Independent Transversals and Independent Coverings in Sparse Partite Graphs

Raphael Yuster

Department of Mathematics

Raymond and Beverly Sackler Faculty of Exact Sciences

Tel Aviv University, Tel Aviv, Israel

Abstract

An $[n, k, r]$-*partite graph* is a graph whose vertex set, V, can be partitioned into n pairwise-disjoint independent sets, $V_1, \ldots, V_n$, each containing exactly k vertices, and the subgraph induced by $V_i \cup V_j$ contains exactly r independent edges, for $1 \leq i < j \leq n$. An *independent transversal* in an $[n, k, r]$-partite graph is an independent set, T, consisting of n vertices, one from each V_i. An *independent covering* is a set of k pairwise-disjoint independent transversals. Let $t(k, r)$ denote the maximal n for which every $[n, k, r]$-partite graph contains an independent transversal. Let $c(k, r)$ be the maximal n for which every $[n, k, r]$-partite graph contains an independent covering. We give upper and lower bounds for these parameters. Furthermore, our bounds are constructive. These results improve and generalize previous results of Erdös, Gyárfás and Łuczak [5], for the case of graphs.

1 Introduction

All graphs considered here are finite, undirected and simple. Let k, r and n be positive integers with $r \leq k$. An $[n,k,r]$-*partite graph* is a graph, $G = (V,E)$, whose vertex set is partitioned into n pairwise-disjoint independent sets, $V_1, \dots, V_n$, where $|V_i| = k$ for $i = 1, \dots n$, and for each $1 \leq i < j \leq n$, the subgraph of G induced by $V_i \cup V_j$ contains exactly r independent edges. Note that an $[n,k,r]$-partite graph contains kn vertices and $r\binom{n}{2}$ edges. Therefore, for any valid fixed value of r, if $k = w(n)$ where $0 < w(n) \to \infty$ is any function, then $|E| = o(|V|^2)$ and hence the graph is sparse.

An *independent transversal* in an $[n,k,r]$-partite graph is an independent set, $T = \{v_1, \dots v_n\}$, where $v_i \in V_i$. An *independent covering* is a set, $C = \{T_1, \dots T_k\}$, of pairwise-disjoint independent transversals. Note that this implies that every vertex of G belongs to exactly one of the T_i's. Given $1 \leq r \leq k$, let $t(k,r)$ denote the maximal n for which every $[n,k,r]$-partite graph contains an independent transversal. Let $c(k,r)$ be the maximal n for which every $[n,k,r]$-partite graph contains an independent covering. The purpose of this paper is to estimate these parameters. We give upper and lower bounds for these parameters, and in some cases, obtain exact values.

In [5], Erdös, Gyárfás and Łuczak considered the value of $t(k,1)$. They have shown that

$$(1 + o(1))(2e)^{-1} < \frac{t(k,1)}{k^2} < (1 + o(1)).$$

We improve the lower bound considerably and show that

$$(1 + o(1))0.65 < \frac{t(k,1)}{k^2}. \tag{1}$$

The proof of [5] uses the Lovász Local Lemma ([6] see also [1]), and it is non-constructive. That is, for a sufficiently large k, given an $[n,k,1]$-partite graph satisfying, say, $n < 0.99k^2/(2e)$, the proof does not yield an efficient algorithm for finding an independent transversal, although we know it exists. Our proof is constructive, and by applying it, we can efficiently find an independent transversal in any $[0.65k^2, k, 1]$-partite graph. We also generalize the result for the case where $r > 1$. In fact, we have the following theorem.

Theorem 1.1 *Let $g(1) = 0.65$, $g(r) = 0.52/r$ for $r \geq 2$. For every $0 < C < g(r)$, there exists a positive integer $K = K(C)$ such that for every $k \geq K$, $t(k,r) \geq \lfloor Ck^2 \rfloor$. Furthermore, given an $[n,k,r]$-partite graph, G, with $n \leq Ck^2$, we can find an independent transversal in G in polynomial (in k) time.*

Note that Theorem 1.1 implies that for all fixed $r \geq 1$

$$\frac{t(k,r)}{k^2} > (1 + o(1))g(r). \tag{2}$$

Note also that Theorem 1.1 applies to fixed values of r. By applying the Local Lemma in a similar way to the proof in [5] we can obtain the following alternative lower bound for $t(k,r)$ which is valid for all $1 \leq r \leq k$

Theorem 1.2

$$t(k,r) > \frac{1}{2re}k^2. \tag{3}$$

The lower bound established in (2) is better than the one established in (3) for all fixed r. In addition, the bound in (2) is constructive, while the bound in (3) is not. The proof of Theorem 1.1 is probabilistic, and in it we introduce an iterative method that enables us to show, like in the Local Lemma, that none of a set of rare events holds. We believe that the method used in the proof of Theorem 1.1 may be applied to other similar combinatorial problems. Our proof is constructive in the sense that it implies an efficient randomized algorithm for finding the independent transversal. We mention how it can be *derandomized*, and hence obtain the second part of Theorem 1.1.

