Second Neighborhood via First Neighborhood in Digraphs

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Abstract

Let D be a simple digraph without loops or digons. For any $v \in V(D)$, the first out-neighborhood $N^+(v)$ is the set of all vertices with out-distance 1 from v and the second neighborhood $N^{++}(v)$ of v is the set of all vertices with out-distance 2 from v. We show that every simple digraph without loops or digons contains a vertex v such that $|N^{++}(v)| \geq \gamma|N^+(v)|$, where $\gamma = 0.657298\ldots$ is the unique real root of the equation $2x^3 + x^2 - 1 = 0$.

1 Introduction

All digraphs considered in this article are finite and without loops or multiple edges. We also assume that all digraphs do not have digons, i.e. both (u, v) and (v, u) are arcs. Let $D = (V, A)$ denote a digraph with vertex set V and arc set A. For any vertex $v \in V(G)$, let $N^+(v) = \{w : (v, w) \in A\}$ and $d^+(v) = |N^+(v)|$ (the outdegree of v). For any subgraph H, let $N^+_H(v) = N^+(v) \cap V(H)$ and $d^+_H = |N^+_H(v)|$. For any $W \subset V$, we let $G[W]$ denote the subgraph induced by W and $N^+_W(v) = N^+_{G[W]}(v)$ and $d^+_W(v) = d^+_{G[W]}(v)$. Let $N^-(v) = \{u : (u, v) \in A\}$. Similarly, we define $d^-(v)$ (the indegree of v), $N^-_H(v)$, $d^-_H(v)$, $N^-_W(v)$, and $d^-_W(v)$. For any $S \subset V$, we define $N^+(S) = \cup_{s\in S} N^+(s) - S$ and $N^-(S) = \cup_{s\in S} N^-(s) - S$. For any $v \in V$, let $N^{++}(v) = N^+(N^+(v))$ and $d^{++}(v) = |N^{++}(v)|$. Let r be a positive integer, a digraph D is named *r-regular* if $d^+(v) = d^-(v) = r$ for all $v \in V(D)$. For any two disjoint vertex sets X, $Y \subseteq V$, we let $E(X, Y)$ denote the arcs from X to Y and $e(X, Y) = |E(X, Y)|$. Since we assume that D does not have any digon, we have that

$$e(X, Y) + e(Y, X) \leq |X| \times |Y|$$

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for any two disjoint vertex sets X and Y.

For the purpose of this article, all cycles considered here are direct cycles. The *girth*, denoted by $g(D)$, of a digraph D is the length of shortest direct cycle if D contains a cycle. Naturally, one can expect the girth of a graph will be small if the digraph either large minimum indegree or large minimum outdegree or both. The following three conjectures express this phenomenon.

Conjecture 1 ([2]) *If D is an r-regular digraph on n vertices, then the girth $g(D) \leq \lceil n/r \rceil$.*

Conjecture 2 *If D is a digraph on n vertices with $\min\{\delta^+(D), \delta^-(D)\} \leq r$, then the girth $g(D) \leq \lceil n/r \rceil$.*

Conjecture 3 ([5]) *If D is a digraph on n vertices with $\delta^+(D) \geq r$, then $g(D) \leq \lceil n/r \rceil$.*

Conjecture 2 implies Conjecture 1 and Conjecture 3 implies Conjecture 2. Conjecture 2 has been well-known although who originally made the conjecture is unknown to us. Behzad, Chartran, and Wall [2] constructed examples showing that Conjecture 1 is best possible, so are Conjectures 2 and 3. While all three conjectures are unsolved, some progresses have been made. Conjecture 1 has been verified for $r = 2$ by Behzad [1], for $r = 3$ by Bermond [3], for vertex-transitive digraph by Hamidoune [10]. Conjecture 2 has been verified for $r \leq 4$ by Hamidoune [11]. Conjecture 3 has been verified for $r = 2$ By Caccetta and Häggkivist [5], for $r = 3$ by Hamidoune [12], and for $r = 4, 5$ by Hoáng and Read [13].

Let D be a digraph on n vertices with $\delta^+(D) \geq r$. Chvátal and Szemerédi [6] proved that $g(D) \leq \min\{2n/(r+1), n/r + 2500\}$. Nishimura [15] proved that $g(G) \leq n/r + 304$. Recently, Shen [17, 18] showed that $g(G) \leq \min\{\lceil n/r \rceil, n/r + 73, 2r - 2\}$. As a consequence, Conjecture 3 is true for $n \geq 2r^2 - 3r + 1$.

