

Packing and decomposition of graphs with trees

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Abstract

Let H be a tree on $h \geq 2$ vertices. It is shown that if n is sufficiently large and $G = (V, E)$ is an n -vertex graph with $\delta(G) \geq \lfloor n/2 \rfloor$, then there are $\lfloor |E|/(h-1) \rfloor$ edge-disjoint subgraphs of G which are isomorphic to H . In particular, if $h-1$ divides $|E|$ then there is an H -decomposition of G . This result is best possible as there are infinitely many examples of trees on h vertices and graphs G with $m(h-1)$ edges, $\delta(G) \geq \lfloor n/2 \rfloor - 1$, for which G has no H -decomposition.

1 Introduction

All graphs considered here are finite, undirected, and have no loops or multiple edges. For the standard graph-theoretic notations the reader is referred to [2]. Let H be a graph without isolated vertices. An H -packing of a graph G is a set $L = \{G_1, \dots, G_s\}$ of edge-disjoint subgraphs of G , where each subgraph is isomorphic to H . The H -packing number of G , denoted by $P(H, G)$, is the maximum cardinality of an H -packing of G . G has an H -decomposition if it has an H -packing with the property that every edge of G appears in *exactly* one member of the H -packing. Note that in order for G to have an H -decomposition, two necessary conditions must hold. The first is that $e(H)$ divides $e(G)$. The second is that $\gcd(H)$ divides $\gcd(G)$ where the \gcd of a graph is the greatest common-divisor of the degrees of its vertices. Note that for any pair of graphs G and H , we can verify in polynomial time if G satisfies these two conditions. We call these conditions the “ H -decomposition divisibility conditions”. An H -covering of a graph G is a set $L = \{G_1, \dots, G_s\}$ of subgraphs of G , where each subgraph is isomorphic to H , such that every edge of G appears in at least one member of L . The H -covering number of G , denoted by $C(H, G)$, is the minimum cardinality of an H -covering of G . Trivially, $C(H, G) = P(H, G)$ iff G has an H -decomposition.

The combinatorial and computational aspects of the H -packing, H -covering and H -decomposition problems have been studied extensively. Wilson in [10] has proved that if $G = K_n$ where $n \geq n_0 = n_0(H)$, and K_n satisfies the H -decomposition divisibility conditions, then K_n has an H -decomposition. Recently, the H -packing and H -covering problems for $G = K_n$ ($n \geq n(H)$) were solved [4, 5], by giving a closed formula for computing $P(H, K_n)$ and $C(H, K_n)$. In case the graph G is not complete, it is known that the H -covering and H -packing problems are, in general, NP-hard, since Dor and Tarsi [7] showed that deciding if G has an H -decomposition is NP-Complete, if H is, say, any connected graph with at least three edges. Wilson's result can be thought of as a *minimum degree* result, where the minimum degree is the highest possible, i.e. $n - 1$. Following Wilson, Gustavsson has shown in [8] that if G is an n -vertex graph, $\delta(G) \geq (1 - \epsilon(H))n$, where $\epsilon(H)$ is some small positive constant depending on H , and G satisfies the H -decomposition divisibility conditions, then G has an H -decomposition. However, the $\epsilon(H)$ in Gustavsson's result is a very small number. For example, if H is a triangle then $\epsilon(H) \leq 10^{-24}$. In general, $\epsilon(H) \leq 10^{-24}/|H|^q$. It is believed, however, that the correct value for $\epsilon(H)$ is much larger. In fact, Nash-Williams conjectured in [9] that when H is a triangle, then $\epsilon(H) = 1/4$, and he also gives an example showing that this would be best possible. The general problem can therefore be expressed as follows:

Problem 1: Determine $f_H(n)$, the smallest possible integer, such that whenever G has n vertices (where $n \geq n_0(H)$), and $\delta(G) \geq f_H(n)$, and G satisfies the H -decomposition divisibility conditions, then G has an H -decomposition.

It is shown in [11] that $f_H(n) \geq \lfloor n/2 \rfloor - 1$ for every connected graph H with at least 3 vertices (if H is a single edge, the decomposition problem becomes trivial). In that same paper, the author came close to solving Problem 1 in case H is a tree. It is proved there that $f_H(n) \leq n/2 + 10h^4\sqrt{n \log n}$ for every tree H . Although this proves $f_H(n)/n$ tends to 0.5 as n tends to infinity, there is still the $\sqrt{n \log n}$ factor which is the gap between the upper and lower bound.

In this paper we show that for every tree H on $h > 2$ vertices, $f_H(n)$ is either $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor - 1$. Moreover, for infinitely many values of n we show that $f_H(n) = \lfloor n/2 \rfloor$. We summarize our exact result in the following Theorems:

Theorem 1.1 *Let H be any tree with $h > 2$ vertices. Let G be a graph on $n \geq (12h)^{10}$ vertices with $\delta(G) \geq \lfloor n/2 \rfloor$. Then $P(H, G) = \lfloor e(G)/(h-1) \rfloor$, and, in particular, if $h-1$ divides $e(G)$, then G has an H -decomposition.*

Note that Theorem 1.1 guarantees that there exists an optimal H -packing in G , in the sense that there are less than $e(H) = h - 1$ edges which are not packed, which is the best one could hope for.

Theorem 1.2 *For every tree H on $h > 2$ vertices, if $n \geq (12h)^{10}$ then*

$$\lfloor \frac{n}{2} \rfloor \geq f_H(n) \geq \lfloor \frac{n}{2} \rfloor - 1.$$

Furthermore, if n is odd and $h - 1$ does not divide $(n - 1)(n - 3)/8$ then $f_H(n) = \lfloor n/2 \rfloor$, and if n is even and $h - 1$ divides $n(n - 2)/4$ but does not divide $n(n - 2)/8$ then $f_H(n) = n/2$.

Although the result in [11] is an asymptotic version of the result in this paper, the proofs are entirely different. In fact, the main result in [11] (see lemma 3.1 in this paper) shows that every n -vertex graph $G = (V, E)$ which has good edge-expansion (in fact $\Theta(\sqrt{n \log n})$ edge-expansion suffices) has an H -decomposition (assuming $h - 1$ divides $|E|$). There are graphs with such edge-expansion that have maximum degree as small as $O(\sqrt{n \log n})$. It is true that graphs with minimum degree at least $n/2 + \Theta(\sqrt{n \log n})$ *always* have the required edge-expansion, and hence are H -decomposable. However, in order to obtain the exact results proved in this paper, one needs to consider, in particular, graphs with, say, $\delta(G) = n/2$ which are *not* good expanders. This requires an entirely different approach to the proof.

