A Ramsey type result for oriented trees

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Abstract

Given positive integers h and k, denote by $r(h,k)$ the smallest integer n such that in any k-coloring of the edges of a tournament on more than n vertices there is a monochromatic copy of every oriented tree on h vertices. We prove that $r(h,k) = (h-1)^k$ for all k sufficiently large ($k = \Theta(h \log h)$ suffices). The bound $(h-1)^k$ is tight. The related parameter $r^*(h,k)$ where some color contains all oriented trees is asymptotically determined. Values of $r(h,2)$ for some small h are also established.

1 Introduction

An *oriented graph* is a digraph such that for every two distinct vertices u, v at most one of the ordered pairs (u,v) or (v,u) is an edge. Stated otherwise, an oriented graph is obtained by assigning a direction to each edge of an undirected graph. The undirected graph is also called the *underlying graph*. A *tournament* is an oriented graph whose underlying graph is complete. An *oriented tree* is an oriented graph whose underlying graph is a tree.

A seminal theorem, so called the Gallai-Roy Theorem asserts that any oriented graph has a directed path whose order is at least as large as the chromatic number of its underlying graph. This theorem was obtained independently by Gallai [7], Hasse [9], Roy [13], and Vitaver [14]. We note that the Gallai-Roy Theorem generalizes Redei's Theorem [12] that states that any tournament has a Hamilton path.

By observing that in any edge coloring of a complete graph with more than $\prod_{i=1}^{k}(h_i - 1)$ vertices with k colors, there is a color i that induces a graph whose chromatic number is at least h_i, Gyárfás and Lehel [8], Bermond [2], and Chvátal [4] deduced that in any k-coloring of the edges of a tournament on more than $\prod_{i=1}^{k}(h_i - 1)$ vertices, there is a directed path of order h_i, all of whose edges are colored i. They also observed that there is a simple construction showing that the bound $\prod_{i=1}^{k}(h_i - 1)$ is tight. The diagonal case, where all h_i are equal, is equivalently stated as the following Ramsey-type parameter. Let P_h denote the directed path of order h. Given a positive integer h, let $r(P_h, k)$ be the smallest integer n such that in any k-coloring of the edges of a tournament with more than n vertices, there is a monochromatic P_h. The aforementioned result states that $r(P_h, k) = (h-1)^k$.

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A natural question which follows is the value of the corresponding Ramsey number of oriented trees other than the directed path. In particular, what bound guarantees a monochromatic copy of any oriented tree on h vertices? Already the case $k = 1$ is interesting, and, in fact, notoriously difficult. A famous conjecture of Sumner from 1971 states that any tournament on $2h - 2$ vertices contains any oriented tree on h vertices (we always assume $h \geq 2$ to avoid the trivial case). If true, then this is best possible since a regular tournament on $2h-3$ vertices has all in-degrees and out-degrees equal to $h - 2$. It therefore has no copy of S_h, the out-directed star on h vertices. Sumner's conjecture is still open, though it has recently been established for very large h by Kuhn, Mycroft, and Osthus [11]. The best bound that applies to all h is $3h - 3$ proved by El-Sahili [6] based on a method of Havet and Thomassé [10].

Let $r(h, k)$ be the smallest integer n such that in any k-coloring of the edges of a tournament with more than n vertices, there is a monochromatic copy of every oriented tree on h vertices. Determining $r(h, 1)$ is thus equivalent to solving Sumner's conjecture. The discussion in the previous paragraphs implies, in particular, that $r(h, k) \geq (h - 1)^k$, that $3h - 4 \geq r(h, 1) \geq 2h - 3$ and that $r(h, 1) = 2h - 3$ for all h sufficiently large. Our first main result is the following.

Theorem 1 *Let $h \geq 2$ be a positive integer and let k be a positive integer satisfying $(1 + 1/(h - 2))^k > 2(h - 2)k + 1$. Then, for every $n > (h - 1)^k$, any edge coloring of an n-vertex tournament with k colors contains a monochromatic copy of every oriented tree on h vertices. In particular, $r(h, k) = (h - 1)^k$.*

