Orthogonal Decomposition and Packing of Complete Graphs

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Abstract

An H-decomposition of a graph G is a partition of the edge-set of G into subsets, where each subset induces a copy of the graph H. A *k-orthogonal H-decomposition* of a graph G is a set of k H-decompositions of G, such that any two copies of H in distinct H-decompositions intersect in at most one edge. In case $G = K_n$ and $H = K_r$, a k-orthogonal K_r-decomposition of K_n is called an (n, r, k) *completely-reducible super-simple design.* We prove that for any two fixed integers r and k, there exists $N = N(k, r)$ such that for every $n > N$, if K_n has a K_r-decomposition, then K_n also has an (n, r, k) completely-reducible super-simple design. If K_n does not have a K_r-decomposition, we show how to obtain a *k-orthogonal optimal K_r-packing* of K_n. Complexity issues of k-orthogonal H-decompositions are also treated.

1 Introduction

All graphs considered here are finite, undirected, and have no loops or multiple edges. For the standard graph-theoretic and design-theoretic notations the reader is referred to [12] and [17] respectively. An *H-subgraph of G* is a subgraph of a graph G, which is isomorphic to a graph H. An *H-decomposition* of a graph G is a set L of edge-disjoint H-subgraphs of G, such that each edge of G appears in some element of L. Thus, L contains $e(G)/e(H)$ elements, where $e(X)$ denotes the number of edges of a graph X. It is straightforward to see that a necessary condition for the existence of an H-decomposition is that $e(H)$ divides $e(G)$. Another obvious requirement is that $gcd(H)$ divides $gcd(G)$ where the *gcd* of a graph is the greatest common divisor of the degrees of its vertices. An *optimal H-packing* of G is a set L of edge-disjoint H-subgraphs of G, with maximum cardinality. The corresponding H-packing number of G, denoted $P(H, G)$, is the cardinality of an optimal H-packing. Clearly, $P(H, G) \le e(G)/e(H)$ with equality achieved if and only if G has an H-decomposition.

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In general, it is NP-Complete to determine whether a given graph G has an H-decomposition for every fixed graph H containing more than two edges in some connected component. This has been proved by Dor and Tarsi [19]. Consequently, it is NP-Hard to determine $P(H,G)$ for every such fixed graph H. However, a seminal result of Wilson [42], is that the existence of the two necessary conditions mentioned above is also sufficient to guarantee an H-decomposition of K_n for every $n > n_0(H)$, and this result holds for every fixed nonempty graph H. In terms of design-theory, Wilson's Theorem states that the necessary conditions are sufficient for the existence of a $2-(v,k,1)$-design, provided that v is sufficiently large (in fact, it is sufficient for the existence of a $2-(v,k,\lambda)$-design). Recently, Caro and Yuster [15, 16] have provided formulas for $P(H,K_n)$, as well as the related covering number $C(H,K_n)$, provided that $n > n_1(H)$, and Alon, Caro and Yuster have shown how to efficiently compute $P(H,G)$ and $C(H,G)$ in polynomial time, for arbitrary *dense* and *large* graphs G [2].

In order to present our result in the exact context we shall switch momentarily to the language of design-theory. Since the appearance of the seminal work of Wilson, the notion of repeated blocks in a $t-(v,k,\lambda)$ design became a central issue in design theory. We refer the reader to [41] and [17] which are major comprehensive sources for design theory and the emergence of the repeated-block issue. For research papers on this subject we refer the reader to [5, 6, 21, 33]. Two main branches developed from the study of designs with non-repeated blocks. These are the *intersection problem* and the theory of *simple designs*.

The intersection problem asks for the existence of a $2-(v,k,2)$ design in which exactly $m \geq 0$ blocks are used twice. Extensions of this problem to $2-(v,k,\lambda)$ designs in which exactly $m \geq 0$ blocks are used λ times while any other block is used at most once were considered as well. In fact, this line of research has been extended to include small graphs and simple structured trees instead of just complete graphs as the blocks of the design. We refer the reader to [6, 7, 8, 9, 14, 18, 21, 27, 30, 33, 37] for various papers on the intersection problem, and to [32] as one of the first papers where the problem was raised explicitly. These works also have an obvious connection to the famous works of Lu [34, 35] and Teirlinck [38, 39, 40] on the existence of large sets of Steiner triple systems where, clearly, $m = 0$ in the above notation.