In the following section we prove several density lemmas which are needed for the proof of Theorem 1.1. This proof, which requires some probabilistic arguments, appears in Section 3. Section 4 contains the proof of Theorem 1.2, and some concluding remarks.

A word about notation used in the rest of this paper. The degree of a vertex v in the graph $G = (V, E)$ is denoted by $d_G(v)$. The number of edges of G is denoted by $e(G)$. For $X \subset V$, we denote by $G[X]$ the subgraph induced by X . The number of edges between X and Y is denoted by $e(X, Y)$, and $e(X)$ denotes the number of edges in $G[X]$. The number of neighbors of v in X is denoted by $d(v, X)$.

2 The lemmas

For the rest of this paper, let H be a fixed tree on $h \geq 3$ vertices. A graph $G = (V, E)$ is called *r edge-expanding* if for every $X \subset V$ with $|X| \leq |V|/2$, there are at least $r|X|$ edges between X and $V \setminus X$. That is, $e(X, V \setminus X) \geq r|X|$.

The following simple lemma shows that if G has large minimum degree, and bad edge-expansion, then there is a large set X demonstrating this fact:

Lemma 2.1 *If $G = (V, E)$ has n vertices, $\delta(G) \geq \lfloor n/2 \rfloor$ and G is not $0.01n - 1$ edge-expanding, then there exists $X \subset V$, $0.49n \leq |X| \leq 0.5n$, such that $e(X, V \setminus X) < (0.01n - 1)|X|$.*

Proof: Since G is not $0.01n - 1$ edge-expanding, there exists a set $X \subset V$, $|X| \leq n/2$, such that $e(X, V \setminus X) < (0.01n - 1)|X|$. Since $\delta(G) \geq \lfloor n/2 \rfloor$ we also know that every $X' \subset V$ with $|X'| < 0.49n$, has $e(X', V \setminus X') \geq (\lfloor n/2 \rfloor - (|X'| - 1))|X'| \geq 0.01n|X'|$. Thus, $|X| \geq 0.49n$. \square

A well-known fact (see, e.g. [2] p. xvii) is the following:

Fact 1: *Every graph S with at least d vertices, and with at least $d|S| - d(d+1)/2 + 1$ edges has a subgraph with minimum degree at least $d + 1$.*

Using this fact and Lemma 2.1, we can prove that the graph G of Lemma 2.1 contains two large disjoint subgraphs with high minimum degree:

Lemma 2.2 *If $G = (V, E)$ is as in Lemma 2.1, then there are two disjoint subsets of vertices $X_0, Y_0 \subset V$, such that $\delta(G[X_0]) \geq 0.4n$ and $\delta(G[Y_0]) \geq 0.4n$.*

Proof: According to Lemma 2.1 there exists $X \subset V$ with $0.49n \leq |X| \leq 0.5n$ which is not $0.01n - 1$ edge-expanding. Put $Y = V \setminus X$. Thus, $0.5n \leq |Y| \leq 0.51n$. Clearly, $\sum_{v \in X} d_G(v) \geq \lfloor n/2 \rfloor |X|$. On the other hand, $e(X, Y) < (0.01n - 1)|X|$. It follows that

$$e(X) > \frac{\lfloor n/2 \rfloor |X| - (0.01n - 1)|X|}{2} \geq 0.245n|X|.$$

Put $d = \lceil 0.4n \rceil$. We shall prove that there exists a subset $X_0 \subset X$ such that $\delta(G[X_0]) \geq d$. According to Fact 1, it suffices to show that $e(X) \geq d|X| - d(d+1)/2 + 1$. Indeed,

$$d|X| - d(d+1)/2 + 1 \leq 0.4n|X| - 0.08n^2 \leq 0.4n|X| - 0.16n|X| < 0.245n|X| < e(X). \quad (1)$$

We now similarly show the existence of Y_0 . The sum of the degrees in G of the vertices of Y is at least $\lfloor n/2 \rfloor |Y|$. Since $e(Y, X) < (0.01n - 1)|X| \leq (0.01n - 1)|Y|$, it follows that

$$e(Y) > \frac{\lfloor n/2 \rfloor |Y| - (0.01n - 1)|Y|}{2} \geq 0.245n|Y|.$$

Using similar arguments as in the case for X_0 (only that now we must use the fact that $|Y| \leq 0.51n$ when we replace $|X|$ with $|Y|$ in (1), but the inequality still holds), we obtain that Y contains a subset Y_0 such that $\delta(G[Y_0]) \geq \lceil 0.4n \rceil$. \square

Our next goal is to show that the sets X_0 and Y_0 of the previous lemma can be extended to a partition of V , with sets $A \supset X_0$ and $B \supset Y_0$, such that both $G[A]$ and $G[B]$ have good edge-expansion, and high minimum degree.

Lemma 2.3 *If $G = (V, E)$ is as in Lemma 2.1, then there is a partition of V into subsets A and B with sizes at least $0.4n$ each, such that both $G[A]$ and $G[B]$ are $0.1n$ edge-expanding, and both have minimum degree at least $0.15n$.*

Proof: Let X_0 and Y_0 be the subsets guaranteed by Lemma 2.2. We define a partition of v into the subsets A and B as follows. For each $v \in V$, if $v \in X_0$ we put v in A . If $v \in Y_0$ we put v in B . If $v \in V \setminus (A \cup B)$ we put v in A if $d(v, X_0) > d(v, Y_0)$. Otherwise we put v in B . Clearly, $|A| \geq |X_0| > 0.4n$ and $|B| \geq |Y_0| > 0.4n$. We still need to show that $G[A]$ and $G[B]$ are at least $0.1n$ edge-expanding, and have minimum degree $0.15n$. Since the proofs in both cases are identical, we prove it only for $G[A]$. Consider $A' \subset A$ with $|A'| \leq |A|/2$. We must show that $e(A', A \setminus A') \geq 0.1n|A'|$. A' can be partitioned into two parts, A_1 and A_2 , where $A_1 = A' \cap X_0$ and $A_2 = A' \setminus X_0$. Since $d_{G[X_0]}(v) \geq 0.4n$ for every $v \in X_0$, and since $A_1 \subset X_0$, we have that

$$d(v, A \setminus A') \geq 0.4n - |A_1| \text{ for every } v \in A_1. \quad (2)$$

