Rainbow decompositions

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Abstract

A *rainbow coloring* of a graph is a coloring of the edges with distinct colors. We prove the following extension of Wilson's Theorem. For every integer k there exists an $n_0 = n_0(k)$ so that for all $n > n_0$, if $n \bmod k(k-1) \in \{1, k\}$ then every properly edge-colored K_n contains $\binom{n}{2}/\binom{k}{2}$ pairwise edge-disjoint rainbow copies of K_k.

Our proof uses, as a main ingredient, a double application of the probabilistic method.

1 Introduction

All graphs considered here are finite, undirected, and simple. For standard graph-theoretic terminology the reader is referred to [2]. For an integer $k \geq 3$, a *Steiner system* $S(2, k, n)$ is a set X of n points, and a collection of subsets of X of size k (called blocks), such that any two points of X are in exactly one of the blocks. We say that the complete graph K_n has a K_k*-decomposition* if K_n contains $\binom{n}{2}/\binom{k}{2}$ pairwise edge-disjoint copies of K_k. Clearly, an $S(2, k, n)$ exists if and only if K_n has a K_k-decomposition. More generally, for a given graph H we say that K_n is H-decomposable if K_n contains $\binom{n}{2}/e(H)$ edge-disjoint copies of H.

For a graph H, let $gcd(H)$ denote the largest integer that divides the degree of each vertex of H. Two obvious necessary conditions for the existence of an H-decomposition of K_n are that $e(H)$ divides $\binom{n}{2}$ and that $gcd(H)$ divides $n - 1$ (notice that $n - 1 = gcd(K_n)$). These trivial divisibility conditions are not always sufficient. For example K_4 does not have a $K_{1,3}$-decomposition. However, a seminal result of Wilson [10] show that, for all n sufficiently large, the two divisibility conditions suffice.

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Theorem 1.1 *[Wilson [10]] For every fixed graph H there exists $n_0 = n_0(H)$ so that if $n > n_0$, $e(H) \mid \binom{n}{2}$, and gcd$(H) \mid n-1$ then K_n has an H-decomposition.*

We note that if $H = K_k$ the divisibility conditions translate to $n \bmod k(k-1) \in \{1, k\}$. The case $k = 3$ (i.e. the existence of a Steiner triple system) is an old result. Such a system exists if and only if $n \bmod 6 \in \{1, 3\}$ [9]. The case $k = 4$ is also known to exist if and only if $n \bmod 12 \in \{1, 4\}$ [3].

Recently there have been a rise of interest in extremal *rainbow-type problems*. A *rainbow coloring* of a graph is a coloring of the edges with distinct colors. An edge coloring of a graph is called *proper* if two edges sharing an endpoint receive distinct colors. Vizing's theorem asserts that there exists a proper edge coloring of a graph G which uses at most $\Delta(G)+1$ colors. In extremal graph theory, one is interested in establishing conditions on a graph G that guarantee the existence of a (possibly induced) set of subgraphs of a specific type (Ramsey and Turán type problems are central problems of this type). In rainbow-type problems one is interested in establishing conditions on a properly edge-colored graph G that guarantee the existence of a (possibly induced) set of rainbow subgraphs of a specific type. Many graph theoretic parameters have corresponding rainbow variants. Erdős and Rado [6] were among the first to consider problems of this type. Jamison, Jiang and Ling [7], and Chen, Schelp and Wei [4] considered Ramsey type variants where an arbitrary number of colors can be used. Alon et al. [1] studied the function $f(H)$ which is the minimum integer n such that any proper edge coloring of K_n has a rainbow copy of H. Keevash et al. [8] considered the rainbow Turán number $ex^*(n, H)$ which is the largest integer m such that there exists a properly edge-colored graph with n vertices and m edges and which has no rainbow copy of H. Yuster [12] gave necessary and sufficient conditions for the existence of rainbow H-factors.

Is Theorem 1.1 still true in the rainbow setting? The main result of this paper shows that, indeed, this is the case.