The theory of simple designs asks for the existence of a $t-(v,k,\lambda)$ design with no repeated blocks (namely the case $m = 0$ in the intersection problem). However, stronger conditions are usually imposed on the design. In case no two blocks have more than one pair (edge) in common, the design is called a *super-simple design* and is denoted $SS(t,v,k,\lambda)$ design, or simply $SS(v,k,\lambda)$ design if $t = 2$. In case that a $SS(t,v,k,\lambda)$ design splits into λ copies of a $SS(t,v,k,1)$-design, the design is called a *completely-reducible super-simple* design, denoted by $CRSS(t,v,k,\lambda)$ or simply

$CRSS(v,k,\lambda)$ if $t = 2$. Recent results on super-simple and completely-reducible super-simple designs can be found in [1, 11, 23, 28, 29]. The requirement that any two blocks have at most one pair in common is called the *orthogonality* property. Many results in design theory concerning orthogonality have appeared in recent years and we refer the reader to the surveys in [3, 17] for details and to [3, 4, 13, 20, 22, 24, 25] for recent developments in this area.

The main result in this paper establishes, in particular, the existence of a $CRSS(v,k,\lambda)$ design, for every v which is sufficiently large, and which satisfies the necessary divisibility conditions. We now switch back to the language of graph theory in order to present our results. A *k-orthogonal* H-decomposition of a graph G is a set of k H-decompositions of G, such that any two copies of H in any two distinct H-decompositions have at most one edge in common. A 2-orthogonal H-decomposition is simply called an *orthogonal* H-decomposition. Similarly, one defines a *k-orthogonal optimal H-packing* as a set of k optimal H-packings of G, such that any two copies of H in any two distinct optimal H-packings have at most one edge in common. Obviously, a k-orthogonal H-decomposition does not necessarily exist, even if an H-decomposition exists. Also, a k-orthogonal optimal H-packing does not always exist, although, by definition, an optimal H-packing always exists. Note that in case both G and H are complete graphs, a k-orthogonal H-decomposition of G is also a $CRSS(n,r,k)$ design.

All values of k and n for which a $CRSS(n,3,k)$-design or a $SS(n,3,k)$-design exists are known [34, 35, 39]. Also, all values of k and n for which a k-orthogonal optimal K_3-packing of K_n exists, are known [36, 31]. For $r = 4$, it is known whenever a $CRSS(n,4,2)$ design exists, and whenever a $SS(n,4,4)$ design exists [1]. Several other sporadic results involving the case $r = 4$ also appear in [11, 18, 23, 29]. The main theorem of this paper solves the $CRSS(n,r,k)$ existence problem completely, for all $n > N(k,r)$. In fact, we prove something stronger, since we prove that if $n > N(k,r)$ then there is *always* a k-orthogonal optimal K_r-packing of K_n:

Theorem 1.1 *Let $r \ge 2$ and $k \ge 1$ be integers. There exists $N = N(k,r)$ such that if $n > N$ then K_n has a k-orthogonal optimal K_r-packing.*

An immediate corollary from Theorem 1.1 and Wilson's Theorem is the following:

Corollary 1.2 *Let $r \ge 2$ and $k \ge 1$ be integers. There exists $N = N(k,r)$ such that if $n > N$ then there exists a $CRSS(n,r,k)$ if and only if $n = 1, r \bmod r(r-1)$.*

In fact, one may view Corollary 1.2 as an extension of Wilson's theorem, for $k > 1$, and Theorem 1.1 as an extension of the above-mentioned result of Caro and Yuster, for $k > 1$. Another interesting corollary is that whenever $n = 1, r \bmod r(r-1)$, the notions of $SS(n,r,k)$ and $CRSS(n,r,k)$

coincide. The proof of Theorem 1.1 relies on several probabilistic and combinatorial arguments. The probabilistic part is handled in Section 2, and the combinatorial part of the proof, which relies on the result proved in Section 2, and on several additional ideas, is proved in Section 3.