The fact that Theorem 1 requires some lower bound on k in order for the value $(h-1)^k$ to hold is, of course, necessary as shown already for the case $k = 1$. It is thus of some interest to determine, for a given h, the value $f(h)$ which is the smallest k for which $r(h, k) = (h-1)^k$. Theorem 1 shows that $f(h) = O(h \log h)$, but we cannot rule out that $f(h)$ is bounded by a value independent of h. Nevertheless, we certainly have $f(h) \geq 2$ for all $h \geq 3$ as demonstrated by the lower bound in Sumner's conjecture. Furthermore, Sumner's conjecture is known to hold for some small h by computer verification. As usual in Ramsey theory, when the number of colors increases, say even $k = 2$ colors, it is not easy to determine $r(h, 2)$ even for very small h. The fact that $r(3, 2) = 5$ and $r(3, k) = 2^k$ for all $k \geq 3$ is a simple exercise. Hence $f(3) = 3$. Already determining the first non-trivial case $r(4, 2)$ turns out to be somewhat involved, as well as determining $f(h)$ for $h \geq 4$. We show that:

Theorem 2 $r(4, 2) = 12$. *Hence, $f(4) \geq 3$. In fact, $f(h) \geq 3$ for all $h \leq 6$.*

Notice that it is hopeless to use computer verification for $r(4, 2)$ as one needs to check all 2-edge colorings of all (non-isomorphic) tournaments on 13 vertices and it is known that there are more than 2^{45} such tournaments.

One may wonder whether Theorem 1 can be strengthened to show that there is some particular color so that there is a monochromatic copy of every oriented tree with that color. Formally, let $r^*(h, k)$ be the smallest integer n such that in any k-coloring of the edges of a tournament with more than n vertices, some color induces a subgraph that contains all oriented trees on h vertices.

Clearly $r^*(h,k) \geq r(h,k)$. However, we show in proposition 3.1 that $r^*(k,h) \geq (2h-3)(h-1)^{k-1}$ so Theorem 1 does not hold for this stronger parameter, and $r^*(h,k)$ is truly separated from $r(h,k)$. Nevertheless, we can prove that the bound $(h-1)^k$ is asymptotically correct in the sense that the base $h-1$ can be replaced with $h-1+\epsilon$.

Theorem 3 *For every $\epsilon > 0$, an integer $h \geq 2$ and a positive integer k satisfying $(1+\epsilon/(h-1))^k > 2(h-2)k+1$ we have $r^*(h,k) \leq (h-1+\epsilon)^k$.*

It is appropriate to mention here the conjecture of Burr [3] that any digraph whose chromatic number is at least $2h-2$ contains every oriented tree on h vertices. It has been proved by Addario-Berry, Havet, Sales, Reed, and Thomassé [1] that chromatic number $h^2/2-h/2+1$ suffices. Hence, if Burr's conjecture is true, then $r^*(h,k) \leq (2h-3)^k$ is true for all h and k. The bound in Theorem 3 which applies to all k sufficiently large, is significantly stronger.

In the remainder of this paper we prove Theorem 1 in Section 2, the case of $r^*(h,k)$, in particular Theorem 3 and Proposition 3.1 are proved in Section 3, and Theorem 2 is proved in Section 4.

2 Proof of Theorem 1

Let H be an oriented tree on h vertices. We will prove a stronger version of Theorem 1 which is based on some graph parameter of H, its *strong radius*, which is now defined. Let r be some vertex of H and consider H rooted at r. Namely, the children of r are all the vertices of H that have an edge in any direction connecting them to r (so r is their parent). Similarly, the children of any other vertex u are all the vertices of H, excluding the parent of u, that are connected to u with an edge in any direction. A leaf is a vertex with no children. We say that a vertex u is in level k, if its parent is in level $k-1$. The level of r is zero. We shall dispense labels to the vertices as follows. The label of r is zero. Assume we have given labels to all vertices in level k and that the maximum label used thus far is t. If all the edges connecting vertices in level k to vertices in level $k+1$ are oriented in the same direction (either all from level k to level $k+1$ or all from level $k+1$ to level k), then all vertices in level $k+1$ receive label $t+1$. Otherwise, all vertices in level $k+1$ that point to their parents receive label $t+1$ and all vertices in level $k+1$ that are pointed to by their parents receive label $t+2$. We let $rad(r)$ denote the value of the largest label used. We let $rad(H)$, *the strong radius* of H denote the smallest value of $rad(v)$ ranging over all vertices v of H.

Some observations follow. The only oriented trees with $rad(H) = 1$ are the out-directed star and the in-directed star. For any other oriented star we have $rad(H) = 2$. The only oriented tree with $rad(H) = h-1$ is the directed path on h vertices. Any other oriented tree has $rad(H) \leq h-2$. This is true since in any other oriented tree there is either a vertex with out-degree at least 2 or a vertex with in-degree at least 2, so letting that vertex be a root, we save at least one label. An anti-directed path (a path which has no directed sub-path of length 2) has $rad(H) = \lfloor h/2 \rfloor$.

