Orthogonal H-decompositions

Yair Caro * Raphael Yuster †

Abstract

An *H-decomposition* of a graph G is a set L of edge-disjoint H-subgraphs of G, such that each edge of G appears in some element of L. A *k-orthogonal* H-decomposition of a graph G is a set of k H-decompositions of G, such that any two copies of H in any two distinct H-decompositions have at most one edge in common.

We prove that for every fixed graph H and every fixed integer $k \geq 1$, if n is sufficiently large then K_n has a k-orthogonal H-decomposition if and only if it has an H-decomposition. This occurs whenever $\binom{n}{2}$ is a multiple of $e(H)$ and $n-1$ is a multiple of the *gcd* of the degrees of H.

1 Introduction

All graphs considered here are finite, undirected, and have no loops, multiple edges, or isolated vertices. For the standard graph-theoretic and design-theoretic notations the reader is referred to [5] and [8] respectively. An *H-subgraph of G* is a subgraph of a graph G, which is isomorphic to a graph H. An *H-decomposition* of a graph G is a set L of edge-disjoint H-subgraphs of G, such that each edge of G appears in some element of L. Thus, L contains $e(G)/e(H)$ elements, where $e(X)$ denotes the number of edges of a graph X. It is straightforward to see that a necessary condition for the existence of an H-decomposition is that $e(H)$ divides $e(G)$. Another obvious requirement is that $gcd(H)$ divides $gcd(G)$ where the *gcd* of a graph is the greatest common divisor of the degrees of its vertices.

In general, it is NP-Complete to determine whether a given graph G has an H-decomposition for every fixed graph H containing more than two edges in some connected component. This has been proved by Dor and Tarsi [10]. However, a seminal result of Wilson [21], is that the existence of the

*Department of Mathematics, University of Haifa-Oranim, Tivon 36006, Israel. email: yairc@macam98.ac.il

†Department of Mathematics, University of Haifa-Oranim, Tivon 36006, Israel. email: raphy@macam98.ac.il

two necessary conditions mentioned above is also sufficient to guarantee an H-decomposition of K_n for every $n > n_0(H)$, and this result holds for every fixed nonempty graph H. In terms of design-theory, Wilson's Theorem states that the necessary conditions are sufficient for the existence of a $2-(v,k,1)$-design, provided that v is sufficiently large (in fact, it is sufficient for the existence of a $2-(v,k,\lambda)$-design).

A *k-orthogonal* H-decomposition of a graph G is a set of k distinct H-decompositions of G, such that any two copies of H in any two distinct H-decompositions have at most one edge in common. A 2-orthogonal H-decomposition is simply called an *orthogonal* H-decomposition. Obviously, a k-orthogonal H-decomposition does not necessarily exist, even if an H-decomposition exists. The notion of k-orthogonal decompositions has a natural design theoretic translation. The theory of simple designs asks for the existence of a $t-(v,k,\lambda)$ design with no repeated blocks. However, stronger conditions are usually imposed on the design. In case no two blocks have more than one pair (edge) in common, the design is called a *super-simple design* and is denoted $SS(t,v,k,\lambda)$ design, or simply $SS(v,k,\lambda)$ design if $t=2$. In case that a $SS(t,v,k,\lambda)$ design splits into λ copies of a $SS(t,v,k,1)$-design, the design is called a *completely-reducible super-simple* design, denoted by $CRSS(t,v,k,\lambda)$ or simply $CRSS(v,k,\lambda)$ if $t=2$. Recent results on super-simple and completely-reducible super-simple designs can be found in [1, 4, 13, 16, 17]. The requirement that any two blocks have at most one pair in common is called the *orthogonality* property. Many results in design theory concerning orthogonality have appeared in recent years and we refer the reader to the surveys in [2, 8] for details and to [2, 3, 6, 11, 12, 14, 15] for recent developments in this area. Clearly, a $CRSS(n,r,k)$ is equivalent to a k-orthogonal K_r-decomposition of K_n.

In [7] the authors have shown the existence of a $CRSS(n,r,k)$ design, for every n which is sufficiently large (as a function of r and k), and which satisfies the necessary divisibility conditions. In other words, this gives necessary and sufficient conditions for the existence of k-orthogonal decompositions in case both H and G are complete (and G is sufficiently large). It should be noted that for $r=3$, all values of k and n for which a $CRSS(n,3,k)$-design or a $SS(n,3,k)$-design exists are known [18, 19, 20]. For $r=4$, it is known whenever a $CRSS(n,4,2)$ design exists [1]. Several other sporadic results involving the case $r=4$ also appear in [4, 9, 13, 17].