As mentioned above, it is NP-Complete to determine whether a general graph G has an H-decomposition, unless H has no connected component with more than two edges. It is, therefore, a plausible conjecture that the decision problem: "Given an input graph G, does it have a k-orthogonal H-decomposition" is also NP-Complete for every fixed k and for every graph H with at least three edges in some connected component. One should notice that the answer to this question *does not* follow directly from the Dor-Tarsi result. We will show, however, that for every fixed star $H = K_{1,r}$ ($r \geq 3$), and for every fixed positive integer k, this problem is, indeed, NP-Complete. The proof is presented in Section 4. Section 4 also contains some concluding remarks and an open problem.

2 Random permutations and semi-orthogonal packings

Consider a labeling of the vertices of K_n with the integers $1, \ldots, n$, and let X be a labeled subgraph of K_n. If π is any permutation of $\{1, \ldots, n\}$, we denote by X^π the labeled subgraph of K_n which is isomorphic to X via the isomorphism π, namely the isomorphism $x \to \pi(x)$ for every vertex x of K_n.

Let L be a set of labeled edge-disjoint subcliques of K_n (a subclique is a subgraph which is a clique), and denote by $L^\pi = \{X^\pi \mid X \in L\}$. A subclique F of $X \in L$, is called *invariant under* π if F has at least three vertices, and there exists $Y \in L$ (it is allowed that $Y = X$), such that F is also a subclique of Y^π. Note that if F_1 and F_2 are two distinct *maximal* (with respect to containment) subcliques of X that are invariant under π, then they must be edge-disjoint. We call an edge $e \in X$ π-*bad* if it appears in a subclique that is invariant under π. We call π an (r, L) *semi-orthogonal permutation* if every $X \in L$ has at most r π-bad edges. Note that L and L^π are orthogonal if and only if π is $(0, L)$ semi-orthogonal (i.e. there are no subcliques that are invariant under π). The crucial argument about semi-orthogonal permutations is given in the following lemma:

Lemma 2.1 *Let $0 < q < 1$ be any real number. Let L be a set of edge-disjoint labeled subcliques of K_n. Assume that each $X \in L$ has at most s vertices, and that $n \geq s^{18}/(1-q)$. Then, a random permutation π of $\{1, \ldots, n\}$ is $(6, L)$ semi-orthogonal with probability at least q.*

Before proving Lemma 2.1 we need the following lemma which analyzes the possible sizes of subcliques that are invariant under π, in case X has more than 6 π-bad edges.

Lemma 2.2 *If $X \in L$ has more than 6 π-bad edges, then at least one of the following cases holds:*

1. *X has a K_5 that is invariant under π.*

2. *X has an $F_1 = K_4$ that is invariant under π, and an $F_2 = K_3$ that is invariant under π, and one of the following two cases holds:*

   (a) *F_1 and F_2 are vertex-disjoint.*

   (b) *F_1 and F_2 share one common vertex.*

3. *X has a three triangles F_1, F_2 and F_3 that are all invariant under π, and one of the following cases holds:*

   (a) *F_1, F_2 and F_3 are pairwise vertex-disjoint.*

   (b) *F_1 and F_2 share a common vertex, and F_3 is vertex-disjoint from both F_1 and F_2.*

   (c) *F_1, F_2 and F_3 all share the same common vertex.*

   (d) *F_1 shares a common vertex with F_2 and another common vertex with F_3, and F_2 is vertex-disjoint from F_3.*

   (e) *F_1 shares a common vertex with F_2 and another common vertex with F_3, and F_2 shares another common vertex with F_3.*

Since the proof of Lemma 2.2 is a simple combinatorial exercise, we omit the obvious proof. We are now ready to prove Lemma 2.1.

Proof of Lemma 2.1: The proof relies on probabilistic arguments. Let π be a random permutation, chosen uniformly from all $n!$ possible permutations. We must prove that with probability at least q, every $X \in L$ has at most 6 π-bad edges. We may assume $s \geq 5$ (otherwise, every element of L contains at most 6 edges, and the lemma trivially holds).

