The number of edge-disjoint transitive triples in a tournament

Raphael Yuster *
Department of Mathematics
University of Haifa at Oranim
Tivon 36006, Israel

Abstract

We prove that a tournament with n vertices has more than $0.13n^2(1+o(1))$ edge-disjoint transitive triples. We also prove some results on the existence of large packings of k-vertex transitive tournaments in an n-vertex tournament. Our proofs combine probabilistic arguments and some powerful packing results due to Wilson and to Frankl and Rödl.

1 Introduction

All graphs and directed graphs considered here are finite and have no loops or multiple edges. For the standard terminology used the reader is referred to [2]. A *tournament* on n vertices is an orientation of K_n. Thus, for every two distinct vertices x and y, either (x, y) or (y, x) is an edge, but not both.

Let TT_k denote the unique transitive tournament on k vertices. TT_3 is also called a *transitive triple* as it consists of some triple $\{(x, y), (x, z), (y, z)\}$. A TT_k*-packing* of a directed graph D is a set of edge-disjoint copies of TT_k subgraphs of D. The TT_k*-packing number* of D, denoted $P_k(D)$, is the maximum size of a TT_k-packing of D. The TT_3-packing number of D_n, the complete directed graph with n vertices and $n(n-1)$ edges, has been studied, e.g., in the papers of Gardner [4], Phelps and Lindner [6] and Skillicorn [9]. A closely related result of Keevash and Sudakov [5] deals with packing monochromatic triangles in a red-blue edge coloring of K_n.

In this paper we consider only TT_k-packings of tournaments. Let $f_k(n)$ denote the minimum possible value of $P_k(T_n)$, where T_n ranges over all possible n-vertex tournaments. For simplicity, put $f(n) = f_3(n)$ and $P(T_n) = P_3(T_n)$. Trivially, $P_k(T_n) \le n(n-1)/(k(k-1))$, and in particular $f(n) \le n(n-1)/6 < 0.167n^2(1+o(1))$. In fact, it is not difficult to show that $f(n) \le \lceil n(n-1)/6 - n/3 \rceil$ (see Section 4 for this and also for a general way to construct an upper bound for $f_k(n)$). We conjecture the following:

*e-mail: raphy@research.haifa.ac.il World Wide Web: http:\\research.haifa.ac.il\~raphy

Conjecture 1.1 $f(n) = \lceil n(n-1)/6 - n/3 \rceil$.

This conjecture was verified for all $n \le 8$ (see Section 4). Our main result is the following lower bound for $f(n)$.

Theorem 1.2 $f(n) > 0.13n^2(1+o(1))$.

We prove Theorem 1.2 in Section 2. A simple application of a result of Frankl and Rödl [3] shows that if T_n is a random tournament on n vertices then $P(T_n) \ge \frac{1}{6}n^2(1-o(1))$ almost surely. In fact, more generally, $P_k(T_n) \ge \frac{1}{k(k-1)}n^2(1-o(1))$ almost surely. For completeness, we describe this application in Section 3. The final section contains some concluding remarks.

2 Proof of the main result

From here on we assume that the vertex set of a tournament with k vertices is $[k] = \{1,\dots,k\}$. Let T_k be any k-vertex tournament. For $v \in [k]$, let $d^+(v)$ denote the out-degree of v in T_k. Let $a(T_k)$ denote the total number of transitive triples in T_k, and let $t(T_k)$ denote the total number of directed triangles in T_k. Clearly, $a(T_k) + t(T_k) = \binom{k}{3}$. We shall also make use of the obvious inequality, which follows from the fact that in a transitive triple there is one source and one sink.

$$a(T_k) = \sum_{v=1}^{k} \frac{1}{2}\left(\binom{d^+(v)}{2} + \binom{k-1-d^+(v)}{2}\right) \ge \frac{k(k-1)(k-3)}{8}. \qquad (1)$$

In the proof of Theorem 1.2 we need the following (special) case of Wilson's Theorem [10].

Lemma 2.1 *There exists a positive integer N such that for all $n > N$, if $n \equiv 1 \pmod{42}$ then K_n decomposes into $\binom{n}{2}/21$ edge-disjoint copies of K_7.* ■

The next lemma quantifies the fact that if $t(T_7)$ is relatively small then $P(T_7)$ is relatively large.

