far from transitive.*

The following lemma shows that in order to prove that a graph W is not unavoidable, it is a enough to construct for every $c > 0$ and some $\epsilon > 0$ a sequence of W-free tournaments that are all (c, ϵ)-far from transitive.

Lemma 2.6 *If W is unavoidable, then for every $\epsilon > 0$ and every $n > n_{2.6}(\epsilon, W)$ every n-vertex tournament that is $(1/c_W, \epsilon)$-far from transitive contains a copy of W.*

Proof. Suppose T is an n-vertex tournament that is $(1/c_W, \epsilon)$-far from transitive. It is enough to show that if n is large enough, then T is ϵ-far from satisfying $\chi(T) \leq c_W$. Indeed, consider any partition of $V(T)$ into c_W sets $V_1, \ldots, V_{c_W}$. Then some set V_i must be of size at least n/c_W so by assumption one should change the direction of at least $\epsilon \binom{n/c_W}{2}$ edges just to make V_i transitive. In particular, T is (ϵ/c_W^2)-far from satisfying $\chi(T) \leq c_W$. Hence, it follows from Definition 1.1 that for large enough n (that depends only on ϵ and W), T must contain a copy of W. ∎

Lemma 2.8 gives a necessary condition for a strong tournament to be unavoidable. Before proving it, we need the following simple claim observed in [6].

Lemma 2.7 *Suppose a tournament T contains three vertex sets A_1, A_2, A_3 each of size r so that for every $1 \leq i \leq 3$ and every $v \in A_i$ and $u \in A_{i+1}$ (where indices are taken (mod 3)) the edge (v, u) belongs to T. Then one should change the direction of at least r^2 edges in order to make T transitive.*

Proof. T contains r^3 copies of C_3 each with one vertex in each of the sets A_1, A_2, A_3. Changing the direction of an edge in T can destroy at most r of these copies of C_3 hence one needs to make at least r^2 modifications to make T transitive. ∎

Lemma 2.8 *Every strong unavoidable tournament with at least three vertices has to be of the form $\Delta(P, Q, R)$ where P, Q, R are nonempty tournaments.*

Proof. Define the following sequence of tournaments, denoted by $U_1, U_2, \ldots$ as follows. $U_1 = C_3$ and $U_k = \Delta(U_{k-1}, U_{k-1}, U_{k-1})$. Notice that U_k has 3^k vertices and is, in fact, a regular tournament[5].

We next show that for any $0 < c \leq 1$, and for all k sufficiently large, the tournament U_k is (c, ϵ)-far from transitive for $\epsilon = 0.01c^6$. Consider some sub-tournament R_k of U_k with $c \cdot 3^k$ vertices. We define the following process starting from $i = 0$. Recall that U_{k-i} is constructed from three disjoint copies of U_{k-i-1}. If all of the three copies contain each at least $0.1|R_{k-i}|$ vertices of R_{k-i}, we halt. Otherwise, some copy of U_{k-i-1} contains at least $0.45|R_{k-i}|$ vertices. Denote the sub-tournament of R_{k-i} induced by this copy by R_{k-i-1} and continue to the next i.

How long can this process continue? Observe that R_{k-i} has at least $0.45^i c \cdot 3^k$ vertices. On the other hand, U_{k-i} has 3^{k-i} vertices and hence we must have $3^{k-i} \geq 0.45^i c \cdot 3^k$. Hence, $1.35^i \leq c^{-1}$ and thus $i \leq \ln(c^{-1})/\ln(1.35)$. When the process halts we have that R_{k-i} has at least $0.1|R_{k-i}|$ vertices in each of the three parts of U_{k-i}. By Lemma 2.7, to make R_{k-i} transitive we need to change the direction of at least $(0.1|R_{k-i}|)^2$ edges. To conclude we must show that $(0.1|R_{k-i}|)^2 \geq \binom{|R_k|}{2}0.01c^6$. Indeed,

$$(0.1|R_{k-i}|)^2 \geq 0.01 \cdot 0.45^{2i}c^2 9^k \geq 0.01 \cdot 0.45^{2\ln(c^{-1})/\ln(1.35)}c^2 9^k \geq 0.01c^8 9^k \geq \binom{|R_k|}{2}0.01c^6$$

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[5]A regular tournament is a tournament where the in-degree and out-degree at each vertex are equal.

as required.