Also note that since

$$|X_0| + |Y_0| \geq 2(0.4n + 1) = 0.8n + 2$$

we have that for $v \in A_2$, $d(v, X_0 \cup Y_0) \geq \lfloor n/2 \rfloor - 0.2n + 2 \geq 0.3n$ and since v was chosen to A we have $d(v, X_0) \geq d(v, Y_0)$ so $d(v, X_0) \geq 0.15n$. This implies that

$$d(v, A \setminus A') \geq 0.15n - |A_1| \text{ for every } v \in A_2. \quad (3)$$

We may also use the obvious fact that $|A'| \leq 0.3n$ since $|A'| \leq |A|/2 = (n - |B|)/2 \leq (n - 0.4n)/2 = 0.3n$. Consider first the case $0.3n \geq |A_1| \geq 0.1n$. Using (2) we obtain

$$e(A', A \setminus A') \geq |A_1|(0.4n - |A_1|) \geq 0.03n^2 \geq 0.1n|A'|.$$

Now consider the case $|A_1| \leq 0.1n$. Using both (2) and (3) we obtain

$$e(A', A \setminus A') \geq |A_1|(0.4n - |A_1|) + (0.15n - |A_1|)(|A'| - |A_1|).$$

We therefore wish to show that

$$|A_1|(0.4n - |A_1|) + (0.15n - |A_1|)(|A'| - |A_1|) \geq 0.1n|A'|.$$

By rearranging the terms in the last inequality we get the identical inequality $|A'|(|A_1| - 0.05n) \leq 0.25n|A_1|$, which trivially holds for all $|A_1| \leq 0.1n$, since $|A'| \leq 0.3n$.

In fact, note that we have shown also that every $v \in X_0$ has $d(v, A) \geq d(v, X_0) \geq 0.4n$, and every $v \in A \setminus X_0$ has $d(v, A) \geq d(v, X_0) \geq 0.15n$. This shows $\delta(G[A]) \geq 0.15n$. \square

Our next step is to show that if n is sufficiently large, then the edge-set of A from Lemma 2.3 can be partitioned into two parts A^α and A^β , such that the spanning subgraph of $G[A]$ consisting of the edges of A^α has large-enough minimum degree, while the spanning subgraph of $G[A]$ consisting of the edges of A^β , still has good edge-expansion. Analogously, we will show the existence of B^α and B^β .

Lemma 2.4 *If $G = (V, E)$, is as in Lemma 2.1, with $n \geq 10^6$ vertices, and A and B are as guaranteed by Lemma 2.3, then the edge sets of $G[A]$ and $G[B]$ can each be partitioned into two sets A^α, A^β and B^α, B^β , which define two spanning subgraphs of $G[A]$ and $G[B]$, respectively (to simplify notation, we shall also denote these subgraphs by $A^\alpha, A^\beta, B^\alpha, B^\beta$) such that $\delta(A^\alpha) \geq 0.01n$ and $\delta(B^\alpha) \geq 0.01n$, and A^β and B^β are $0.01n$ edge-expanding.*

Proof: Since the proofs for A and B are identical, we prove only for A . We shall prove the existence of the desired A^α and A^β using a probabilistic argument. Each edge of $G[A]$ chooses to be in A^α with probability $1/12$, otherwise, it is in A^β (with probability $1 - 1/12 = 11/12$). All the choices are independent. Consider a vertex $v \in A$. The expected degree of v in A^α , is exactly

$$\mu_v = E[d_{A^\alpha}(v)] = \frac{d_{G[A]}(v)}{12}.$$

Note that $d_{A^\alpha}(v)$ is a random variable with binomial distribution, since it is the sum of $d_{G[A]}(v)$ independent indicator random variables with probability $1/12$. It therefore follows from the large deviation result of Chernoff (cf. e.g. [1]) that:

$$\text{Prob}[|d_{A^\alpha}(v) - \mu_v| \geq \frac{\mu_v}{5}] \leq 2e^{\frac{-2\mu_v^2}{25d_{G[A]}(v)}} = 2e^{\frac{-d_{G[A]}(v)}{1800}} \leq 2e^{\frac{-0.15n}{1800}} < 1/n$$

where the last inequality holds since $n \geq 10^6$. This shows that with positive probability, for every $v \in A$,

$$|d_{A^\alpha}(v) - \mu_v| < \frac{\mu_v}{5}$$

or equivalently,

$$\frac{d_{G[A]}(v)}{15} < d_{A^\alpha}(v) < \frac{d_{G[A]}(v)}{10}.$$

Since, according to Lemma 2.3, $d_{G[A]}(v) \geq 0.15n$ and also since $|A| \leq 0.6n$, it follows that

$$0.01n < d_{A^\alpha}(v) < 0.06n$$

which shows, in particular, that $\delta(A^\alpha) > 0.01n$, and also since $d_{A^\beta}(v) = d_{G[A]}(v) - d_{A^\alpha}(v)$ it follows from the fact that $G[A]$ is $0.1n$ edge-expanding that for every $X \subset A$ with $|X| \leq |A|/2$:

$$e_{A^\beta}(X, A \setminus X) \geq e_{G[A]}(X, A \setminus X) - \sum_{v \in X} d_{A^\alpha}(v) \geq 0.1n|X| - 0.06n|X| > 0.01n|X|$$

which proves that A^β is $0.01n$ edge-expanding. \square

Another useful fact (mentioned, e.g., in [3]) is the following:

Fact 2: *Every graph S with $\delta(S) \geq h-1$ contains a copy of every tree H on h vertices. Furthermore,*

if v is a vertex of S and t is a vertex of H , there is a copy of H in S which maps t to v . In particular, every graph S with at least $(h - 1)|S|$ edges contains every tree on h vertices as a subgraph.

Fact 2 holds since one can embed any tree with h vertices, vertex by vertex, using the greedy algorithm, in a graph with minimum degree $h - 1$. The initial embedding may start with any vertex t of H , and this vertex can be mapped to any vertex v of S . The “in particular” part of Fact 2 follows from Fact 1, since a graph with $|S|(h - 1)$ edges has a subgraph with minimum degree $h - 1$.

3 Proof of Theorem 1.1

The following Theorem which is proved in [11], shows that if $G = (V, E)$ has good edge-expansion, and $h - 1$ divides $|E|$, then G has an H -decomposition.

Lemma 3.1 *If $G = (V, E)$ has n vertices, $h - 1$ divides $|E|$, and G is $10h^4\sqrt{n \log n}$ edge-expanding, then G has an H -decomposition. \square*

Unfortunately, it is not difficult to construct graphs with $\delta(G) \geq n/2 + \Omega(\sqrt{n \log n})$ which are not $10h^4\sqrt{n \log n}$ edge-expanding, so Lemma 3.1, by itself, cannot be used to prove Theorem 1.1.

