Packing transitive triples in a tournament

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Abstract

We prove that a tournament with n vertices has more than $\frac{41}{300}n^2(1-o(1))$ arc-disjoint transitive triples, and more than $\frac{113}{3000}n^2(1-o(1))$ arc-disjoint transitive quadruples, improving earlier bounds. In particular, 82 percent of the arcs of a tournament can be packed with transitive triples and 45 percent of the arcs of a tournament can be packed with transitive quadruples. Our proof is obtained by examining the fractional version of the problem and utilizing a connection between the integral and fractional versions.

Keywords: Tournament, Packing, Fractional
AMS Subject classification: 05C20, 05C70

1 Introduction

Throughout this paper we follow the graph-theoretic notations of [2]. Given an undirected graph, an *orientation* is obtained by assigning a direction to each edge. Thus, orientations are simple directed graphs with no 2-cycles. A *tournament* is an orientation of the complete graph. Namely, for every two distinct vertices x and y, either (x,y) or (y,x) is an arc, but not both. Tournaments play an important role in combinatorics and in social choice theory.

There are many non-isomorphic tournaments with n vertices, but only one *transitive* tournament. It is obtained by labeling the vertices $\{1,\dots,n\}$, where (i,j) is an arc if and only if $i<j$. We denote the transitive tournament with n vertices by TT_n. The tournament TT_3 is also called a *transitive triple* as it consists of some triple $\{(x,y),(x,z),(y,z)\}$. A *TT_k-packing* of a directed graph D is a set of arc-disjoint subgraphs of D, each of them isomorphic to TT_k. The *TT_k-packing number* of D, denoted $\nu_k(D)$, is the maximum size of a TT_k-packing of D. The TT_3-packing number of D_n, the complete directed graph with n vertices and $n(n-1)$ arcs, has been studied, e.g., in the papers of Gardner [7], Phelps and Lindner [10] and Skillicorn [12].

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Given an n-vertex tournament T, how small can $\nu_k(T)$ be? We denote this parameter by $\nu_k(n)$. Namely, $\nu_k(n) = \min_T \nu_k(T)$ where T ranges over all n-vertex tournaments. Clearly, the arc-disjointness requirement implies that $\nu_k(n) \le \binom{n}{2}/\binom{k}{2}$, and, in fact, for $k > 3$ there are examples showing that this bound cannot be reached even asymptotically. However, for $k = 3$, it has been conjectured by Yuster in [14] that $\nu_3(n) = \frac{1}{6}n^2(1 - o_n(1))$. That is, all but a negligible fraction of the arcs can be packed with transitive triples. In fact, the conjecture is sharper:

Conjecture 1.1 $\nu_3(n) = \lceil n(n-1)/6 - n/3 \rceil$.

He verified this conjecture for all $n \le 8$ using a computer. There are (many) examples of tournaments T with n vertices for which $\nu_k(T) = \lceil n(n-1)/6 - n/3 \rceil$. Currently, the best lower bound for $\nu_3(n)$ is $\frac{51}{392}n^2(1 - o(1))$, given in [14].

The main result of this paper improves this bound. We show that $\nu_3(n) \ge 0.1366n^2(1 + o(1))$ implying that, approximately, 82 percent of the arcs of a tournament can be packed with transitive triples. More precisely, we prove:

Theorem 1.2 $\nu_3(n) \ge \frac{41}{300}n^2(1 - o(1))$.

The proof technique is completely different from that of [14]. The original proof in [14] uses probabilistic arguments combined with some decomposition results for undirected graphs. Our proof, however, uses a fractional relaxation of the problem. We obtain a lower bound for the fractional version of the problem and then utilize a result of Nutov and Yuster [9], based on a technique of Haxell and Rödl [6], enabling us to deduce the same bound for our (integral) version, with only a minor loss in the error term. We also use an idea similar to the one used by Keevash and Sudakov in [8] to improve our constants even further. The big advantage in using the fractional version is the ability to investigate small (but not too small) cases, using a computer, and then *glue* the solutions of the small cases into the large n-vertex tournament. Still, in order to obtain meaningful results we need to examine all tournaments with, say, 10 vertices, and for each of them solve a linear program.

In Section 2, we define the fractional relaxation of the problem, and show how the solution to the integral version can be deduced from the solution to the fractional version. We then reduce the general fractional problem to a specific fractional problem on a constant number of vertices. For any r, solving the fractional problem for tournaments with r vertices (via linear programming) implies a solution for arbitrary large tournaments. The larger r is, the better the final general solution will be. Our computational resources enable us to solve the fractional problem for $r = 10$. This is already far from trivial as there are 2^{45} labeled tournaments with 10 vertices. Thus, in order to solve it in our lifetime, we must be able to discard isomorphic tournaments, while still computing the solution of the linear program for at least one tournament in each isomorphism class. We show how to achieve this goal in Section 3.

Section 4 contains a further application of our method. We begin by showing that our method yields a nontrivial lower bound for $\nu_4(n)$. It is not difficult to construct examples showing that $\nu_4(n) \leq \frac{1}{14}n^2$. Thus, one cannot expect, in general, to pack more than 86 percent of the arcs of a tournament with copies of TT_4. We prove that one can always pack more than 45 percent of the arcs with copies of TT_4. Next, we consider orientations of not necessarily complete graphs. We prove that for any $\delta > 0$, there are orientations of graphs with minimum degree at least $n(1-\delta)$ that cannot be almost fully packed with transitive triples. Thus, one cannot replace the tournament in (the asymptotic version of) Conjecture 1.1 with an arbitrary dense orientation.