Suppose now that W is a strong unavoidable tournament. Then, by Lemma 2.6, every suffi-ciently large tournament that is $(c_W, 0.01c_W^6)$-far from transitive must contain W. In particular, for all k sufficiently large, U_k must contain a copy of W. Let k be the smallest integer such that U_k contains W. We may assume that $k > 1$ (otherwise $W = C_3$ and we are trivially done). Recall that $U_k = \Delta(A_1, A_2, A_3)$ where A_i is isomorphic to U_{k-1} for $i = 1, 2, 3$. Let B denote a copy of W in U_k and let $B_i = A_i \cap B$ for $i = 1, 2, 3$ and thus $B = \Delta(B_1, B_2, B_3)$. We cannot have that B is entirely contained in some A_i by the minimality of k. We can also not have that B intersects precisely two of the A_i's since B is strong. Hence each B_i is nonempty. ∎

Lemma 2.9 *The tournament $\Delta(C_3, C_3, T_1)$ is not unavoidable.*

Proof. Set $W = \Delta(C_3, C_3, T_1)$ and let c be a positive integer. By Lemma 2.6, it is enough to show that there is some $\epsilon > 0$ so that for any large enough n there is a tournament that is (c, ϵ)-far from transitive and is yet W-free. Let t be the smallest integer such that there exists an undirected graph G_t on t vertices with girth at least 8 and maximum independent set smaller than $ct/4$. The existence of G_t follows from a result of Erdős [11]. Set $\epsilon = 1/4t^2$.

The following tournament is constructed in [5]. Take t vertices $v_1, \ldots, v_t$ and a copy of G_t on these vertices. Orient every edge of G_t from the endpoint with higher index to the endpoint with lower index (so that all these edges go backwards from "right to left"). Every non-edge of G_t becomes an edge from left to right. Denote the resulting tournament by R_t and notice that G_t is the "back-edge graph" of R_t.

As any subgraph on seven vertices of G_t is a forest, we have that any induced subgraph on seven vertices of R_t is a union of two transitive sets (which one obtains from a bipartition of the forest induced by these seven vertices). But since $W = \Delta(C_3, C_3, T_1)$ is not the union of two transitive sets, it follows that R_t does not contain W. Furthermore, as G_t is triangle-free, any transitive set of R_t is the union of two independent sets of G_t (one consisting of all the vertices that are not the head of any back-edge and the other consisting of all the vertices that are not the tail of any back-edge). Hence, the largest transitive set of R_t has size smaller than $ct/2$.

Now, let G^* be a transitive blowup of R_t obtained by replacing each vertex with a transitive set of size n/t. Then G^* has n vertices and we denote the t parts of G^* by $V_1, \ldots, V_t$ where V_i is the blowup of v_i and $|V_i| = n/t$. We claim that G^* does not contain W as a sub-tournament. Indeed, assume there is some sub-tournament X of G^* isomorphic to W. We cannot have each of the seven vertices of X in a distinct part of G^* since otherwise X would have been a sub-tournament of R_t, contradicting the fact that R_t does not contain W. But we also cannot have two or more vertices of X in the same part as W is *not* a transitive blowup of any tournament with fewer than seven vertices, since its chromatic number is 3 while the chromatic number of any smaller tournament is at most 2.

We next claim that every set of size cn in G^* is far from transitive, in the sense that one needs to change the direction of more than $c^2n^2/(4t^2)$ edges to make it transitive. To see this, consider some set A of cn vertices of G^*. Let $A_i = A \cap V_i$ for $i = 1, \ldots, t$. Let S be the sub-tournament

of R_t such that $v_s \in S$ if $|A_s| \geq cn/(2t)$ and let $A^* = \cup_{v_s \in S} A_s$. Notice that the total number of vertices of $A \setminus A^*$ is at most $cn/2$ so $|A^*| \geq cn/2$ and hence we also have that $|S| > ct/2$, so by the construction, S is not a transitive set. This means that A^* is not a transitive set and hence, as S contains a directed triangle, A^* contains the $cn/(2t)$-blowup of a directed triangle. By Lemma 2.7, in order to make A^* transitive one must change the direction of at least $(cn/(2t))^2 = c^2 n^2/(4t^2)$ edges from A^* which is more than an ϵ fraction of the $\binom{cn}{2}$ edges of A. As we can take n to be arbitrary large, this proves that W is not unavoidable. ∎