Using Lemma 2.4, Lemma 3.1 and Fact 2, we are now ready to prove Theorem 1.1. Let $G = (V, E)$ be a graph with $n \geq (12h)^{10}$ vertices, $\delta(G) \geq \lfloor n/2 \rfloor$, and $|E| = m(h - 1) + h_0$, where m is a positive integer, and $0 \leq h_0 \leq h - 2$. We need to show that there exists a set $L = \{T_1, \dots, T_m\}$ of edge-disjoint subgraphs of G , which are isomorphic to H . A rough sketch of the proof is the following: We distinguish two possibilities. The first, and easy case, occurs when G has good edge-expansion. In this case we use Lemma 3.1 (with slight modification) to obtain L . Otherwise, G is not a good expander, in the sense that it satisfies the conditions of Lemma 2.4. In this case we show how to select a set of edge-disjoint copies of H which absorb all the edges between A and B (the sets from Lemma 2.4), and (unfortunately) some edges from $G[A]$ and $G[B]$, such that after deleting these copies, the remains of $G[A]$ and $G[B]$ are still good expanders, in the sense that we can apply Lemma 3.1 to each of them (this process turns out to be rather complicated). Uniting the decompositions of the remains of $G[A]$ and $G[B]$ with the initial set of edge disjoint copies yields L . We now turn to the details of the proof.

The first, and easy case to consider, is when G is $10h^4\sqrt{n \log n} + 1$ edge-expanding. In this case, we can pick a set of h_0 independent edges of G , and consider the graph G' obtained from G by omitting these edges. (The fact that $\delta(G) \geq \lfloor n/2 \rfloor$ guarantees that G has a Hamiltonian path, and since $n > 2h_0$, there is a set of h_0 independent edges.) Note that G' has $m(h - 1)$ edges, and for every $X \subset V$ with $|X| \leq n/2$ there are $e(X, V \setminus X) \geq (10h^4\sqrt{n \log n} + 1)|X|$ between X and $V \setminus X$, in G .

Since $d_{G'}(v) \geq d_G(v) - 1$ for every $v \in V$ it follows that there are $e(X, V \setminus X) \geq 10h^4\sqrt{n \log n}|X|$ between X and $V \setminus X$ in G' . Thus, G' is $10h^4\sqrt{n \log n}$ edge-expanding, and according to Lemma 3.1, G' has an H -decomposition with m members, as required.

We can now assume that G is not $10h^4\sqrt{n \log n} + 1$ edge-expanding. Since $n \geq (12h)^{10}$ it follows, in particular, that $10h^4\sqrt{n \log n} + 1 \leq 0.01n - 1$. Thus, G satisfies the conditions of Lemma 2.4, and we can find a partition of V into A and B with sizes at least $0.4n$ each, and create the spanning graphs $A^\alpha, A^\beta, B^\alpha$ and B^β , as guaranteed by Lemma 2.4. Now let C be the (bipartite) subgraph of G consisting only of the edges which connect a vertex of A with a vertex of B . Thus, C has $e(A, B)$ edges, and

$$e(A, B) + e(A^\alpha) + e(A^\beta) + e(B^\alpha) + e(B^\beta) = m(h - 1) + h_0.$$

Our initial claim is that $e(A, B)$ is not too small:

Claim 1: $e(A, B) \geq \min\{|A|, |B|\} \geq 0.4n > 2h > h_0 + h - 1$.

Proof: Since $|A| + |B| = n$, we can assume, e.g., that $|B| \leq \lfloor n/2 \rfloor$. Since $\delta(G) \geq \lfloor n/2 \rfloor$, it follows that every vertex of $v \in B$ has a neighbor in A . This proves the claim.

Let us designate a set C' of $h - 1$ edges of C , and a set C'' of h_0 edges of C , where C' and C'' are disjoint. This can be done, by Claim 1. The edges of C'' will be the edges which will not participate in any member of the packing L , while the edges of C' will be used later on to overcome a defect in the decomposition that we create, due to divisibility problems. Put $C^* = C \setminus (C' \cup C'')$.

Let $L_1 = \{T_1, \dots, T_t\}$ be a maximal set of edge-disjoint subgraphs of C^* , which are isomorphic to H . Let D be the subgraph of C^* consisting of the edges of C^* which are not in any T_i . Since D has no copy of H , we have, according to Fact 2:

$$(h - 1)n > e(D) = e(C^*) - t(h - 1) = e(A, B) - (h - 1 + h_0) - t(h - 1) = e(A, B) - (t + 1)(h - 1) - h_0.$$

We now perform the following process in D , which decomposes the edge-set of D into *connected* edge-disjoint subgraphs that are isomorphic to connected subgraphs of H . In order to describe our process we need some notations. Let q be a vertex of H having degree one (a leaf), and let e_q denote the unique edge of H incident with q . If P is any subtree of H which contains e_q , we say that a vertex of P is *terminal* if its degree in P is less than its degree in H . The other vertices of P are called *non-terminal*. Note that q is a non-terminal vertex of P .

We find in D a *maximal* subtree S_1 which is isomorphic to a subtree of H which contains e_q . The term *maximal* here means that there is no way to add an edge of D to S_1 and still obtain a subtree of H which contains e_q . Deleting the edges of S_1 from D , we now similarly find a maximal subtree S_2 in the remaining subgraph of D , and so forth. We repeat this process as long as there remain

edges in D which do not belong to any S_i . Since a single edge is isomorphic to the single-edge subtree of H obtained from e_q by itself, the process can be completed. Let $\mathcal{S} = \{S_1, \dots, S_p\}$ be the set of subtrees obtained by this process. Clearly,

$$p \leq e(D) < (h-1)n. \quad (4)$$

We claim the following:

Claim 2: Every $v \in V$ appears in at most $h-2$ members of \mathcal{S} as a terminal vertex.

Proof: Let v be a terminal vertex of S_i , where i is minimal. S_i is isomorphic to some subtree P of H , where $e_q \in P$. The vertex v is mapped, under the isomorphism, to a vertex q' of P , which is terminal, i.e. $d_H(q') > d_P(q') = d_{S_i}(v)$. The number of edges incident with v in D which do not appear in any of the subtrees S_1, \dots, S_i is therefore at most $|S_i| - (1 + d_P(q')) \leq h-3$, since otherwise, there would have been an edge (v, u) , with $u \notin S_i$, such that (v, u) could be added to S_i , and form a subtree isomorphic to a subtree of H , which is obtained from P by adding an edge (q', q'') , where $q'' \notin P$, thereby contradicting the maximality of S_i . This shows that there are at most $h-3$ members of \mathcal{S} , which appear after S_i , and which contain v . In particular, there are at most $h-2$ members of \mathcal{S} in which v is terminal. This completes the proof of the claim.

