Equitable coloring of k-uniform hypergraphs

Raphael Yuster
Department of Mathematics
University of Haifa at Oranim
Tivon 36006, Israel.
e-mail: raphy@research.haifa.ac.il

Abstract

Let H be a k-uniform hypergraph with n vertices. A *strong r-coloring* is a partition of the vertices into r parts, such that each edge of H intersects each part. A strong r-coloring is called *equitable* if the size of each part is $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$. We prove that for all $a \geq 1$, if the maximum degree of H satisfies $\Delta(H) \leq k^a$ then H has an equitable coloring with $\frac{k}{a \ln k}(1 - o_k(1))$ parts. In particular, every k-uniform hypergraph with maximum degree $O(k)$ has an equitable coloring with $\frac{k}{\ln k}(1 - o_k(1))$ parts. The result is asymptotically tight. The proof uses a double application of the non-symmetric version of the Lovász Local Lemma.
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1 Introduction

All hypergraphs considered here are finite. For standard terminology the reader is referred to [5]. Let H be a k-uniform hypergraph with n vertices. A *strong r-coloring* is a partition of the vertices of H into r parts, such that each edge of H intersects each part. (A *weak* r-coloring is a coloring where no edge is monochromatic.) A strong r-coloring is called *equitable* if the size of each part is $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$. The study of equitable colorings is motivated by scheduling applications in which some tasks are required to perform at the same time. A good survey on equitable colorings is given in [8]. See also [4, 7] for other related results in the graph-theoretic case. Let $c(H)$ denote the maximum possible number of parts in a strong coloring of H. Let $ec(H)$ denote the maximum possible number of parts in an equitable coloring of H. Trivially, $1 \leq ec(H) \leq c(H) \leq k$. In general, k could be large and still $ec(H) = c(H) = 1$, if we do not impose upper bounds on the maximum degree. Consider the complete k-uniform hypergraph on $2k$ vertices. Trivially, it has $c(H) = 1$, and the maximum degree is less than 4^k. In this paper we prove that $c(H)$ and $ec(H)$ are quite large if the maximum degree is bounded by a polynomial in k. In fact, we get the following asymptotically tight result:

Theorem 1.1 *If $a \geq 1$, and H is a k-uniform hypergraph with maximum degree at most k^a, then $ec(H) \geq \frac{k}{a \ln k}(1 - o_k(1))$. The lower bound is asymptotically tight. For all $a \geq 1$, there exist k-uniform hypergraphs H with maximum degree at most k^a and $c(H) \leq \frac{k}{a \ln k}(1 + o_k(1))$.*

The tightness is shown by exhibiting a random hypergraph with appropriate parameters. Alon [1] has shown that there exist k-uniform hypergraphs with n vertices and maximum degree at most k that do not have a vertex cover (transversal) of size less than $(n \ln k/k)(1 - o_k(1))$. In particular, no strong coloring (moreover an equitable one) could have more than $(k/\ln k)(1 + o_k(1))$ parts. For completeness, we give a general argument valid for all $a \geq 1$ in Section 3. The proof of the main result appears in Section 2. The final section contains some concluding remarks.

2 Proof of the main result

In the proof of Theorem 1.1 we need to use the Lovász Local Lemma [6] in its strongest form, known as the *nonsymmetric version*. Here it is, following the notations in [2] (which also contains a simple proof of the lemma). Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = [n]$ is called a *dependency digraph* for the events $A_1, \ldots, A_n$ if for each i, $i = 1, \ldots, n$, the event A_i is mutually independent of all the events $\{A_j \ : \ (i,j) \notin E\}$.