Corollary 2.10 *Every strong unavoidable tournament with at least three vertices is of the form* $\Delta(P, T_k, T_\ell)$ *where P is unavoidable and k, ℓ are positive integers.*

Proof. By Lemma 2.8, if W is a strong tournament that is unavoidable, it must be of the form $\Delta(P, Q, R)$ where P, Q, R are nonempty tournaments. Notice, however, that every subtournament of an unavoidable tournament is unavoidable. Thus, we cannot have more than one of P, Q, R non-transitive, as otherwise W contains $\Delta(C_3, C_3, T_1)$ which is not unavoidable by Lemma 2.9. Hence, without loss of generality, both Q and R are transitive tournaments, say $Q = T_k$ and $R = T_\ell$. ∎

The following lemma gives a necessary condition for a tournament to be unavoidable.

Lemma 2.11 *If W is unavoidable, then it is a transitive blowup of a hero.*

Proof. We prove the lemma by induction on the size of W. If W has at most three vertices, the result trivially holds, so assume $|V(W)| > 3$. Assume first that W is not strong, and let $W_1, \ldots, W_k$ be its strong components (we can assume that for $i < j$, all edges between W_i and W_j point from W_i to W_j). As each W_i is unavoidable (since they are subtournaments of W) we have, by the induction hypothesis, that W_i is a transitive blowup of some hero H_i for $i = 1, \ldots, k$. Now, by Theorem 3, the tournament H whose strong components are $H_1, \ldots, H_k$ is also a hero. But observe that W is a transitive blowup of H, hence the claim holds.

Assume next that W is strong. By Corollary 2.10, W is of the form $\Delta(P, T_k, T_\ell)$ where P is unavoidable. As P has fewer vertices than W, the induction hypothesis implies that P is a transitive blowup of some hero H. But by Theorem 3, the tournament $\Delta(H, T_1, T_1)$ is a hero. Since W is a transitive blowup of $\Delta(H, T_1, T_1)$, the claim holds in this case as well. ∎

Proof. (of Theorem 1) Immediate from Lemmas 2.4 and 2.11. ∎

3 Weakly Unavoidable Tournaments

In this section we prove that every tournament is weakly unavoidable. We begin with a definition and a few lemmas. The density of an *ordered* pair of nonempty disjoint vertex sets (A, B) in a digraph is defined to be $d(A, B) = e(A, B)/(|A||B|)$ where $e(A, B)$ is the number of edges pointing from A to B (note that we do *not* necessarily have $d(A, B) = d(B, A)$).

Definition 3.1 *A bipartite digraph (A, B) is (δ, γ)-dense if for every pair of subsets $A' \subseteq A$ and $B' \subseteq B$ satisfying $|A'| \geq \delta|A|$ and $|B'| \geq \delta|B|$ we have $d(A', B') \geq \gamma$ and $d(B', A') \geq \gamma$.*

Lemma 3.2 *Let $\gamma < \frac{1}{2}$ and let W be a tournament on w vertices. If $U_1, \ldots, U_w$ are disjoint vertex sets satisfying $|U_i| \geq (1/\gamma)^w$ for every $1 \leq i \leq w$, and each pair of them is (γ^w, γ)-dense, then they contain a copy of W. In particular, if U is a set of at least $w(1/\gamma)^w$ vertices and for each pair of disjoint subsets $A, B \subseteq U$ of size at least $(\gamma^w/w)|U|$ we have $d(A, B) \geq \gamma$ and $d(B, A) \geq \gamma$, then U contains a copy of W.*

Proof. We prove the first part of the lemma by induction on w. Denote the vertices of W by $V(W) = \{v_1, \ldots, v_w\}$. We will prove that $U_1, \ldots, U_w$ contain a copy of W where the vertex of U_i in that copy plays the role of v_i. Notice that the statement is trivially true for $w = 1$.