Recently, people became particularly interested in a special case of Conjecture 3: Any digraph with n vertices and minimum outdegree at least $n/3$ contains a triangle. By a *triangle*, we shall mean a directed cycle of length 3. Let c be the minimum positive real number such that every digraph D on n vertices with $\delta^+ \geq cn$ contains a triangle. Caccetta and Haggkvist [5] showed that $c \leq (3 - \sqrt{5})/2 = 0.3819\ldots$, Bondy [4] showed that $c \leq (2\sqrt{6} - 3)/5 = 0.3797\ldots$, Shen [16] showed that $c \leq 3 - \sqrt{7} = 0.3542\ldots$. Let β be the minimum positive real number such that every digraph D on n vertices with $\min\{\delta^+(D), \delta^-(D)\} \geq \beta n$ contains a triangle. Graaf, Schrijver, and Seymour [9] showed that $\beta \leq 0.3487\ldots$. In fact, they showed that a upper bound of β can be obtained from the inequality:

$$\left(\frac{4}{\alpha^2} - \frac{2}{\alpha}\right) x^2 - \left(\frac{24}{\alpha^2} - \frac{16}{\alpha}\right) x + \left(\frac{36}{\alpha^2} - \frac{30}{\alpha} + 1\right) > 0,$$

where α can be chosen to be any number greater than or equal to c. By choosing $\alpha = 3 - \sqrt{7}$ in the above inequality, Shen [16] showed that $\beta \leq 0.3477\ldots$.

2 The Second Neighborhood Conjecture

Seymour (see [7]) put forward the following conjecture which would implies the case $r = \lceil n/3 \rceil$ of Conjecture 2.

Conjecture 4 ([7]) *For any digraph D, there exists a vertex v such that $d^{++}(v) \geq d^{+}(v)$.*

Fisher [8] showed that Conjecture 4 is true if D is a tournament, which is conjectured to be true by Dean [7]. Kaneko and Locke [14] and others verified Conjecture 4 for digraphs with maximum degree at most 6. Another approach to Conjecture 4 is to determinate the maximum value of c such that there is a vertex v satisfying $d^{++}(v) \geq cd^{+}(v)$ for every digraph D. The relation between this parameter and minimum outdegree condition for a digraph containing a triangle is stated below.

Proposition 5 *If β is a positive real number such that, for every digraph D, there exists a vertex v such that $d^{++}(v) \geq \beta d^{+}(v)$, then any digraph D on n vertices has a triangle if* $\min\{\delta^{+}(D), \delta^{+}(D)\} \geq \frac{n}{2+\beta}$.

Proof: Let D be a digraph on n vertices with minimum outdegree $\delta^{+}(D) \geq \frac{n}{2+\beta}$. Since

$$\sum_{v \in V} d^{-}(v) = \sum_{v \in V} d^{+}(v) \geq \frac{n^2}{2+\beta},$$

there is a vertex u such that $d^{-}(u) \geq \frac{n}{2+\beta}$. Thus, we have that

$$\begin{aligned} |N^{+}(u)| &\geq \frac{n}{2+\beta}, \\ |N^{-}(u)| &\geq \frac{n}{2+\beta}, \text{ and} \\ |N^{++}(u)| &\geq \beta |N^{+}(u)| \geq \frac{\beta n}{2+\beta}. \end{aligned}$$

Hence,

$$|N^{+}(u)| + |N^{-}(u)| + |N^{++}(u)| \geq n,$$

which implies that $N^{-}(u) \cap N^{++}(u) \neq \emptyset$ since $N^{+}(u) \cap (N^{-}(u) \cup N^{++}(u)) = \emptyset$. Then, D contains a triangle. $\square$

Taking $\beta = 1$, we see that Conjecture 4 implies the case of $n/r \leq 3$ in Conjecture 2. Let $\gamma = 0.657298\ldots$ be the unique real root of $2x^3 + x^2 - 1 = 0$. The purpose of this paper is to prove the following result.

Theorem 6 *For any digraph D, there exists a vertex $v \in V(D)$ such that $d^{++}(v) \geq \gamma d^{+}(v)$, where $\gamma = 0.657298\ldots$ is the unique real root of $2x^3 + x^2 - 1 = 0$.*

Proof: We will prove Theorem 6 by induction on the number of vertices. Theorem 6 is trivial for digraphs with 1 or 2 vertices. Suppose that D is a digraph on n vertices. Assume, to the contrary, D does not contain a vertex v such that $|N^{++}(v)| \geq \gamma |N^{+}(v)|$.

Let u be a vertex of D with minimum outdegree, i.e. $d^+(u) = \delta^+(D)$. Let D^* be the sub-digraph induced by $N^+(u)$. For convenience, let $A = N^+(u)$, $B = N^{++}(u)$, $a = |A|$, and $b = |B|$. We will show that $e(A,B)+e(B,A) > ab$, a contradiction to that D does not contain any digon.