Lemma 2.1 (The Local Lemma, nonsymmetric version) *If $x_1, \ldots, x_n$ are real numbers so that $0 \leq x_i < 1$ and $\Pr[A_i] \leq x_i \prod_{(i,j)\in E}(1 - x_j)$ for all $i = 1, \ldots, n$, then, with positive probability no event A_i occurs.* $\square$

If the maximum outdegree in D is at most $d \geq 1$ and each A_i has $\Pr[A_i] \leq p$ then, by assigning $x_i = 1/(d+1)$ we immediately obtain:

Corollary 2.2 (The Local Lemma, symmetric version) *If $p(d+1) \leq 1/e$ then with positive probability no event A_i occurs.* $\square$

Proof of Theorem 1.1: Let $a \geq 1$ be any real number, and let $\epsilon > 0$ be small. Throughout the proof we assume k is sufficiently large as a function of a and ϵ. Let k be so large that there is an integer between $\frac{k}{(1+\epsilon^2/4)a \ln k}$ and $\frac{k}{(1+\epsilon^2/8)a \ln k}$. Thus, for some $\gamma \in [\epsilon^2/8 \ , \ \epsilon^2/4]$, the number $t = \frac{k}{(1+\gamma)a \ln k}$ is an integer. Now, let $H = (V, E)$ be a hypergraph with n vertices and $\Delta(H) \leq k^a$. We will show that there exists an equitable coloring of H with $\frac{k}{(1+\gamma)a \ln k} - \lceil \sqrt{\gamma}\frac{k}{a \ln k} \rceil > (1 - \epsilon)\frac{k}{a \ln k}$ colors.

Assume that we have the set of colors $\{1, \ldots, t\}$. It will be convenient to deal with the finite set of hypergraphs having $n < 2k \ln k$ separately. We begin with the general case.

2.1 The general case: $n > 2k \ln k$

In the first phase of the proof we color most of the vertices (that is, we obtain a partial coloring) such that certain specific properties hold. In the second phase we color the vertices that were not colored in the first phase and show that we can do it carefully enough to obtain a strong t-coloring. In the third phase we show how to modify our coloring to obtain an equitable coloring.

2.1.1 First Phase

Our goal in this phase is to achieve a partial coloring with several essential properties:

Lemma 2.3 *There exists a partial coloring of H with the colors $\{1, \ldots, t\}$ such that the following four conditions hold:*

1. *Every edge contains at least $k\gamma/5$ uncolored vertices.*
2. *Every edge has at most $\lceil 10/\gamma \rceil$ colors that do not appear on its vertex set.*
3. *Put $z = \lceil k^{1-a\gamma/4} \rceil$. For each $v \in V$, and for each sequence of z* **distinct** *colors $c_1, \ldots, c_z$ and for each sequence of z* **distinct** *edges containing v denoted $f_1, \ldots, f_z$, at least one f_i has an element colored c_i.*
4. *Every color appears on at least $n\frac{(1+\gamma/4)a \ln k}{k}$ vertices.*

Proof: We let each vertex $v \in V$ choose a color from $\{1, \ldots, t\}$ randomly. The probability to choose color i is $p = \frac{(1+\gamma/2)a \ln k}{k}$ for $i = 1, \ldots, t$ and the probability of remaining uncolored is, therefore, $q = 1 - pt = \frac{\gamma}{2(1+\gamma)}$. For an edge f, let A_f denote the event that f contains less than $k\gamma/5$ uncolored vertices. Let B_f denote the event that f has more than $\lceil 10/\gamma \rceil$ colors missing from its vertex set. For a vertex v, let C_v denote the event that there exist z distinct edges $f_1, \ldots, f_z$ each f_i contains v, and there exist z distinct colors $c_1, \ldots, c_z$, such that c_i is missing from f_i for each $i = 1, \ldots, z$. For a color c, let D_c denote the event that the color c appears on less than $n\frac{(1+\gamma/4)a \ln k}{k}$ vertices. We must show that with positive probability, none of the $2|E| + |V| + t$ events above hold. The following four claims provide upper bounds for the probabilities of the events A_f,B_f, C_v, D_c.

Claim 2.4 $\Pr[A_f] < \frac{1}{k^{5a}}$.

