The rainbow connection of a graph is (at most) reciprocal to its minimum degree

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Abstract

An edge-colored graph G is *rainbow edge-connected* if any two vertices are connected by a path whose edges have distinct colors. The *rainbow connection* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow edge-connected. We prove that if G has n vertices and minimum degree δ then $rc(G) < 20n/\delta$. This solves open problems from [5] and [3].

A vertex-colored graph G is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. One cannot upper-bound one of these parameters in terms of the other. Nevertheless, we prove that if G has n vertices and minimum degree δ then $rvc(G) < 11n/\delta$. We note that the proof in this case is different from the proof for the edge-colored case, and we cannot deduce one from the other.

1 Introduction

All graphs in this paper are finite, undirected and simple. We follow the notation and terminology of [2]. The following interesting connectivity measure of a graph has recently attracted the attention of several researchers. An edge-colored graph G is *rainbow edge-connected* if any two vertices are connected by a path whose edges have distinct colors. Clearly, if a graph is rainbow edge-connected, then it is also connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow edge-connected; just color each edge with a distinct color. Thus, the following natural graph parameter was defined by Chartrand et al. in [4]. Let the *rainbow connection* of a connected graph G, denoted by $rc(G)$, be the smallest number of colors that are needed in order to make G rainbow edge-connected.

An easy observation is that if G has n vertices then $rc(G) \le n-1$, since one may color the edges of a given spanning tree with distinct colors (and leave the remaining edges uncolored). It

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is easy to verify that $rc(G) = 1$ if and only if G is a clique, that $rc(G) = n - 1$ if and only if G is a tree, and that a cycle with $k > 3$ vertices has rainbow connection $\lceil k/2 \rceil$. Also notice that, clearly, $rc(G) \geq diam(G)$ where $diam(G)$ denotes the diameter of G. The parameter $rc(G)$ is monotone non-increasing in the sense that if we add an edge to a graph we cannot increase its rainbow connection.

Caro et al. [5] observed that $rc(G)$ can be bounded by a function of $\delta(G)$, the minimum degree of G. They have proved that if $\delta(G) \geq 3$ then $rc(G) \leq \alpha n$ where $\alpha < 1$ is a constant and $n = |V(G)|$. They conjecture that $\alpha = 3/4$ suffices and prove that $\alpha < 5/6$ (a solution to this conjecture was recently announced by Zsolt Tuza). Clearly, we cannot obtain a similar result if we only assume that $\delta(G) \geq 2$. Just consider two vertex-disjoint triangles connected by a long path of length $n-5$. The diameter of this graph, as well as its rainbow connection, is $n-3$. More generally, it is proved in [5] that if $\delta = \delta(G)$ then $rc(G) \leq \frac{\ln \delta}{\delta} n(1 + o_\delta(1))$. An easier non-asymptotic bound $rc(G) \leq n\frac{4 \ln \delta + 3}{\delta}$ is also proved there. They also construct an example of a graph G with minimum degree δ for which $diam(G) = \frac{3n}{\delta+1} - \frac{\delta+7}{\delta+1}$. Naturally, they raise the open problem of determining the true behavior of $rc(G)$ as a function of $\delta(G)$. The lower bound construction suggests that the logarithmic factor in their upper bound may not be necessary and that, in fact $rc(G) \leq Cn/\delta$ where C is a universal constant.

If true, notice that for graphs with a linear minimum degree ϵn, this implies that $rc(G)$ is at most C/ϵ. However, the result from [5] does not even guarantee the weaker claim that $rc(G)$ is a constant. This was proved recently by Chakraborty et al. in [3]. They prove that for every fixed $\epsilon > 0$ there exists a constant $K = K(\epsilon)$ so that if G is a connected graph with minimum degree at least ϵn then $rc(G) \leq K$. We note that the constant $K = K(\epsilon)$ they obtain is *a tower function* in $1/\epsilon$ and in particular extremely far from being reciprocal to $1/\epsilon$.

Our main result in this paper determines the behavior of $rc(G)$ as a function of $\delta(G)$ and in particular resolves the above-mentioned open problem.

Theorem 1.1 *A connected graph G with n vertices has $rc(G) < 20n/\delta(G)$.*

We note that the constant 20 obtained by our proof is not optimal and can be slightly improved with additional effort. However, by the construction from [5] one cannot expect to replace C by a constant smaller than 3.