Theorem 1.2 *For every fixed graph H there exists $n_1 = n_1(H)$ so that if $n > n_1$, $e(H) \mid \binom{n}{2}$, and gcd$(H) \mid n-1$ then a properly edge-colored K_n has an H-decomposition so that each copy of H in the decomposition is rainbow colored.*

We note that the case $H = K_3$ is trivial since every properly edge-colored triangle is also rainbow colored. However, already for K_4, the analogue of Brouwer's result [3] trivially does not hold, as a properly edge-colored K_4 need not be rainbow colored. The proof of Theorem 1.2 appears in Section 3 and is based upon a double application of the probabilistic method. A few lemmas that are needed for the proof of Theorem 1.2 follow in the next section. The final section contains some concluding remarks and open problems.

2 Preliminary lemmas

Let F be a family of positive integers. We say that K_n is *F-decomposable* if we can color the edges of K_n such that each color class induces a complete graph whose order belongs to F.

Let H be a fixed graph, and let t be a positive integer. We say that a finite set of positive integers F is an *(H,t)-complete decomposition set* ((H,t)-CDS for short) if the following holds:

1. If $k \in F$ then $k \geq t$ and K_k is H-decomposable.
2. There exists N such that for all $n > N$, K_n is H-decomposable if and only if K_n is F-decomposable.

We prove that, for all H and t, an (H,t)-CDS always exists. For this purpose we need a theorem of Wilson on F-decompositions. For a (possibly infinite) set of positive integers F, let $gcd(F)$ denote the largest positive integer which divides each number in F, let $F_1 = \{n-1 \mid n \in F\}$ and let $F_2 = \{n(n-1)/2 \mid n \in F\}$. In [11] Wilson has proved the following:

Lemma 2.1 (Wilson [11]) *Let F be a finite set of positive integers. Then, there exists $n_0 = n_0(F)$ such that if $n > n_0$, $gcd(F_1)$ divides $n-1$ and $gcd(F_2)$ divides $\binom{n}{2}$ then there exists an F-decomposition of K_n.* ■

We can now show the following:

Lemma 2.2 *Let H be a graph, and let t be a positive integer. Then, an (H,t)-CDS exists.*

Proof: Let

$$S = \{s \mid s \geq t \text{ and } K_s \text{ is } H-decomposable\}.$$

S is infinite but, obviously, $gcd(S_1)$ and $gcd(S_2)$ are finite. Thus, there are finite subsets $S^\alpha \subset S$ and $S^\beta \subset S$ such that $\gcd(S_1^\alpha) = gcd(S_1)$ and $\gcd(S_2^\beta) = gcd(S_2)$. Let $F = S^\alpha \cup S^\beta$. Note that since $S^\alpha \subset F$ then $gcd(F_1)$ divides $gcd(S_1)$. Similarly, since $S^\beta \subset F$ we have that $gcd(F_2)$ divides $gcd(S_2)$. We claim that F is an (H,t)-CDS. First note that, by definition, every $s \in F$ satisfies $s \geq t$ and K_s is H-decomposable. Now let $N = \max\{n_0, t\}$ where $n_0 = n_0(F)$ is the constant defined in the statement of Lemma 2.1. It suffices to show that for every $n > N$, if K_n is H-decomposable then it is also F-decomposable. Indeed, if K_n is H-decomposable, then $n \in S$, so $gcd(F_1) \mid gcd(S_1) \mid n-1$, and $gcd(F_2) \mid gcd(S_2) \mid \binom{n}{2}$. Thus, by Lemma 2.1, K_n is F-decomposable. ■

A properly colored forest T will be called a *weed* if it contains three distinct edges e_1, e_2, e_3, so that for each e_i there is an edge $f_i \notin \{e_1, e_2, e_3\}$ having the same color as e_i (the f_i need not be distinct; in particular, if e_1, e_2, e_3 have the same color we can have $f_1 = f_2 = f_3$). Furthermore, we assume that T has no sub-weed. Trivially, every weed contains at most 6 edges and at least 4 edges. In fact, is is not difficult to check that, up to color isomorphism, there is precisely one weed with four edges, 8 weeds with five edges, and 41 weeds with six edges. The 50 weeds are shown in Figure 1.

A properly colored graph is called *multiply colored* if no color appears only once in the graph. We need the following combinatorial lemma.