Proof: Let X_f denote the random variable counting the uncolored elements of f. The expectation of X_f is $E[X_f] = kq$. Since each vertex chooses its color independently we have by a common Chernoff inequality (cf. [2])

$$\Pr[A_f] = \Pr\left[X_f < \frac{k\gamma}{5}\right] \le \Pr\left[X_f < \frac{kq}{2}\right] = \Pr\left[X_f < \frac{E[X_f]}{2}\right] <$$

$$e^{-2(E[X_f]/2)^2/k} = e^{-k^2q^2/(2k)} = e^{-kq^2/2} << \frac{1}{k^{5a}}. \quad \square$$

Claim 2.5 $\Pr[B_f] < \frac{1}{k^{5a}}$.

Proof: Fix $s = \lceil 10/\gamma \rceil$ distinct colors. The probability that none of them appear on f is precisely $(1-sp)^k$. Now,

$$(1-sp)^k = \left(1 - \frac{s(1+\frac{\gamma}{2})a \ln k}{k}\right)^k < \frac{1}{k^{as+as\gamma/2}}.$$

As there are $\binom{t}{s} < k^s$ possible sets of s distinct colors we get that

$$\Pr[B_f] < \binom{t}{s}\frac{1}{k^{as+as\gamma/2}} < \frac{1}{k^{as\gamma/2}} \le \frac{1}{k^{5a}}. \quad \Box$$

Claim 2.6 $\Pr[C_v] < \frac{1}{k^{5a}}$.

Proof: If the degree of v is less than z there is nothing to prove. Otherwise, fix a set of z distinct colors $\{c_1, \ldots, c_z\}$ and z distinct edges containing v, denoted $\{f_1, \ldots, f_z\}$. We begin by computing the probability that for each $i = 1, \ldots, z$, c_i does not appear on an element of f_i. Denote this probability by $\rho = \rho(v, f_1, \ldots, f_z, c_1, \ldots, c_z)$. For every vertex u let d_u be the number of edges f_i, $1 \le i \le z$, that contain u. By the definition of the event C_v we know that if C_v holds then there is a set of d_u colors none of which was assigned to u. The probability of this is $1 - d_u p$. Thus

$$\rho = \prod_u (1 - d_u p) \le e^{-p\Sigma_u d_u} = e^{-p\Sigma_i |f_i|} = e^{-pkz} = \frac{1}{k^{a(1+\gamma/2)z}}.$$

There are exactly $(t)_z < (k/\ln k)^z$ ordered sets of z distinct colors. Thus, the probability that $f_1, \ldots, f_z$ each miss a distinct color is less than $(k/\ln k)^z / k^{a(1+\gamma/2)z}$. There are at most $\binom{\lfloor k^a \rfloor}{z}$ distinct subsets of z edges containing v. This, together with Stirling's formula, gives

$$\Pr[C_v] < \binom{\lfloor k^a \rfloor}{z} \frac{k^z}{(\ln k)^z \, k^{a(1+\gamma/2)z}} < \left(\frac{ek^a}{z}\frac{k}{k^{a(1+\gamma/2)} \ln k}\right)^z \le \left(\frac{e}{k^{a\gamma/4}\ln k}\right)^z << \frac{1}{k^{5a}}. \quad \Box$$

Claim 2.7 $\Pr[D_c] < \frac{1}{e^{n/k}}$.