Consider an element $X \in L$ with more than 6 edges. We will prove that the probability that X has more than 6 π-bad edges is less than $20(1-q)/n^2$. This suffices, as the number of elements of L containing more than 6 edges (and thus, at least 10 edges), is at most $\binom{n}{2}/10 < n^2/20$. By Lemma 2.2, it suffices to show that each of the 8 cases described there, occurs with probability less than $20(1-q)/(8n^2)$. We now consider each of these cases, and show that, indeed, each case occurs with probability smaller than $20(1-q)/(8n^2)$. Let $x = |X|$ denote the number of vertices of X, and recall that $x \leq s$.

1. Let F be a K_5-subclique of X, and let $Y \in L$. Put $y = |Y|$. The probability that F is a subclique of Y^π is exactly

$$\text{Prob}[F \subset Y^\pi] = \frac{y}{n}\frac{y-1}{n-1}\frac{y-2}{n-2}\frac{y-3}{n-3}\frac{y-4}{n-4} < \frac{y^5}{n^5} \leq \frac{s^5}{n^5}.$$

Since there are less than n^2 elements in L, and since there are $\binom{x}{5}$ K_5-subcliques of X we get that

$$\text{Prob[Case 1 occurs]} < n^2 \binom{x}{5} \frac{s^5}{n^5} < \frac{s^{10}}{n^3} < 20 \frac{1-q}{8n^2}.$$

2a. Let F_1 be a K_4-subclique of X, and let F_2 be a K_3-subclique of X which is vertex-disjoint from F_1. Let Y_1 and Y_2 be two distinct elements of L. $Y_1 \cup Y_2$ has at most $2s$ vertices. The probability that F_1 is a subclique of Y_1 and that F_2 is a subclique of Y_2 is, at most, the probability that the 7 vertices of $F_1 \cup F_2$ all appear in $Y_1^\pi \cup Y_2^\pi$. Therefore:

$$\text{Prob}[F_1 \subset Y_1^\pi \text{ and } F_2 \subset Y_2^\pi] \le \frac{2s}{n} \frac{2s-1}{n-1} \cdots \frac{2s-6}{n-6} < 128 \frac{s^7}{n^7}.$$

Since there are less than n^4 possible pairs Y_1 and Y_2, and since there are $\binom{x}{4} \cdot \binom{x-4}{3}$ possible choices for F_1 and F_2 in X, we get that:

$$\text{Prob[Case 2a occurs]} < n^4 \binom{x}{4} \binom{x-4}{3} 128 \frac{s^7}{n^7} < \frac{s^{14}}{n^3} < 20 \frac{1-q}{8n^2}.$$

2b. Let F_1 be a K_4-subclique of X, and let F_2 be a K_3-subclique of X which has a common vertex with F_1. Let Y_1 and Y_2 be two distinct elements of L. Note that if Y_1 and Y_2 are vertex-disjoint, then the probability that F_i is a subclique of Y_i^π for $i = 1, 2$ is 0. Thus, we may assume that Y_1 shares a vertex with Y_2. Consequently, $Y_1 \cup Y_2$ has at most $2s-1$ vertices. The probability that F_1 is a subclique of Y_1 and that F_2 is a subclique of Y_2 is, at most, the probability that the 6 vertices of $F_1 \cup F_2$ all appear in $Y_1^\pi \cup Y_2^\pi$. Therefore:

$$\text{Prob}[F_1 \subset Y_1^\pi \text{ and } F_2 \subset Y_2^\pi] \le \frac{2s-1}{n} \frac{2s-2}{n-1} \cdots \frac{2s-6}{n-5} < 64 \frac{s^6}{n^6}.$$

Since the elements of L are edge-disjoint, every vertex of K_n appears in at most $n-1$ elements of L. Thus, the number of pairs Y_1 and Y_2 which share a common vertex is at most $n \cdot \binom{n-1}{2}$. There are $x \cdot \binom{x-1}{3} \cdot \binom{x-4}{2}$ possible choices for F_1 and F_2 in X, so we obtain:

$$\text{Prob[Case 2b occurs]} < n \cdot \binom{n-1}{2} x \binom{x-1}{3} \binom{x-4}{2} 64 \frac{s^6}{n^6} < 3 \frac{s^{12}}{n^3} < 20 \frac{1-q}{8n^2}.$$