Lemma 2.3 *Every multiply colored graph with at least* 29 *edges contains a weed.*

Proof: Let G be a multiply colored graph with 29 edges. If some color appears four times in G then it forms a matching with four edges which is the weed W_1. Otherwise, suppose that some color c appears three times in the edges $(v_1, v_2), (v_3, v_4), (v_5, v_6)$. Since six vertices induce at most 15 edges, there is some edge (x, y) colored with c', and $x \notin \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let (w, z) be another edge colored with c'. Since the coloring of G is proper, the five edges $(v_1, v_2), (v_3, v_4), (v_5, v_6), (x, y), (w, z)$ form a weed.

We remain with the case where each color appears precisely twice in G. Let c be the color of (v_1, v_2) and of (v_3, v_4). Let c' be a color not appearing in the subgraph induced by $\{v_1, v_2, v_3, v_4\}$. Denote the edges colored with c' by (v_5, v_6) and (v_7, v_8). Assume first that $\{v_1, v_2, v_3, v_4\} \cap \{v_5, v_6, v_7, v_8\} = \emptyset$. Since 8 vertices induce at most 28 edges, there is some edge (x, y) colored with c'', and $x \notin \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. Let (w, z) be another edge colored with c''. Since the coloring of G is proper, the six edges $(v_1, v_2), (v_3, v_4), (v_5, v_6), (v_7, v_8), (x, y), (w, z)$ form a weed.

Assume next that $|\{v_1, v_2, v_3, v_4\} \cap \{v_5, v_6, v_7, v_8\}| = 1$. Without loss of generality, $v_1 = v_5$. Since 7 vertices induce at most 21 edges, there is some edge (x, y) colored with c'' so that c'' is not the color of the edge (v_2, v_6) (if the latter even exists), and $x \notin \{v_1, v_2, v_3, v_4, v_6, v_7, v_8\}$. Let (w, z) be another edge colored with c''. The six edges $(v_1, v_2), (v_3, v_4), (v_1, v_6), (v_7, v_8), (x, y), (w, z)$ form a weed.

We remain with the case $|\{v_1, v_2, v_3, v_4\} \cap \{v_5, v_6, v_7, v_8\}| = 2$. There are two sub-cases here, up to isomorphism. The first is $v_1 = v_5$ and $v_3 = v_7$. The second is $v_1 = v_5$ and $v_2 = v_7$. In the first sub-case, since six vertices induce at most 15 edges, there is some edge (x, y) colored with c'' so that c'' is not the color of the edge (v_2, v_6) nor of the edge (v_4, v_8) (if any of these edges even exist), and $x \notin \{v_1, v_2, v_3, v_4, v_6, v_8\}$. Let (w, z) be another edge colored with c''. Since the coloring of G is proper, the six edges $(v_1, v_2), (v_3, v_4), (v_1, v_6), (v_3, v_8), (x, y), (w, z)$ form a weed. Similarly, in the second sub-case, there is some edge (x, y) colored with c'' so that c'' is not the color of the edge (v_2, v_6) nor of the edge (v_1, v_8) nor of the edge (v_6, v_8) (if any of these edges even exist), and $x \notin \{v_1, v_2, v_3, v_4, v_6, v_8\}$. Let (w, z) be another edge colored with c''. The six edges $(v_1, v_2), (v_3, v_4), (v_1, v_6), (v_2, v_8), (x, y), (w, z)$ form a weed. ■

It is possible to improve the constant 29 in Lemma 2.3 at the price of a tedious case analysis. This, however, is not important for our purposes. A multiple coloring of K_6 (which has 15 edges) having no weed is easily obtained by coloring a matching of three edges red, and the other 12 edges with 6 distinct colors, where the union of each of these colors with the red edges contains a C_4.

3 Proof of the main result

The proof of Theorem 1.2 is established by the following two lemmas.