Our next step is to extend each S_i to a tree R_i , which is isomorphic to H , and such that *almost* all the trees R_1, \dots, R_p are edge-disjoint. The edges of $R_i \setminus S_i$ will be taken from $A^\alpha \cup B^\alpha$. We now describe the process of creating R_i . The process uses probabilistic arguments, and we will *not* care whether the R_i 's are edge-disjoint. However, we will show that with high probability, there is a set of at most $O(h^8)$ R_i 's, such that all the other R_i 's are edge-disjoint. Before we begin, it will be convenient to consider *orientations* of A^α and B^α , denoted by \vec{A}^α and \vec{B}^α , such that the indegree and outdegree of every vertex differ by at most 1. That is, if $v \in A$, and $d^+(v)$ and $d^-(v)$ denote, respectively, the indegree and outdegree of v in \vec{A}^α , then $|d^+(v) - d^-(v)| \leq 1$ (and the same holds for $v \in B$). Since $d^+(v) + d^-(v) = d_{A^\alpha}(v) \geq 0.01n$, (and, similarly, for $v \in B$) it follows that

$$d^+(v) \geq n/200 - 1/2 \geq n/201.$$

The fact that every non-directed graph can be oriented such that the indegree and outdegree differ by at most 1 is a well-known consequence of Euler's Theorem (cf. e.g. [2]).

Consider S_i , and let v_1, \dots, v_k be the set of terminal vertices of S_i . Let P be the subtree of H which is isomorphic to S_i , and let q_1, \dots, q_k be the corresponding terminal vertices of P . Let Q_j be the subtree of H which consists of all the vertices of H which are reachable from q_j with paths that contain only edges which do not appear in P . Clearly, P, Q_1, \dots, Q_k are all edge disjoint subtrees of H , and their union is H . We can view Q_j as a rooted (directed) subtree, whose root is q_j . For

each $v_j \in A$, we will find a rooted subtree W_i^j of \vec{A}^α , whose root is v_j , which is isomorphic (as a directed graph) to Q_j , and with v_j being isomorphic to q_j . Similarly, if $v_j \in B$, we will find the rooted subtree W_i^j in \vec{B}^α . Furthermore, W_i^1, \dots, W_i^k will be vertex disjoint, and each W_i^j will share with S_i only the vertex v_j . Thus, the union of S_i, W_i^1, \dots, W_i^k is a *tree* isomorphic to H , and will be denoted by R_i . In the next paragraph we describe how to create W_i^j . We shall assume that $v_j \in A$, since the process in case $v_j \in B$ is similar.

Let $\{q_j = x_1, x_2, \dots, x_r\}$ be an ordering of the vertices of Q_j in which the children of a vertex appear after it in the ordering (breadth first search and depth first search are examples of such orderings). Let y_z be the number of children of x_z for $z = 1, \dots, r$. We create W_i^j in r stages, where in stage z we select the y_z children of the vertex corresponding to x_z (if x_z is a leaf then $y_z = 0$ and we do nothing). In the first stage we must select y_1 edges which emanate from v_j in \vec{A}^α , and such that every chosen edge is directed toward a vertex which does not appear in S_i , nor in $W_i^1 \cup \dots \cup W_i^{j-1}$. There are $d^+(v_j)$ edges emanating from v_j in \vec{A}^α . However, some of these edges may lead to vertices which already appear in $S_i \cup W_i^1 \cup \dots \cup W_i^{j-1}$. Denote by $f(v_j, i)$ the number of edges emanating from v_j which lead to vertices in $S_i \cup W_i^1 \cup \dots \cup W_i^{j-1}$. Clearly, $f(v_j, i) \leq h - 2$, since $S_i \cup W_i^1 \cup \dots \cup W_i^{j-1}$ is isomorphic to a proper subtree of H , consisting of the parts P, Q_1, \dots, Q_{j-1} , and thus containing at most $h - 2$ edges. Thus, there are at least $d^+(v_j) - f(v_j, i)$ edges emanating from v_j which are plausible candidates for the y_1 edges we wish to select. We will choose this set of y_1 edges *randomly*. That is, each of the $\binom{d^+(v_j) - f(v_j, i)}{y_1}$ sets of y_1 edges is equally likely to be chosen, and the choice is made uniformly. Suppose now we have already made $z - 1$ stages, and we now wish to perform stage z . Let $v \in \vec{A}^\alpha$ be the vertex which corresponds to x_z . We need to select y_z edges which emanate from v . These edges can be selected from a set of $d^+(v) - f(v, i)$ edges, where $f(v, i)$ is the number of edges of v which lead to vertices in $S_i \cup W_i^1, \dots \cup W_i^{j-1} \cup X$ where X denotes the $z - 1$ vertices of W_i^j which have already been selected in previous stages. (In stage 1, X was empty, so we did not consider it). As in the first stage, $f(v, i) \leq h - 2$, and the selection of the y_z edges is made at random. This completes the description of the randomized process which creates W_i^j . Note that the process shows that, indeed, W_i^1, \dots, W_i^k are all vertex-disjoint, and W_i^j shares with S_i only the vertex v_j . Thus, the union of S_i, W_i^1, \dots, W_i^k , which we denote by R_i , is, indeed, a tree isomorphic to H .

The random process which creates R_i , is completely independent of the process which created R_j , for $i \neq j$. It may therefore happen that R_i and R_j are not edge disjoint, a situation we wish to avoid. We will show that, with positive probability, we can delete a set of at most $O(h^8)$ trees from $\mathcal{R} = \{R_1, \dots, R_p\}$, such that the remaining trees are edge-disjoint. The following claim essentially proves this fact:

Claim 3: For each directed edge (a, b) of $\vec{A}^\alpha \cup \vec{B}^\alpha$, the probability that (a, b) appears in more than one member of \mathcal{R} is at most $35^6 h^8 / n^2$.