A constructive upper bound of $(1+o(1))k^2$ for $t(k,1)$ is described in [5]. We conjecture that $t(k,r) \le (1+o_k(1))k^2/r$ for all $1 \le r \le k$. (Here $o_k(1)$ denotes a quantity tending to zero as $k \to \infty$.) We are able to prove this for a very wide range of values of r.

Theorem 1.3 *For every $\epsilon > 0$, if $\frac{432}{\epsilon^2} \log k \le r \le \frac{\epsilon}{3} k$ then $t(k,r) < (1+\epsilon)k^2/r$.*

Turning our attention to independent coverings, it turns out that in this case we can establish the *exact* values of $c(k,1)$ and $c(k,2)$. In fact, proving that $c(k,1)=k$ is rather simple. Somewhat surprising is the fact that $c(k,2)=k$ as well, but the proof in this case is more difficult. In fact, we show the following:

Theorem 1.4 *If $1 \le r \le k$, then $k \ge c(k,r) \ge \min\{k, k-r+2\}$.*

Note that when r approaches k, the theorem is not tight. In fact, we conjecture that for all $k \ge 4$, $c(k,r)=k$ for all $r = 1,\dots,k$ [1]. Note that since, clearly, $c(k,r)$ is a monotone decreasing function of r, it suffices to prove the following.

Conjecture 1.5 *For all $k \ge 4$ it holds that $c(k,k)=k$.*

Currently, we do not even know the exact value of $c(k,3)$ for all k.

The remainder of this paper is organized as follows. In section 2 we prove the lower bounds on $t(k,r)$, namely Theorems 1.1 and 1.2. In section 3 we discuss upper bounds for independent transversals and prove Theorem 1.3. In section 4 we study independent coverings and prove Theorem 1.4.

[1]In the original version of this paper we conjectured this also for $k=3$, but this is easily seen wrong as observed by MacKeigan; a simple computer program verifies shows that already $c(4,4)=4$)

2 Lower bounds for independent transversals

We begin this section by proving Theorem 1.2, since we require it as an ingredient in the proof of Theorem 1.1. The proof of Theorem 1.2 is based on the Lovász Local Lemma and, in fact, it is merely a generalization of the proof presented in [5] for the case $r = 1$, and is also very similar to the proof of Proposition 5.3 in Chapter 5 of [1].

Proof of Theorem 1.2 Let G be an $[n, k, r]$-partite graph with $n < 1 + (2re)^{-1}k^2$. We pick, from each vertex class of G, randomly and independently a single vertex according to a uniform distribution. Let T be the random set of vertices picked. We must show that with positive probability, T is an independent transversal. For each edge e of G, let A_e be the event that T contains both endpoints of T. We clearly have $\text{Prob}[A_e] = 1/k^2$. Note that A_e is independent of all the events corresponding to edges whose endpoints do not lie in any of the two vertex classes of the endpoints of e. Hence, A_e is independent of all but at most $2r(n-2)+r-1$ other events. Now, since $k^{-2}e(2r(n-2)+r) = k^{-2}er(2n-3) < 1$, we infer from the Local Lemma that with positive probability, no event A_e holds. This means that, with positive probability, T is an independent transversal. □

The following lemma establishes the properties that we require from $g(r)$ in Theorem 1.1.

Lemma 2.1 *For an integer $r \geq 1$ and $0 < x < g(r)$, we have*

$$(1 - r + re^{-x})^2 x > x - 1/r + e^{-rx}/r.$$

Proof Put $x = c/r$. For $r \geq 2$ it suffices to prove that if $0 < c < 0.52$ then

$$1 - r + re^{-c/r} > (1 - 1/c + e^{-c}/c)^{1/2}.$$

The r.h.s. is a monotone increasing function of c in the interval $(0,\infty)$ and is less than 0.48 for $c = 0.52$. For the l.h.s. we can use the inequality $e^{-y} > 1-y$, and obtain that $1-r+re^{-c/r} > 1-c > 0.48$. Note that the constant 0.52 is not tight for small r (since $e^{-y} > 1-y$ is not tight for large y). In fact, for $r=1$ we may even choose the constant 0.65, which respects the definition of $g(1)$. $\square$

The next lemma can easily be proved by applying l′Hôpital′s rule.

Lemma 2.2 *For every $\epsilon > 0$ and every $r \geq 1$, there exists a positive real $M = M(\epsilon, r)$ such that if $k > M$ then*

$$1-(1-\frac{1}{k})^{r/k} < \frac{r}{(1-\epsilon)k^2}.$$

Proof of Theorem 1.1 Let $r \geq 1$, and $0 < C < g(r)$ be fixed. We must show that there exists an integer $K = K(C)$ such that for every $k \geq K$, we have $t(k,r) \geq Ck^2$.