Theorem 4 *Let H be an oriented tree with $rad(H) = s$. Let k be any integer satisfying $(1+1/(h-2))^k > 2(h-2)k+1$. Then, for every $n \geq (s+1)^k$, any edge coloring of an n-vertex tournament with k colors contains a monochromatic copy of H.*

Theorem 1 is a consequence of Theorem 4 and the Gallai-Roy Theorem. Indeed, if H is a directed path, Theorem 1 is valid for H by the Gallai-Roy Theorem. Since $rad(H) \le h-2$ for any H which is not a directed path, Theorem 1 is valid for H by Theorem 4.

Proof of Theorem 4. It suffices to prove the theorem for tournaments on $n = (s+1)^k$ vertices. Fix some oriented tree H on h vertices and suppose that $rad(H) = s \le h-2$ (we may assume this since Theorem 4 trivially holds for the directed path which is the only oriented tree with $rad(H) = h-1$).

Let $G = (V, E)$ be a tournament on n vertices whose edges have been colored by the colors $1, \dots, k$. We denote the spanning oriented graph consisting of the edges colored i by $G_i = (V, E_i)$.

Consider a vertex $r \in V(H)$ with $rad(r) = s$. Fixing r, this defines children, parents, levels and labels. Let the label of a vertex be denoted by $\ell(v)$ and recall that $1 \le \ell(v) \le s$ for all $v \ne r$. Let $H_t = \{v \mid \ell(v) = t\}$. So, $V(H) = \{r\} \cup H_1 \cup \cdots \cup H_s$. We say that H_t is *backward* if all vertices in H_t point to their parents. We say that H_t is *forward* if all vertices in H_t are pointed to by their parents. By definition of the labels, each H_t is either forward or backward.

For each oriented graph G_i, we define disjoint sets of vertices of G, denoted by $V_{i,1}, \dots, V_{i,s}$ as follows. We start from s downwards, first defining $V_{i,s}$. If H_s is forward, then $V_{i,s}$ are all the vertices whose out-degree in G_i is at most $h-2$. If H_s is backward, then $V_{i,s}$ are all the vertices whose in-degree in G_i is at most $h-2$. Let $U_{i,s} = V \setminus V_{i,s}$. Let $G_{i,s} = G_i[U_{i,s}]$ be the subgraph of G_i induced by $U_{i,s}$. Assume that we have already defined $V_{i,t}$ and $U_{i,t}$, we now define $V_{i,t-1}$ and $U_{i,t-1}$ as follows. If H_{t-1} is forward, then $V_{i,t-1}$ are all the vertices whose out-degree in $G_{i,t}$ is at most $h-2$. If H_{t-1} is backward, then $V_{i,t-1}$ are all the vertices whose in-degree in $G_{i,t}$ is at most $h-2$. Let $U_{i,t-1} = U_{i,t} \setminus V_{i,t-1}$. Let $G_{i,t-1} = G_i[U_{i,t-1}]$ be the subgraph of G_i induced by $U_{i,t-1}$.

Lemma 2.1 *If $U_{i,1} \ne \emptyset$, then G_i contains a copy of H.*

Proof. We embed a copy of H in G_i in $s+1$ steps starting from step 0 where in step t we embed all the vertices of H_t (vertices having label t). We maintain the property that the copy of the subgraph of H consisting of all vertices having label at most t is in $G_{i,t+1}$ (for completeness, define $G_{i,s+1} = G_i$). For $v \in V(H)$, we denote by $f(v)$ the vertex of G_i to which v was embedded.

Step 0 simply consists of embedding r to any vertex of $U_{i,1}$ (thus $f(r) \in U_{i,1}$). This is possible since $U_{i,1} \ne \emptyset$. We describe Step 1. Now H_1 is either forward or backward. Assume without loss of generality that it is forward (namely all the vertices with label 1 are out-neighbors of r). By construction, $U_{i,1}$ consists of all the vertices in $G_{i,2}$ having out-degree at least $h-1$ in $G_{i,2}$. So, in particular, $f(r)$ has at least $h-1$ out-neighbors in $G_{i,2}$ so we pick q of them where $q = |H_1| \le h-1$ and embed the vertices of H_1 arbitrarily to them. Observe that all the images of the vertices having labels 0 or 1 are in $G_{i,2}$ as required.