2 Packing transitive triples

We begin this section by defining the fractional relaxation of the TT_k-packing problem, and define the parameter $\nu_k^*(n)$ that is the fractional analogue of $\nu_k(n)$. We then utilize a result of Nutov and Yuster to obtain that $\nu_k^*(n) \leq \nu_k(n) + o(n^2)$. This, in effect, enables us to consider only the fractional parameter. We next show how optimal solutions of the fractional TT_k-packing problem of small tournaments can be glued together to obtain a (not necessarily optimal) solution of the fractional TT_k-packing problem of a large tournament, thereby achieving a lower bound for $\nu_k(n)$ based on the exact solution of $\nu_k^*(r)$ for some *fixed* r. Next, we show how to exploit the existence of TT_3-factors (that exist in every tournament whose number of vertices is a multiple of 3) in order to achieve even better bounds for $\nu_3(n)$ based on the exact solution of $\nu_3(r)$ for some *fixed* r. Finally we state a conjecture analogous to that of Conjecture 1.1 for the fractional version.

2.1 Fractional relaxation

Let R_+ denote the set of nonnegative reals. A *fractional TT_k-packing* of a directed graph D is a function ψ from the set $\mathcal{F}_k$ of copies of TT_k in D to R_+, satisfying $\sum_{e \in X \in \mathcal{F}_k} \psi(X) \leq 1$ for each arc $e \in E(D)$. Letting $|\psi| = \sum_{X \in \mathcal{F}_k} \psi(X)$, the *fractional TT_k packing number*, denoted $\nu_k^*(D)$, is defined to be the maximum of $|\psi|$ taken over all fractional TT_k packings ψ. Since a TT_k-packing is also a fractional-TT_k packing (by letting $\psi = 1$ for elements of $\mathcal{F}_k$ in the packing and $\psi = 0$ for the other elements), we always have $\nu_k^*(D) \geq \nu_k(D)$. However, the two parameters may differ. Consider, for example, $D = TT_4$. Trivially, $\nu_3(TT_4) = 1$ as two triangles of K_4 share an edge. However $\nu_3^*(TT_4) = 2$ as can be seen by assigning $\psi = 1/2$ to each of the four transitive triples of TT_4, and noticing that each edge of K_4 belongs to two triangles.

It is well known that computing $\nu_k(D)$ (and hence finding a maximum TT_k-packing) is an NP-Hard problem. Even the very special case of deciding whether a directed graph has a decomposition into transitive triples is known to be NP-Complete (see, e.g. [4] for a more general theorem on the NP-Completeness of such decomposition problems). On the other hand, computing $\nu_k^*(D)$ (for k fixed) amounts to solving a linear programming problem of polynomial size, and hence can be done in polynomial time. Thus, it is interesting to find out when $\nu_k(D)$ and $\nu^*_k(D)$ are "close" as, in particular, for such instances this immediately yields an efficient approximation algorithm for this NP-Hard problem. Indeed, a result of Nutov and Yuster [9] asserts that, for orientations, these two parameters differ by at most $o(n^2)$, thus giving an approximation algorithm with an $o(n^2)$ additive error term. In fact, their result is much more general. Let $\mathcal{S}$ be any given (finite or infinite) family of orientations. For an orientation D, let $\nu_{\mathcal{S}}(D)$ denote the maximum number of arc-disjoint copies of elements of $\mathcal{S}$ that can be found in D, and let $\nu^*_{\mathcal{S}}(D)$ denote the respective fractional relaxation. The following is proved in [9]

Theorem 2.1 *For any given family S of orientations, if D is an n-vertex orientation then $\nu^*_{\mathcal{S}}(D) - \nu_{\mathcal{S}}(D) = o(n^2)$. Furthermore, a set of at least $\nu_{\mathcal{S}}(D) - o(n^2)$ arc-disjoint elements of $\mathcal{S}$ can be generated in randomized polynomial time.*

We note that an *undirected* version of Theorem 2.1 has been recently proved by Yuster [15] extending an earlier result of Haxell and Rödl [6] dealing with single element families. The proof of Theorem 2.1 makes use of the *directed* version of Szemerédi's regularity lemma [13] that has been used implicitly in [3] and proved in [1].

By considering the single-element family $\mathcal{S} = \{TT_k\}$ we obtain the following immediate corollaries of Theorem 2.1.

Corollary 2.2 *Let $k \geq 3$ be a fixed integer. If D is an n-vertex orientation then $\nu^*_k(D) - \nu_k(D) = o(n^2)$. Furthermore, a set of at least $\nu_k(D) - o(n^2)$ arc-disjoint copies of TT_k in D can be generated in randomized polynomial time.*

Let $\nu^*_k(n)$ be the minimum possible value of $\nu^*_k(T)$ ranging over all n-vertex tournaments T. Obviously, $\nu^*_k(n) \geq \nu_k(n)$. Together with Corollary 2.2 we have:

Corollary 2.3 *Let $k \geq 3$ be a fixed integer. Then, $\nu^*_k(n) \geq \nu_k(n) \geq \nu^*_k(n) - o(n^2)$.*

2.2 Gluing fractional solutions

The goal of this subsection is to prove the following lemma.

Lemma 2.4 *Let $2 < k < r < n$ be integers. Then,*

$$\nu^*_k(n) \geq \frac{n(n-1)}{r(r-1)} \nu^*_k(r).$$

Proof: Clearly, the complete graph K_n has precisely $\binom{n}{r}$ distinct copies of K_r (not necessarily edge-disjoint). Also, for any edge (u, v) of K_n, we can select any set of $r-2$ vertices, other than u, v and obtain a copy of K_r containing (u, v). Thus, every edge of K_n appears in precisely

$$\binom{n-2}{r-2}$$

copies of K_r.