For $i = 1, \ldots, w - 1$, we say that a vertex u of U_w is *bad for i* if u has fewer than $\gamma|U_i|$ out-neighbors in U_i in case $(v_w, v_i) \in E(W)$ or fewer than $\gamma|U_i|$ in-neighbors in U_i in case $(v_i, v_w) \in E(W)$. Let $U_{w,i} \subset U_w$ denote the set of vertices that are bad for i. Clearly, $d(U_{w,i}, U_i) < \gamma$ in case $(v_w, v_i) \in E(W)$ or $d(U_i, U_{w,i}) < \gamma$ in case $(v_i, v_w) \in E(W)$. Hence, by the assumption that (U_w, U_i) is (γ^w, γ)-dense, we must have $|U_{w,i}| < \gamma^w|U_w|$. We therefore have that there exists $X \subset U_w$ with $|X| \geq |U_w| - (w-1)\gamma^w|U_w|$ such that each $u \in X$ has at least $\gamma|U_i|$ out-neighbors in U_i in case $(v_w, v_i) \in E(W)$ and at least $\gamma|U_i|$ in-neighbors in U_i in case $(v_i, v_w) \in E(W)$. Notice that X is not empty since $\gamma^w < 1/(w-1)$.

Pick some $u \in X$ and let $R_i \subset U_i$ be the set of out-neighbors of u in U_i in case $(v_w, v_i) \in E(W)$ or the set of in-neighbors of u in U_i in case $(v_i, v_w) \in E(W)$. Let $W' = W \setminus \{v_w\}$. It suffices to prove that $R_1, \ldots, R_{h-1}$ contain a copy of W' where the vertex of R_i in that copy plays the role of v_i. Notice first that

$$|R_i| \geq \gamma|U_i| \geq \gamma(1/\gamma)^w \geq (1/\gamma)^{w-1} \ .$$

Also notice that $\gamma^{w-1}|R_i| \geq \gamma^w|U_i|$, so each pair (R_i, R_j) is (γ^{w-1}, γ)-dense. By the induction hypothesis, there is a copy of W' in $R_1, \ldots, R_{w-1}$ where the vertex of R_i in that copy plays the role of v_i.

For the second part of the lemma just partition U arbitrarily into w sets of equal size and observe that each pair of them must be (γ^w, γ)-dense. ∎

Lemma 3.3 *Suppose $U_1, \ldots, U_t$ are vertex-disjoint subsets of a tournament where $t = 3/\epsilon$ and $|U_i| \geq q$ for all $i = 1, \ldots, t$. Furthermore, for every $1 \leq i < j \leq t$ we have $d(U_j, U_i) \leq \epsilon/2$. Then there are subsets $R_1, \ldots, R_t$ with $R_i \subset U_i$ and $|R_i| = q$, such that the sub-tournament spanned by $\cup_{i=1}^t R_i$ is ϵ-close to transitive.*

Proof. Consider a random subset $R_i \subset U_i$ of size q. Since $d(U_j, U_i) \leq \epsilon/2$ whenever $j > i$, we have that the expected number of edges pointing from R_j to R_i is at most $q^2\epsilon/2$. Thus, by linearity of expectation, there is a choice of subsets $R_1, \ldots, R_t$ such that one can change the direction of at most $\binom{t}{2}q^2\epsilon/2$ edges such that no edge points from any R_j to any R_i whenever $j > i$. Remove also

all edges with both endpoints in R_i for $i = 1, \ldots, t$. Hence the sub-tournament $R = \cup_{i=1}^t R_i$ can be made transitive by changing the direction of at most $t\binom{q}{2} + \binom{t}{2}q^2\epsilon/2$ edges. But

$$t\binom{q}{2} + \binom{t}{2}q^2\epsilon/2 \leq \frac{tq^2}{2} + \frac{t^2q^2\epsilon}{4} \leq q^2\left(\frac{3}{2\epsilon} + \frac{9}{4\epsilon}\right) < q^2\frac{4}{\epsilon} < \binom{tq}{2}\epsilon \ .$$

Since the number of edges of R is $\binom{tq}{2}$, the latter inequality proves that R is ϵ-close to transitive.