Let $\epsilon > 0$ be chosen such that the following holds:

$$(1-\epsilon)(1-r+r(1-\epsilon)^2e^{-C})^2C \geq C-(1-\epsilon)^3(1/r-e^{-rC}/r). \tag{4}$$

The existence of ϵ is guaranteed by Lemma 2.1. For $i \geq 0$, put $C_i = C(1-\epsilon)^i$, and let $l \geq 0$ be the minimal integer for which $C_l < (2er)^{-1}$. For $i \geq 0$ and positive integer x, we define the function $k_i(x)$ recursively, as follows: $k_0(x) = x$ and for $i \geq 1$ we define

$$k_i(x) = \lceil k_{i-1}(x)(1-r+r(1-\epsilon)^2e^{-C_{i-1}})\rceil. \tag{5}$$

Note that we have $0 < k_i(x) \leq k_{i-1}(x)$ for all $i \geq 1$ and for all $x \geq 1$. Furthermore,

$$k_i(x) \geq x(1-r+r(1-\epsilon)^2e^{-C})^i. \tag{6}$$

We can therefore define the following three constants K_1,K_2 and K_3.

1. K_1 is the least positive integer for which $k_l(K_1) > M(\epsilon, r)$ where $M(\epsilon, r)$ is defined in Lemma 2.2.

2. K_2 is the least positive integer for which

$$(1 - \frac{1}{k_l(K_2)})^{k_l(K_2)} > \frac{1-\epsilon}{e}.$$

3. K_3 is the least positive integer such that for all $k > k_l(K_3)$ and all $1 \le i \le l$

$$(1-\epsilon)(1-e^{-rC_i}) \le 1 - (1 - \frac{1}{k})^{r\lfloor C_i k^2 \rfloor / k}$$

holds. (Note that if we did not insist on the floor function in the above inequality, the inequality would hold for all $k > 0$ even without the $(1-\epsilon)$ factor).

Next, we put

$$\gamma = \frac{\epsilon(1-\epsilon)^2(1-e^{-rC})}{rC - (1-\epsilon)^3(1-e^{-rC})}. \tag{7}$$

Note that if we replace C with C_i for any $i > 0$ in (7) we obtain a value which is greater than γ. Let K_4 be the least positive integer which has the property that for every $k \ge K_4$

$$rCk^2 e^{-k_l(k)(2\epsilon^2(1-\epsilon)^2 e^{-2C})} < \frac{\gamma}{2}. \tag{8}$$

By (6) K_4 exists. Finally, we put $K = \max\{K_1, K_2, K_3, K_4\}$.

Let $k \ge K$, and put $k_i = k_i(k)$. We will show that $t(k_i, r) \ge \lfloor C_i k_i^2 \rfloor$ for $i = 0, \ldots l$, which, for $i = 0$, implies the theorem. We will show this by induction on i, starting from $i = l$ and descending toward $i = 0$. For the basis of our induction, we need to show that $t(k_l, r) \ge C_l k_l^2$. This is indeed the case since $C_l < (2er)^{-1}$, and Theorem 1.2 applies. We now assume that $t(k_{i+1}, r) \ge \lfloor C_{i+1} k_{i+1}^2 \rfloor$, and we show that this implies that $t(k_i, r) \ge \lfloor C_i k_i^2 \rfloor$. Let G be an $[n, k_i, r]$-partite graph with $n = \lfloor C_i k_i^2 \rfloor$, and denote its vertex classes by $V_1, \ldots, V_n$, where $V_j = \{v_{j1}, \ldots, v_{jk_i}\}$ for $j = 1, \ldots, n$. Let $d(v_{jt})$ denote the degree of the vertex v_{jt} in G, and let $d^{j-1}(v_{jt})$ denote the number of neighbors of v_{jt} in $V_1 \cup \ldots \cup V_{j-1}$. We pick from each vertex class of G, randomly and independently, a single vertex according to a uniform distribution. Let $T = \{u_1, \ldots, u_n\}$ be the random set of vertices picked. We now construct the independent set $T' \subset T$ which contains all vertices u_j such that none of $u_1, \ldots, u_{j-1}$ are adjacent to u_j. We call a vertex class V_j *good* if $u_j \in T'$. Let p_j denote the probability that V_j is good. Clearly

$$p_j = \frac{1}{k_i} \sum_{t=1}^{k_i} (1 - \frac{1}{k_i})^{d^{j-1}(v_{jt})}.$$

Since $\sum_{t=1}^{k_i} d^{j-1}(v_{jt}) = r(j-1)$, we have by convexity,

$$p_j \geq (1 - \frac{1}{k_i})^{\frac{r(j-1)}{k_i}}.$$

If X is the number of good vertex classes we have

$$E(X) \geq \sum_{j=1}^{n} (1 - \frac{1}{k_i})^{\frac{r(j-1)}{k_i}} = \frac{1 - (1 - 1/k_i)^{rn/k_i}}{1 - (1 - 1/k_i)^{r/k_i}} \geq (1-\epsilon)\frac{k_i^2}{r}(1 - (1 - 1/k_i)^{rn/k_i}) \geq (1-\epsilon)^2 \frac{k_i^2}{r}(1 - e^{-rC_i}) \tag{9}$$

The third inequality in (9) follows from Lemma 2.2 by the fact that $k \geq K_1$ and therefore $k_i = k_i(k) \geq k_l(k) \geq k_l(K_1) > M(\epsilon, r)$. The rightmost inequality in (9) similarly follows from the fact $k \geq K_3$ and hence $k_i \geq k_l(K_3)$.