Let $\mathcal{T}_r$ be the set of all r-vertex tournaments of an n-vertex tournament T. Suppose that for each $X \in \mathcal{T}_r$ we have computed $\nu_k^*(X)$ realized by some fractional TT_k-packing ψ_X. We define a fractional TT_k-packing ψ of T as follows. Let $\mathcal{F}_k$ be the set of all copies of TT_k in T. For $Y \in \mathcal{F}_k$ let $R(Y) \subset \mathcal{T}_r$ be those r-vertex tournaments that contain Y. Notice that

$$|R(Y)| = \binom{n-k}{r-k}$$

and notice that for each $X \in R(Y)$, $\psi_X(Y)$ is well defined. We now define

$$\psi(Y) = \frac{1}{\binom{n-2}{r-2}} \sum_{X \in R(Y)} \psi_X(Y).$$

We claim that ψ is a proper fractional TT_k-packing of T. Indeed, notice first that ψ is non-negative and its domain is only $\mathcal{F}_k$. It remains to show that for each arc (u,v) of T,

$$\sum_{(u,v) \in Y} \psi(Y) \le 1.$$

Indeed, since all the ψ_X are TT_k-packings we have, for all $(u,v) \in X$,

$$\sum_{(u,v) \in Y \subset X} \psi_X(Y) \le 1.$$

Summing the last inequality over all X containing (u,v) we have

$$\sum_{(u,v) \in X} \sum_{(u,v) \in Y \subset X} \psi_X(Y) \le \binom{n-2}{r-2}.$$

Therefore,

$$\sum_{(u,v) \in Y} \psi(Y) = \sum_{(u,v) \in Y} \frac{1}{\binom{n-2}{r-2}} \sum_{X \in R(Y)} \psi_X(Y)$$

$$= \frac{1}{\binom{n-2}{r-2}} \sum_{(u,v) \in X} \sum_{(u,v) \in Y \subset X} \psi_X(Y) \le \frac{\binom{n-2}{r-2}}{\binom{n-2}{r-2}} \le 1.$$

We have shown that ψ is a proper fractional TT_k-packing of T. Furthermore, the value of ψ is

$$|\psi| = \frac{1}{\binom{n-2}{r-2}} \sum_{X \in \mathcal{T}_r} \nu_k^*(X).$$

Since, by definition, $\nu_k^*(X) \ge \nu_k^*(r)$ we have

$$|\psi| \ge \frac{1}{\binom{n-2}{r-2}} \binom{n}{r} \nu_k^*(r) = \frac{n(n-1)}{r(r-1)} \nu_k^*(r).$$

Sine T was an arbitrary n-vertex tournament we have $\nu_k^*(n) \ge \frac{n(n-1)}{r(r-1)} \nu_k^*(r)$, as required. ■

2.3 Recursively exploiting a TT_3-factor

Keevash and Sudakov [8] discovered a simple trick that combines factors with fractional solutions in order to achieve slightly better approximations than the ones obtained using the blowup technique of the previous subsection. They used their method on an edge-coloring problem. We shall use a similar method for our TT_3-packing problem. We first need the following simple lemma.

Lemma 2.5 *Let $n > 1$ and let T be tournament with $3n$ vertices. Then, T contains a factor of transitive triples (namely, n vertex-disjoint transitive triples).*

Proof: Using the greedy algorithm, the problem reduces to showing that a tournament with six vertices has two vertex-disjoint transitive triples. A tournament with 6 vertices has $\binom{6}{3} = 20$ triples, partitioned into 10 pairs of vertex-disjoint triples. Thus, it suffices to show that any tournament T with 6 vertices has more than 10 transitive triples. Let, as usual, $d^+(v)$ and $d^-(v)$ denote the in-degree and out-degree of a vertex, respectively. Clearly, $\binom{d^+(v)}{2} + \binom{d^-(v)}{2} \geq 4$. Thus,

$$\sum_{v \in T} \left[\binom{d^+(v)}{2} + \binom{d^-(v)}{2} \right] \geq 24.$$

Since each transitive triple contains precisely one source and one sink, it follows that T has at least $24/2 = 12$ transitive triples. ■

The following lemma is similar to lemma 2.2 in [8].

Lemma 2.6 *Let $r > 1$. Then, $\nu_3^*(3r) \geq 9\nu_3^*(r) + r$.*

Proof: Consider a tournament T with $3r$ vertices. By Lemma 2.5, let $T_1, \ldots, T_r$ be r vertex-disjoint transitive triples in T. Consider the r-partite tournament with each vertex class j consisting of the vertices of T_j. By definition, for any one of the 3^r distinct tournaments on r vertices with exactly one vertex in each T_j, we can find a fractional TT_3-packing ψ_i with value at least $\nu_3^*(r)$. Note also that every arc of the r-partite tournament is contained in precisely 3^{r-2} such tournaments. Therefore, as in the previous subsection, $3^{-(r-2)} \sum_i \psi_i$ is a valid fractional TT_3-packing of this r-partite tournament. The fractional TT_3-packing number of the r-partite tournament is, therefore, at least $3^{-(r-2)} 3^r \nu_3^*(r) = 9\nu_3^*(r)$. As we have not used the arcs of the T_j, this implies that $\nu_3^*(3r) \geq 9\nu_3^*(r) + r$, as required. ■

Corollary 2.7

$$\frac{\nu_3^*(n)}{n(n-1)} \geq \frac{\nu_3^*(r)}{r^2} + \frac{1}{6r} - o_n(1).$$

Proof: Iterating the result of Lemma 2.6 we obtain that

$$\nu_3^*(3^{p+1}r) \geq 9^{p+1}\nu_3^*(r) + \sum_{i=0}^{p} 9^{p-i}3^i r = 9^{p+1}\nu_3^*(r) + 9^p r\frac{3}{2}(1 - 3^{-p-1})$$

for every $p \geq 0$. This implies that

$$\frac{\nu_3^*(3^{p+1}r)}{3^{p+1}r(3^{p+1}r - 1)} \geq \frac{\nu_3^*(3^{p+1}r)}{9^{p+1}r^2} \geq \frac{\nu_3^*(r)}{r^2} + \frac{1}{9r}\frac{3}{2}(1 - 3^{-p-1}).$$