Lemma 2.2 *If $t(T_7) \le 4$ then $P(T_7) = 7$. If $t(T_7) \le 11$ then $P(T_7) \ge 6$. If $t(T_7) \ge 12$ then $P(T_7) \ge 5$.*

Proof Let T_7 be a tournament on seven vertices. Consider a Steiner triple system (Fano plane) randomly placed on the same vertex set. Clearly, the expected number of directed triangles of T_7 contained in triples of such a random Steiner triple system is

$$7 \cdot \frac{t(T_7)}{t(T_7)+a(T_7)} = \frac{7}{35}t(T_7).$$

Hence, if $t(T_7) < 5$ then this expectation is less than 1. Thus, there is a Steiner triple system with no directed triangle. Namely, $P(T_7) = 7$ in this case. Similarly, by (1), we always have $a(T_7) \ge 21$

and so $t(T_7) \leq 14$. Therefore, the expectation above is always at most $14 \cdot \frac{7}{35} \leq 2.8$. Thus, there is always a Steiner triple system with at most two directed triangles. Namely, $P(T_7) \geq 5$ always.

We remain with the case where $t(T_7) \leq 11$. Notice that we may assume $t(T_7) = 11$ or $t(T_7) = 10$ since otherwise the above expectation argument yields $P(T_7) \geq 6$. Assume first that $t(T_7) = 11$. Hence $a(T_7) = 24$ and by (1) the only possible scores (sorted out-degree sequence) of such a T_7 are $(4,4,4,3,2,2,2)$, $(5,3,3,3,3,2,2)$ and $(4,4,3,3,3,3,1)$. The last two scores are complementary (namely, reversing the edges of a T_7 with one of these scores yields a tournament with the other score) and the first score is self-complementary. Hence, one needs only to check the first two scores.

There are precisely 18 non-isomorphic tournaments with the score $(4,4,4,3,2,2,2)$, and each can be checked to have at least 6 edge-disjoint transitive triples. A convenient way to enumerate these 18 non-isomorphic tournaments is as follows. Let A_i be the set of vertices with out-degree i, $i = 2,3,4$. $|A_2| = |A_4| = 3$, $|A_3| = 1$. First case: The subgraph induced by A_2 is a directed triangle and the subgraph induced by A_4 is also a directed triangle. There are four non-isomorphic tournaments with this restriction. Second case: The subgraph induced by A_2 is a directed triangle and the subgraph induced by A_4 is a transitive triple. There are four non-isomorphic tournaments with this restriction. Third case: The subgraph induced by A_2 is a transitive triple and the subgraph induced by A_4 is a directed triangle. There are four non-isomorphic tournaments with this restriction. Fourth case: Both A_2 and A_4 induce a transitive triple. There are six non-isomorphic tournaments with this restriction. Altogether there are $4+4+4+6 = 18$ possibilities.

There are precisely 15 non-isomorphic tournaments with the score $(5,3,3,3,3,2,2)$, and each can be checked to have at least 6 edge-disjoint transitive triples. A convenient way to enumerate these 18 non-isomorphic tournaments is as follows. Let A_i be the set of three vertices with out-degree i, $i = 2,3,5$. $A_2 = \{a,b\}$ $A_5 = \{c\}$, $A_3 = \{d,e,f,g\}$. We may assume the edge inside A_2 is (a,b). First case: (a,c) is an edge. There is a unique tournament with this restriction. Second case: (c,a), (a,d) and (d,c) are edges. There are four non-isomorphic tournaments. Third case: (c,a), (a,d) (c,d) and (b,d) are edges. There are three non-isomorphic tournaments. Fourth case: (c,a), (a,d) (c,d) and (d,b) are edges. There are 7 non-isomorphic tournaments. Altogether there are $1+4+3+7 = 15$ possibilities.

In case $t(T_7) = 10$ the expected number of directed triangles in a random Steiner triple system is precisely 2. However, the distribution is easily seen to be non-constant (e.g., the variance is positive). Thus, there is a Steiner triple system with less than two directed triangles. Namely, $P(T_7) \geq 6$ in this case. ■

Note: A much simpler version of Lemma 2.2 that circumvents the case $t(T_7) = 11$ obviously holds by assuming only that if $t(T_7) \leq 10$ then $P(T_7) \geq 6$. Using such a version leads to a slightly weaker constant in Theorem 1.2, namely 0.128 instead of 0.13.

Fix T_n, and let $3 \leq m \leq n$. Let T_m be a randomly chosen m-vertex induced subgraph of T_n. Let $X = a(T_m)$ denote the random variable corresponding to the number of transitive triples of

T_m, and let $E[X]$ denote the expectation of X.

Proposition 2.3 $E[X] \geq \frac{3}{4}\frac{n-3}{n-2}\binom{m}{3}$.