We will need to use the following easily proved probabilistic fact:

Claim: Let X be a random variable where $0 \leq X \leq n$ always holds, and $E(X) = \mu$. Then

$$\text{Prob}[X \leq (1 - \epsilon)\mu] \leq \frac{n - \mu}{n - (1 - \epsilon)\mu}.$$

Proof of claim: Put $p = \text{Prob}[X \leq (1 - \epsilon)\mu]$. Then $\mu = E(x) \leq (1 - p)n + p(1 - \epsilon)\mu$. The claim clearly follows.

Let A be the event that there are at least $(1-\epsilon)^3(k_i^2/r)(1-e^{-rC_i})$ good vertex classes. By the last claim and by (9) with $\mu = E(X)$,

$$\mathrm{Prob}[A] \geq \mathrm{Prob}[X \geq (1-\epsilon)\mu] \geq 1 - \frac{n-\mu}{n-(1-\epsilon)\mu} = \tag{10}$$

$$\frac{\epsilon(\mu/n)}{1-(1-\epsilon)(\mu/n)} \geq \frac{\epsilon(1-\epsilon)^2(1-e^{-rC_i})}{rC_i-(1-\epsilon)^3(1-e^{-rC_i})} \geq \gamma.$$

We can easily partition G into a union of r spanning graphs of G, $G_1,\ldots G_r$, each being an $[n,k_i,1]$-partite graph. That is, each edge of G appears in exactly one of the graphs G_s, for $s=1,\ldots,r$. Let $d^{(s)}(v_{jt})$ denote the degree of the vertex v_{jt} in G_s. Let $X_j^{(s)}$ be the number of vertices of V_j that are not adjacent in G_s to any vertex of T. Let $X_{jt}^{(s)}$ be the indicator random variable whose value is 1 if no neighbor of v_{jt} in G_s is in T, and 0 otherwise. Clearly, $X_j^{(s)} = \sum_{t=1}^{k_i} X_{jt}^{(s)}$ and

$$E(X_j^{(s)}) = \sum_{t=1}^{k_i} \mathrm{Prob}[X_{jt}^{(s)} = 1] = \sum_{t=1}^{k_i} (1-\frac{1}{k_i})^{d^{(s)}(v_{jt})}.$$

Since $\sum_{t=1}^{k_i} d^{(s)}(v_{jt}) = n-1$, we again have by convexity that

$$E(X_j^{(s)}) \geq k_i(1-\frac{1}{k_i})^{\frac{n-1}{k_i}} \geq k_i(1-\frac{1}{k_i})^{C_ik_i} \geq k_i(1-\epsilon)^{C_i}e^{-C_i} \geq k_i(1-\epsilon)e^{-C_i}.$$

The third inequality follows from the fact that $k > K_2$ and hence $k_i \geq k_l(K_2)$. The rightmost inequality follows follows from $C_i \leq C < g(r) < 1$. The crucial point to observe is that the r.v's $X_{j1}^{(s)},\ldots,X_{jk_i}^{(s)}$ are independent since in G_s there is only one edge between any two vertex classes. We may therefore use the large deviation result of Chernoff [4] (see also [1] appendix A), to obtain that

$$\mathrm{Prob}[X_j^{(s)} < (1-\epsilon)^2k_ie^{-C_i}] < e^{-\frac{2(\epsilon(1-\epsilon)k_ie^{-C_i})^2}{k_i}} = e^{-k_i(2\epsilon^2(1-\epsilon)^2e^{-2C_i})}. \tag{11}$$

Let $B_j^{(s)}$ be the event that $X_j^{(s)} \geq (1-\epsilon)^2k_ie^{-C_i}$, and let $B = \cap_{j,s}B_j^{(s)}$. By (11) we have that

$$\mathrm{Prob}[\overline{B}] \leq rne^{-k_i(2\epsilon^2(1-\epsilon)^2e^{-2C_i})}. \tag{12}$$

We will now show that $\text{Prob}[A \cap B] > 0$. In fact, we will show something slightly stronger, namely that $\text{Prob}[\overline{B}] < 0.5\text{Prob}[A]$. Indeed, by (12) and (10) it suffices to show that

$$rne^{-k_i(2\epsilon^2(1-\epsilon)^2 e^{-2C_i})} < \frac{\gamma}{2}. \tag{13}$$

This is true however since $k \geq K_4$ and therefore

$$rne^{-k_i(2\epsilon^2(1-\epsilon)^2 e^{-2C_i})} \leq rC_i k_i^2 e^{-k_l(2\epsilon^2(1-\epsilon)^2 e^{-2C_i})} \leq rCk^2 e^{-k_l(k)(2\epsilon^2(1-\epsilon)^2 e^{-2C})} < \frac{\gamma}{2}$$

where the last inequality follows from (8).