Taking the limit as p tends to infinity yields the required result. ■

Proof of Theorem 1.2: In the next section we will show that $\nu_3^*(10) = 12$. This fact, together with Corollary 2.7 gives that

$$\frac{\nu_3^*(n)}{n(n-1)} \geq \frac{12}{100} + \frac{1}{60} - o_n(1) = \frac{41}{300} - o_n(1).$$

Thus, $\nu_3^*(n) \geq \frac{41}{300}n^2 - o(n^2)$. Together with Corollary 2.3 we obtain $\nu_3(n) \geq \frac{41}{300}n^2(1 - o(1))$ as well. ■

2.4 A conjecture for fractional TT_3-packings

Since $\nu_3^*(n) \geq \nu_3(n)$, it may be that a sharp inequality holds. We show that if Conjecture 1.1 holds, then it implies $\nu_3^*(n) = \nu_3(n)$ for all n. Indeed, it suffices to show that there exist tournaments T on n vertices with $\nu_3^*(T) \leq \lceil n(n-1)/6 - n/3 \rceil$. In fact, these would be the same tournaments having $\nu_3(T) \leq \lceil n(n-1)/6 - n/3 \rceil$ constructed in [14].

Let $T_3(n)$ be the complete 3-partite Turán graph with n vertices. It is well-known that $T_3(n)$ has $\binom{n}{2} - \lceil n(n-1)/6 - n/3 \rceil$ edges. Denote the partite classes by V_1, V_2, V_3. Orient all edges between V_1 and V_2 from V_1 to V_2. Orient all edges between V_2 and V_3 from V_2 to V_3. Orient all edges between V_1 and V_3 from V_3 to V_1. Complete this orientation to a tournament T by adding arcs between any two vertices in the same partite class in any arbitrary way. Notice that each transitive triple in T contains at least one arc with both endpoints in the same vertex class. Hence, the total weight of any fractional TT_3-packing cannot exceed the number of arcs with both endpoints in the same vertex class, which is precisely $\lceil n(n-1)/6 - n/3 \rceil$.

3 Computer verification

As shown in the previous section, the bound 41/300 in the proof of Theorem 1.2 relies on the fact that $\nu_3^*(10) = 12$. The fact that $\nu_3^*(10) \leq 12$ is a special case of the construction in subsection 2.4. Specifically, Let V_1 consist of four vertices, V_2 and V_3 consist of three vertices each. All the arcs go from V_1 to V_2, from V_2 to V_3 and from V_3 to V_1. The 12 arcs with both endpoints in the same vertex class are oriented arbitrarily. As each transitive triple contain one of these 12 arcs, we have that $\nu_3^*(T) \le 12$ for such tournaments.

Proving that $\nu_3^*(10) \ge 12$ is a nontrivial computational problem. As K_{10} has $\binom{10}{2} = 45$ edges, there are 2^{45} distinct labeled tournaments on 10 vertices. Thus, simply generating each one, and proving $\nu_3^*(T) \ge 12$ for each such labeled tournament T is not computationally feasible. Fortunately, there are less than 2^{45} *isomorphic* tournaments. In fact, the number of non-isomorphic tournaments on 10 vertices is small enough (though still in the millions) so that the problem becomes computationally feasible. We begin this section by describing the procedure that generates 10-vertex tournaments without missing any isomorphism class. Next, we show that, for many generated tournaments T, there is no need to solve the costly linear program that computes $\nu_3^*(T)$. By examining some aspects of the structure of T we can be sure, in many cases, that $\nu_3^*(T) \ge 12$. Such tournaments are filtered out. Finally, we describe the linear program for those generated tournaments which pass our filter.

3.1 Generating all non-isomorphic 10-vertex tournaments

For a tournament T let T^t denote the *transpose* of T. Namely, T^t is obtained from T by reversing the direction of each arc. Since the transpose of a transitive tournament is a transitive tournament we have that $\nu_k^*(T) = \nu_k^*(T^t)$.

Let F_k be the family of k-vertex tournaments having the property that every labeled tournament with k vertices is isomorphic to some element of F_k and no two elements of F_k are isomorphic. The *score* of a tournament is the sorted out-degree sequence. The score of TT_k is $(k-1, k-2, \ldots, 1, 0)$. In fact, this is the only k-vertex tournament with this score. Clearly, F_3 consists of only two elements, a transitive triple whose score is $(2,1,0)$ and a 3-cycle whose score is $(1,1,1)$. It is also not difficult to see that F_4 consists of four elements whose scores are $(3,2,1,0)$, $(3,1,1,1)$, $(2,2,2,0)$ and $(2,2,1,1)$. They are shown in Figure 1. A more extensive case analysis is needed to determine F_5. In fact, it consists of 12 elements. Four of those can simply be obtained from 4-vertex tournaments by adding a new vertex with out-degree 4. The obtained scores are $(4,3,2,1,0)$, $(4,3,1,1,1)$, $(4,2,2,2,0)$ and $(4,2,2,1,1)$. Seven other tournaments have maximum out-degree 3. One of them has score $(3,3,3,1,0)$, one of them has score $(3,3,2,2,0)$, two of them have score $(3,3,2,1,1)$, and three of them have score $(3,2,2,2,1)$. Finally, one (regular) tournament has score $(2,2,2,2,2)$. Figure 2 exhibits the 12 elements of F_5.