∎

Lemma 3.4 *Let $\epsilon > 0$ and let w be a positive integer. Every tournament T with $n \geq n_{3.4}(\epsilon, w)$ vertices contains one of the following:*

1. *A set U of at least $(\epsilon/8w)^{3w/\epsilon}n$ vertices satisfying the assertion of Lemma 3.2 with $\gamma = \epsilon/4$.*

2. *A collection of $3/\epsilon$ subsets $U_1, \ldots, U_t$ each of size at least $(\epsilon/8w)^{3w/\epsilon}n$ satisfying the assertion of Lemma 3.3 with $q = (\epsilon/8w)^{3w/\epsilon}n$.*

Proof. Define

$$n_{3.4}(\epsilon, w) = w\left(\frac{4}{\epsilon}\right)^w\left(\frac{8w}{\epsilon}\right)^{3w/\epsilon} . \tag{2}$$

For an integer $t \geq 0$, let us say that a sequence $\{U_1, \ldots, U_t, V_{t+1}\}$ of $t + 1$ disjoint sets of vertices of T is *nice* if it satisfies the following conditions:

1. $|V_{t+1}| \geq (\epsilon/8w)^{wt}n$.

2. For each $1 \leq i \leq t$ we have $|U_i| \geq (\epsilon/8w)^{iw}n$.

3. For every $1 \leq i < j \leq t$ we have $d(U_j, U_i) \leq \epsilon/2$.

4. For every $1 \leq i \leq t$, every vertex in V_{t+1} has at most $\epsilon|U_j|/2$ out-neighbors in U_j.

Note that for $t = 0$ we can get a nice collection by setting $V_1 = V(T)$. Suppose first that T contains a nice collection $U_1, \ldots, U_t, V_{t+1}$ with $t = 3/\epsilon$. Then it is clear that $U_1, \ldots, U_t$ satisfy the second assertion of the lemma and we are done.

Otherwise, let $U_1, \ldots, U_t, V_{t+1}$ be a nice collection with the largest possible t, which we assume satisfies $t < 3/\epsilon$. We now claim that in this case we can take V_{t+1} as the set U in the first assertion of the lemma with $\gamma = \epsilon/4$. To see this, first note that by (2),

$$|V_{t+1}| \geq (\epsilon/8w)^{wt}n \geq w(1/\gamma)^w$$

so it satisfies the assumption of Lemma 3.2 and it is also of the right size since it has size at least $(\epsilon/8w)^{3w/\epsilon}n$. We now need to show that for every pair of disjoint subset $A, B \subseteq V_{t+1}$ of size at least $(\gamma^w/w)|V_{t+1}|$ we have $d(A, B) \geq \gamma$ and $d(B, A) \geq \gamma$. So suppose this is not the case and that A, B is such a pair satisfying $d(B, A) < \gamma$.

Let V_{t+2} be the set of vertices of B that have at most $2\gamma|A|$ edges pointing to A. Observe that $|V_{t+2}| \geq |B|/2$ as otherwise that would give $d(B, A) \geq \gamma$. Define $U_{t+1} = A$ and observe that we have found sets U_{t+1} and V_{t+2} satisfying

$$|U_{t+1}| \geq \frac{\gamma^w}{w}|V_{t+1}| \geq \frac{\gamma^w}{w}\left(\frac{\epsilon}{8w}\right)^{wt} n \geq \left(\frac{\epsilon}{8w}\right)^{w(t+1)} n .$$

and

$$|V_{t+2}| \geq \frac{|B|}{2} \geq \frac{1}{2}\frac{\gamma^w}{w}|V_{t+1}| \geq \frac{1}{2}\frac{\gamma^w}{w}\left(\frac{\epsilon}{8w}\right)^{wt} n \geq \left(\frac{\epsilon}{8w}\right)^{w(t+1)} n .$$

Furthermore, note that our assumption that $U_1, \ldots, U_t, V_{t+1}$ is a nice sequence, implies that $U_1, \ldots, U_{t+1}, V_{t+2}$ is also a nice sequence, contradicting the maximality of t. ∎

We now derive a weaker version of Theorem 2, showing that if a tournament is W-free then some set of vertices of linear size must be close to transitive. We then use this weaker version iteratively in order to prove Theorem 2.