Lemma 3.1 *For a positive integer r, and for a graph H, there is a constant $C = C(r, H)$ so that if $k > C$ and K_k is H-decomposable, then for any given set U of r edges of K_k, there is an H-decomposition of K_k in which no two elements of U appear together in the same H-copy of the decomposition.*

Proof: Suppose K_k is H-decomposable, and fix an H-decomposition of K_k, denoted L. Let π be a permutation of $\{1, \ldots, k\}$. The permutation π and L naturally define another H-decomposition of K_k, denoted L_π. Indeed, if $Q \in L$ is a copy of H then Q is mapped to a copy $Q_\pi \in L_\pi$ by the automorphism π on the vertex set of K_k.

Fix a set U of r edges of K_k. Picking π uniformly at random, consider the probability that two edges $e, f \in U$ are in the same copy of H in L_π. If e and f do not share an endpoint, the probability of this event is at most

$$\frac{e(H) - 1}{\binom{k-2}{2}} \ .$$

If e and f share an endpoint, the probability of this event is at most

$$\frac{v(H) - 2}{k - 2} \ .$$

As there are $\binom{r}{2}$ possible pairs of edges of U, we have that, as long as

$$\binom{r}{2} \max \left\{ \frac{e(H) - 1}{\binom{k-2}{2}} \ , \ \frac{v(H) - 2}{k - 2} \right\} < 1, \qquad (1)$$

with positive probability, no two elements of U appear together in the same H-copy of L_π. Thus, for sufficiently large k, as a function of r and H, inequality (1) holds, and hence the lemma follows. ■