We have shown that with some constant small probability, which depends only on C, both events A and B occur. We now fix a transversal T for which both A and B occur. Let $I \subset \{1, \ldots, n\}$ be the set of indices of the non-good vertex classes. Note that since event A occurs, we have

$$n' = |I| \leq n - (1-\epsilon)^3 \frac{k_i^2}{r}(1 - e^{-rC_i}) \leq C_i k_i^2 (1 - (1-\epsilon)^3 \frac{1}{rC_i}(1 - e^{-rC_i})).$$

We claim that each non-good vertex class V_j for $j \in I$, contains a subset $W_j \subset V_j$ of cardinality exactly k_{i+1}, such that every vertex of W_j has no neighbor in T. This is true since the fact that event B occurs implies, in particular, that the events $B_j^{(s)}$ occur for $s = 1, \ldots, r$, which implies that there are at most $k_i - (1-\epsilon)^2 k_i e^{-C_i}$ vertices in V_j that have a neighbor in G_s that is also in T, and hence there are at least

$$\lceil k_i - r(k_i - (1-\epsilon)^2 k_i e^{-C_i}) \rceil = k_{i+1}$$

vertices in V_j that have no neighbor in T. Let us denote by G' the induced n'-partite subgraph of G on the vertex classes W_j for $j \in I$. G' is a spanning subgraph of some $[n', k_{i+1}, r]$-partite graph. The crucial point is that any independent transversal of G' may be extended to an independent transversal of G by taking, for each $j \notin I$, the vertex of V_j that appears in T (recall that these are the vertices of T'). To complete the proof we only need to show that G' contains an independent transversal. We can use the induction hypothesis for $i+1$ if we can show that $n' \leq \lfloor C_{i+1} k_{i+1}^2 \rfloor$. Indeed, by (4) and from the fact that $C_i < C$ we obtain

$$n' \leq C_i k_i^2 (1-(1-\epsilon)^3 \frac{1}{rC_i}(1-e^{-rC_i})) \leq C_i k_i^2 (1-\epsilon)(1-r+r(1-\epsilon)^2 e^{-C_i})^2$$

$$= C_{i+1}(k_i(1-r+r(1-\epsilon)^2 e^{-C_i}))^2 \leq C_{i+1} k_{i+1}^2.$$

This completes the induction step and the proof of the non-algorithmic part of the theorem.

A polynomial ($O(k^4)$ time) randomized algorithm proceeds as follows. We randomly select the transversal T, and check in $O(V+E) = O(k^3)$ time whether events A and B both occur. Recall that we have shown that they both occur with a small (but constant, depending only on C) probability. Hence the expected number of trials until we get a transversal T for which both events A and B occur, is constant. We now apply the inductive step using recursion. The number of recursion steps is exactly l, which is, again, a constant depending only on C. However, at the lowest level of recursion (with k_l vertices in each vertex class), we need to apply the Local Lemma, which is non-algorithmic. Fortunately, it was shown by Beck in [2] that in some cases (including our Local Lemma proof), a constructive algorithmic version of the lemma can be obtained (with running time $o(k_l^4) = o(k^4)$ in our case), but at the price of a significant decrease in the constants. That is, the $(2re)^{-1}$ constant in Theorem 1.2 is replaced by a much smaller constant $c = c(r)$. However, we could easily have modified the definition of l to be the smallest nonnegative integer such that $C_l < c(r)$, (with the remainder of the proof intact).

To obtain a deterministic version of our randomized algorithm, we need to show how to build T deterministically so that events A and B occur. Once again, this can be done by a standard technique of derandomization, namely that of *conditional probabilities* (cf. [1]). This is due to the fact that the algorithm needs to make n choices (select a vertex from each vertex class), where prior to the selection of the first vertex, the probability that the event $A \cap B$ will occur is a positive constant. Assuming that the algorithm selected vertices from $V_1, \ldots, V_j$ in such a way that the probability that $A \cap B$ will occur in a random selection of vertices from the remaining vertex classes $V_{j+1}, \ldots, V_n$ is some constant c, a vertex is selected from V_{j+1} in such a way that the probability that $A \cap B$ will occur in a random selection of vertices from $V_{j+2}, \ldots, V_n$ will be *at least* c. Clearly, at least one vertex of V_{j+1} must have this property. Therefore, each vertex of V_{j+1} is examined, and the conditional probability corresponding to it is estimated, and the vertex with the largest conditional probability estimate (which must be larger than c) is selected. □

3 Upper bounds for independent transversals

As mentioned in the introduction, it is shown in [5] that $t(k,1) \le (1+o(1))k^2$, where the bound is obtained by an explicit construction based on affine planes of order $k+1$, whenever they exist. However, even for $r = 2$ we have no explicit construction which achieves a non-trivial upper bound.