Let S be the set of 10-vertex orientations obtained as follows. For every pair of tournaments (X, Y) where $X \in F_4$ and $Y \in F_5$, create an orientation by adding a new vertex t, and adding the arcs (t, y) for each $y \in Y$ and the arcs (x, t) for each $x \in X$. Notice that S consists of precisely $|F_4| \cdot |F_5| = 4 \times 12 = 48$ orientations. Each one of these orientations has one articulation point, t. Furthermore, no two elements of S are isomorphic. Also, in each element of S, no vertex of the

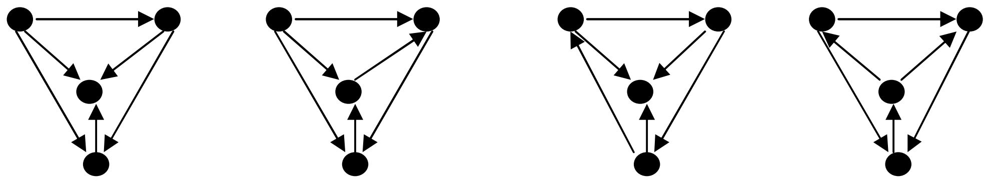

Figure 1: The tournaments on four vertices with their scores

out-neighborhood of t is connected to no vertex of the in-neighborhood of t. Thus, each element of t has 20 missing arcs, and hence 2^{20} ways to transform it into a tournament (some of them will be isomorphic, however). Thus, let S^* be the set of $48 \cdot 2^{20} = 50331648$ tournaments obtained by adding the arcs to each element of S, in every possible way. We therefore have:

Observation 3.1 *S^* contains all the elements of F_{10} that have a vertex with out-degree* 5.

Notice that it is very easy to generate all the elements of S^*, since F_4 and F_5 are explicitly known.

It is a very tedious work to establish the elements of F_6. Instead, let F_6' be the set of 6-vertex tournaments obtained by taking each element of F_5, adding a new vertex, and adding the 5 arcs from the new vertex to the other vertices in any possible way. Clearly, F_6' is easily constructed, and has $2^5 \cdot |F_5| = 384$ elements. Furthermore, $F_6' \supset F_6$.

Let R be the set of 10-vertex orientations obtained as follows. For every pair of tournaments (X, Y) where $X \in F_3$ and $Y \in F_6'$, create an orientation by adding a new vertex t, and adding the arcs (t, y) for each $y \in Y$ and the arcs (x, t) for each $x \in X$. Notice that R consists of precisely $|F_3| \cdot |F_6'| = 2 \times 384 = 768$ orientations. Each one of these orientations has one articulation point, t. In each element of R, no vertex of the out-neighborhood of t is connected to no vertex of the in-neighborhood of t. Thus, each element of t has 18 missing arcs, and hence 2^{18} ways to transform it into a tournament (some of them will be isomorphic, however). Thus, let R^* be the set of $768 \cdot 2^{18} = 201326592$ tournaments obtained by adding the arcs to each element of R in every possible way. We therefore have:

Observation 3.2 *R^* contains all the elements of F_{10} that have a vertex with out-degree* 6.

Notice that it is very easy to generate all the elements of R^*, since F_3 and F_6' are explicitly known.

Let T_0 be the 10-vertex tournament obtained by taking two vertex-disjoint copies of the (unique) 5-vertex tournament with score $(2, 2, 2, 2, 2)$ and orienting the 25 arcs between the copies all in the same direction. Notice that the score of T_0 is $(7, 7, 7, 7, 7, 2, 2, 2, 2, 2)$. In fact:

Observation 3.3 *T_0 is the unique 10-vertex tournament with score* $(7, 7, 7, 7, 7, 2, 2, 2, 2, 2)$.

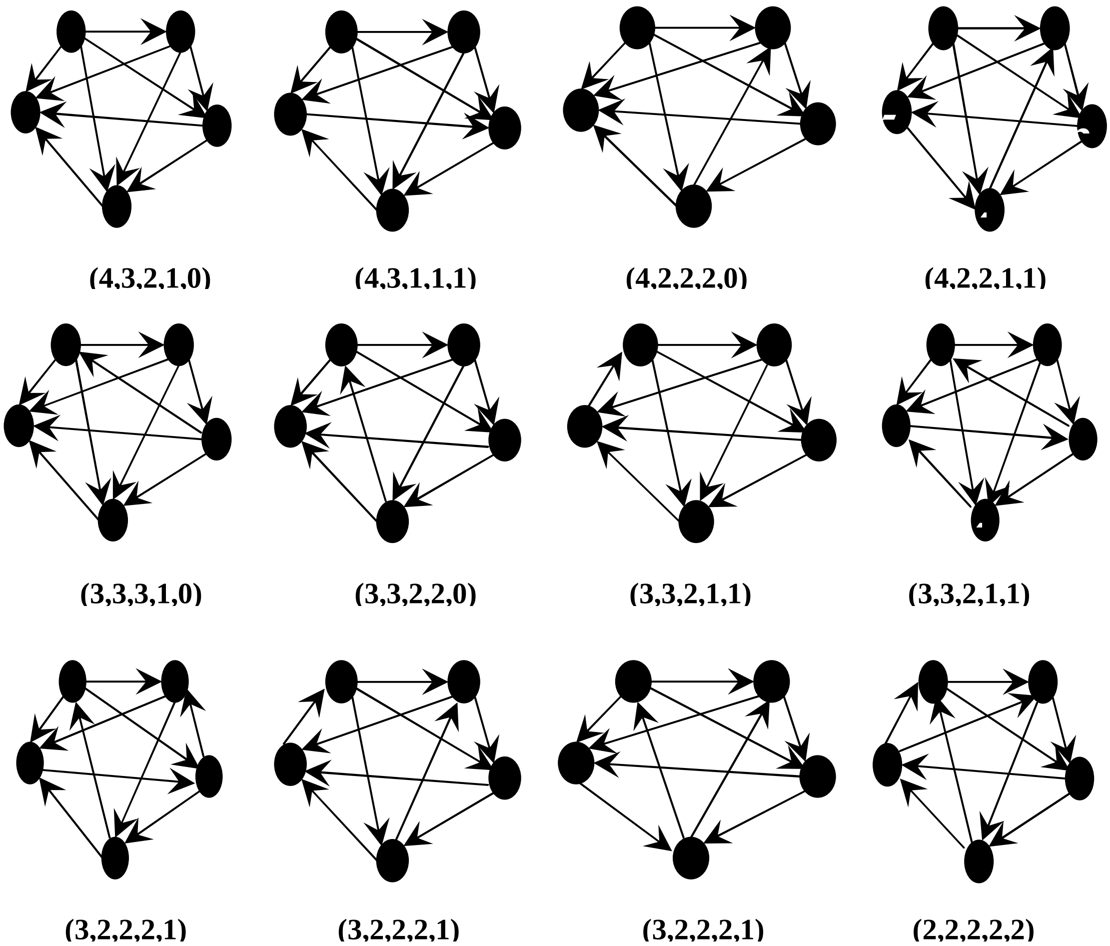

Figure 2: The tournaments on five vertices with their scores

Indeed, consider the sub-tournament A induced on the five vertices with out-degree 2, and the sub-tournament B induced on the five vertices with out-degree 7. Since each of A and B contains 10 internal arcs, we must have that all the arcs go from B to A.