A vertex-colored graph G is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. Obviously, we always have $rvc(G) \leq n - 2$ (except for the singleton graph), and $rvc(G) = 0$ if and only if G is a clique. Also, clearly, $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2.

In some cases $rvc(G)$ may be much smaller than $rc(G)$. For example, $rvc(K_{1,n-1}) = 1$ while $rc(K_{1,n-1}) = n - 1$. On the other hand, in some other cases, $rc(G)$ may be much smaller than

$rvc(G)$. Take n vertex-disjoint triangles and, by designating a vertex from each of them, add a complete graph on the designated vertices. This graph has n cut-vertices and hence $rvc(G) \geq n$. In fact, $rvc(G) = n$ by coloring only the cut-vertices with distinct colors. On the other hand, it is not difficult to see that $rc(G) \leq 4$. Just color the edges of the K_n with, say, color 1, and color the edges of each triangle with the colors $2, 3, 4$. These examples show that there is no upper bound for one of the parameters in terms of the other. Nevertheless, we are able to prove a theorem analogous to Theorem 1.1 for the rainbow vertex-connected case.

Theorem 1.2 *A connected graph G with n vertices has $rvc(G) < 11n/\delta(G)$.*

In the next two sections prove Theorem 1.1 and Theorem 1.2, respectively.

2 Proof of Theorem 1.1

We start this section with several lemmas that are needed in order to establish Theorem 1.1. The first lemma is a simple consequence of Euler's Theorem.

Lemma 2.1 *A graph with minimum degree δ has two edge-disjoint spanning subgraphs, each with minimum degree at least $\lfloor(\delta-1)/2\rfloor$.*

Proof: We can obviously assume that the graph is connected. As there are an even number of vertices with odd degree, we can add a matching to G and obtain a (multi)graph G' which is Eulerian. By coloring the edges of an Eulerian cycle with alternating red and blue colors (starting, say, with a vertex v, with the color blue, and with a non-original edge incident with v if there is such an edge) we obtain that for each vertex u other than v, the number of red edges incident with u is equal to the number of blue edges incident with u. At most one of these edges is not an original edge of G. For the vertex v, if the total number of edges of G' is odd we will have that the number of blue edges incident with v is larger by two than the number of red edges incident with v. This difference of two is also at most the difference in G, since we started with a non-original edge incident with v if there is such an edge. ■

A set of vertices S of a graph G is called a 2-*step dominating set* if every vertex of $V(G) \setminus S$ has either a neighbor in S or a common neighbor with a vertex in S.

Lemma 2.2 *If H is a graph with n vertices and minimum degree k, then H has a 2-step dominating set S whose size is at most $n/(k+1)$.*

Proof: Initialize $H_0 = H$, $S = \emptyset$, and then for as long as $\Delta(H_0) \geq k$, take a vertex v of degree at least k in H_0, add it to S and update H_0 by deleting v and its neighbors from the vertex set of H_0. Observe that when the process has stopped each remaining vertex has lost in its degree and therefore has a neighbor in the set of deleted vertices. Since the latter is dominated by S, we have that S eventually dominates the whole of $V(G)\setminus S$ in two steps. Clearly the process lasted at most $n/(k+1)$ rounds. ■

Lemma 2.3 *If S is a 2-step dominating set of a connected graph G then there is a set of vertices $S' \supset S$ so that $G[S']$ is connected and $|S'| \leq 5|S|-4$.*

Proof: Let c denote the number of connected components of $G[S]$. If $c=1$ we are done, as we may take $S'=S$. Otherwise, consider a shortest path connecting two vertices in distinct components of S, say vertices x and y. This path has at most four internal vertices, as if there were more then there would be a vertex on this path whose distance to any vertex of S is at least 3, contradicting the fact that S is a 2-step dominating set. Thus, by adding four vertices to S we can decrease the number of components. Henceforth, by adding at most $4(c-1)$ vertices to S we obtain a set $S' \supset S$ so that $G[S']$ is connected. ■