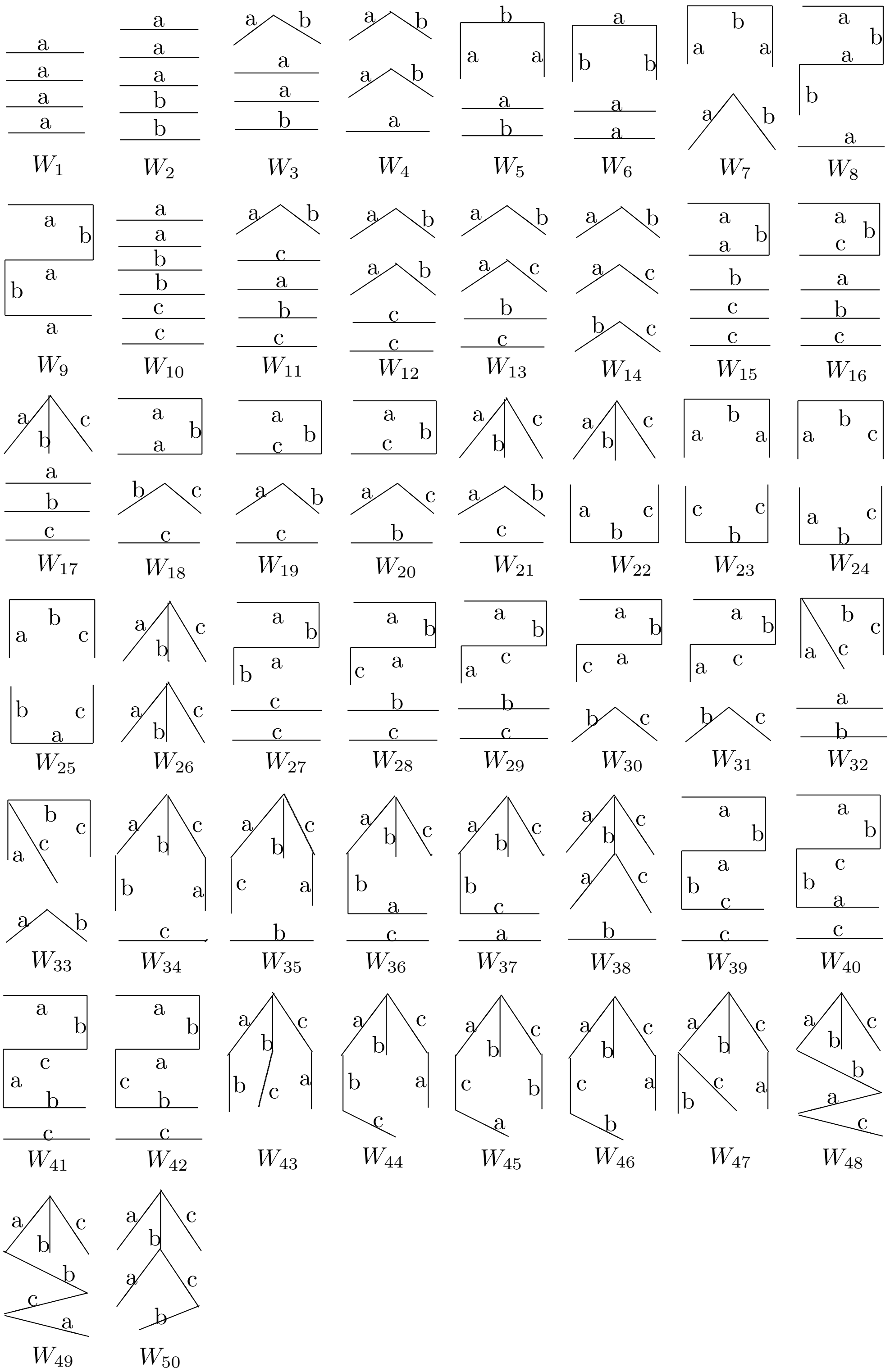

Figure 1: The 50 weeds. Labels on edges represent colors

Lemma 3.2 *For every finite set of positive integers F there exists a constant $M = M(F)$, so that for every $n > M$, if K_n is a properly edge colored and F-decomposable, then K_n also has an F-decomposition so that every element of the decomposition contains no weed.*

Proof: Suppose that K_n is properly edge colored and F-decomposable, and fix an F-decomposition of K_n, denoted L. Notice that, clearly, $|L| \le \binom{n}{2}$. As in the proof of Lemma 3.1, let π be a permutation of $\{1, \ldots, n\}$, and let L_π be the F-decomposition naturally defined by π and L. It suffices to prove that if π is picked uniformly at random, then for each $Q \in L$, the mapped copy $Q_\pi \in L_\pi$, contains a weed with probability less than $1/n^2$ (since $1/n^2 < 1/|L|$), assuming n is sufficiently large as a function of F. Let z be the largest integer in F.

Fix $Q \in L$, and suppose that $|Q| = q \le z$. Without loss of generality, Q is a K_q induced by the vertices $\{1, \ldots, q\}$. We will prove that, for each possible weed W_i for $i = 1, \ldots, 50$, the probability that Q_π (which is a properly colored K_q subgraph of K_n) contains a forest which is color isomorphic to W_i is less than $1/(50n^2)$. For the sake of space compactness, we will prove this for W_1 (the only weed with four edges), for W_7 (a representative weed with five edges), and for W_{33} (a representative weed with six edges). The proof for all other 47 weeds is practically the same.

Consider a labeled copy of a forest that is isomorphic to the uncolored forest W_1 in Q. As W_1 contains 8 vertices, there are less than q^8 such labeled forests. Fixing one such forest P, w.l.o.g. the one defined by the edges $(1, 2), (3, 4), (5, 6), (7, 8)$, we will prove that all the edges of P_π are colored the same with probability less than $1/(50n^2q^8)$. This will show that the probability that Q_π contains the weed W_1 is less than $1/(50n^2)$, as required. Let a be the color of $(\pi(1), \pi(2))$. Given a, the probability that $(\pi(3), \pi(4))$ is also colored a is precisely

$$\frac{n_a - 1}{\binom{n-2}{2}} \le \frac{n/2 - 1}{\binom{n-2}{2}} = \frac{1}{n-3}$$

where n_a is the number of edges colored a in the given proper edge coloring of K_n. Similarly, given that $(\pi(1), \pi(2))$ and $(\pi(3), \pi(4))$ are both colored a, the probability that $(\pi(5), \pi(6))$ is also colored a is precisely $(n_a - 2)/\binom{n-4}{2} \le 1/(n-5)$. Finally, given that $(\pi(1), \pi(2))$, $(\pi(3), \pi(4))$, and $(\pi(5), \pi(6))$ are all colored a, the probability that $(\pi(7), \pi(8))$ is also colored a is precisely $(n_a - 3)/\binom{n-6}{2} \le 1/(n-7)$. Overall, the edges of P_π are colored the same with probability at most

$$\frac{1}{(n-3)(n-5)(n-7)} \le \frac{1}{50n^2z^8} \le \frac{1}{50n^2q^8}$$

if n is sufficiently large, as required.