A probabilistic construction achieving $t(k,r) \le (1+o(1))2k^2 \ln k/r$ is obtained by randomly constructing $[n,k,r]$-partite graphs, where r independent edges between every two vertex classes are selected randomly and independently among all possible choices of r independent edges. Each transversal has a probability of $(1 - r/k^2)^{\binom{n}{2}}$ of being independent, which is less than $1/k^n$ when $n > 2k^2 \ln k/r$. Hence, there exists such a graph with no independent transversal. This construction is far from being optimal for large r. In particular, it is easy to see that $t(k,k) = k$. Taking k vertex-disjoint complete $k+1$ vertex graphs, one uniquely obtains a $[k+1,k,k]$-partite graph having no independent transversal. A greedy algorithm can easily construct an independent transversal in very $[k,k,k]$-partite graph. In fact, we can show that whenever $\log k = o(r)$, $t(k,r) \leq Ck^2/r$. Theorem 1.3 says that with the additional restriction that $r = o(k)$, we have $C = 1 + o(1)$. When r is close to k we can show that $C = 9/8 + o(1)$. We do not prove this here since we conjecture that $t(k,r) = (1+o(1))k^2/r$ for all $1 \leq r \leq k$. Note that Theorem 1.3 establishes this for many values of r. However, it does not give an explicit construction.

Proof of Theorem 1.3. Let $\epsilon > 0$. (We also implicitly assume $\epsilon \leq 1$). Let k and r be positive integers such that $432 \log k/\epsilon^2 \leq r \leq \epsilon k/3$. Note that we may assume $k \geq 432$. Let n be a positive integer satisfying $n = 0 \bmod k+1$ and

$$(1+\frac{\epsilon}{2})\frac{k^2}{r} + 1 \leq n \leq (1+\epsilon)\frac{k^2}{r}.$$

Such an n exists since $\epsilon k^2/(2r) \geq k+2$. Consider an $[n,k,k]$-partite graph G whose vertex classes are $V_1,\ldots,V_n$ and $V_i = \{v_{i1},\ldots,v_{ik}\}$. Let $L_j = \{v_{1j},\ldots,v_{nj}\}$, $j = 1,\ldots,k$. We partition each L_j, randomly and independently into $n/(k+1)$ pairwise-disjoint subsets of size $k+1$ each. Let $K_{j,1},\ldots,K_{j,n/(k+1)}$ be the random partition of L_j. Construct a complete graph on the vertices of each $K_{j,p}$, $j = 1,\ldots,k$, $p = 1,\ldots,n/(k+1)$. These cliques define the edges of G. Clearly, the independence number of G is $kn/(k+1)$. Hence, it does not contain an independent transversal. We now show that with positive probability G is an $[n,k,r]$-partite graph. Clearly, the edges between each two vertex classes are independent. For two vertex classes V_x, V_y, the probability that $v_{x,j}$ and $v_{y,j}$ are connected is $k/(n-1)$. Hence if $X_{x,y}$ is the number of edges between them, then $E(X_{x,y}) = k^2/(n-1)$. Our aim is to show that $\mathrm{Prob}[X_{x,y} > r] < 1/\binom{n}{2}$, which implies the theorem. By our choice of n we have

$$\frac{r}{1+\epsilon} \leq \frac{k^2}{n} \leq \mu = E(X_{x,y}) = \frac{k^2}{n-1} \leq \frac{r}{1+\epsilon/2} = \frac{2r}{2+\epsilon}.$$

By the Chernoff estimates [4] we have that if $a > 0$,

$$\text{Prob}[X_{x,y} - \mu > a] < e^{-a^2/(2\mu)+a^3/(2\mu^2)}.$$

Hence if $a = r\epsilon/(2+\epsilon)$ we have

$$\text{Prob}[X_{x,y} > r] = \text{Prob}[X_{x,y} - 2r/(2+\epsilon) > r\epsilon/(2+\epsilon)] \le \text{Prob}[X_{x,y} - \mu > r\epsilon/(2+\epsilon)] <$$

$$e^{-\frac{r^2\epsilon^2(2+\epsilon)}{(2+\epsilon)^2 4r}+\frac{r^3\epsilon^3(1+\epsilon)^2}{(2+\epsilon)^3 2r^2}} = e^{-r(\frac{\epsilon^2}{4(2+\epsilon)}-\frac{\epsilon^3(1+\epsilon)^2}{2(2+\epsilon)^3})} < e^{-432(\frac{1}{8+4\epsilon}-\frac{\epsilon(1+\epsilon)^2}{2(2+\epsilon)^3})\log k} <$$

$$e^{-432\frac{1}{4\cdot 3^3}\log k} = k^{-4} \le \frac{r^2}{k^4(1+\epsilon)^2} \le n^{-2} < 1/\binom{n}{2}.$$