Proof A specific triple of T_n belongs to precisely $\binom{n-3}{m-3}$ induced subgraphs on m vertices. Thus, by (1),

$$E[X] = \frac{a(T_n)\binom{n-3}{m-3}}{\binom{n}{m}} = a(T_n)\frac{m(m-1)(m-2)}{n(n-1)(n-2)} \geq \frac{3}{4}\frac{n-3}{n-2}\binom{m}{3}.$$

■

Proof of Theorem 1.2: Let $n > N+41$ where N is the constant from Lemma 2.1. Let T_n be a fixed n-vertex tournament. We may assume that $n \equiv 1 \pmod{42}$, since otherwise we may delete at most 41 vertices, apply the theorem on the smaller graph, and this will not affect the claimed asymptotic number of transitive triples in the original graph. By Proposition 2.3, the expected number of transitive triples in a random T_7 of T_n is at least $26.25(n-3)/(n-2) = 26.25(1-o_n(1))$. (Here $o_n(1)$ denotes a function tending to zero as n tends to infinity.) Hence, the expected number of directed triangles is at most $8.75(1+o_n(1))$.

Let p_1 denote the probability that a random T_7 has $t(T_7) \leq 4$. Let p_2 denote the probability that a random T_7 has $5 \leq t(T_7) \leq 11$. Let p_3 denote the probability that a random T_7 has $t(T_7) \geq 12$. Clearly, $p_1+p_2+p_3 = 1$ and

$$5p_2 + 12p_3 \leq 8.75(1+o_n(1)).$$

Let Y denote the random variable corresponding to $P(T_7)$. By definition of p_1, p_2, p_3 and by Lemma 2.2 we have

$$E[Y] \geq 7p_1 + 6p_2 + 5p_3.$$

Minimizing $E[Y]$ subject to $p_1+p_2+p_3 = 1$, $p_i \geq 0$ and $5p_2+12p_3 \leq 8.75(1+o_n(1))$ yields $p_1 = 0$, $p_2 = 13/28(1-o_n(1))$, $p_3 = 15/28(1+o_n(1))$ and $E[Y] \geq \frac{153}{28}(1-o_n(1))$.

Let S be a fixed K_7-decomposition of K_n into $\binom{n}{2}/21$ edge-disjoint copies of K_7. By Lemma 2.1 such an S exists. Each $s \in S$ corresponds to a $7-set$ of $[n]$. Let σ be a random permutation of $[n]$ and let S_σ denote the T_7-decomposition of T_n corresponding to S and σ. Namely, for each $s \in S$ the corresponding T_7-subgraph of T_n, denoted s_σ, consists of the 7 vertices $\{\sigma(i) \ : \ i \in s\}$. Notice that since σ is a random permutation, s_σ is a random T_7 of T_n. Thus, the expected number of edge-disjoint transitive triples of s_σ is at least $\frac{153}{28}(1-o_n(1))$. By linearity of expectation we get that

$$P(T_n) \geq \frac{\binom{n}{2}}{21}\frac{153}{28}(1-o_n(1)) = \frac{51}{392}n^2(1+o_n(1)) > 0.13n^2(1+o_n(1)).$$

■

3 Edge-disjoint transitive triples in a random tournament

A random tournament with n vertices is obtained by selecting the orientation of each edge by flipping an unbiased coin, where all $\binom{n}{2}$ choices are independent. As mentioned in the introduction, the following simple application of a result of Frankl and Rödl [3] shows that if T_n is a random tournament on n vertices then $P_k(T_n) \ge \frac{1}{k(k-1)} n^2(1-o(1))$ almost surely.

Proposition 3.1 *Let T_n be a random tournament on n vertices. Then,*

$$\text{Prob}\left[P_k(T_n) \ge \frac{1}{k(k-1)} n^2(1-o_n(1))\right] \ge 1-o_n(1).$$

Proof Let (x,y) be any edge of T_n. Clearly, each K_k containing (x,y) induces a TT_k with probability $k!/2^{\binom{k}{2}}$. Hence, letting $n(x,y)$ denote the number of transitive k-vertex tournaments containing (x,y), we have $E[n(x,y)] = \binom{n-2}{k-2} k!/2^{\binom{k}{2}}$. As any two k-vertex tournaments containing (x,y) share at most $k-3$ vertices (other than x and y) there is limited dependence between the tournaments containing (x,y) (in fact, for $k=3$ there is complete independence). Hence, standard large deviation arguments for limited dependence (see, e.g., [1]) yield that for every $0.5 > \epsilon > 0$,

$$\text{Prob}\left[\left|n(x,y) - \binom{n-2}{k-2}\frac{k!}{2^{\binom{k}{2}}}\right| > n^{k-2-\epsilon}\right] = o(n^{-2}).$$

Thus, with probability $1-o_n(1)$, all edges of T_n lie on at least $(k(k-1)/2^{\binom{k}{2}})n^{k-2}(1-o_n(1))$ copies of TT_k and at most $(k(k-1)/2^{\binom{k}{2}})n^{k-2}(1+o_n(1))$ copies of TT_k.