Consider a labeled copy of a forest which is isomorphic to the uncolored forest W_7 in Q. As W_7 contains 7 vertices, there are less than q^7 such labeled forests. Fixing one such forest P, w.l.o.g. the one defined by the edges $(1,2),(2,3),(3,4),(5,6),(6,7)$ (notice that this is indeed a labeled copy of W_7), we will prove that the probability that all the edges $(\pi(1),\pi(2)),(\pi(3),\pi(4)),(\pi(5),\pi(6))$ are colored the same, and that the edges $(\pi(2),\pi(3))$ and $(\pi(6),\pi(7))$ are also colored the same is less than $1/(50n^2q^7)$. This will show that the probability that Q_π contains the weed W_7 is less than $1/(50n^2)$, as required. Let a be the color of $(\pi(1),\pi(2))$ and let b be the color of $(\pi(2),\pi(3))$. Notice that $a \neq b$ since the given coloring of K_n is proper. Given a and b, the probability that $(\pi(3),\pi(4))$ is colored a is at most $1/(n-3)$. This is because at most one edge $(\pi(3),x)$ incident with $\pi(3)$ is colored a, and the probability that $x = \pi(4)$ is precisely $1/(n-3)$. Similarly, given that $(\pi(1),\pi(2))$ and $(\pi(3),\pi(4))$ are both colored a, and that $(\pi(2),\pi(3))$ is colored b, the probability that $(\pi(5),\pi(6))$ is colored a is precisely $(n_a-2)/\binom{n-4}{2} \leq 1/(n-5)$. Finally, given that $(\pi(1),\pi(2))$, $(\pi(3),\pi(4))$, and $(\pi(5),\pi(6))$ are colored a, and that $(\pi(2),\pi(3))$ is colored b, the probability that $(\pi(6),\pi(7))$ is colored b is at most $1/(n-6)$. This is because at most one edge $(\pi(6),x)$, where $x \neq \pi(i)$ for $i = 1,\ldots,5$, incident with $\pi(6)$, is colored b, and the probability that $x = \pi(7)$ is precisely $1/(n-6)$. Overall, the probability that $(\pi(1),\pi(2)),(\pi(3),\pi(4)),(\pi(5),\pi(6))$ are colored the same, and that $(\pi(2),\pi(3))$ and $(\pi(6),\pi(7))$ are also colored the same is at most

$$\frac{1}{(n-3)(n-5)(n-6)} \leq \frac{1}{50n^2z^7} \leq \frac{1}{50n^2q^7}$$

if n is sufficiently large, as required.

Consider a labeled copy of a forest which is isomorphic to the uncolored forest W_{33} in Q. As W_{33} contains 8 vertices, there are less than q^8 such labeled forests. Fixing one such forest P, w.l.o.g. the one defined by the edges $(1,2),(2,3),(3,4),(2,5),(6,7),(7,8)$ (notice that this is indeed a labeled copy of W_{33}), we will prove that the probability that the edges $(\pi(1),\pi(2))$ and $(\pi(6),\pi(7))$ are colored the same, the edges $(\pi(2),\pi(3))$ and $(\pi(7),\pi(8))$ are colored the same, and the edges $(\pi(3),\pi(4))$ and $(\pi(2),\pi(5))$ are colored the same, is less than $1/(50n^2q^8)$. This will show that the probability that Q_π contains the weed W_{33} is less than $1/(50n^2)$, as required. Let a be the color of $(\pi(1),\pi(2))$, let b be the color of $(\pi(2),\pi(3))$, and let c be the color of $(\pi(2),\pi(5))$. Notice that a, b, and c are distinct colors since the given coloring of K_n is proper. Given a, b, and c, the probability that $(\pi(3),\pi(4))$ is colored c is at most $1/(n-4)$. Similarly, given that $(\pi(2),\pi(5))$ and $(\pi(3),\pi(4))$ are both colored c, that $(\pi(1),\pi(2))$ is colored a, and that $(\pi(2),\pi(3))$ is colored b, the probability that $(\pi(6),\pi(7))$ is colored a is at most $((n-5)/2)/\binom{n-5}{2} \leq 1/(n-6)$. Finally, given that $(\pi(2),\pi(5))$ and $(\pi(3),\pi(4))$ are both colored c, that $(\pi(1),\pi(2))$ and $(\pi(6),\pi(7))$ are both colored a, and that $(\pi(2),\pi(3))$ is colored b, the probability that $(\pi(7),\pi(8))$ is colored b is at most $1/(n-7)$. Overall, the probability that the edges $(\pi(1),\pi(2))$ and $(\pi(6),\pi(7))$ are colored the same, the edges $(\pi(2),\pi(3))$ and $(\pi(7),\pi(8))$ are colored the same, and the edges $(\pi(3),\pi(4))$