Assume we have already embedded all vertices with label at most $t-1$ and satisfy the property that all their images are in $G_{i,t}$. We show how to embed the vertices with label t. Assume without loss of generality that H_t is forward (namely all the vertices with label t are out-neighbors of their parents). By construction, $U_{i,t}$ consists of all the vertices in $G_{i,t+1}$ having out-degree at least $h-1$ in $G_{i,t+1}$. So, in particular, each parent of a vertex with label t, say v, has at least $h-1$

out-neighbors in $G_{i,t+1}$ so we pick as many out-neighbors of v as needed, not reusing previously embedded vertices and make them the children of v with label t. (The reason we can pick the required amount of vertices that have not yet been embedded is straightforward: Suppose we need to pick q out-neighbors of v. This means that up until now, we have only embedded at most $h-1-q$ vertices not including v, as we did not yet embed any out-neighbors of v. So, out of the at least $h-1$ out-neighbors of v in $G_{i,t+1}$, there are at least $h-1-(h-1-q)=q$ out-neighbors that have not yet been embedded, so these can be picked.) Observe that all the images of the vertices having labels at most t are in $G_{i,t+1}$ as required. ∎

It remains to prove that for some color i we indeed have $U_{i,1} \neq \emptyset$. Recall that $V_{i,1}, \ldots, V_{i,s}$ are disjoint subsets of V (some of them may be empty) and that $U_{i,1} = V \setminus (\cup_{t=1}^{s} V_{i,t})$. So $U_{i,1}, V_{i,1}, \ldots, V_{i,s}$ is a partition of V into $s+1$ vertex classes (some may be empty). Let $F_i \subset E_i$ be the set of edges of G_i where the endpoints of F_i are not in the same vertex class, except for the edges with both endpoints in $U_{i,1}$ which are retained in F_i. Let $G_i^* = (V, F_i)$ be the spanning subgraph of G_i consisting of these edges.

Lemma 2.2 *If $\chi(G_i^*) \geq s+1$, then $U_{i,1} \neq \emptyset$.*

Proof. If $U_{i,1} = \emptyset$, then G_i^* is an s-partite graph so in particular, has $\chi(G_i^*) \leq s$. ∎

How many edges colored i are there with two endpoints in $V_{i,t}$? We claim that there are no more than $(h-2)|V_{i,t}|$ such edges. Indeed, otherwise, there would have been a vertex x inside $V_{i,t}$ with at least $h-1$ out-neighbors in $V_{i,t}$ and all edges having color i. But then this vertex x has, in particular, out-degree at least $h-1$ in $G_{i,t+1}$. Similarly, there would have been a vertex y inside $V_{i,t}$ with at least $h-1$ in-neighbors in $V_{i,t}$ and all edges having color i. But then this vertex y has, in particular, in-degree at least $h-1$ in $G_{i,t+1}$. But then by construction (depending on whether H_t is forward or backward) one of x or y would not have been in $V_{i,t}$.

Since G_i^* is obtained from G_i by removing only edges with both endpoints in the same $V_{i,t}$ for $t=1,\ldots,s$, we have by the previous paragraph that

$$|F_i| \geq |E_i| - (h-2)\sum_{t=1}^{s} |V_{i,t}| \geq |E_i| - (h-2)n \; .$$

Consider now the union of all of the G_i^*, namely $G^* = (V, \cup_{i=1}^{k} F_i)$. This is a graph on n vertices, at least $\binom{n}{2} - k(h-2)n$ edges, and which decomposes into k spanning subgraphs $G_1^*, \ldots, G_k^*$ and we wish to prove that one of these subgraphs has chromatic number at least $s+1$.

Assume otherwise, and fix an s-coloring of G_i^* with color classes $X_{i,1}, \ldots, X_{i,s}$ for $i=1,\ldots,k$ (possibly some vertex classes are empty). So each $v \in V$ is associated with a k-tuple where the i'th coordinate equals j if $v \in X_{i,j}$. So there are n vertices associated with at most s^k possible tuples. We will prove that at least two vertices that are connected by an edge are associated with the same tuple, a contradiction.