Proof: Fix (a, b) , and assume, w.l.o.g., that $(a, b) \in \vec{A}^\alpha$. We first compute the probability that (a, b) is an edge of R_i , namely, $\text{Prob}[(a, b) \in R_i]$. There are three cases:

1. b appears in S_i . In this case, (a, b) cannot be in R_i at all, since whenever we create the edges of the components W_i^j we always take care not to select an edge which leads to a vertex of S_i . Thus, in this case $\text{Prob}[(a, b) \in R_i] = 0$.
2. b does not appear in S_i and a is a terminal vertex of S_i . In this case, we select y_1 edges (recall that y_1 , in this case, is the number of children of q_j in Q_j , where q_j is the vertex of P isomorphic to a , using the notations above) emanating from a , randomly, from a set of $d^+(a) - f(a, i)$ edges. Thus, every edge emanating from a in \vec{A}^α is selected with probability at most $y_1 / (d^+(a) - f(a, i))$. Hence,

$$\text{Prob}[(a, b) \in R_i] \leq \frac{y_1}{d^+(a) - f(a, i)} \leq \frac{h - 2}{n/201 - (h - 2)} \leq \frac{202h}{n}.$$

3. b does not appear in S_i and a is not a terminal vertex of S_i . If a appears in S_i as a non-terminal vertex then, similar to case 1, $\text{Prob}[(a, b) \in R_i] = 0$. Otherwise, both a and b do not appear in S_i . The only way that (a, b) can be a vertex of R_i is that there is some other edge (c, a) of \vec{A}^α which also appears in R_i (c may or may not be a terminal vertex of S_i). Suppose we are given in advance, the set $W = \{c_1, \dots, c_w\}$ of vertices of \vec{A}^α , which appear in R_i and such that $(c_1, a), \dots, (c_w, a)$ are all edges of \vec{A}^α . Clearly, $w < h$, $b \notin W$ (since $(a, b) \in \vec{A}^\alpha$ we cannot have $(b, a) \in \vec{A}^\alpha$), and at most one member $c \in W$ has $(c, a) \in R_i$ (this is because R_i is a directed tree). We shall compute the probability that $(a, b) \in R_i$ given that we know W . Clearly,

$$\begin{aligned} \text{Prob}[(a, b) \in R_i \mid W] &\leq \sum_{w'=1}^w \text{Prob}[(c_{w'}, a) \in R_i \wedge (a, b) \in R_i \mid W] = \\ &\sum_{w'=1}^w \text{Prob}[(c_{w'}, a) \in R_i \mid c_{w'} \in R_i] \cdot \text{Prob}[(a, b) \in R_i \mid a \in R_i]. \end{aligned}$$

As in Case 2,

$$\text{Prob}[(c_{w'}, a) \in R_i \mid c_{w'} \in R_i] \leq \frac{202h}{n}.$$

Similarly,

$$\text{Prob}[(a, b) \in R_i \mid a \in R_i] \leq \frac{202h}{n}.$$

Thus,

$$\text{Prob}[(a, b) \in R_i \mid W] \leq w \frac{(202h)^2}{n^2} < \frac{(35h)^3}{n^2}.$$

The last inequality is independent of W , and hence

$$\text{Prob}[(a, b) \in R_i] \leq \frac{(35h)^3}{n^2}.$$

We now show that the probability that (a, b) appears in more than one member of \mathcal{R} is at most $35^6 h^8 / n^2$. Let $1 \leq i < j \leq p$ be fixed. The event that (a, b) is in R_i is independent of the event that a is in R_j , since each member of \mathcal{R} is created independently. If a is a terminal vertex of both S_i and S_j then according to case 2,

$$\text{Prob}[(a, b) \in R_i \wedge (a, b) \in R_j] \leq \frac{202h}{n} \frac{202h}{n}.$$

If a is a terminal vertex of exactly one of S_i or S_j then according to cases 1,2,3:

$$\text{Prob}[(a, b) \in R_i \wedge (a, b) \in R_j] \leq \frac{202h}{n} \frac{(35h)^3}{n^2}$$

If a is not a terminal vertex of both S_i and S_j then according to cases 1 and 3:

$$\text{Prob}[(a, b) \in R_i \wedge (a, b) \in R_j] \leq \frac{(35h)^3}{n^2} \frac{(35h)^3}{n^2}.$$

According to Claim 2, there are at most $\binom{h-2}{2}$ pairs i, j in which a is a terminal vertex of both S_i and S_j . Since, by (4), $p < (h-1)n$, there are less than $(h-2)(h-1)n$ pairs in which a is a terminal vertex of exactly one of S_i or S_j , and there are less than $\binom{(h-1)n}{2}$ pairs where a is not a terminal vertex of both S_i and S_j . Thus,

$$\begin{aligned} \text{Prob}[\exists i < j \ (a, b) \in R_i \wedge (a, b) \in R_j] &\leq \binom{(h-2)}{2} \frac{202^2 h^2}{n^2} + (h-2)(h-1)n \frac{202h}{n} \frac{(35h)^3}{n^2} + \\ &\quad \binom{(h-1)n}{2} \frac{35^6 h^6}{n^4} \leq \frac{35^6 h^8}{n^2}. \end{aligned}$$

This completes the proof of Claim 3. \square

An immediate consequence of Claim 3 is that there *exists* a set \mathcal{R} where at most $e(A^\alpha)35^6 h^8 / n^2$ edges of A^α appear in more than one member of \mathcal{R} . Since $|A| \leq 0.6n$, we have that $e(A^\alpha) \leq 0.18n^2$. This means that there exists a set \mathcal{R} in which at most $(12h)^8$ edges appear in more than one member of \mathcal{R} . Deleting from such \mathcal{R} the members which contain at least one edge which appears in more than one member we obtain a set $L_2 = \{R_1, \dots, R_{p'}\}$ of edge-disjoint trees which are isomorphic to H , and $p' \geq p - (12h)^8$.