We can now prove the following lemma:

Lemma 3.4 *Let $Q = S^* \cup R^* \cup \{T_0\}$. If $T \in F_{10}$ then either $T \in Q$ or $T^t \in Q$.*

Proof: Let $T \in F_{10}$. If the score of T contains the number 5 then $T \in S^*$. If the score of T contains the number 6 then $T \in R^*$. If the score of T contains the number 4 then $T^t \in S^*$. If the score of T contains the number 3 then $T^t \in R^*$. Thus, we can assume that the score of T contains only the numbers $0, 1, 2, 7, 8, 9$. Let A be the subtournament induced on the vertices with out-degree $0, 1, 2$ and let B be the subtournament induced on the vertices with out-degree $7, 8, 9$. The sum of the out-degrees of the vertices of B is at most $\binom{|B|}{2} + |B|(10 - |B|)$ and at least $7|B|$. Similarly, the sum of the in-degrees of the vertices of A is at most $\binom{10-|B|}{2} + |B|(10-|B|)$ and at least $7(10-|B|)$. It follows that we must have $|A| = |B| = 5$, and that the score of T is $(7,7,7,7,7,2,2,2,2,2)$ which means that $T = T_0$. ■

The main loop of our program simply generates each element of Q. We then verify, for each generated element T, that $\nu_3^*(T) \geq 12$. By Lemma 3.4, this suffices to establish that $\nu_3^*(10) \geq 12$, as required.

3.2 Filtering the non-essential tournaments

Since Q has $768 \cdot 2^{18} + 48 \cdot 2^{20} + 1 = 251658241$ elements, it is quite infeasible to run a linear program for each element of Q. We have used the lp_solve package developed by Michel Berkelaar as our linear programming kit. Our tests show that an average run of a linear program corresponding to an element of Q requires 0.1 seconds on our computing equipment (including the input file creation). This means that it would take roughly 290 days of continuous run to complete the process. In this section we show that, in many cases (in fact, most cases), it is easy to determine that $\nu_3^*(T) \geq 12$ without actually running the linear program.

The first and simplest filter is to eliminate obvious isomorphisms inside Q.

Observation 3.5 *If an element of R^* has a vertex with out-degree 4 or out-degree 5, then it is isomorphic to an element of S^*, or its transpose is isomorphic to an element of S^*.*

It turns out that from the 201326592 elements of R^*, less than ten million *do not* have a vertex with out-degree 4 nor a vertex with out-degree 5.

For an arc e, let $\alpha(e)$ denote the number of transitive triples containing e. For a tournament T, let $\alpha(T)$ be the maximum of $\alpha(e)$ ranging over the arcs of T. If T has 10 vertices, then, trivially, $\alpha(T) \leq 8$. Let $\beta(T)$ be the number of (not necessarily arc-disjoint) transitive triples in T. If T has 10 vertices, then, trivially, $\beta(T) \leq 120$. Let $\gamma(T)$ be the number of arcs with $\alpha(e) = 8$. Notice that $\alpha(T)$, $\beta(T)$, and $\gamma(T)$ can be computed by simple counting.

Lemma 3.6 *If $T \in Q$ and $\beta(T) \geq 12\alpha(T)$ then $\nu_3^*(T) \geq 12$.*

Proof: Assign the weight $1/\alpha(T)$ to each transitive triple of T. Thus, for each arc, the sum of the weights of the transitive triples containing it is at most 1. The total weight is $\beta(T)/\alpha(T) \geq 12$. ■

A special case of Lemma 3.6 holds when $\beta(T) \geq 96$ in which case we always have $\beta(T) \geq 12\alpha(T)$. So is the case, e.g., for T_0, that has more than 100 transitive triples.

Lemma 3.7 *If $T \in Q$ and $\beta(T) - \gamma(T) \geq 84$ then $\nu_3^*(T) \geq 12$.*

Proof: For each arc with $\alpha(e) = 8$, arbitrarily pick one transitive triple containing e. Let W be the set of triples picked, and notice that $|W| \le \gamma(T)$ (it may be that two incident arcs picked the same triple). Assign the weight 0 to the triples of W and assign the weight 1/7 to the other transitive triples. For each arc, the sum of the weights of the transitive triples containing it is at most $7/7 = 1$. The total weight is $\frac{1}{7}(\beta(T) - |W|) \ge 84/7 = 12$. ■

It turns out that the number of elements of Q that pass the filters of Observation 3.5, Lemma 3.6 and Lemma 3.7 is less than four million. Indeed, our program completes its run in less than five days.

3.3 Generating the linear program

Let T be an element of Q that does not pass the filters of the previous subsection. We must verify that $\nu_3^*(T) \ge 12$ by explicitly computing an optimal fractional TT_3-packing for T. As mentioned earlier, this can easily be done by solving a corresponding linear program. We now describe this program.