We wish to lower bound the number of pairs of vertices of V that share the same k-tuple. There are s^k tuples and n vertices, so the number of pairs sharing a tuple is at least

$$s^k \cdot \frac{\frac{n}{s^k} \cdot (\frac{n}{s^k} - 1)}{2} = \frac{n}{2} \cdot (\frac{n}{s^k} - 1) \; .$$

As the number of edges removed from K_n is only at most $k(h-2)n$, we will arrive at the required contradiction if we can show that

$$\frac{n}{2} \cdot (\frac{n}{s^k} - 1) > k(h-2)n \; .$$

Equivalently, we must prove that

$$\frac{n}{s^k} > 2k(h-2) + 1 \; .$$

Equivalently, and using $n = (s+1)^k$, it suffices to prove that

$$\left(1 + \frac{1}{s}\right)^k > 2k(h-2) + 1$$

which indeed holds by the assumption in the statement of Theorem 4 since $s \le h-2$. This completes the proof of Theorem 4. ■

3 Bounds for $r^*(h,k)$

We begin with the following proposition that provides a lower bound for $r^*(h,k)$ showing that it is larger than $r(h,k)$.

Proposition 3.1 *For all $h \ge 2$ and for all $k \ge 1$ there is a k-coloring of the edges of a tournament on $(2h-3)(h-1)^{k-1}$ vertices so that no colored subgraph contains all oriented trees on h vertices.*

Proof. The construction is inductive by k. For the case $k = 1$ we take a regular tournament G_1 on $2h-3$ vertices and color all its edges with color 1. Observe that G_1 has no copy of S_h, the out-directed star on h vertices, as the out-degree of each vertex is $h-2$.

Now assume that $k \ge 2$ and that we have already constructed a coloring of a tournament G_{k-1} on $(2h-3)(h-1)^{k-2}$ vertices using colors $1, \ldots, k-1$. Take $h-1$ vertex-disjoint copies of the colored G_{k-1}, denoting them by $X_1, \ldots, X_{h-1}$. Now orient all edges between X_i and X_j from X_i towards X_j whenever $i < j$ and color each of these edges with the color k. This results in a tournament on $(2h-3)(h-1)^{k-1}$ vertices. Observe that any directed path all of whose edges use color k contains at most one vertex from each X_i, and hence has at most $h-1$ vertices. As this holds in all steps of the induction, there is no monochromatic copy of P_h in the colors $2, \ldots, k$. In fact, notice that in the final graph G_k, color 1 induces a subgraph with $(h-1)^{k-1}$ components, each of which is a G_1. So color 1 still does not contain a monochromatic copy of S_h. So there is no color that contains all oriented trees on h vertices. ■

We now proceed to prove Theorem 3. The proof is similar to the proof of Theorem 4 except that we will be using a universal labeling which applies to all oriented trees. We only outline the differences between the two proofs.

Proof of Theorem 3. Consider the underlying undirected tree H^u of any oriented tree H with h vertices. It is well-known that we can always root H^u in a vertex r such that the rooted tree distance between v and any other vertex in H^u is at most $\lfloor h/2 \rfloor$. Hence, the level of each vertex is an integer in $1, \ldots, \lfloor h/2 \rfloor$. In fact, the only case where we need to use level $h/2$ is when h is even and H^u is an oriented path.

We shall dispense labels to the vertices as follows. The label of r is zero. Assume we have given a label to a parent and we now need to give a label to its child. If the edge in H points from the child to the parent, we assign the child the first odd label larger than the parent's label. If the edge in H points from the parent to the child, we assign the child the first even label larger than the parent's label. Observe that it is possible for vertices with the same label to occur in different levels. Notice that in this labeling, the largest label that is used is at most $h-1$, since vertices in level t never receive a label larger than $2t$. There is one exception though, which is the case where h is even and H is an oriented path, in which case there could be a single vertex v in level $h/2$ and we might assign it the label h. Observe that this will happen only if there is a directed path in H from the root r to v, of length $h/2$. Let r' be the vertex immediately after r on this path. We can re-root H at r' and now r' will receive label 0, r will receive label 1, v will receive label $h-2$, and any other vertex will receive label at most $h-1$.

It suffices to prove the theorem for tournaments on $n = (h-1+\epsilon)^k$ vertices. Let $G = (V, E)$ be a tournament on n vertices whose edges have been colored by the colors $1, \ldots, k$. We denote the spanning oriented graph consisting of the edges colored i by $G_i = (V, E_i)$.