The goal of this note is to give necessary and sufficient conditions for the existence of a k-orthogonal H-decomposition of K_n where H is an arbitrary fixed graph, and n sufficiently large. We state this formally as follows:

Theorem 1.1 *Let H be a fixed nonempty graph and let $k \geq 1$ be an integer. There exists $N = N(k,H)$ such that if $n > N$ and K_n has an H-decomposition, then K_n also has a k-orthogonal H-decomposition.*

The proof of Theorem 1.1 is based partly on the result in [7] mentioned above, together with additional ideas and a theorem of Wilson on block designs, and is given in the next section.

2 Proof of the main result

As mentioned in the introduction, the following is proved in Theorem 1.1 of [7]:

Lemma 2.1 *Let $r \geq 2$ and $k \geq 1$ be integers. There exists $N = N(k, r)$ such that if $n > N$ and K_n has a K_r-decomposition then K_n also has a k-orthogonal K_r-decomposition.*

Let F be a family of positive integers. We say that K_n is *F-decomposable* if we can color the edges of K_n such that each color class induces a complete graph whose order belongs to F. Note that if $F = \{r\}$ this simply means that K_n has a K_r-decomposition.

Let H be a fixed graph, and let t be a positive integer. We say that a *finite* family of positive integers F is an (H, t) *complete decomposition set* ((H, t)-CDS for short) if the following holds:

1. If $k \in F$ then $k \geq t$ and K_k is H-decomposable.
2. There exists N such that for all $n > N$, K_n is H-decomposable if and only if K_n is F-decomposable.

Our next goal is to show that an (H, t)-CDS always exists. For this purpose we need to state a theorem of Wilson concerning F-decompositions. For a (possibly infinite) family of positive integers F, let:

$$\alpha(F) = \gcd(\{r - 1 \mid r \in F\})$$
$$\beta(F) = \gcd(\{\binom{r}{2} \mid r \in F\})$$

(the gcd of a set is the greatest common divisor of all the elements of the set.) In [22] Wilson has proved the following:

Lemma 2.2 (Wilson [22]) *Let F be a finite family of positive integers. Then, there exists $n_0 = n_0(F)$ such that if $n > n_0$, $\alpha(F)$ divides $n - 1$ and $\beta(F)$ divides $\binom{n}{2}$ then K_n is F-decomposable.* $\Box$

We can now show the following:

Lemma 2.3 *Let H be a graph, and let t be a positive integer. Then, an (H,t)-CDS exists. Furthermore, if $H = K_r$ there is an (H,t)-CDS consisting of two elements r_1 and r_2 satisfying $r_2 - r_1 = r-1$, $r_1 \equiv 1 \bmod r(r-1)$, $\gcd(r_1-1, r_2-1) = r-1$ and $\gcd(\binom{r_1}{2}, \binom{r_2}{2}) = \binom{r}{2}$.*

