A Note on Packing Trees into Complete Bipartite Graphs

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Abstract

Let $\{T_2, \ldots, T_t\}$ be a sequence of trees, where T_i has i vertices. We show that if $t < 0.79n$, then the sequence can be packed into the complete bipartite graph $K_{n-1,n/2}$ ($K_{n,(n-1)/2}$), where n is even (odd). This significantly improves the result which appears in [2].

Let $\{T_2, \ldots, T_t\}$ be a sequence of trees, where T_i has i vertices. We call such a sequence a t-sequence. We say that the sequence can be packed into the graph G, if G contains edge disjoint subgraphs $H_2, \ldots, H_t$ such that H_i is isomorphic to T_i, $i = 2, \ldots, t$. Clearly G must contain at least $\binom{t}{2}$ edges. Gyárfás and Lehel [3] conjectured that any n-sequence can be packed into K_n. Hobbs, Bourgeois and Kasiraj [4] conjectured that any n-sequence can be packed into the complete bipartite graph $K_{n-1,n/2}$ when n is even, and in $K_{n,(n-1)/2}$ when n is odd. Both conjectures, if true, are best possible, and both are still open. Bollobás observed in [1] that any $\lfloor n/\sqrt{2} \rfloor$-sequence can be packed into K_n. Using a similar observation, Caro and Roditty showed in [2] that any $\lfloor 0.3n \rfloor$-sequence can be packed into $K_{n-1,n/2}$. In this note we significantly improve their result and show that any $\lfloor \sqrt{5/8}n \rfloor$-sequence can be packed into $K_{n-1,n/2}$. Note that $\sqrt{5/8} > 0.79$.

Bollobás [1] gave a simple procedure for embedding a tree T in a graph H with sufficiently many edges. The following lemma shows that if H is bipartite, significantly less edges are needed in order to guarantee the existence of T in H.

Lemma 1.1 *Let H be a bipartite graph with vertex classes H_1 and H_2 of sizes h_1 and h_2 respectively, $h_1 \leq h_2$. Let T be a tree whose bipartite vertex classes are of sizes k_1 and k_2. If $k_1 \leq h_1$ and $k_2 \leq h_2$ and $e(H) \geq k_2h_1 + k_1h_2 + k_1 + k_2 - h_1 - h_2 - k_1k_2$ then H contains a subgraph isomorphic to T.*

Proof We perform the following procedure on the vertices of H. If there exists a vertex in H_1 whose degree is less than k_2, we delete it from the graph. Otherwise, if there exists a vertex in H_2 whose degree is less than k_1 we delete it from the graph. Otherwise, we halt. We claim that when we halt, H_i contains at least k_i vertices, $i = 1, 2$. Note that otherwise, at some stage in the procedure, H_i contains exactly $k_i - 1$ vertices, for some $i = 1, 2$. Suppose this happens first for $i = 1$. Every remaining vertex of H_2 has, therefore, a degree less than k_1 and we can therefore delete them all in the next stages of the procedure until H becomes empty. We have thus deleted at most $(k_2 - 1)(h_1 - (k_1 - 1)) + (k_1 - 1)h_2$ edges during the process, and remained with an empty graph. Hence, $e(H) \leq k_2h_1 + k_1h_2 + k_1 + k_2 - h_1 - h_2 - k_1k_2 - 1$, which is a contradiction. A similar argument holds for $i = 2$. Having shown this, note that when the procedure ends, every remaining vertex of H_1 has degree at least k_2, and every remaining vertex of H_2 has degree at least k_1. Hence, one can construct a copy of T in the subgraph of H induced on the remaining vertices.

□

Corollary 1.2 *Let H be a subgraph of $K_{n-1,n/2}$ and suppose that $n \geq 2k$. If $e(H) > (k-1)(3n/2 - k)$ then H contains every tree on $2k$ vertices.*

Proof Let T be a tree on $2k$ vertices, whose vertex classes are of sizes k_1 and k_2, where $k_1 \leq k_2$. We apply Lemma 1.1 to T and H with $h_1 = n/2$, $h_2 = n-1$. Note that $k_1 + k_2 \leq n$ and therefore $k_2 \leq h_2$ and $k_1 \leq h_1$. If H does not contain T then, according to the Lemma, $e(H) \leq k_2 n/2 + k_1(n-1) + k_1 + k_2 - k_1 k_2 - 3n/2$. Replacing k_2 with $2k - k_1$ we obtain that

$$e(H) \leq (2k - k_1)n/2 + k_1 n + 2k - k_1 - k_1(2k - k_1) - 3n/2 =$$

$$nk + k_1 n/2 + 2k - 2k \cdot k_1 + k_1^2 - k_1 - 3n/2.$$

Since $1 \leq k_1 \leq k$, the maximum of the last inequality is obtained when $k_1 = k$, which implies that $e(H) \leq (k-1)(3n/2 - k)$, a contradiction. $\square$

Theorem 1.3 *Any $\lfloor \sqrt{5/8}n \rfloor$-sequence can be packed into $K_{n-1,n/2}$ (n even).*

Proof Put $t = \lfloor \sqrt{5/8}n \rfloor$ and let $\{T_2, \ldots, T_t\}$ be a t-sequence. Clearly, $K_{n-1,n/2}$ contains a copy of T_t (in fact, it contains any tree on n vertices). Suppose that we have already packed $T_t, \ldots, T_{k+1}$ into $K_{n-1,n/2}$ for some $t > k > 1$. Let H be the spanning subgraph of $K_{n-1,n/2}$ which contains all the edges that do not appear in this packing. Clearly,

$$e(H) = \binom{n}{2} - \binom{t}{2} + \binom{k}{2}.$$

Suppose first that k is even. If we can show that $e(H) > (k/2 - 1)(3n/2 - k/2)$, then according to Corollary 1.2, we may find a copy of T_k in H, and add T_k to the packing. Hence we need to show that

$$\frac{n^2}{2} + n - \frac{t^2}{2} + \frac{t}{2} + \frac{3k^2}{4} - k - \frac{3nk}{4} > 0.$$

The minimum of the l.h.s. is attained when $k = n/2 + 2/3$. Replacing k with $n/2 + 2/3$ we have

$$\frac{5n^2}{16} + \frac{n}{2} - \frac{1}{3} - \frac{t^2}{2} + \frac{t}{2} > 0$$

which holds for our chosen value of t. A similar computation yields the result when k is odd (in which case we need to show that $e(H) > (k/2-1/2)(3n/2-k/2-1/2)$. □

A smilar proof shows that any $\lfloor\sqrt{5/8}n\rfloor$-sequence can be packed into $K_{n,(n-1)/2}$ (n odd).

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