Let H be any oriented tree on h vertices and consider a rooting with root r and labeling as described above. As in the proof of Theorem 4 we define $H_t = \{v \mid \ell(v) = t\}$, this time for $t = 1, \ldots, h-1$. So, $V(H) = \{r\} \cup H_1 \cup \cdots \cup H_{h-1}$ and notice that it is possible that some H_t are empty. We say that H_t is *backward* if t is odd and that H_t is *forward* if t is even.

For each oriented graph G_i, we define disjoint sets of vertices of G, denoted by $V_{i,1}, \ldots, V_{i,h-1}$ as in the proof of Theorem 4 (so we use $s = h-1$ in Theorem 4). Recall that this also defines $G_{i,t} = G_i[U_{i,t}]$ for $t = 1, \ldots, h-1$. Notice the crucial fact that the $V_{i,t}, G_{i,t}, U_{i,t}$ are now independent of H (unlike in Theorem 4), because being backward or forward now only depends on parity.

As in Lemma 2.1, it suffices to prove that $U_{i,1} \neq \emptyset$ to obtain that G_i has a copy of H, and as H is an arbitrary oriented tree, this applies to *all* H (the proof is identical).

As in Theorem 4, it remains to prove that for some color i we indeed have $U_{i,1} \neq \emptyset$. Using the same notation, it remains to prove that $\chi(G_i^*) \geq h$ for some i. We again have the inequality $|F_i| \geq |E_i| - (h-2)\sum_{t=1}^{h-1} |V_{i,t}| \geq |E_i| - (h-2)n$. Consider now the union of all of the G_i^*, namely $G^* = (V, \cup_{i=1}^k F_i)$. This is a graph on n vertices, at least $\binom{n}{2} - k(h-2)n$ edges, and which decomposes into k spanning subgraphs $G_1^*, \ldots, G_k^*$ and we wish to prove that one of these subgraphs has chromatic number at least h.

Assume otherwise, and fix an $(h-1)$-coloring of G_i^* with color classes $X_{i,1}, \ldots, X_{i,h-1}$ for $i = 1, \ldots, k$ (possibly some vertex classes are empty). So each $v \in V$ is associated with a k-tuple where the i'th coordinate equals j if $v \in X_{i,j}$. So there are n vertices associated with at most $(h-1)^k$ possible tuples. We will prove that at least two vertices that are connected by an edge are associated with the same tuple, a contradiction.

As in Theorem 4, we lower bound the number of pairs of vertices of V that share the same k-tuple. There are $(h-1)^k$ tuples and n vertices, so the number of pairs sharing a tuple is at least

$$(h-1)^k \cdot \frac{\frac{n}{(h-1)^k} \cdot (\frac{n}{(h-1)^k} - 1)}{2} = \frac{n}{2} \cdot (\frac{n}{(h-1)^k} - 1) \ .$$

As the number of edges removed from K_n is only at most $k(h-2)n$, we will arrive at the required contradiction if we can show that

$$\frac{n}{(h-1)^k} > 2k(h-2) + 1 \ .$$

Equivalently, and using $n = (h-1+\epsilon)^k$, it suffices to prove that

$$\left(1 + \frac{\epsilon}{h-1}\right)^k > 2k(h-2) + 1$$

which indeed holds by the assumption in the theorem. ■

4 Two colors

In this section we prove Theorem 2. We first need the following lemma. Let $S_{a,i}$ be the star on a vertices where the root has out-degree i and in-degree $a-1-i$.

Lemma 4.1 *In any red-blue edge coloring of a tournament with more than $8a-20$ vertices there is a copy of $S_{a,\lfloor a/2 \rfloor}$. There exists a red-blue coloring of a tournament with $8a-20$ vertices with no copy of $S_{a,\lfloor a/2 \rfloor}$.*

Proof. We start with the construction. We consider first the case where a is odd. Consider four disjoint vertex classes A,B,C,D each with $2a-5$ vertices. Each of them will induce a red-blue edge colored tournament G' on $2a-5$ vertices as follows. Since the complete graph on $2a-5$ vertices decomposes into $a-3$ Hamilton cycles, we can make each cycle into a directed one, and color half of the cycles (that is, $(a-3)/2$ cycles) red, and half blue. Hence, G' has no monochromatic copy of $S_{a,\lfloor a/2 \rfloor}$. Now orient all edges from C to A, from D to B, and from D to A and make all of them red. Orient all edges from A to B, from C to D, and from C to B and make all of them blue. We thus have a coloring of a tournament on $8a-20$ vertices with no monochromatic $S_{a,\lfloor a/2 \rfloor}$.