Proof: Let X_c denote the number of vertices that received the color c. Clearly, $E[X_c] = pn = n\frac{(1+\gamma/2)a \ln k}{k}$. Put $\beta = n\frac{a\gamma \ln k}{4k}$. We shall use the Chernoff inequality (cf. [2])

$$\Pr[X_c - pn < -\beta] < e^{-\beta^2/(2pn)}.$$

In our case

$$\Pr[D_c] = \Pr[X_c - pn < -\beta] < e^{-\beta^2/(2pn)} = e^{-\frac{na \ln k}{k}\left(\frac{\gamma^2}{32(1+\gamma/2)}\right)} <$$
$$e^{-\frac{na\ln k}{k}\left(\frac{\gamma^2}{33}\right)} = \frac{1}{k^{(an/k)(\gamma^2/33)}} \le \frac{1}{k^{(n/k)(\gamma^2/33)}} < \frac{1}{e^{n/k}}. \quad \Box$$

We now construct a dependency digraph for all the events of the form A_f, B_f, C_v, D_c (we refer to the events as "type A", "type B", "type C", and type "D" respectively). Consider an event A_f.

Let $E(f)$ denote the set of edges of H that are disjoint from f. Let $V(f)$ denote the set of vertices of H that do not appear on any edge that intersects f. Clearly A_f is mutually independent of all the $2|E(f)|+|V(f)|$ events of the form A_g, B_g or C_v which correspond to the elements of $E(f)$ and $V(f)$. Since there are at most k^{a+1} edges intersecting f and since there are at most k^{a+2} vertices in these edges, the outdegree in the dependency graph from A_f to other events of type A is at most k^{a+1}. Similarly the outdegree in the dependency graph from A_f to other events of type B is at most k^{a+1}, and to events of type C it is at most k^{a+2}. Since A_f depends on all events of type D, we have that the outdegree is t. This explains the first line of Table 1 (the dependency table). The other elements in the table are determined similarly. Note that events of type D depend on all other events (the fourth line in Table 1). In order to apply Lemma 2.1 we need to assign a

source \ target	A_f	B_f	C_v	D_t
A_f	k^{a+1}	k^{a+1}	k^{a+2}	t
B_f	k^{a+1}	k^{a+1}	k^{a+2}	t
C_v	k^{2a+1}	k^{2a+1}	k^{2a+2}	t
D_t	$\|E\|$	$\|E\|$	n	t

Table 1: The maximum possible outdegrees in the dependency digraph

coefficient to each event in the dependency digraph (the coefficients correspond to the x_i in Lemma 2.1). To each event of type A, B or C we assign the coefficient $3/k^{5a}$. To each event of type D we assign the coefficient $1/e^{n/2k}$. It remains to show that the conditions in Lemma 2.1 hold for each event. For events of type A we must show that

$$\Pr[A_f] < \frac{3}{k^{5a}}\left(1-\frac{3}{k^{5a}}\right)^{k^{a+1}}\left(1-\frac{3}{k^{5a}}\right)^{k^{a+1}}\left(1-\frac{3}{k^{5a}}\right)^{k^{a+2}}\left(1-\frac{1}{e^{n/2k}}\right)^{t}. \tag{1}$$

Indeed, recall that $n > 2k\ln k$ so $(1-1/e^{n/2k})^{k-1} > e^{-1}$. Using Claim 2.4 and the relation $t < k-1$, we find that the right side of (1) exceeds

$$\frac{3}{k^{5a}}\left(1-\frac{3}{k^{5a}}\right)^{3k^{a+2}} e^{-1} > \frac{3}{k^{5a}}\cdot 0.99\cdot e^{-1} > \frac{1}{k^{5a}} > \Pr[A_f].$$

The analogous inequalities for events of type B and C follow similarly from Claims 2.5 and 2.6 respectively. Finally, consider events of type D. We must show that

$$\Pr[D_c] < \frac{1}{e^{n/2k}}\left(1-\frac{3}{k^{5a}}\right)^{2|E|+n}\left(1-\frac{1}{e^{n/2k}}\right)^{t}. \tag{2}$$