Assume that the vertices of T are labeled $\{0, \ldots, 9\}$. The linear program has $\beta(T)$ variables, each corresponding to a transitive triple of T, and 45 constraints, each corresponding to an arc of T. For a transitive triple induced on the vertices $\{i, j, k\}$ with $i < j < k$, we create the variable x_{ijk}. For any arc (u, v), let $C(u, v)$ be the sum of all the variables corresponding to the transitive triples that contain (u, v) (in case (u, v) does not appear on any transitive triple, we define $C(u, v) = 0$). The set of 45 constraints is, therefore, $\{C(u, v) \le 1 \ : \ (u, v) \in E(T)\}$. Finally, the objective function is to maximize the sum of all variables. Clearly, the value of an optimal solution to this linear program corresponds to $\nu_3^*(T)$.

As mentioned earlier, we have used the linear programming package lp_solve available from the site **http://www.cs.sunysb.edu/~algorith/implement/lpsolve/implement.shtml**). The source code of our program is available to the readers upon request. The following is an example of an input file expected by lp_solve, corresponding to some tournament with 10 vertices, and the corresponding (truncated) output file generated by lp_solve.

```
max:
x012+x013+x014+x015+x019+x023+x024+x025+x028+x029+x034+x035+x036+
x037+x045+x049+x058+x067+x068+x069+x078+x079+x089+x123+x124+x125+
x126+x127+x128+x129+x134+x135+x136+x137+x138+x145+x146+x147+x148+
x149+x156+x157+x158+x167+x168+x178+x234+x235+x236+x237+x245+x246+
x247+x249+x256+x257+x258+x267+x289+x348+x349+x356+x357+x359+x367+
x368+x369+x378+x379+x389+x456+x457+x458+x467+x468+x478+x567+x569+
x579+x589+x678+x679+x689+x789;

c1:
x012+x013+x014+x015+x019<=1;
```

```
x012+x023+x024+x025+x028+x029<=1;
x013+x023+x034+x035+x036+x037<=1;
x014+x024+x034+x045+x049<=1;
x015+x025+x035+x045+x058<=1;
x036+x067+x068+x069<=1;
x037+x067+x078+x079<=1;
x028+x058+x068+x078+x089<=1;
x019+x029+x049+x069+x079+x089<=1;
x012+x123+x124+x125+x126+x127+x128+x129<=1;
x013+x123+x134+x135+x136+x137+x138<=1;
x014+x124+x134+x145+x146+x147+x148+x149<=1;
x015+x125+x135+x145+x156+x157+x158<=1;
x126+x136+x146+x156+x167+x168<=1;
x127+x137+x147+x157+x167+x178<=1;
x128+x138+x148+x158+x168+x178<=1;
x019+x129+x149<=1;
x023+x123+x234+x235+x236+x237<=1;
x024+x124+x234+x245+x246+x247+x249<=1;
x025+x125+x235+x245+x256+x257+x258<=1;
x126+x236+x246+x256+x267<=1;
x127+x237+x247+x257+x267<=1;
x028+x128+x258+x289<=1;
x029+x129+x249+x289<=1;
x034+x134+x234+x348+x349<=1;
x035+x135+x235+x356+x357+x359<=1;
x036+x136+x236+x356+x367+x368+x369<=1;
x037+x137+x237+x357+x367+x378+x379<=1;
x138+x348+x368+x378+x389<=1;
x349+x359+x369+x379+x389<=1;
x045+x145+x245+x456+x457+x458<=1;
x146+x246+x456+x467+x468<=1;
x147+x247+x457+x467+x478<=1;
x148+x348+x458+x468+x478<=1;
x049+x149+x249+x349<=1;
x156+x256+x356+x456+x567+x569<=1;
x157+x257+x357+x457+x567+x579<=1;
x058+x158+x258+x458+x589<=1;
x359+x569+x579+x589<=1;
x067+x167+x267+x367+x467+x567+x678+x679<=1;
x068+x168+x368+x468+x678+x689<=1;
x069+x369+x569+x679+x689<=1;
x078+x178+x378+x478+x678+x789<=1;
x079+x379+x579+x679+x789<=1;
x089+x289+x389+x589+x689+x789<=1;
```

Example of an input file

```
Value of objective function:               15
x012                      0
x013                      0
x014                0.29545
x015               0.068182
```

```
x019                0.63636
x023                0.20455
              .
              .
              .
x579                0.52273
x589               0.022727
x678                      0
x679                0.38636
x689                      0
x789                      0
```

Example of an output file

4 Related results

In this section we consider problems related to TT_3-packing. We first study TT_4-packings, and show that the methods we used to prove Theorem 1.2 can be applied to this case as well in order to obtain nontrivial lower bounds. We then consider the problem of TT_3-packings in orientations of not necessarily complete graphs. We show that orientations of random graphs behave similar to orientations of a tournament, while, on the other hand, minimum degree requirements are not enough to guarantee such behavior.

4.1 Packing TT_4

Determining $\nu_4(n)$ and $\nu_4^*(n)$ is, obviously, a more difficult problem than (the already very difficult problem of) determining $\nu_3(n)$ and $\nu_3^*(n)$. First, as mentioned in the introduction, we cannot hope, in general, to pack almost all of the edges of a tournament with edge-disjoint copies of TT_4. The following construction appears in [14]. It is well-known (cf. [11]) that there is a unique tournament T_7 with seven vertices, and with *no* TT_4. Consider the complete 7-partite orientation with n vertices obtained by blowing up each vertex of T_7 with $n/7$ vertices. Add arbitrary arcs connecting two vertices in the same vertex class to obtain a tournament T with n vertices. Clearly, any TT_4 of T must contain an arc with both endpoints in the same vertex class. Hence, $\nu_4(n) \le \nu_4(T) \le 7\binom{n/7}{2} = \frac{1}{14}n^2(1+o(1))$. Hence at least $\binom{n}{2} - 6\nu_4(T) \ge \frac{1}{14}n^2(1+o(1))$ arcs must be unpacked. The same example also shows that $\nu_4^*(n) \le \frac{1}{14}n^2(1+o(1))$. Similar constructions exist for all $k \ge 4$, where the fraction of packed edges tends to zero as k increases.