3a. Let F_1, F_2 and F_3 be three vertex-disjoint triangles of X, and let Y_1, Y_2 and Y_3 be three distinct elements of L. $Y_1 \cup Y_2 \cup Y_3$ has at most $3s$ vertices. The probability that F_i is a subclique of Y_i for $i = 1, 2, 3$ is at most the probability that the 9 vertices of $F_1 \cup F_2 \cup F_3$ all appear in $Y_1^\pi \cup Y_2^\pi \cup Y_3^\pi$. Therefore:

$$\text{Prob}[F_i \subset Y_i^\pi \; i = 1, 2, 3] \le \frac{3s}{n} \frac{3s-1}{n-1} \cdots \frac{3s-8}{n-8} < 3^9 \frac{s^9}{n^9}.$$

Obviously, we may assume that each Y_i, $i = 1, 2, 3$ has at least 3 vertices (otherwise, the last computed probability is 0). There are at most $\binom{n}{2}/3$ elements of L with at least three vertices. Thus, the number of possible triples Y_i $i = 1, 2, 3$ to consider is less than $(\binom{n}{2}/3)^3$. There are $\binom{x}{3} \cdot \binom{x-3}{3} \cdot \binom{x-6}{3}$ choices for F_1, F_2 and F_3 in X. Therefore, we get that:

$$\text{Prob[Case 3a occurs]} < (\frac{\binom{n}{2}}{3})^3 \binom{x}{3}\binom{x-3}{3}\binom{x-6}{3} 3^9 \frac{s^9}{n^9} < \frac{s^{18}}{n^3} < 20\frac{1-q}{8n^2}.$$

3b. Let F_1, F_2 and F_3 be three triangles of X, where F_1 and F_2 share a common vertex, and F_3 is vertex-disjoint from both F_1 and F_2. Let Y_1, Y_2 and Y_3 be three distinct elements of L. If Y_1 and Y_2 are vertex-disjoint then the probability that $F_i \subset Y_i^\pi$ for $i = 1, 2, 3$ is 0. We therefore assume that Y_1 and Y_2 share a common vertex (Y_3 may or may not be vertex-disjoint from Y_2 or Y_1). Thus, $Y_1 \cup Y_2 \cup Y_3$ has at most $3s - 1$ vertices. The probability that F_i is a subclique of Y_i for $i = 1, 2, 3$ is at most the probability that the 8 vertices of $F_1 \cup F_2 \cup F_3$ all appear in $Y_1^\pi \cup Y_2^\pi \cup Y_3^\pi$. Therefore:

$$\text{Prob}[F_i \subset Y_i^\pi \; i = 1, 2, 3] \le \frac{3s-1}{n}\frac{3s-2}{n-1}\cdots\frac{3s-8}{n-7} < 3^8\frac{s^8}{n^8}.$$

As explained in case 2b, there are at most $n \cdot \binom{n-1}{2}$ pairs Y_1 and Y_2 which share a vertex, and since the number of elements of L which contain at least three vertices is at most $\binom{n}{2}/3$, we obtain that there are less than $(\binom{n}{2}/3)n \cdot \binom{(n-1)}{2}$ triples Y_1, Y_2 and Y_3 where Y_1 and Y_2 share a vertex. There are $x\binom{x-1}{2}\binom{x-3}{2}\binom{x-5}{3}$ possible choices for F_1, F_2 and F_3 in X. Thus,

$$\text{Prob[Case 3b occurs]} < \frac{\binom{n}{2}}{3} n\binom{n-1}{2} x\binom{x-1}{2}\binom{x-3}{2}\binom{x-5}{3} 3^8\frac{s^8}{n^8} < 23\frac{s^{16}}{n^3} < 20\frac{1-q}{8n^2}.$$

3c. Let F_1, F_2 and F_3 be three triangles of X, which all share the same common vertex. If Y_1, Y_2 and Y_3 are three distinct elements of L, they must also share a common vertex, if we are to have any chance that F_i is a subclique of Y_i^π for $i = 1, 2, 3$. Thus, $Y_1^\pi \cup Y_2^\pi \cup Y_3^\pi \le 3s - 2$, and since $F_1 \cup F_2 \cup F_3$ has 7 vertices, a similar computation to the one given in case 3b yields:

$$\text{Prob}[F_i \subset Y_i^\pi \; i = 1, 2, 3] \le \frac{3s-2}{n}\frac{3s-3}{n-1}\cdots\frac{3s-8}{n-6} < 3^7\frac{s^7}{n^7}.$$