In any k-uniform hypergraph we have $|E| \le n\Delta/k$. Thus, in our case, $2|E|+n \le 3k^{a-1}n$. Using Claim 2.7 and again the relation $(1-1/e^{n/2k})^{k-1} > e^{-1}$, we find that the right side of (2) exceeds

$$\frac{1}{e^{n/2k}}\left(1-\frac{3}{k^{5a}}\right)^{3k^{a-1}n} e^{-1} > \frac{1}{e^{n/2k}}\left(1-\frac{3}{k^{5a}}\right)^{(\frac{k^{5a}}{3}-1)\frac{18n}{k^{4a+1}}} e^{-1} >$$

$$\frac{1}{e^{n/2k}}e^{-\frac{18n}{k^{4a+1}}-1} > \frac{1}{e^{n/2k}}\frac{1}{e^{n/2k}} = \frac{1}{e^{n/k}} > \Pr[D_c].$$

According to Lemma 2.1, with positive probability, none of the events in the dependency digraph hold. We have completed the proof of Lemma 2.3.

2.1.2 Second Phase

Fix a partial coloring satisfying the four conditions in Lemma 2.3. For an edge f, let $M(f)$ denote the set of missing colors from f. By Lemma 2.3 we know that $|M(f)| \le \lceil 10/\gamma \rceil$. For a vertex v, let $S(v) = \cup_{v \in f} M(f)$. We claim that $|S(v)| \le \lceil 10/\gamma \rceil (z-1) \le 11z/\gamma$. To see this, notice that if $|S(v)| > \lceil 10/\gamma \rceil (z-1)$ then there must be at least z distinct edges containing v, say, $f_1, \ldots, f_z$ and z distinct colors $c_1, \ldots, c_z$ such that c_i does not appear on f_i for $i = 1, \ldots, z$. However, this is impossible by the third requirement in Lemma 2.3. In the second phase we only color the vertices that are uncolored after the first phase. Let v be such a vertex. We let v choose a random color from $S(v)$ with uniform distribution. The choices made by distinct vertices are independent. (In case $S(v) = \emptyset$ we can assign an arbitrary color to v.) Let $f \in E$ be any edge, and let $c \in M(f)$. Let $A_{f,c}$ denote the event that after the second phase, c still does not appear as a color on a vertex of f. Our goal is to show that with positive probability, none of the events $A_{f,c}$ for $f \in E$ and $c \in M(f)$ hold. This will give a strong t-coloring of H (although not necessarily an equitable one).

Let $T(f)$ be the subset of vertices of f that are uncolored after the first phase. By Lemma 2.3 we have $|T(f)| \ge k\gamma/5$. If $c \in M(f)$ we have that for each $u \in T(f)$, the color c appears on $S(u)$. Hence,

$$\Pr[A_{f,c}] = \Pi_{u \in T(f)} \left(1 - \frac{1}{|S(u)|}\right) \le \Pi_{u \in T(f)} \left(1 - \frac{\gamma}{11z}\right) \le$$
$$\left(1 - \frac{\gamma}{11z}\right)^{k\gamma/5} < e^{-\frac{k\gamma^2}{55z}} < e^{-k^{a\gamma/4}\frac{\gamma^2}{110}} << \frac{1}{k^{a+2}}.$$

Since each event $A_{f,c}$ is mutually independent of all other events but those that correspond to edges that intersect f, we have that the dependency digraph of the events has maximum outdegree at most $\lceil 10/\gamma \rceil k^{a+1} < k^{a+2}/e - 1$. Since $\frac{1}{k^{a+2}}((k^{a+2}/e - 1) + 1) = 1/e$ we have, by Corollary 2.2, that with positive probability none of the events of the form $A_{f,c}$ hold. In particular, there exists a strong t-coloring of H.