Lemma 3.5 *Suppose W has w vertices, $\epsilon > 0$ and $n > n_{3.4}(\epsilon, w)$. Then every n-vertex tournament T that is $((\epsilon/8w)^{3w/\epsilon}, \epsilon)$-far[6] from transitive contains a copy of W.*

Proof. If T satisfies the second assertion of Lemma 3.4, then Lemma 3.3 implies that T contains a subset of size at least $(\epsilon/8w)^{3w/\epsilon}n$ that is ϵ-close to transitive, contradicting the assumption of the lemma. Therefore T must satisfy the first assertion of Lemma 3.4. Lemma 3.2 now implies that T has a copy of W. ∎

Proof. (of Theorem 2) We prove that there is a function $c_W(\epsilon)$ such that if T is a W-free tournament, then T is ϵ-close to having chromatic number at most $c_W(\epsilon)$. So, suppose T is W-free. If T has fewer than $n_{3.4}(\epsilon, w)$ vertices, then we can just take the trivial partition into $n_{3.4}(\epsilon, w)$ vertices. Otherwise, by Lemma 3.5, we can find a subset S_1 of size $(\epsilon/8w)^{3w/\epsilon}n$ that is ϵ-close to transitive. We can now keep pulling subsets $S_2, S_3, \ldots$ on an $(\epsilon/8w)^{3w/\epsilon}$-fraction of the remaining vertices until we are either left with fewer than ϵn vertices or with fewer than $n_{3.4}(\epsilon, w)$ vertices. This clearly happens after at most $\log(1/\epsilon) \cdot (8w/\epsilon)^{3w/\epsilon} \leq (8w/\epsilon)^{4w/\epsilon}$ iterations. We can then remove from each of the sets S_i the ϵ-fraction of vertices which make it transitive. Finally, if we were left with a set of size less than ϵn we can remove all edges in the set, and if we were left with fewer than $n_{3.4}(\epsilon, h)$ vertices, we partition it into sets of size 1. In any case we remove fewer than $\epsilon\binom{n}{2}$ edges, and thus partition T into at most $n_{3.4}(\epsilon, h) + (8w/\epsilon)^{4w/\epsilon} \leq (8w/\epsilon)^{5w/\epsilon}$ sets that are transitive. ∎

4   Concluding Remarks and Open Problems

A removal lemma for tournaments: An n-vertex graph or digraph G is ϵ-far from being H-free if one should remove from G at least ϵn^2 edges in order to make it H-free. The famous graph

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[6]Recall Definition 2.5.

removal lemma states that in this case G must contain $f_H(\epsilon)n^h$ copies of H, where $h = |V(H)|$. As is well known, the proof of this general result uses Szemerédi's regularity lemma and as a result the bound on $f_H(\epsilon)$ is extremely poor. Alon [1] characterized the graphs H for which $f_H(\epsilon)$ can be bounded from below by ϵ^C for some $C = C(H)$. This suggests the following problem:

Problem 4.1 *For a tournament H on h vertices and $\epsilon > 0$ let $f_H(\epsilon)$ be the largest real so that any tournament T that is ϵ-far from being H-free contains $f_H(\epsilon)n^h$ copies of H. For which tournaments can $f_H(\epsilon)$ be bounded from below by ϵ^C for some $C = C(H)$?*

We note that if T in the above problem is not required to be a tournament, that is if T is allowed to be an arbitrary digraph, then a characterization is given in [2]. However, the proof in [2] showing that $f_H(\epsilon)$ is not polynomial in ϵ critically relies on the fact that T is not a tournament. For example, it follows from [2] that $f_{C_3}(\epsilon)$ is not polynomial in ϵ when T is an arbitrary digraph, while Lemma 2.1 shows that $f_{C_3}(\epsilon)$ is polynomial when T is required to be a tournament. As of now we can show that there are tournaments H for which $f_H(\epsilon)$ is not polynomial (in fact, we can show that as h grows, almost all h-vertex tournaments are such) but we are still not able to resolve Problem 4.1 completely. As a special case of Problem 4.1, is it true that $f_H(\epsilon)$ is polynomial in ϵ whenever H is a hero? Lemma 2.4 shows that a positive answer to this question would actually imply that $f_H(\epsilon)$ is polynomial whenever H is a transitive blowup of a hero, that is, whenever H is unavoidable.

A characterization of c-unavoidable tournaments: Given a tournament W let c_W denote the smallest c for which W is c-unavoidable, and let c'_W be the smallest c so that any W-free tournament T satisfies $\chi(T) \leq c$ (i.e. the smallest c for which W satisfies the condition of being a hero). Then as we mentioned in Section 1, Fox and Sudakov [15] characterized the tournaments that satisfy $c_W = 1$. Furthermore, this characterization is "efficient" in the sense that given W it is possible to determine in polynomial time whether $c_W = 1$. The result of Berger et al. [5] gives a characterization of the tournaments satisfying $c'_W = O(1)$ and our main result is a characterization of the tournaments satisfying $c_W = O(1)$. It is thus natural to ask if given a tournament W it is possible to compute c_W. To this end we have the following partial answer.