Proof of Theorem 1.1: Suppose that G is a connected graph with n vertices and minimum degree δ. Set $k = \lfloor(\delta-1)/2\rfloor$. We first apply Lemma 2.1 to obtain two edge-disjoint spanning subgraphs of G, denote G_1 and G_2, with $\delta(G_i) \geq k$. We next apply Lemma 2.2 to each of the G_i to obtain a 2-step dominating set S_i of G_i with $|S_i| \leq n/(k+1)$ for $i=1,2$. Since S_i is also a 2-step dominating set of G, and since G is connected, we can apply Lemma 2.3 and obtain $S'_i \supset S_i$ with $|S'_i| \leq 5n/(k+1)-4$ and so that $G[S'_i]$ is connected. Now consider $S' = S'_1 \cup S'_2$. Notice that we have that either $G[S']$ is connected (this happens, for example, if S'_1 and S'_2 intersect) or else, using the fact that S_1 is a 2-step dominating set, we can add at most one vertex to S' to obtain a set $S \supset S'$ so that $G[S]$ is connected. In any case, we have constructed a set S with $|S| \leq 10n/(k+1)-7$ vertices so that $G[S]$ is connected and $S_i \subset S$ for $i=1,2$.

Let T be a spanning tree of S. Let $W = V \setminus S$ and consider the following subsets of W. Let $D_i \subset W$ be the vertices of W having a neighbor of S_i for $i=1,2$ and notice that each vertex of $L_i = W \setminus D_i$ has a neighbor in D_i for $i=1,2$. We color the edges of G as follows. Each edge of T receives a fresh distinct color. All edges between S_1 and D_1 belonging to G_1 receive the same fresh color. All edges between D_1 and L_1 belonging to G_1 receive the same fresh color. All edges between S_2 and D_2 belonging to G_2 receive the same fresh color. All edges between D_2 and L_2 belonging to G_2 receive the same fresh color. The remaining edges of G may stay uncolored. The overall number of colors used is $|S|+3 \leq 10n/(k+1)-4$.

It remains to show that the coloring makes G rainbow edge-connected. Indeed, let x,y be two vertices of G. If $x \in S$ and $y \in S$ then we can use the path in T connecting them. Otherwise if $x \in S$ and $y \in D_1 \cup D_2$ then we can use an edge of G_1 from y to some vertex $z \in S$ and then the path in T connecting z and x. Otherwise if $x \in S$ then we must have $y \in L_1 \cap L_2$ and hence we can use an edge of G_1 from y to $z \in D_1$, an edge of G_1 from z to $u \in S$ and the path from u to x in T. Otherwise, we may assume that both x and y are in W. If $x \in D_1$ and $y \in D_2$ then let $u \in S$ be a neighbor of x so that $(x,u) \in E(G_1)$ and let $z \in S$ be a neighbor of y so that $(y,z) \in E(G_2)$. These two edges together with the path in T connecting u and z form a rainbow path between x and y. Otherwise if $x \in D_1$ and $y \in L_2$ then we can reduce to the last argument by adding another edge of $E(G_2)$ from y to a vertex of D_2. Otherwise we can assume that both x and y are in $L_1 \cap L_2$ and we can reduce to the last argument by adding another edge of $E(G_1)$ from x to a vertex of D_1. ■

3 Proof of Theorem 1.2

The proof of Theorem 1.2 also requires us to find a relatively small 2-step dominating set. However, we need additional important requirement from it. We call a 2-step dominating set *k-strong* if every vertex that is not dominated by it has at least k neighbors that are dominated by it.

Lemma 3.1 *If H is a graph with n vertices and minimum degree δ, then H has a $\delta/2$-strong 2-step dominating set S whose size is at most $2n/(\delta+2)$.*

Proof: Initialize $H_0 = H$, $S = \emptyset$, and then for as long as $\Delta(H_0) \geq \delta/2$, take a vertex v of degree at least $\delta/2$ in H_0, add it to S and update H_0 by deleting v and its neighbors from the vertex set of H_0. Observe that when the process has stopped each remaining vertex has lost more than $\delta/2$ in its degree and therefore has more than $\delta/2$ neighbors in the set of deleted vertices. Clearly the process lasted at most $n/(\delta/2+1)$ rounds. ■
Notice the obvious, but important fact: adding vertices to a 2-step dominating set does not decrease its strength.