□

4 Independent coverings in sparse partite graphs

In this section we prove Theorem 1.4. Let us start with the upper bound, which is easy. We need to show that $c(k,r) \le k$. Since, clearly, $c(k,r+1) \le c(k,r)$, it suffices to show that $c(k,1) \le k$. Indeed consider an $[n,k,r]$-partite graph that contains a K_n (a clique of order n). That is, in every vertex class there is only one non-isolated vertex and the set of all non-isolated vertices forms a clique. Clearly, any independent covering must contain at least n independent transversals. This is possible only if $n \le k$. Consequently, $c(k,1) \le k$.

In order to prove the lower bound, we will show that for all $k \ge r \ge 2$, any $[k-r+2,k,r]$-partite graph contains an independent covering. This implies that any $[n,k,r]$-partite graph with $n \le k-r+2$ also contains an independent covering (since it is an induced subgraph of some $[k-r+2,k,r]$-partite graph). In particular, we will have $c(k,r) \ge k-r+2$. For $r = 2$, this implies that $c(k,2) \ge k$. On the other hand, we have seen that $k \ge c(k,1) \ge c(k,2)$, so $c(k,1) = c(k,2) = k$. (Thus we have $c(k,1) = k$ without proving it directly).

Let $k \geq 3$ and let $k \geq r \geq 2$ (the case $k = r = 2$ is trivial). Let G be a $[k-r+2, k, r]$-partite graph with vertex classes $V_1, \ldots, V_{k-r+2}$. Note that the maximum degree of a vertex in G is at most $k-r+1$, and any vertex class has at most r vertices with degree $k-r+1$. We distinguish between two cases.

Case 1: every vertex class has exactly r vertices of degree $k-r+1$ each. Let H be the induced subgraph of G consisting of all these $r(k-r+2)$ vertices. H is a $(k-r+2)$-partite graph. Let H' be obtained from H by adding edges that connect vertices in the same vertex class. The degree of every vertex in H' is exactly $(r-1)+(k-r+1) = k$. However, H' does not contain a clique of order $\max\{r+1, k-r+3\} \leq k+1$ since the set of edges connecting two distinct vertex classes is independent (it is a matching). Hence by a theorem of Brooks (cf. [3]), H' is k-colorable. Assume that the colors are $\{1, \ldots, k\}$. Consider a vertex class V_j. The r vertices of it with degree $k-r+1$ received r distinct colors in H'. Color the remaining $k-r$ vertices of V_j (which must all be isolated vertices) arbitrarily by the $k-r$ remaining colors. This is done for $j = 1, \ldots k-r+2$. It is easy to see that for $i = 1, \ldots, k$, the set of vertices colored by the color i is an independent transversal.

Case 2: at least one vertex class has at most $r-1$ vertices of degree $k-r+1$. We may assume that V_{k-r+2} is such a class. We start coloring the vertex classes with k colors, beginning with V_1. Our coloring has the property that each vertex class that has been colored has exactly one vertex colored by each color, and the set of vertices colored by a specific color is independent. Assume we have already colored $V_1, \ldots, V_j$, where $j < k-r+2$. We now need to match k colors to the k vertices of V_{j+1}, and remain with a proper coloring. Define a bipartite graph H as follows: One vertex class of H is V_{j+1}, and the other is the set of colors $K = \{1, \ldots, k\}$. A vertex $v \in V_{j+1}$ is connected in H to a color $i \in K$ iff no neighbor of v is already colored by i. Clearly, our aim is to obtain a perfect matching in H. Let $S \subset V_{j+1}$ be nonempty. Let $T \subset K$ be the set of neighbors in H of the vertices of S. Put $s = |S|$ and $t = |T|$. Our aim is to show that $s \leq t$, which implies, by Hall's condition (cf. [3]), that H contains a perfect matching. There are at least $s(k-t)$ edges adjacent to the vertices of S in G, and whose other endpoint is already colored. This is because each vertex of S is adjacent in G to at least one vertex whose color is i for each $i \in K \setminus T$. On the other hand, there are at most $j \cdot s$ vertices adjacent to the vertices of S in G, and whose other endpoint is colored. Hence, we must have

$$j \geq k - t. \tag{14}$$

If $s \leq r$, we must have $s \leq t$. Otherwise, we would have $j \leq k-r+1 \leq k-s+1 \leq k-t$. According to (14) this is possible only if $j = k-r+1$, $s = r$ and $t = s-1$. However, this means that there are r vertices in V_{k-r+2} (namely the set S), each of degree exactly $k-r+1$, which contradicts our assumption for case 2.