By our assumption, the following inequality holds,

$$b = d^{++}(u) < \gamma d^+(u) = \gamma a. \tag{1}$$

Since $a = d^+(u) = \delta^+(D)$, then $d^+_A(x) + d^+_B(x) = d^+(x) \geq d^+(u) = a$ for every vertex $x \in A$. Since D does not contain any digon, we have that $\sum_{x\in A} d^+(x) \leq a(a-1)/2$. Thus,

$$e(A,B) = \sum_{x\in A} d^+_B(x) \geq \sum_{x\in A} (a - d_A(x)) \geq a^2 - a(a-1)/2 > a^2/2. \tag{2}$$

Since $|A| = a < n$, by induction hypothesis, there is a vertex $x \in N(u)$ such that $|N^{++}_A(x)| \geq \gamma |N^+_A(x)|$. Let $X = N^+_A(x)$, $Y = N^+(x) - A = N^+(x) \cap B$, and $d = |Y|$. Since $|A - X| \geq |N^{++}_A(x)| \geq \gamma |X|$, then $(1+\gamma)|X| \leq a$. Thus,

$$|X| \leq \frac{1}{1+\gamma} a \leq \frac{2a}{3},$$

where the last inequality follows since $\gamma \geq 1/2$. Since $d^+(x) \geq \delta^+(D) = d^+(u)$,

$$d = |Y| = |N^+(x)| - |X| \geq a - \frac{2a}{3} = \frac{a}{3}. \tag{3}$$

For every $y \in Y$, since $d^{++}(x) < \gamma d^+(x)$ and $d^{++}_A(x) \geq \gamma d^+_A(x)$, we have

$$d^+_{V-A-Y}(y) \leq d^{++}(x) - d^{++}_A(x) < \gamma d^+(x) - \gamma d^+_A(x) = \gamma |Y| = \gamma d$$

Using the inequalities

$$\begin{aligned} d^+(y) \geq \delta^+(D) &= d^+(u) = a, \text{ and} \qquad (4)\\ \sum_{y\in Y} d^+_Y(y) &\leq d(d-1)/2, \qquad (5) \end{aligned}$$

we obtain the following inequalities.

$$\begin{aligned} e(Y,A) &= \sum_{y\in Y} d^+_A(y) &(6)\\ &\geq \sum_{y\in Y} (a - d^+_{V-A-Y}(y) - d^+_Y(y)) \quad \text{(by 4)} &(7)\\ &\geq (a - \gamma d)d - \sum_{y\in Y} d^+(y) &(8)\\ &\geq (a-\gamma d)d - d(d-1)/2 \quad \text{(by 5)} &(9)\\ &> (a - \gamma d - d/2)d. &(10) \end{aligned}$$

Combining (1), (2) and (10), we obtain that

$$\begin{aligned}
\gamma a^2 &> ab && (11)\\
&\geq e(A,B)+e(B,A) && (12)\\
&\geq e(A,B)+e(Y,A) && (13)\\
&\geq a^2/2+(a-\gamma d-d/2)d., && (14)
\end{aligned}$$

where $a/3 \leq d \leq \gamma a$.

Let $f(z)=a^2/2+(a-\gamma z-z/2)z=-(\gamma+\frac{1}{2})z^2+az+\frac{a^2}{2}$. Since $f(z)$ is a quadratic function with a negative leading coefficient, the following inequality holds.

$$f(z) > \min\{f(a/3), f(\gamma a)\} \text{ for all } z \in (a/3, \gamma a). \quad (15)$$

Thus, $\gamma a^2 > \min\{f(a/3), f(\gamma a)\}$.

A simple calculation gives us that

$$f(a/3)=\frac{a^2(7-\gamma)}{9}.$$

Solving $\gamma a^2 > \frac{a^2(7-\gamma)}{9}$, we obtain that $\gamma > 0.7$, a contradiction.

Also, a simple calculation gives us that

$$f(\gamma a)=\frac{a^2(-2\gamma^3-\gamma^2+2\gamma+1)}{2}.$$

Simplify the inequality

$$\gamma a^2 > \frac{a^2(-2\gamma^3-\gamma^2+2\gamma+1)}{2},$$

we obtain that $2\gamma^3+\gamma^2-1>0$, which contradicts that γ is the unique real root of the equation $2x^3+x^2-1=0$. □

Corollary 7 *If D is a digraph on n vertices and* $\min\{\delta^+(D), \delta^+(D)\} \geq 0.3764n$, *then D contain a triangle.*

Proof: By Theorem 6 and Proposition 5, we have that D contains a triangle if $\delta^+(D) \geq \frac{n}{2+\beta}$. Corollary 7 follows immediately from the fact that $\beta = 0.657298\ldots$. □

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