Consider the $\binom{k}{2}$-uniform hypergraph H whose $N = \binom{n}{2}$ vertices are the edges of T_n and whose edges are the (edge sets of) TT_k copies of T_n. The degree of all the vertices in this hypergraph is $(k(k-1)/2^{\binom{k}{2}})n^{k-2}(1 \pm o_n(1)) = 2^{k/2-1-k(k-1)/2}k(k-1)N^{k/2-1}(1 \pm o_n(1))$, (i.e. the hypergraph is almost regular). Furthermore, the co-degree of any two vertices in this hypergraph is at most $O(n^{k-3}) = O(N^{k/2-1.5}) = o(N^{k/2-1})$. By the result of Frankl and Rödl [3], this hypergraph has a matching that covers all but at most $N(1-o_N(1))$ vertices. Such a matching corresponds to a set of $\frac{1}{k(k-1)}n^2(1-o_n(1))$ edge-disjoint copies of TT_k in T_n. ■

4 Concluding remarks

- Whenever $P(T_n) = n(n-1)/6$ we say that T_n has a *transitive* Steiner triple system. Clearly, this may occur only if K_n has a Steiner triple system, namely, when $n \equiv 1,3 \pmod 6$. It would be interesting to characterize the tournaments that have a transitive Steiner triple system.

- Conjecture 1.1, if true, would be best possible. We show $f(n) \le \lceil n(n-1)/6 - n/3 \rceil$. Let $T_3(n)$ be the complete 3-partite Turán graph with n vertices. It is well-known that $T_3(n)$ has

$\binom{n}{2} - \lceil n(n-1)/6 - n/3 \rceil$ edges. Denote the partite classes by V_1, V_2, V_3. Orient all edges between V_1 and V_2 from V_1 to V_2. Orient all edges between V_2 and V_3 from V_2 to V_3. Orient all edges between V_1 and V_3 from V_3 to V_1. Complete this oriented graph to a tournament T_n by adding directed edges between any two vertices in the same partite class in any arbitrary way. Notice that each transitive triple in T_n contains at least one edge with both endpoints in the same vertex class. Hence, $P(T_n) \le \lceil n(n-1)/6 - n/3 \rceil$.

- Conjecture 1.1 has been verified for $n \le 8$. The values $f(1) = f(2) = f(3) = 0$ and $f(4) = 1$ are trivial. The values $f(5) = 2$, $f(6) = 3$ are easy exercises. The value $f(7) \ge 5$ is a consequence of Lemma 2.2, and thus $f(7) = 5$ by the above Turán graph argument. The value $f(8) \ge 7$ (and hence $f(8) = 7$) is computer verified.

- Conjecture 1.1 claims, in particular, that one can cover almost all edges of T_n with edge-disjoint transitive triples. Proposition 3.1 asserts that this is true for the random tournament and that, in fact, the random tournament can be covered almost completely with edge-disjoint copies of TT_k for every fixed k. However, for $k \ge 4$ there are constructions showing that a significant amount of edges must be uncovered by any set of edge-disjoint TT_k. Consider TT_4. It is well-known (cf. [7]) that there is a unique T_7 with no TT_4. Consider the complete 7-partite directed graph with n vertices obtained by blowing up each vertex of this unique T_7 with $n/7$ vertices. Add arbitrary directed edges connecting two vertices in the same vertex class to obtain a T_n. Clearly, any TT_4 of this T_n must contain an edge with both endpoints in the same vertex class. Hence, $f_4(n) \le P_4(T_n) \le 7\binom{n/7}{2} = O(\frac{1}{14}n^2)$. Hence at least $\binom{n}{2} - 6f_4(T_n) \ge \frac{1}{14}n^2(1+o(1))$ must be uncovered. Similar constructions exist for all $k \ge 4$, where the fraction of covered edges tends to zero as k increases.

- It is possible to slightly improve the constant appearing in Theorem 1.2. Recall that the proof of Theorem 1.2 assumed a worst case of $p_1 \ge 0$, where p_1 is the probability that a random T_7 has at most four directed triangles. However, it is very easy to prove that for n sufficiently large, $p_1 > c > 0$ where c is some (small) absolute constant. This follows from the fact that every T_{54} contains a TT_7 [8]. Thus there exists a positive constant c' such that for n sufficiently large, T_n has at least $c'n^7$ copies of TT_7. Hence, a random induced 7-vertex subgraph of T_n is a TT_7 with constant positive probability. This improvement for p_1 immediately implies a (very small) improvement for the constant appearing in Theorem 1.2.

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