2.1.3 Third Phase

Assume the color classes of the strong t-coloring obtained after the second phase are $V_1, \ldots, V_t$ where $|V_i| \ge |V_{i+1}|$, $i = 1, \ldots, t-1$. By Lemma 2.3 we know that $|V_i| \ge n\frac{(1+\gamma/4)a \ln k}{k}$, $i = 1, \ldots, t$. Let $s = \lceil \sqrt{\gamma} k/(a \ln k) \rceil$ and let $W = V_1 \cup \cdots \cup V_s$. Clearly

$$n - |W| = |V \setminus W| = |V_{s+1} \cup \cdots \cup V_t| \ge (t-s)n\frac{(1+\frac{\gamma}{4})a \ln k}{k} = n\left(\frac{1+\frac{\gamma}{4}}{1+\gamma}\right) - \frac{sn(1+\frac{\gamma}{4})a \ln k}{k}.$$

Hence,

$$|W| \le n\left(1-\frac{1+\frac{\gamma}{4}}{1+\gamma}\right)+\frac{sn(1+\frac{\gamma}{4})a\ln k}{k} < \gamma n+\frac{sn(1+\frac{\gamma}{4})a\ln k}{k}.$$

In particular, $|V_s| \le |W|/s < \gamma n/s + n(1+\gamma/4)a\ln k/k$. It follows that $||V_i|-|V_j|| < \gamma n/s$ for all $s+1 \le i < j \le t$. Hence, it suffices to show that $|W| \ge (t-s)\gamma n/s$ since we can then transfer all the vertices in the color classes $V_1,\dots,V_s$ to the color classes $V_{s+1},\dots,V_t$ such that after the transfer, the $t-s$ remaining classes form an equitable partition (the strong coloring stays proper, of course). Indeed,

$$|W| > sn\frac{a\ln k}{k} = s^2 n\frac{a\ln k}{sk} \ge n\gamma\frac{k^2}{a^2(\ln k)^2}\frac{a\ln k}{sk} = n\gamma\frac{k}{sa\ln k} > n\frac{t\gamma}{s} > (t-s)\frac{n\gamma}{s}.$$

We have shown how to obtain an equitable coloring with $t-s = \frac{k}{(1+\gamma)a\ln k} - \lceil\sqrt{\gamma}\frac{k}{a\ln k}\rceil > (1-\epsilon)\frac{k}{a\ln k}$ colors.

2.2 The finite case: $n < 2k\ln k$

As in the proof for the general case, let each vertex choose a color randomly and independently, each color with probability p where $p = \frac{(1+\gamma/2)a\ln k}{k}$ for $i=1,\dots,t$ and the probability of remaining uncolored is $q = 1-pt = \frac{\gamma}{2(1+\gamma)}$. As in the proof of Claim 2.4, the probability that an edge contains less than $k\gamma/5$ uncolored vertices is less than $1/k^{5a}$. There are $|E| \le nk^a/k \le 2k^a\ln k$ edges. Hence, the expected number of edges with less than $k\gamma/5$ edges is less than $1/k^3$. Thus, with probability at least $1-1/k^3$, all edges have at least $k\gamma/5$ uncolored vertices. As in the proof of Claim 2.7, the probability that a color appears on less than $na\ln k(1+\gamma/4)/k$ vertices is less than $\frac{1}{k^{(n/k)(\gamma^2/33)}}$. Unlike Claim 2.7 we cannot bound this number from above by $e^{-n/k}$; instead, since $n \ge k$ (otherwise there are no edges at all), we can bound it with $k^{-\gamma^2/33}$. Since there are $t < k$ colors, the expected number of colors that appear on less than $na\ln k(1+\gamma/4)/k$ vertices is less than $k^{1-\gamma^2/33}$. Thus, with probability at least $2/3$ there are less than $3k^{1-\gamma^2/33}$ such colors. Finally, let X count the number of pairs (e,c) where $e \in E$ and c is a color that is missing from e. Clearly,

$$E[X] = |E|t(1-p)^k < 2k^a\ln k\cdot k\cdot k^{-a(1+\gamma/2)} = 2k^{1-a\gamma/2}\ln k < 2k^{1-\gamma/4} < \frac{k\gamma}{15}.$$

Hence, with probability at least $2/3$, $X < k\gamma/5$.