Up to this stage, we have designated $p' + t$ edge-disjoint copies of H in G , namely the sets L_1 and L_2 . We need to extend this with an additional set of $m - p' - t$ edge-disjoint copies. Let us call

the edges of G which are not in any member of $L_1 \cup L_2$, nor in C'' , the *non-covered* edges. The number of non-covered edges is exactly $(m - p' - t)(h - 1)$ and they are formed by all the edges of A^β and B^β , some of the edges of A^α and B^α (namely those that do not appear in any member of L_2), the $h - 1$ edges of C' , and those edges of $D \subset C^*$ which are in some S_i whose corresponding R_i is not a member of L_2 . Let F denote the set of non-covered edges of D . Consider the edge-set $F \cup C'$. Clearly,

$$|F| + |C'| \leq (h - 2) \cdot (12h)^8 + h - 1 \leq 12^8 h^9.$$

(We used here the fact that every member of \mathcal{S} contains at most $h - 2$ edges.) Let a be the number of non-covered edges with both endpoints in A . Clearly, $e(A^\beta) \leq a \leq e(A^\beta) + e(A^\alpha)$. Similarly, define b as the number of non-covered edges with both endpoints in B . Clearly, $e(B^\beta) \leq b \leq e(B^\beta) + e(B^\alpha)$. Note that $a + b + (h - 1) + |F| = (m - p' - t)(h - 1)$. Partition $F \cup C'$ arbitrarily into two subsets F_1, F_2 such that $|F_1| + a \equiv 0 \pmod{h - 1}$. This forces $|F_2| + b \equiv 0 \pmod{h - 1}$. Note that the partition can be done since $|F \cup C'| \geq h - 1$ (this explains why we designated C' in the beginning). Note that $(a + |F_1|) + (b + |F_2|) = (m - p' - t)(h - 1)$. Let G_1 be the subgraph of G composed of the edge-set F_1 and the a non-covered edges of A . Similarly define G_2 as the subgraph of G composed of F_2 and the b non-covered edges of B . It remains to show that both G_1 and G_2 have an H -decomposition, which together with L_1 and L_2 gives m edge-disjoint copies of H in G . We now show that G_1 has an H decomposition. This suffices, since the proof for G_2 is identical. Let $G_1[A]$ be the subgraph of G_1 induced by A . $G_1[A]$ has a edges, but, since it contains A^β , we know by Lemma 2.4 that $\delta(G_1[A]) \geq \delta(A^\beta) \geq 0.01n$, since an r edge-expanding graph must have, in particular, a minimum degree of r . Our first goal is to find a set of $|F_1|$ edge-disjoint copies of H in G_1 , such that each copy contains exactly one edge of F_1 and the other $h - 2$ edges have both endpoints in A . Define a tree H' on $h - 1$ vertices obtained from H by deleting a leaf y' and its incident edge (x', y') . Consider we have already found a set of $f < |F_1|$ edge-disjoint copies of H covering f edges of F_1 . Let (x, y) be an edge of F_1 which is not yet covered by any of the f copies. We show how to find in G_1 another edge-disjoint copy, containing (x, y) . Since $(x, y) \in F_1 \subset F \cup C' \subset C$, we know that (x, y) connects a vertex of A and a vertex of B . We may therefore assume that $x \in A$. There are $f(h - 2)$ edges of $G_1[A]$ which are used by the previous f copies. Thus, the non-used edges of $G_1[A]$ form a subgraph G' with $a - f(h - 2)$ edges whose minimum degree is at least $0.01n - f(h - 2)$. Note that

$$\begin{aligned} \delta(G') &\geq 0.01n - f(h - 2) \geq 0.01n - (|F_1| - 1)(h - 2) \geq \\ &0.01n - (|F \cup C'| - 1)(h - 2) > 0.01n - 12^8 h^9 (h - 2) > h - 2. \end{aligned}$$

In the last inequality we used the fact that $n \geq (12h)^{10}$. According to fact 2, G' contains a copy of the tree H' where x is mapped to the vertex x' of H' . We can extend this copy to a copy of H by

adding the edge (x, y) where y is mapped to y' .

Let G' be the subgraph of $G_1[A]$ which contains the $a - |F_1|(h - 2)$ edges which are not covered by the above mentioned $|F_1|$ edge-disjoint copies. Note that $a - |F_1|(h - 2) \equiv 0 \pmod{h - 1}$ since $a + |F_1| \equiv 0 \pmod{h - 1}$. It remains to show that G' has an H -decomposition. We will show this by using Lemma 3.1. The graph A^β is $0.01n$ edge-expanding, by Lemma 2.4. Thus, trivially, the graph $G_1[A]$ which contains A^β is also $0.01n$ edge-expanding. Since G' is obtained from $G_1[A]$ by omitting only $|F_1|(h - 2)$ edges, it follows trivially that G' is $0.01n - |F_1|(h - 2)$ edge-expanding. However,

$$0.01n - |F_1|(h - 2) \geq 0.01n - 12^8 h^9 (h - 2) \geq 0.002n \geq 10h^4 \sqrt{n \log n} \geq 10h^4 \sqrt{|A| \log |A|}.$$

(We have used here the fact that $n \geq (12h)^{10}$). Thus, according to Lemma 3.1, G' has an H -decomposition. \square

4 Proof of Theorem 1.2 and some concluding remarks

We begin this section with a proof of Theorem 1.2. Let H be a fixed tree on $h > 2$ vertices, and let $n \geq (12h)^{10}$. We first note that Theorem 1.1 shows that $f_H(n) \leq \lfloor n/2 \rfloor$. As mentioned in the introduction, the fact that $f_H(n) \geq \lfloor n/2 \rfloor - 1$ is shown in [11]. However, Theorem 1.2 is stronger since it establishes many cases in which $f_H(n) = \lfloor n/2 \rfloor$. For two integers s and $t \leq s/2$, let $K(s, t)$ denote the complete graph on s vertices from which t independent edges were removed. Note that $\delta(K(s, t)) = s - 2$. We consider the following cases:

1. n is odd, and $h - 1$ does not divide $(n - 1)(n - 3)/8$. We shall create a graph G on n vertices as follows. G has two connected components. The first component is $K_{(n-1)/2}$, and the second component is the graph $K((n + 1)/2, t)$ where $0 \leq t \leq h - 2$ satisfies $(n - 1)^2/4 \equiv t \pmod{h - 1}$. Clearly, $\delta(G) = (n + 1)/2 - 2 = \lfloor n/2 \rfloor - 1$. The number of edges of G is exactly $(n - 1)(n - 3)/8 + (n + 1)(n - 1)/8 - t \equiv 0 \pmod{h - 1}$. However, G does not have an H -decomposition since the first component, $K_{(n-1)/2}$, does not have an H -decomposition, because it has $(n - 1)(n - 3)/8$ edges, and this is not divisible by $h - 1$. This shows $f_H(n) > \lfloor n/2 \rfloor - 1$, and hence, $f_H(n) = \lfloor n/2 \rfloor$.
2. n is odd, and $h - 1$ divides $(n - 1)(n - 3)/8$. We create a graph G with two components, where the first is $K((n - 1)/2, 1)$ and the second is $K((n + 1)/2, t)$, where $0 \leq t \leq h - 2$ satisfies $(n - 1)(n - 3)/8 - 1 + (n - 1)(n + 1)/8 - t \equiv 0 \pmod{h - 1}$. Clearly, $\delta(G) = \lfloor n/2 \rfloor - 2$, its number of edges is divisible by $h - 1$, but it has no H -decomposition, since the first component,

$K((n-1)/2, 1)$, has $(n-1)(n-3)/8 - 1$ edges, which is not divisible by $h-1$. This shows that $f_H(n) > \lfloor n/2 \rfloor - 2$, and therefore $f_H(n) \geq \lfloor n/2 \rfloor - 1$.