Lemma 3.2 *If G is a connected graph with minimum degree δ then it has a connected spanning subgraph with minimum degree δ and with less than $n(\delta + 1/(\delta+1))$ edges.*

Proof: By deleting from G edges that connect two vertices with degree greater than δ as long as there are any we obtain a spanning subgraph with minimum degree δ and less than δn edges. Each connected component of this spanning subgraph has at least $\delta+1$ vertices. Thus, by adding back at most $n/(\delta+1)-1$ edges we can make it connected. ■

Proof of Theorem 1.2: The statement of the theorem is trivial for $\delta \leq 11$ so we assume that $\delta > 11$. Suppose that G is a connected graph with n vertices and minimum degree δ. By Lemma 3.2 we may assume that G has less than $n(\delta + 1/(\delta+1))$ edges. We use Lemma 3.1 to construct a set S which is a $\delta/2$-strong 2-step dominating set of size $|S| \leq 2n/(\delta+2)$. From Lemma 2.3 we can add at most $4(|S|-1)$ vertices to S and obtain $S' \supset S$ so that $G[S']$ is connected and S' is also a $\delta/2$-strong 2-step dominating set. Observe that $|S'| \leq 10n/\delta - 5$.

Let $W = V(G) \setminus S'$ and consider the partition $W = D \cup L$ where D is the set of vertices directly dominated by S' and L is the set of vertices not dominated by S'. Since S' is $\delta/2$-strong, each $v \in L$ has at least $\delta/2$ neighbors in D. We further partition D into two parts D_1 and D_2 where D_1 are those vertices with at least $\delta(\delta+1)$ neighbors in L. Notice that $|D_1| < n/\delta$ since otherwise

G would have had at least $n(\delta+1)$ edges, contradicting our assumption. We also partition L into two parts L_1 and L_2 where L_1 are those vertices that have at least one neighbor in D_1.

We are now ready to describe our coloring. The vertices of $S \cup D_1$ are each colored with a distinct color. The vertices of D_2 are colored only with *five* fresh colors so that each vertex of D_2 chooses its color randomly and independently from all other vertices of D_2. The vertices of L remain uncolored. The overall number of colors used is less than $11n/\delta$.

It remains to show that, with positive probability, our coloring yields a rainbow vertex-connected graph. We first need to establish the following claim.

Claim 3.3 *With positive probability, every vertex of L_2 has at least two neighbors in D_2 colored differently.*

Consider a vertex $v \in L_2$. As it has no neighbor in D_1, it has at least $\delta/2$ neighbors in D_2. Fix, therefore a set $X(v) \subset D_2$ of neighbors of v with $|X(v)| = \lceil \delta/2 \rceil$. The probability of the bad event B_v that all of the vertices of $X(v)$ receive the same color is $5^{-\lceil \delta/2 \rceil+1}$. As each vertex of D_2 has less than $\delta(\delta+1)$ neighbors in L we have that the event B_v is independent of all other events B_u for $u \neq v$ but at most $(\delta(\delta+1)-1)\lceil \delta/2 \rceil$ of them. Since

$$e \cdot 5^{-\lceil \delta/2 \rceil+1}((\delta(\delta+1)-1)\lceil \delta/2 \rceil + 1) < 1$$

for all $\delta \geq 11$, we have by the Lovász Local Lemma (cf. [1]) that, with positive probability, none of the bad event B_u hold.

Having proved the claim we can now fix a coloring of D_2 with five colors so that each vertex of L_2 has at least two neighbors in D_2 colored differently. We now show that this coloring, together with the coloring of $S' \cup D_1$ with distinct colors, yields a rainbow vertex-connected graph. As $S' \cup D_1$ is connected, and since each vertex of D_2 has a neighbor in S', we only need to show that pairs of vertices of L have a rainbow path connecting them. Each $v \in L$ has (at least) two neighbors in D colored differently. This is true for $v \in L_2$ as v has two such neighbors already in D_2. This is also trivially true for $v \in L_1$ since the vertices of D_1 are colored distinctly, and with colors that are not one of the five colors used in D_2. Now let $u, v \in L$. Let $x \in D$ be a neighbor of u and let $y \in D$ be a neighbor of v whose color is different from the color of x. As there is a rainbow path from x to y whose internal vertices are only taken from S', the result follows. ■

References

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