We have proved that with probability at least $1-1/k^3-1/3-1/3 > 0$ all the following occur simultaneously:

1. All edges have at least $k\gamma/5$ uncolored vertices.
2. At least $t-3k^{1-\gamma^2/33}$ colors appear each on at least $na\ln k(1+\gamma/4)/k$ vertices.
3. The number of pairs (e,c) of edges e and colors c such that c is missing from e is less than $k\gamma/5$.

Fix a partial coloring with all these properties. Trivially we can make it a strong coloring by assigning a color c that is missing from an edge e to one of the uncolored vertices of e, and we can do it greedily to all such (e, c) pairs. We therefore obtain a strong t-coloring of H, where, in addition, at least $t - 3k^{1-\gamma^2/33}$ colors appear each on at least $na \ln k(1 + \gamma/4)/k$ vertices. We can now use the same arguments as in the third phase of the general case and obtain an equitable coloring. The only difference is that instead of t we only use $t - r$ colors where r is the number of color classes having less than $na \ln k(1+\gamma/4)/k$ vertices. Thus, $t-r \geq t-3k^{1-\gamma^2/33} > t(1-\gamma/33)$, and it is easily seen that all computations in the third phase hold when replacing t with $t(1-\gamma/33)$. $\square$

3 A random hypergraph "construction"

Let $a \geq 1$ and let $\epsilon > 0$. Let $n = k^{2a}$. For simplicity we assume n is an integer in order to avoid floors and ceilings. We select k sufficiently large to justify this assumption and the assumptions that follow. Let $m = (1-\epsilon)k^{3a-1}$ (again, assume m is an integer). Consider the random k-uniform hypergraph on the vertex set $[n]$ with m randomly selected edges $f_1, \ldots, f_m$. Each edge f_i is chosen uniformly from all $\binom{n}{k}$ possible edges. The m choices are independent (thus, the same edge can be selected more than once). The expected degree of a vertex v (including multiplicities) is $mk/n = (1-\epsilon)k^a$. Notice that for k sufficiently large we have, using a Chernoff inequality, that the degree of v is greater than k^a with probability less than $1/(2k^{2a}) = 1/(2n)$. Hence, with probability greater than 0.5 the maximum degree is at most k^a. Put $t = (1-2\epsilon)na \ln k/k$. Again, we assume t is an integer. We show that with probability greater than 0.5, no t-subset of vertices is a vertex cover. This proves the existence of hypergraphs H with $\Delta(H) \leq k^a$ and $c(H) \leq (1 + o_k(1))k/(a \ln k)$.

Fix $X \subset [n]$ with $|X| = t$. For each edge f_i we have, assuming k is sufficiently large,

$$\Pr[f_i \cap X = \emptyset] = \frac{(n-t)(n-t-1)\cdots(n-t-k+1)}{n(n-1)\cdots(n-k+1)} > \left(1 - \frac{t}{n-k+1}\right)^k > \left(1 - \frac{t}{(1-\epsilon)n}\right)^k =$$

$$\left(1 - \frac{(1-2\epsilon)a\ln k}{(1-\epsilon)k}\right)^k > \left(1 - \frac{(1-\epsilon)a\ln k}{k}\right)^k > \frac{1}{2}e^{-(1-\epsilon)a\ln k} = \frac{1}{2k^{a(1-\epsilon)}}.$$

Since each edge is selected independently we have

$$\Pr[X \text{ is a vertex cover}] < \left(1 - \frac{1}{2k^{a(1-\epsilon)}}\right)^m.$$