and $(\pi(2), \pi(5))$ are colored the same is at most

$$\frac{1}{(n-4)(n-6)(n-7)} \leq \frac{1}{50n^2z^8} \leq \frac{1}{50n^2q^8}$$

if n is sufficiently large, as required. ■

Proof of Theorem 1.2: Fix a graph H, and let $t = C(28, H)$ be the constant from Lemma 3.1. Let F be an (H,t)-CDS, whose existence is guaranteed by Lemma 2.2. Let $M = M(F)$ be the constant from lemma 3.2. Since F is an (H,t)-CDS, there exists $N = N(F)$ so that for all $n > N$, K_n is H-decomposable if and only if K_n is F-decomposable. We define

$$n_1 = n_1(H) = \max\{M\ ,\ N\ ,\ n_0(H)\}$$

where $n_0(H)$ is the constant from Theorem 1.1.

Suppose that $n > n_1$, $e(H) \mid \binom{n}{2}$, and $gcd(H) \mid n-1$ and consider a properly edge-colored K_n. Since $n_1 \geq n_0$ we have, by Theorem 1.1, that K_n is H-decomposable. Since $n_1 \geq N$ we have, by the definition of F, that K_n is also F-decomposable. Since $n_1 \geq M$ we have, by Lemma 3.2, that there is an F-decomposition of K_n so that every element of the decomposition contains no weed. Consider some K_k element of such an F-decomposition. Thus, $k \in F$ and hence $k \geq t$. Furthermore, K_k is H-decomposable. Let U be a maximal multiply colored subgraph of K_k. We identify U with its set of edges. Since K_k contains no weed, we have, by Lemma 2.3, that $|U| \leq 28$. Since $k \geq t = C(28, H)$ we have, by Lemma 3.1, that K_k has an H-decomposition so that no two edges of U appear together in the same H-copy of the decomposition. But this implies that each copy of H in such a decomposition is rainbow colored. Repeating this process for each element of the F-decomposition yields an H-decomposition of K_n in which each element is rainbow colored.
■

4 Concluding remarks

- The statement of Theorem 1.2 remains valid even if K_n is not necessarily properly colored, but, instead, no color appears more than constantly many times in the edges incident with a vertex. More precisely, an edge coloring is C-proper if the subgraph induced by each color has maximum degree at most C. The proof is essentially the same, although one has to broaden the definition of weeds to forests that are not necessarily properly colored, but C-properly colored (there are still finitely many such weeds). The following extension of Theorem 1.2 is:

Theorem 4.1 *For every fixed graph H and positive integer C there exists $n_1 = n_1(H, C)$ so that if $n > n_1$, $e(H) \mid \binom{n}{2}$, and $gcd(H) \mid n - 1$ then a C-properly edge-colored K_n has an H-decomposition so that each copy of H in the decomposition is rainbow colored.*

- The exact smallest possible value of $n_1 = n_1(H)$ is extremely difficult to determine even for the smallest non-trivial cases. For example, is it true that every properly edge colored K_{13} contains 13 rainbow copies of K_4? It is not difficult to show that this is true if each color is used at most twice, but already in K_{13} we can have each color appearing 6 times. We leave this as an open problem.

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