Proof: Let

$$S = \{s \mid s \geq t \text{ and } K_s \text{ is } H-decomposable\}.$$

S is infinite but since $\alpha(S)$ and $\beta(S)$ are finite we have finite subsets $S_\alpha \subset S$ and $S_\beta \subset S$ such that $\alpha(S_\alpha) = \alpha(S)$ and $\alpha(S_\beta) = \beta(S)$. Let $F = S_\alpha \cup S_\beta$. Note that since $S_\alpha \subset F$ then $\alpha(F)$ divides $\alpha(S)$. Similarly, since $S_\beta \subset F$ we have that $\beta(F)$ divides $\beta(S)$. We claim that F is an (H,t)-CDS. First note that, by definition, every $s \in F$ satisfies $s \geq t$ and K_s is H-decomposable. Now let $N = \max\{n_0, t\}$ where $n_0 = n_0(F)$ is the constant defined in the statement of Lemma 2.2. It suffices to show that for every $n > N$, if K_n is H-decomposable then it is also F-decomposable. Indeed, if K_n is H-decomposable, then $n \in S$, so $\alpha(F) \mid \alpha(S) \mid n-1$, and $\beta(F) \mid \beta(S) \mid \binom{n}{2}$. Thus, by Lemma 2.2, K_n is F-decomposable.
For the second part of the lemma, let $H = K_r$. By Wilson's decomposition theorem, there exists $N = N(H)$ such that for all $n > N$, if $r-1|n-1$ and $\binom{r}{2}|\binom{n}{2}$, then K_n is K_r-decomposable. Now, let r_1 be the smallest prime satisfying $r_1 > N$, $r_1 > t$, and $r_1 \equiv 1 \bmod r(r-1)$. By Dirichlet's Theorem on primes in arithmetic progressions, r_1 exists. Note that K_{r_1} is K_r-decomposable, and also K_{r_2} is K_r-decomposable for $r_2 = r_1 + r - 1$. Let $F = \{r_1, r_2\}$ and let $n_0 = n_0(F)$ be as in Lemma 2.2. In order to prove that F is an (H,t)-CDS it suffices to show that for all $n > n_0$, if $r-1|n-1$ and $\binom{r}{2}|\binom{n}{2}$ then K_n is F-decomposable. Indeed, $\alpha(F) = r-1$ since $r_2 - r_1 = r-1$ and both are divisible by $r-1$. Also, $\beta(F) = \binom{r}{2}$ since r_1 is a prime. Hence, $\alpha(F)|n-1$ and $\beta(F)|\binom{n}{2}$, so by Lemma 2.2, K_n is F-decomposable. □

Proof of Theorem 1.1: Let H be a graph and let k be a positive integer. Let F be an $(H,1)$-CDS. Thus, there exists N such that for $n > N$, if K_n is H decomposable it is also F-decomposable. For each $r \in F$ let $F_r = \{r_1, r_2\}$ be a $(K_r, N(k,r))$-CDS where $N(k,r)$ is defined in the statement of Lemma 2.1. Let $F' = \cup_{r \in F} F_r$. We claim that F' is an $(H,1)$-CDS. Indeed, let $n_0 = n_0(F')$ be as defined in Lemma 2.2. Let $n > \max\{n_0, N\}$ such that K_n is H-decomposable. We need to show that it is F'-decomposable. First note that, since $n > N$, K_n is F-decomposable. Thus, $\alpha(F)|n-1$ and $\beta(F)|\binom{n}{2}$. In order to apply Lemma 2.2 for n and F' it suffices to show that $\alpha(F')|\alpha(F)$ and $\beta(F')|\beta(F)$. In fact, we can show that $\alpha(F') = \alpha(F)$ and $\beta(F') = \beta(F)$. Put $F = \{r^{(1)}, \ldots, r^{(t)}\}$. Then $F' = \{r_1^{(1)}, r_2^{(1)}, \ldots, r_1^{(t)}, r_2^{(t)}\}$ and by Lemma 2.3 and the properties of the gcd we have:

$$\alpha(F') = \gcd\left(r_1^{(1)}-1, r_2^{(1)}-1, \ldots, r_1^{(t)}-1, r_2^{(t)}-1\right) =$$

$$\gcd\left(\gcd\left(r_1^{(1)}-1, r_2^{(1)}-1\right), \ldots, \gcd\left(r_1^{(t)}-1, r_2^{(t)}-1\right)\right) =$$

$$\gcd\left(r^{(1)}-1, \ldots, r^{(t)}-1\right) = \alpha(F)$$

and similarly,

$$\beta(F') = \gcd\left(\binom{r_1^{(1)}}{2}, \binom{r_2^{(1)}}{2}, \ldots, \binom{r_1^{(t)}}{2}, \binom{r_2^{(t)}}{2}\right) =$$

$$\gcd\left(\gcd\left(\binom{r_1^{(1)}}{2}, \binom{r_2^{(1)}}{2}\right), \ldots, \gcd\left(\binom{r_1^{(t)}}{2}, \binom{r_2^{(t)}}{2}\right)\right) =$$