For the even case, we will use $|A| = 2a-5$, $|B| = 2a-3$, $|C| = 2a-7$ and $|D| = 2a-5$. The tournament induced by A will consist of $(a-2)/2$ red directed Hamilton cycles and $(a-4)/2$ blue directed Hamilton cycles. The tournament induced by B will consist of $(a-2)/2$ red directed Hamilton cycles and $(a-2)/2$ blue directed Hamilton cycles. The tournament induced by C will consist of $(a-4)/2$ red directed Hamilton cycles and $(a-4)/2$ blue directed Hamilton cycles. The tournament induced by D will consist of $(a-4)/2$ red directed Hamilton cycles and $(a-2)/2$ blue directed Hamilton cycles. Now orient all edges from C to A, from D to B, and from D to A and make all of them red. Orient all edges from A to B, from C to D, and from C to B and make all of them blue. We thus have a coloring of a tournament on $8a-20$ vertices with no monochromatic $S_{a,a/2}$.

We next show that in any red-blue edge coloring of a tournament G on $8a-19$ vertices there is a monochromatic $S_{a,\lfloor a/2\rfloor}$. We consider the case where a is even. The case where a is odd is similar and a bit simpler. Let $X \subset V(G)$ be the set of vertices with red in-degree at most $(a-4)/2$. Let $Y \subset (V(G) \setminus X)$ be the set of vertices with red out-degree at most $(a-2)/2$. If $X \cup Y \neq V(G)$ we are done as a vertex $v \notin X \cup Y$ has red in-degree at least $(a-2)/2$ and red out-degree at least $a/2$, so there is a red $S_{a,a/2}$. We are therefore left with the case $|X|+|Y| = 8a-19$. We assume by way of contradiction that G has no monochromatic $S_{a,\lfloor a/2\rfloor}$ and show that this assumption implies that $|X|+|Y| < 8a-19$, a contradiction.

Let G_1 be the spanning subgraph on vertex set X induced by the blue edges. Let $C \subset X$ be the set of vertices whose in-degree in G_1 is at most $(a-4)/2$. Let $D \subset (X \setminus C)$ be the set of vertices whose out-degree in G_1 is at most $(a-2)/2$. If $C \cup D \neq X$ we are done as a vertex $x \in X$ where $x \notin C \cup D$ has in-degree at least $(a-2)/2$ in G_1 and out-degree at least $a/2$ in G_1, so there is a blue $S_{a,a/2}$. So we may assume $|C|+|D| = |X|$. By the definitions of X and C, the subtournament of G induced by C has red in-degree at most $(a-4)/2$ and blue in-degree at most $(a-4)/2$, so any vertex in this sub-tournament has in-degree at most $a-4$. This implies that $|C| \le 2a-7$. By the definitions of X and D, the subtournament of G induced by D has red in-degree at most $(a-4)/2$. Hence it has at most $|D|(a-4)/2$ red edges. It also has blue out-degree at most $(a-2)/2$ so it has at most $|D|(a-2)/2$ blue edges. Altogether, it has at most $|D|(a-3)$ edges. This implies that $|D| \le 2a-5$.

Let G_2 be the spanning subgraph on vertex set Y induced by the blue edges. Let $A \subset Y$ be the set of vertices whose in-degree in G_2 is at most $(a-4)/2$. Let $B \subset (Y \setminus A)$ be the set of vertices whose out-degree in G_2 is at most $(a-2)/2$. If $A \cup B \neq Y$ we are done as a vertex $y \in Y$ where $y \notin A \cup B$ has in-degree at least $(a-2)/2$ in G_2 and out-degree at least $a/2$ in G_2, so there is a blue $S_{a,a/2}$. So we may assume $|A|+|B| = |Y|$. By the definitions of Y and A, the subtournament of G induced by A has red out-degree at most $(a-2)/2$. Hence it has at most $|A|(a-2)/2$ red edges. It also has blue in-degree at most $(a-4)/2$ so it has at most $|A|(a-4)/2$ blue edges. Altogether, it has at most $|A|(a-3)$ edges. This implies that $|A| \le 2a-5$. By the definitions of Y and B, the subtournament of G induced by B has red out-degree at most $(a-2)/2$ and blue out-degree at most $(a-2)/2$, so any vertex in this sub-tournament has out-degree at most $a-2$. This implies that $|B| \le 2a-3$.