We now remain with the case $s > r$. Assume first that $s \geq k-r+2$. In this case we have at least $s(k-t)$ edges going from S to the set of $j(k-t)$ vertices whose colors are in $K \setminus T$. Hence, at least one vertex colored by a color from $K \setminus T$ is adjacent to at least s/j vertices of S. But $s \geq k-r+2 > k-r+1 \geq j$, and this is impossible in a sparse partite graph. Next, assume that $s < k-r+1$. We will show that if $t < s$, then $s(k-t) > r(k-r+1)$ which is clearly a contradiction as there are at most $r(k-r+1)$ edges adjacent to V_{j+1}. Indeed, if $t < s$ then $(k-r+2)(s-r) > (s+1)(s-r) \geq s(t+2-r) - r$, which is equivalent to $s(k-t) > r(k-r+1)$. Finally, we remain with the case that $s = k-r+1$. If $j < k-r+1$ or $t < s-1$ we may again derive a contradiction by showing, in the same way, that $s(k-t) > rj$. The only remaining case to handle is

$$s = k - r + 1, \;\; t = s - 1 = k - r, \;\; j = k - r + 1. \tag{15}$$

Our aim is to show that this case could have been avoided by a careful choice of the coloring of V_1 and V_2 (we assume $k - r + 2 > 2$, since the case $r = k$ is trivial, a perfect matching in the bipartite complement of $V_1 \cup V_2$ is also an independent covering in this case). Let us denote by $W_l \subset V_l$ for $l = 1, \ldots, k - r + 1$, the set of vertices of V_l colored by a color from $K \setminus T$. Note that $|W_l| = k - t = k - (k - r) = r$. There are $r(k - r + 1)$ vertices in $\cup_{l=1}^{k-r+1} W_l$ and each has at most one neighbor in S. On the other hand, each vertex of S has at least $k - t = r$ neighbors in $\cup_{l=1}^{k-r+1} W_l$. Since $s = k - r + 1$ it follows that each vertex of $\cup_{l=1}^{k-r+1} W_l$ has exactly one neighbor in S, and each vertex of S has exactly r neighbors in $\cup_{l=1}^{k-r+1} W_l$, one from each color of $K \setminus T$. Furthermore, since there are exactly $r(k - r + 1) = rs$ edges adjacent to V_{k-r+2}, it follows that the vertices of $V_{k-r+2} \setminus S$ are isolated, and that there are no edges between S and $\cup_{l=1}^{k-r+1} V_l \setminus W_l$. Note that this is an explicit configuration that could have been recognized prior to the beginning of the coloring of V_1. That is, we could have checked whether, in fact, V_{k-r+2} has exactly $k - r + 1$ non-isolated vertices, each of degree exactly r, (and denote this set by S) and that there are exactly r neighbors of S in each vertex class V_l for $l = 1, \ldots, k - r + 1$ (and denote this set by W_l). If such a configuration does not exist, we are done. If it does exist (and we have shown how to recognize it prior to the beginning of the coloring of the first vertex class), we will show how we could have colored the vertex classes in such a way that $\cup_{l=1}^{k-r+1} W_l$ contains vertices with more than r colors. In fact, as we mentioned before, we will show how we can color V_1 and V_2 in such a way that $W_1 \cup W_2$ contains more than r colors. Let $w \in V_2 \setminus W_2$. Since $r \geq 2$ there exists $u \in W_1$ such that w and v are not connected. Color v and w with the same color. Now we note that the bipartite complement of $V_1 \setminus \{v\}$ and

$V_2 \setminus \{w\}$ has minimum degree $k-2$, and $k-1$ vertices in each vertex class. Since $k \geq 3$, it contains a perfect matching, which determines the coloring of the other $k-1$ pairs. Note that $W_1 \cup W_2$ contains at least $r+1$ colors. The coloring of V_3 etc. proceeds as before, and we are guaranteed that the situation in (15) will not occur. □

5 Acknowledgment

The author wishes to thank Noga Alon for meaningful discussions, and the anonymous referee for his valuable comments leading to an improvement of the result obtained in Theorem 1.1.

References

[1] N. Alon and J. H. Spencer, *The Probabilistic Method*, John Wiley and Sons Inc., New York, 1991.

[2] J. Beck, *An algorithmic approach to the Lovász Local Lemma*, Random structures and algorithms 2 (1991), 343-365.

[3] B. Bollobás, *Extremal graph theory*, Academic press, New York 1978.

[4] H. Chernoff, *A measure of the asymptotic efficiency for tests of a hypothesis based on the sum of observations*, Ann. Math. Stat. 23 (1952), 493-509.

[5] P. Erdös, A. Gyárfás and T. Łuczak, *Independent transversals in sparse partite hypergraphs*, Combinatorics, Probability and Computing 3 (1994), 293-296.

[6] P. Erdös and L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, Infinite and Finite Sets (A. Hajnal et al., eds.), North-Holland, Amsterdam (1975), 609-628.