Some Remarks on Domination

by

D. Archdeacon[1], J. Ellis-Monaghan[2], D. Fisher[3], D. Froncek[4], P.C.B. Lam[5]
S. Seager[6], B. Wei[7], and R. Yuster[8]

We give two results on domination in graphs, including a proof of a conjecture of Favaron, Henning, Mynhart and Puech [2]. Corollary 2 was found by four separate subsets of the authors. We decided to give this joint presentation of our results. We first offer a result about bipartite graphs.

Lemma 1 *Let G be a bipartite graph with partite sets (X, Y) whose vertices in Y are of minimum degree at least 3. Then there exists a set $A \subset X$ of size at most $|X \cup Y|/4$ such that every vertex in Y is adjacent to a vertex in A.*

Proof: The proof is by induction on $|V(G)| + |E(G)|$. The smallest graph as described in the lemma is $K_{1,3}$, for which the statement holds. This gives the start of our induction. Let $x = |X|$ and $y = |Y|$. If there exists a vertex v in Y of degree at least 4, then delete any edge e incident to v. The subset A of $G - e$ guaranteed by the inductive hypothesis is adjacent in G to every vertex in Y as desired. So we may assume that the vertices in Y are all of

[1]Department of Math. and Stat., University of Vermont, Burlington VT 05405, e-mail: `dan.archdeacon@uvm.edu`

[2]Department of Mathematics, St. Michael's College, Winooski Park, Colchester VT 05439, e-mail: `jellis-monaghan@smcvt.edu`

[3]Department of Mathematics, Univ. of Colorado at Denver, Denver, CO 80217, e-mail: `dfisher@orphan.cudenver.edu`

[4]Department of Math. and Stat., University of Minnesota Duluth, Duluth MN 55812, and Technical University Ostrava, 708 33 Ostrava, Czech Republic, e-mail: `dfroncek@d.umn.edu`

[5]Department of Mathematics, Hong Kong Baptist University, Hong Kong, e-mail: `cblam@hkbu.edu.hk`. Partially supported by FRG, HKBU.

[6]Department of Mathematics, Mount Saint Vincent University, Halifax, NS, Canada B3M 2J2, e-mail: `suzanne.seager@msvu.ca`

[7]Department of Mathematics, University of Mississippi, University, MS 38677, e-mail: `bwei@olemiss.edu`. Partially supported by the National Natural Science Foundation of China and the Croucher Foundation of Hong Kong.

[8]Department of Mathematics, University of Haifa-Oranim, Tivon 36006, Israel, e-mail: `raphy@research.haifa.ac.il`

degree exactly 3. If there exists a vertex v in X of degree at least 3, then delete that vertex and all of its neighbors. Adding v into the subset A from this smaller graph yields our desired subset for G. So we may assume that every vertex in X is of degree at most 2. If there exists an isolated vertex $v \in X$, then again the set A in $G - v$ is adjacent to every vertex in Y. So we may assume that every vertex in X is of degree at least 1.

We now know enough about G to prove the existence of the desired subset A directly. Let X_i denote the vertices in X of degree i, $i = 1, 2$, and let $x_i = |X_i|$. Under the conclusions of the previous paragraph, $x = x_1 + x_2$, and since all vertices in Y are of degree three, $3y = x_1 + 2x_2$. The desired result is a set $A \subset X$ of cardinality at most $(x + y)/4 = x_1/3 + 5x_2/12$.

Form the graph G' from G where $V(G') = X_2$, and $uv \in E(G')$ if and only if u, v are adjacent with a common vertex in G. Since the maximum degree in G of a vertex $x \in X$ is 2, G' is of maximum degree 4. Hence, by Brooks' Theorem, G' has an independent set of size at least $x_2/4$ (the possible exceptional case $G' = K_5$ cannot arise by this construction). The corresponding vertices in G have disjoint neighborhoods, hence they are adjacent to at least $x_2/2$ different vertices in Y. The set A uses these vertices, and for each remaining vertex in Y an adjacent vertex in X. Now $|A| \le x_2/4 + y - 2x_2/4 = x_1/3 + 5x_2/12$ as desired. ∎

A *total dominating set* in a graph H is a subset A of vertices such that every vertex in H is adjacent to a vertex in A.

Corollary 2 *Every graph H of order n and of minimum degree at least 3 has a total dominating set of size at most $n/2$.*

Proof: Construct a bipartite graph G from H as follows. Each vertex v_i in H gives two vertices x_i, y_i in G. Each edge $v_i v_j$ in H gives two edges $x_i y_j$ and $x_j y_i$ in G. The bipartition is $(X, Y) = (\{x_i\}, \{y_i\})$. A total dominating set A in H corresponds to an $A \subset X$ in G adjacent to every vertex in Y. The result now follows by the previous lemma. ∎

This corollary settles a conjecture of Favoron et al. [2], which also contains some history of the problem. As shown in [2] the statement in the corollary is tight. For every n divisible by four there are 3-regular graphs of order n having no dominating set of size less than $n/2$. In fact, using a somewhat more complicated argument, it is possible to show [6] that any extremal graph must be 3-regular and n must be divisible by 4. Reference [4] extends

Corollary 2 to the set of graphs of minimum degree at least 2 where no degree-2 vertex is adjacent to two other degree-2 vertices. Reference [3] shows that every connected graph on n vertices and $e \geq 2$ edges and maximum degree at most 3 is totally dominated by a set of $n - e/3$ vertices, from which Collorary 2 follows. They also examine when the bound is tight.

The result can also be phrased in terms of transversals of rank 3 hyper-gaphs. In this context Lemma 1 is related to work by Chvátal and McDiarmid [1]. This relation and Corollary 2 was noted by Thomasse and Yeo [5].

The advantage to our approach is studying total domination through the corresponding bipartite graph. This allows more subtle inductive steps. These techniques may have other applications to total domination.

References

[1] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs, *Combinatorica* **12** no. 1 (1992) 19–26.

[2] O. Favaron, M.A. Henning, C.M. Mynhart and J. Puech, Total domination in graphs of minimum degree three, *J. Graph Theory* **34** (2000) 9–19.

[3] D. Fisher and S. Seager, The total domination number of graphs of maximum degree 3, (manuscript).

[4] P.C.B. Lam and B. Wei, On the total domination number of graphs, (manuscript).

[5] S. Thomasse and A. Yeo, Total domination of graphs and small transversals of hypergraphs, (manuscript).

[6] R. Yuster, Graphs with minimum degree three are totally dominated by half of the vertices, (manuscript).