3. n is even, and $h-1$ divides $n(n-2)/4$ but does not divide $n(n-2)/8$. Consider the graph G obtained from two vertex-disjoint copies of $K_{n/2}$. Clearly, $\delta(G) = n/2 - 1$, G has $n(n-2)/4$ edges, and thus G satisfies the H -decomposition divisibility conditions, but G does not have an H -decomposition since $K_{n/2}$ does not have an H -decomposition. This shows $f_H(n) > n/2 - 1$ and hence $f_H(n) = n/2$.
4. n is even, and the divisibility conditions in the previous case do not hold. We create G from two vertex-disjoint components $K(n/2, a)$ and $K(n/2, b)$ where $n(n-2)/4 - a - b \equiv 0 \pmod{h-1}$, and $n(n-2)/8 - a$ is not divisible by $h-1$, and $a \in \{0, 1\}$ and $0 \leq b \leq h-1$ (note that this can always be done). Note that $\delta(G) = n/2 - 2$, G satisfies the H -decomposition divisibility conditions, but G does not have an H -decomposition since $K(n/2, a)$ does not have an H -decomposition. This shows $f_H(n) > n/2 - 2$ and thus $f_H(n) \geq n/2 - 1$. \square

We end this section with a few concluding remarks:

1. Although Theorem 1.2 shows that $f_H(n)$ is, either $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor - 1$, (and, in many cases, the exact value is known), it is still interesting to determine the exact value of $f_H(n)$ for all n . This would not be so easy, since it is not only a function of $|H| = h$ and n but also a function of the structure of H . To see this, consider, e.g. two trees on $h = 8$ vertices. The first is S_8 , the star with 8 vertices, and the second is P_8 , the path with 8 vertices. Now, if n is an even integer satisfying $n(n-2)/4 + 1 \equiv 7 \pmod{14}$ (e.g. $n \equiv 10 \pmod{14}$ satisfies this), then we can create the connected graph G composed from two vertex-disjoint copies of $K_{n/2}$ which are joined by a single edge, denoted by (a, b) . This graph has $n(n-2)/4 + 1$ edges, and thus satisfies the S_8 -decomposition divisibility conditions. It is clear, however, that G has no S_8 -decomposition, since if (a, b) is covered by a copy of S_8 then we can assume that a is the root. Hence 6 more edges of this copy are all in the $K_{n/2}$ which contains a . The remaining $n(n-2)/8 - 6$ edges in this $K_{n/2}$ cannot be decomposed to edge-disjoint copies of a tree on 7 edges since $n(n-2)/8 - 6$ is *not* divisible by 7. However, it is not too difficult to show that G has a decomposition into P_8 , since one can take a copy of P_8 whose middle edge covers (a, b) , and the remaining edges form a graph with two identical components, each is $K_{n/2}$ with a P_4 missing, and the number of edges in this component is $n(n-2)/8 - 3$ which is a multiple of 7, and according to Theorem 1.1, if n is large enough, there is a P_8 decomposition for each of the components. Thus, for these values of n , $f_{S_8}(n) = n/2$, while it is not difficult to show that $f_{P_8}(n) = n/2 - 1$.

2. It is a challenging open problem to find other families of graphs, which are not trees, for which $f_H(n)$ can be computed exactly (or even asymptotically) for every member H of the family.
3. Although we did not try to optimize the constant $(12h)^{10}$ appearing in the statement of Theorems 1.1 and 1.2, it is worth noting that this constant is only polynomial in h , while the constants in Wilson's and Gustavsson's Theorems are exponential in $h = |H|$ [6].
4. By slightly modifying the proof of Theorem 1.1 one can obtain that the H -covering number of G satisfying the conditions of Theorem 1.1 is $C(H, G) = \lceil e(G)/(h-1) \rceil$. This is done by first selecting one copy T_0 of H in G , and then deleting from this copy some $0 < h_0 < h-1$ edges such that the remaining graph G' has a $m(h-1)$ edges, and now applying Theorem 1.1 for G' , to obtain a decomposition. Adding T_0 to this decomposition we get an H -covering of G with exactly $h-1-h_0$ edges which are covered twice. A slight modification to the lemmas of Section 2 is needed since it is no longer true that $\delta(G') \geq \lfloor n/2 \rfloor$, although this is not an obstacle since there are at most h_0 vertices with degree less than $\lfloor n/2 \rfloor$, and their degree is not less than $\lfloor n/2 \rfloor - h_0$.

References

- [1] N. Alon and J. H. Spencer, *The Probabilistic Method*, John Wiley and Sons Inc., New York, 1991.
- [2] B. Bollobás, *Extremal Graph Theory*, Academic Press, 1978.
- [3] B. Bollobás, *Some remarks on packing trees*, Discrete Math. 46 (1983), 203-204.
- [4] Y. Caro and R. Yuster, *Packing graphs: The packing problem solved*, Elect. J. Combin. 4 (1997), #R1.
- [5] Y. Caro and R. Yuster, *Covering graphs: The covering problem solved*, J. Combin. Theory Ser. A 83 (1998), 273-282.
- [6] Y. Chang, *A bound for Wilson's Theorem - III*, J. Combinatorial Designs 4 (1996), 83-93.
- [7] D. Dor and M. Tarsi, *Graph decomposition is NPC - A complete proof of Holyer's conjecture*, Proc. 20th ACM STOC, ACM Press (1992), 252-263.
- [8] T. Gustavsson, *Decompositions of large graphs and digraphs with high minimum degree*, Doctoral Dissertation, Dept. of Mathematics, Univ. of Stockholm, 1991.

- [9] C. St. J. A. Nash-Williams, *An unsolved problem concerning decomposition of graphs into triangles*, Combinatorial Theory and its Applications III. ed. P. Erdős, P. Rényi and V.T. Sós. North Holland (1970), 1179-1183.
- [10] R. M. Wilson, *Decomposition of complete graphs into subgraphs isomorphic to a given graph*, Congressus Numerantium XV (1975), 647-659.
- [11] R. Yuster, *Tree packing of graphs*, Random Structures and Algorithms 12 (1998), 237-251.