There are $\binom{n}{t}$ possible choices for X. It suffices to show that

$$\binom{n}{t}\left(1 - \frac{1}{k^{2a(1-\epsilon)}}\right)^m < \frac{1}{2}.$$

Indeed, for k sufficiently large

$$\binom{n}{t}\left(1-\frac{1}{2k^{a(1-\epsilon)}}\right)^m < \left(\frac{en}{t}\right)^t\left(1-\frac{1}{2k^{a(1-\epsilon)}}\right)^{(1-\epsilon)k^{3a-1}} =$$

$$\left(\frac{ek}{(1-2\epsilon)a\ln k}\right)^{(1-2\epsilon)k^{2a-1}\ln k}\left(1-\frac{1}{2k^{a(1-\epsilon)}}\right)^{(1-\epsilon)k^{3a-1}} =$$

$$\left(\left(\frac{ek}{(1-2\epsilon)a\ln k}\right)^{(1-2\epsilon)\ln k}\left(1-\frac{1}{2k^{a(1-\epsilon)}}\right)^{(1-\epsilon)k^a}\right)^{k^{2a-1}} < \left(e^{\ln^2 k}e^{-k^{a\epsilon}(1-\epsilon)/2}\right)^{k^{2a-1}} << \frac{1}{2}. \quad \square$$

4 Concluding remarks

In the proof of Theorem 1.1 we require that $\Delta(H) \le k^a$ for some fixed $a \ge 1$. It is possible (although the computations get somewhat more complicated) to prove Theorem 1.1 when a is not necessarily a constant but satisfies $a = a(k) = o(k/\ln k)$. In other words, $\Delta(H)$ is allowed to be any subexponential function of k.

The proof of Theorem 1.1 is not algorithmic. It is, however, possible to obtain a polynomial time (in the number of vertices of the hypergraph, and not in its uniformity) algorithm that yields an equitable partition with $(1-o_k(1))ck/(a\ln k)$ parts where c is a fixed small constant (depending only on a). This can be done by using the method of Beck for the two coloring of hypergraphs [3] and generalizing it to more colors. We also need to take care that the coloring obtained be equitable (Beck's algorithm does not guarantee this). However, Beck's algorithm can be modified so as to guarantee that all colors use *roughly* the same number of colors, and then we can use the approach from the third phase of our proof to show that by sacrificing only a small fraction of the colors we can make the partition equitable using the remaining colors. Notice that the third phase can easily be implemented in polynomial time.

A special case of Theorem 1.1 yields an interesting result about graphs. Let G be a k-regular graph. If k is sufficiently large, then G has an equitable coloring with $(1-o_k(1))(k/\ln k)$ colors such that each color class is a total dominating set (a total dominating set D is a subset of the vertices that has the property that each vertex $v \in G$ has a neighbor in D). To see this, we can construct a hypergraph H from the graph G as follows. For each vertex $v \in G$, let $N(v)$ denote the neighborhood of v. The vertices of H are those of G and the edges are all the sets $N(v)$. Note that H is k-uniform and $\Delta(H) = k$. Theorem 1.1 applied to H gives the desired result about G.

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References

[1] N. Alon, *Transversal numbers of uniform hypergraphs*, Graphs and Combinatorics 6 (1990), 1–4.

[2] N. Alon and J. H. Spencer, *The Probabilistic Method*, John Wiley and Sons Inc., New York, 1991.

[3] J. Beck, *An algorithmic approach to the Lovász Local Lemma*, Random structures and algorithms 2 (1991), 343–365.

[4] B. Bollobás and R. K. Guy, *Equitable and proportional coloring of trees*, J. Combin. Theory Ser. B 34 (1983), 177–186.

[5] J. A. Bondy and U.S. R. Murty, *Graph Theory with Applications*, Macmillan Press, London, 1976.

[6] P. Erdös and L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, Infinite and Finite Sets (A. Hajnal et al., eds.), North-Holland, Amsterdam (1975), 609-628.

[7] A. V. Kostochka, K. Nakprasit and S. Pemmaraju, *Equitable colorings with constant number of colors*, Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (Baltimore, MA, 2003), ACM, New York, 2003.

[8] K.-W. Lih, *The equitable coloring of graphs*, Handbook of combinatorial optimization, Vol. 3, 543–566, Kluwer Acad. Publ., Boston, MA, 1998.