Each vertex of K_n appears in at most $n-1$ elements of L, and therefore the number of triples Y_1, Y_2 and Y_3 sharing a common vertex is at most $n \cdot \binom{n-1}{3}$. There are $x\binom{x-1}{2}\binom{x-3}{2}\binom{x-5}{2}$ possible choices for F_1, F_2 and F_3 in X. Thus,

$$\text{Prob[Case 3c occurs]} < n\binom{n-1}{3} x\binom{x-1}{2}\binom{x-3}{2}\binom{x-5}{2} 3^7\frac{s^7}{n^7} < 46\frac{s^{14}}{n^3} < 20\frac{1-q}{8n^2}.$$

3d. Let F_1, F_2 and F_3 be three triangles of X, such that F_1 shares a common vertex with F_2 and another common vertex with F_3, and F_2 and F_3 are vertex-disjoint. If Y_1, Y_2 and Y_3 are three distinct elements of L, then Y_1 must also share a common vertex with Y_2 and another common vertex with Y_3, if we are to have any chance that F_i is a subclique of Y_i^π for $i = 1, 2, 3$. (Note however, that Y_2 does not have to be vertex-disjoint from Y_3). Thus, $Y_1^\pi \cup Y_2^\pi \cup Y_3^\pi \leq 3s - 2$, and since $F_1 \cup F_2 \cup F_3$ has 7 vertices, a computation identical to the one given in case 3c yields:

$$\text{Prob}[F_i \subset Y_i^\pi \;\, i = 1, 2, 3] \leq \frac{3s-2}{n} \frac{3s-3}{n-1} \cdots \frac{3s-8}{n-6} < 3^7 \frac{s^7}{n^7}.$$

Consider two distinct vertices of K_n, which appear in some $Y_1 \in L$. Each of these vertices may also appear in at most $n-2$ other elements of L, in addition to L_1. Thus, the number of triples L_1, L_2 and L_3 such that L_1 shares a vertex with L_2 and another vertex with L_3 is at most $\binom{n}{2}(n-2)^2$. There are $\binom{x}{2}(x-2)\binom{x-3}{2}\binom{x-5}{2}$ possible choices for F_1, F_2 and F_3 in X. Therefore, using the fact that $s \geq 5$,

$$\text{Prob}[\text{Case 3d occurs}] < \binom{n}{2}(n-2)^2 \binom{x}{2}(x-2)\binom{x-3}{2}\binom{x-5}{2} 3^7 \frac{s^7}{n^7} < 137 \frac{s^{14}}{n^3} < 20 \frac{1-q}{8n^2}.$$

3e. Let F_1, F_2 and F_3 be three triangles of X, such that each pair share a common vertex, but not the same common vertex. If Y_1, Y_2 and Y_3 are three distinct elements of L, then each pair must also share a distinct common vertex, if we are to have any chance that F_i is a subclique of Y_i^π for $i = 1, 2, 3$. Thus, $Y_1^\pi \cup Y_2^\pi \cup Y_3^\pi \leq 3s - 3$. the probability that F_i is a subclique of Y_i^π for $i = 1, 2, 3$, is at most the probability that the 6 vertices of $F_1 \cup F_2 \cup F_3$ all appear in $Y_1^\pi \cup Y_2^\pi \cup Y_3^\pi$. Therefore,

$$\text{Prob}[F_i \subset Y_i^\pi \;\, i = 1, 2, 3] \leq \frac{3s-3}{n} \frac{3s-4}{n-1} \cdots \frac{3s-8}{n-5} < 3^6 \frac{s^6}{n^6}.$$