We have shown that $|X|+|Y| = |A|+|B|+|C|+|D| \le (2a-5)+(2a-3)+(2a-7)+(2a-5) = 8a-20 < 8a-19$, a contradiction. ■

Proof of Theorem 2. We start with the first part of the theorem, showing that $r(4,2) = 12$. Let $\mathcal{H} = \{P_4, Q_1, Q_2, Q_3, S_{4,0}, S_{4,1}, S_{4,2}, S_{4,3}\}$ denote the 8 distinct oriented trees on four vertices. Here P_4 is the path on four vertices, $S_{4,i}$ is the star on four vertices where the root has out-degree i and in-degree $3-i$ for $i=0,1,2,3$. Q_1, Q_2 and Q_3 are the following oriented paths:

$$Q_1 = (1 \rightarrow 2 \leftarrow 3 \rightarrow 4)$$
$$Q_2 = (1 \rightarrow 2 \rightarrow 3 \leftarrow 4)$$
$$Q_3 = (1 \leftarrow 2 \rightarrow 3 \rightarrow 4)$$

Lemma 4.1 gives, in particular, a construction of a tournament with 12 vertices and a red-blue coloring of its edges with no monochromatic $S_{4,2}$. Hence we have $r(4,2) \geq 12$.

We next prove that every red-blue edge coloring of any tournament G on 13 vertices has all elements of $\mathcal{H}$ as monochromatic subgraphs. Let us first consider the easy cases. A monochromatic P_4 exists since $13 > (4-1)^2 = 9$ (by the Gallai-Roy Theorem already every red-blue edge colored 10-vertex tournament contains a monochromatic P_4). There is a vertex of G with out-degree at least 6 hence there must be a monochromatic $S_{4,3}$. Similarly, there is a vertex of G with in-degree at least 6 hence there must be a monochromatic $S_{4,0}$.

We next show that there is a monochromatic $S_{4,2}$ (the proof for $S_{4,1}$ is symmetric). This indeed holds by Lemma 4.1 by using $a=4$.

For the remaining graphs, Q_1, Q_2, Q_3, we can use a more general result of El Sahili [5] which states that any oriented graph whose underlying graph has chromatic number at least 4 contains every path on four vertices. Since in any red-blue edge coloring of K_{13} (in fact, K_{10}) one of the colors induces a graph that has chromatic number at least 4, it also contains Q_1, Q_2, Q_3 (and of course P_4). In our case we can use a simpler, more direct argument, as follows.

We show that there is a monochromatic Q_1 (the anti-directed path). Assume without loss of generality that at least half of the edges of a given red-blue edge coloring of G are blue. Let G_1 be the spanning subgraph on the blue edges. Consider a vertex u of G_1 with maximum out-degree t. Then we must have $t \geq 3$ as the average out-degree in G_1 is at least 3. If there are two out-neighbors of u in G_1, say, v, w where (v,w) is a blue edge, then the three edges $(v,w),(u,w),(u,x)$ where x is another out-neighbor of u in G_1 form a blue copy of Q_1. Otherwise, the out-neighborhood of u in G_1 induces a red sub-tournament. If $t \geq 4$ then we have a red copy of Q_1 since any tournament on 4 vertices contains Q_1. If $t=3$, then all vertices of G_1 have out-degree 3 and in-degree 3. In this case the three edges $(u,x),(u,v),(y,v)$ where y is an in-neighbor of v distinct from u and x form a blue copy of Q_1.

We show that there is a monochromatic copy of Q_3. The proof for Q_2 is symmetric. As in the previous case we define G_1, u, and $t \geq 3$. If there are two out-neighbors of u in G_1, say, v, w where (v,w) is a blue edge, then the three edges $(u,x),(u,v),(v,w)$ where x is another out-neighbor of u in G_1 form a blue copy of Q_3. Otherwise, the out-neighborhood of u in G_1 induces a red sub-tournament. If $t \geq 4$ then we have a red copy of H_1 since any tournament on 4 vertices contains Q_3. If $t=3$ then all vertices of G_1 have out-degree 3 and in-degree 3. In this case the three edges $(u,x),(u,v),(v,y)$ where y is an out-neighbor of v distinct from u and x form a blue copy of Q_3.

For the second part of the theorem, notice that Lemma 4.1 implies, in particular that $r(h,2) \geq 8h-20$. Since $8h-20 > (h-1)^2$ for $h<7$, we have that $f(h) \geq 3$ for all $h \leq 6$. ■

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