In order to obtain a lower bound for $\nu_4^*(n)$ we shall require a lemma analogous to Lemma 2.6. Unlike the case for transitive triples, tournaments with $4n$ vertices do not necessarily have n vertex-disjoint copies of TT_4. But they do have $n-1$ such copies, since it is well known (see, e.g. [11]) that every tournament with 8 vertices has a TT_4. Thus, the following lemma is a straightforward analogue of Lemma 2.6.

Lemma 4.1 *Let $r > 1$. Then, $\nu_4^*(4r) \geq 16\nu_4^*(r) + r - 1$.*

Similarly, the following corollary is analogous to Corollary 2.7.

Corollary 4.2
$$\frac{\nu_4^*(n)}{n(n-1)} \geq \frac{\nu_4^*(r)}{r^2} + \frac{1}{12r} - \frac{1}{15r^2} - o_n(1).$$

Proof: Iterating the result of Lemma 4.1 we obtain that

$$\nu_4^*(4^{p+1}r) \geq 16^{p+1}\nu_4^*(r) + \sum_{i=0}^{p} 16^{p-i}(4^i r - 1)$$

$$= 16^{p+1}\nu_4^*(r) + 16^p r\frac{4}{3}(1 - 4^{-p-1}) - 16^p\frac{16}{15}(1 - 16^{-p-1})$$

for every $p \geq 0$. This implies that

$$\frac{\nu_4^*(4^{p+1}r)}{4^{p+1}r(4^{p+1}r-1)} \geq \frac{\nu_4^*(4^{p+1}r)}{16^{p+1}r^2} \geq \frac{\nu_4^*(r)}{r^2} + \frac{1}{16r}\frac{4}{3}(1 - 4^{-p-1}) - \frac{1}{16r^2}\frac{16}{15}(1 - 16^{-p-1}).$$

Taking the limit as p tends to infinity yields the required result. ■

Proposition 4.3 $\nu_4(n) \geq \frac{113}{3000}n^2(1 - o(1))$.

Modifying our computer program we were able to compute $\nu_4^*(10) = 3$. This fact, together with Corollary 4.2 gives that

$$\frac{\nu_4^*(n)}{n(n-1)} \geq \frac{3}{100} + \frac{1}{120} - \frac{1}{1500} - o_n(1) = \frac{113}{3000} - o_n(1).$$

Thus, $\nu_4^*(n) \geq \frac{113}{3000}n^2 - o(n^2)$. Together with Corollary 2.3 we obtain $\nu_4(n) \geq \frac{113}{3000}n^2(1 - o(1))$ as well. Notice that this means that one can always pack approximately 45 percent of the arcs of a tournament with arc-disjoint copies of TT_4, while the example in the beginning of this subsection shows that, in general, we cannot expect to pack more than $100 \cdot (12/14) \approx 86$ percent of the arcs.
■

4.2 Packing transitive triples in orientations of non-complete graphs

Proposition 4.4 *For every $\delta > 0$, there exist orientations of graphs with minimum degree at least $n(1-\delta)$ so that in every packing with transitive triples, at least $\delta n^2(1 - o(1))$ arcs are unpacked.*

Proof: Since we are concerned with asymptotics, we may assume that n is a multiple of 6 and that δn is an even integer. Let H be an undirected graph with $n/3$ vertices which is $n/3 - \delta n$ regular. Consider the undirected graph G obtained by taking three vertex-disjoint copies of H, denoted H_1, H_2 and H_3, and connecting with an edge any two vertices belonging to different copies. Notice that G has n vertices and is $n(1-\delta)$-regular. Orient the edges from H_1 to H_2, from H_2 to H_3 and from H_3 to H_1. Orient the edges inside the H_i arbitrarily. Denote the oriented graph by $\vec{G}$. Since each transitive triple of $\vec{G}$ contains an arc with both endpoints in the same vertex class we have

$$\nu_3(\vec{G}) \le 3\frac{n}{6}(\frac{n}{3} - \delta n) = \frac{n^2}{6} - \delta\frac{n^2}{2}.$$

The total number of edges of the elements of an optimal TT_3-packing is, therefore, at most $\frac{n^2}{2} - \frac{3}{2}\delta n^2$. On the other hand, the number of arcs of T is $\frac{n^2}{2}(1-\delta)$. it follows that in every TT_3-packing of $\vec{G}$, at least δn^2 arcs remain unpacked. ■

We note that it is proved in [17] that every oriented graph with n vertices can be *edge-decomposed* into at most $\lfloor n^2/3 \rfloor$ transitive subtournaments, and this is tight.

The situation in Proposition 4.3 should be compared to the very different situation in the related undirected problem. For an undirected graph G, let $\rho(G)$ be the maximum number of edge-disjoint triangles that can be packed into G. It is shown in [16] that for $\delta < 3^{-12}$, every graph with minimum degree $n(1-\delta)$ can be packed with triangles so that only $o(n^2)$ edges remain unpacked.

Let $0 < p < 1$ be a constant, and let $G(n,p)$ be the random graph with n vertices and edge probability p. Namely, for any two vertices, there is an edge between them with probability p. All $\binom{n}{2}$ choices are performed independently. By the result of Frankl and Rödl [5], it is well known that for any t, if n is sufficiently large, the random graph $G(n,p)$ almost surely has a set of edge-disjoint copies of K_t so that only $o(n^2)$ edges remain unpacked. Now, in any orientation of the random graph, each such K_t becomes a t-vertex tournament, which, in turn, by Theorem 1.2, can be packed with $\frac{41}{300}t^2(1-o_t(1))$ edge-disjoint transitive triples. It follows that for any p, almost surely, any orientation of $G(n,p)$ has a packing with transitive triples so that approximately 82 percent of the edges are packed. In fact, if Conjecture 1.1 is true, then, almost surely, any orientation of $G(n,p)$ has a packing with transitive triples so that only $o(n^2)$ arcs remain unpacked.

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