$$\gcd\left(\binom{r^{(1)}}{2}, \ldots, \binom{r^{(t)}}{2}\right) = \beta(F).$$

It is now not difficult to see that whenever $n > \max\{n_0, N\}$ and K_n is H-decomposable, then K_n also has a k-orthogonal H-decomposition. Indeed, for such an n, we know that K_n is F'-decomposable. Let $Q_1, \ldots, Q_m$ be the elements of an F'-decomposition. Each Q_j is a clique whose order is either r_1 or r_2 where $r \in F$. Since $r_2 > r_1 > N(k,r)$ we know, by Lemma 2.1, that Q_j has a k-orthogonal decomposition of K_r. Thus, Q_j has k distinct K_r-decompositions $L_j^1, \ldots, L_j^k$, where any two elements of any two of these decompositions share at most one edge. Since $r \in F$, each element of L_j^i is H-decomposable. The union of the H-decompositions of all the elements of L_j^i is an H-decomposition of Q_j which we denote by M_j^i. Trivially, $M_j^1, \ldots, M_j^k$ form a k-orthogonal H-decomposition of Q_j. The union of the H-decompositions M_j^i for $j = 1, \ldots, m$ forms an H-decomposition R^i of K_n. Thus, $R^1, \ldots, R^k$ form a k-orthogonal H-decomposition of K_n □

Acknowledgment

The authors thank Daniel Berend for useful discussions.

References

[1] P. Adams, D.E. Bryant and A. Khodkar, *On the existence of super-simple designs with block size 4*, Aequationes Math. 51 (1996), 230-246.

[2] B. Alspach, K. Heinrich and G. Liu, *Orthogonal factorizations in graphs*, In: *Contemporary Design Theory, J.H. Dinitz and D.R. Stinson: A Collection of Surveys*, pp. 13-40, J. Wiley, 1992.

[3] R.P. Anstee and L. Caccetta, *Orthogonal matchings*, Discrete Math. 179 (1998), 37-47.

[4] I. Bluskov and K. Heinrich, *Super-simple designs with* $v \leq 32$, manuscript.

[5] B. Bollobás, *Extremal Graph Theory*, Academic Press, 1978.

[6] D.E. Bryant and A. Khodkar, *On orthogonal double covers of graphs*, Design, Codes and Cryptography 13 (1998), 103-105.

[7] Y. Caro and R. Yuster, *Orthogonal decomposition and packing of complete graphs,* Journal of Combinatorial Theory, Series A 88 (1999), 93-111.

[8] C.J. Colbourn and J.H. Dinitz, CRC Handbook of Combinatorial Design*,* CRC press 1996.

[9] C.J. Colbourn, D.G. Hoffman and C.C. Lindner, Intersections of $S(2,4,v)$ designs*,* Ars Combinat. 33 (1992), 97-111.

[10] D. Dor and M. Tarsi, Graph decomposition is NPC - A complete proof of Holyer's conjecture*,* Proc. 20th ACM STOC, ACM Press (1992), 252-263.

[11] B. Ganter, H.-D.O.F. Gronau and R.C. Mullin, On orthogonal double covers of K_n*,* Ars Combinat. 37 (1994), 209-221.

[12] A. Granville, H.-D.O.F. Gronau and R.C. Mullin, On a problem of Hering concerning orthogonal covers of K_n*,* J. Combin. Theory, Ser. A 72 (1995), 345-350.

[13] H.-D.O.F. Gronau and R.C. Mullin, On super-simple $2-(v,4,\lambda)$ designs*,* J. Combin. Math. Combin. Comput. 11 (1992), 113-121.

[14] H.-D.O.F. Gronau, R.C. Mullin and A. Rosa, Orthogonal double covers of complete graphs by trees*,* Graphs and Combinatorics 13 (1997), 251-262.

[15] H.-D.O.F. Gronau, R.C. Mullin and P.J. Schellenberg, On orthogonal double covers of K_n and a conjecture of Chung and West*,* J. Combin. Designs 3 (1995), 213-231.

[16] C. Kejun, On the existence of super-simple (v,4,3) BIBDs, J. Combin. Math. Combin. Comput., to appear.

[17] A. Khodkar, Various super-simple designs with block size four, Australas. J. Combin. 9 (1994), 201-210.

[18] J.-X. Lu, On large sets of disjoint Steiner-triple systems I-III, J. Combin. Theory, Ser. A 34 (1983), 140-182.

[19] J.-X. Lu, On large sets of disjoint Steiner-triple systems IV-VI, J. Combin. Theory, Ser. A 37 (1984), 136-192.

[20] L. Teirlinck, *A completion of Lu's determination of the spectrum for large sets of disjoint Steiner triple systems,* J. Combin. Theory, Ser. A. 57 (1991), 302-305.

[21] R. M. Wilson, Decomposition of complete graphs into subgraphs isomorphic to a given graph, Congressus Numerantium XV (1975), 647-659.

[22] R. M. Wilson, *An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures*, J. Combin. Theory, Ser. A 13 (1972), 246-273.