Lemma 4.2 *For an unavoidable tournament W, let H be a minimal (in terms of number of vertices) sub-tournament of W with the property that W is transitive blowup of H. Then $c_W = c'_H$. Furthermore, given W it is possible to find a minimal H as above in polynomial time.*

Proof. (sketch) Let H be as above. Since W is unavoidable, Theorem 1 tells us that W is a transitive blowup of some hero H', implying that H' is a subgraph of W. Hence H' is also a transitive blowup of H and the minimality of H implies that H must be a subgraph of H'. Now, clearly a subgraph of a hero is also a hero, so H must be a hero as well. The proof of Lemma 2.4 thus implies that $c_W \leq c'_H$.

For the other direction, the definition of c'_H implies that there is an H-free tournament R on r vertices satisfying $\chi(R) = c'_H$. Let T be a k-transitive[7] blowup of R, for some arbitrary k. We now wish to show that T is W-free and r^{-2}-far from satisfying $\chi(T) \leq c'_H - 1$ implying that $c_W \geq c'_H$. First, it is easy to check that the minimality of H implies that T is also H-free and thus also W-free. Second, we claim that T is r^{-2}-far from satisfying $\chi(T) \leq c'_H - 1$. Indeed, T has $n = kr$ vertices and suppose that Q is some set of fewer than $n^2/r^2 = k^2$ edges. We need to show that after changing the direction of the edges in Q the resulting graph cannot be partitioned into $c'_H - 1$ transitive sets. For every $1 \leq i \leq r$ let V_i denote the vertex set of the copy of T_k that replaced vertex i of R. Let $Q_{i,j}$ denote the edges of Q that connect V_i to V_j. Randomly and uniformly pick a vertex v_i from each of the sets V_i. Then the probability that the edge connecting v_i and v_j belongs to Q is $|Q_{i,j}|/k^2$, hence the probability that some pair (v_i, v_j) belongs to Q is bounded by $|Q|/k^2 < 1$ so there is a choice of r vertices so that all pairs of vertices are connected by edges that do not belong to Q and thus span a copy of R. But R cannot be partitioned into $c'_H - 1$ transitive sets, implying that after changing the direction of the edges in Q the resulting graph still cannot be partitioned into $c'_H - 1$ transitive sets.

As to the task of finding H, let us say that a pair of vertices x, y in W are *identical* if they form a homogenous set of size 2 [7], namely for any other vertex z, either (x, z) and (y, z) are both edges of W or both non-edges of W. Now, as long as there is an identical pair of vertices (x, y) in W, remove x and continue. It is easy to check that once we end up with a graph that has no identical pair we get the required graph H. ∎

It follows from the above lemma that computing c_W reduces to the task of computing c'_H. Unfortunately, we do not know how to compute c'_H efficiently. It would be interesting to determine how hard is this task.

A stronger version of Theorem 1: Note that the proof of Theorem 1 shows that if T is ϵ-far from satisfying $\chi(T) \leq c_W$ then T contains $\epsilon^C n^w$ copies of W where n and w denote the number of vertices of T and W respectively and C is a constant that depends only on w. Hence there is some $\delta > 0$ so that even if T is $n^{-\delta}$-far from satisfying $\chi(T) \leq c_W$, then T still contains at least one copy of W. Indeed, this follows from the fact that under the assumption of Lemma 2.3, the h-graph actually contains $C\epsilon^{t^h} n^{ht}$ copies of the complete h-partite h-graph with t vertices in each part. See the statement of Lemma 8.1 in [14]. This justifies the comment we made after the statement of Theorem 1.

A better bound in Theorem 2: It would be interesting to determine the best dependence of $c_W(\epsilon)$ on ϵ and w (= number of vertices of W) in Theorem 2. Our proof gives a bound that is (roughly) exponential in w/ϵ and we can show that the dependence should indeed be exponential in w. Is the exponential dependence on ϵ necessary or can it be made polynomial?

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[7]Namely, a transitive blowup where each vertex is replaced with a copy of T_k

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