Every triangle of K_n is either completely contained is some element of L or uniquely defines three elements of L, Y_1, Y_2 and Y_3 where each pair shares a distinct common vertex. Thus, the overall number of such triples is at most $\binom{n}{3}$. There are $\binom{x}{3}(x-3)(x-4)(x-5)$ possible choices for F_1, F_2 and F_3 in X. Therefore,

$$\text{Prob}[\text{Case 3e occurs}] < \binom{n}{3}\binom{x}{3}(x-3)(x-4)(x-5) 3^6 \frac{s^6}{n^6} < 21 \frac{s^{12}}{n^3} < 20 \frac{1-q}{8n^2}.$$

□

3 Optimal k-orthogonal K_r-Packings

In order to complete the proof of Theorem 1.1 we need several definitions and lemmas. A K_r-decomposable graph S is called *t-evasive* if for any set T of t edges of S, there exists a K_r-decomposition of S such that each copy of K_r in the decomposition contains at most one edge from T. Note that the definition holds for every $t \geq 1$, and that, trivially, every K_r-decomposable graph is 1-evasive. Our first goal is to show that given t and r, every large-enough complete graph which is K_r-decomposable, is also t-evasive. Let $Z_{p,r}$ denote the graph which is composed by taking a K_p, which is called the *center* of $Z_{p,r}$, and for each edge (x, y) of the center, constructing a copy of K_r whose vertices are x and y and $r-2$ new vertices. Note that $Z_{p,r}$ has $p + (r-2)\binom{p}{2}$ vertices, and $\binom{p}{2}\binom{r}{2}$ edges, and $Z_{p,r}$ can be decomposed into $\binom{p}{2}$ copies of K_r, each containing exactly one edge from the center.

Lemma 3.1 *Let $p \geq 2$ and $r \geq 3$ be integers. If $n > r^3p^4$ and K_n has a K_r-decomposition, then there exists a set of $\binom{p}{2}$ elements of the decomposition whose union is $Z_{p,r}$.*

Proof: The proof is by induction on p. For $p = 2$ there is nothing to prove since $Z_{2,r} = K_r$. Assume the lemma holds for $p-1$. Let L be a K_r-decomposition of K_n. By the induction hypothesis, there is a set of $\binom{p-1}{2}$ elements of L whose union forms $Z_{p-1,r}$. Let X be the set of vertices of this $Z_{p-1,r}$, and let X_0 be the center. Recall that $|X| = p - 1 + (r-2)\binom{p-1}{2}$ and that $|X_0| = p - 1$. Thus, there are at most $\binom{|X|}{2}$ elements of L containing an edge with both endpoints in X. Since $n > \binom{|X|}{2}r$, there is a vertex v of K_n having the property that every element of L containing v, has no edge with both endpoints in X. Let us add to $Z_{p-1,r}$ the $p-1$ elements of L which contain an edge joining v to some vertex of X_0. Note that the choice of v guarantees that this addition forms a $Z_{p,r}$, whose center is $X_0 \cup \{v\}$. $\square$

By Lemma 3.1 we have that if $n > r^3p^4$ then K_n has a K_r-decomposition if and only if the graph $K_n \setminus Z_{p,r}$ has a K_r-decomposition ($K_n \setminus Z_{p,r}$ is the graph obtained by deleting the edge set of a copy of $Z_{p,r}$ in K_n). Also note that given any set T of t edges, they span at most $2t$ vertices. Thus, we may create a $Z_{2t,r}$ whose center contains all the edges of T, and, by definition, $Z_{2t,r}$ has a K_r-decomposition in which every element of K_r contains at most one edge from T. We therefore obtain the following corollary:

Corollary 3.2 *Let $t \geq 1$ and $r \geq 2$ be positive integers. If $n \geq r^3(2t)^4$ then if K_n is K_r-decomposable, then K_n is also t-evasive. In particular, by Wilson's Theorem, there exists $M = M(t, r)$ such that for every $n > M$, if $r-1$ divides $n-1$ and $\binom{r}{2}$ divides $\binom{n}{2}$ then K_n is K_r-decomposable and t-evasive.*

Let $s = s(k,r)$ be the smallest integer satisfying $s-r+1 > M(6(k-1),r)$ and $s = r \bmod r(r-1)$, where M is the constant defined in Corollary 3.2. By Corollary 3.2 both K_s and K_{s-r+1} are K_r-decomposable and $6(k-1)$-evasive. Put $H(k,r) = K_s \cup K_{s-r+1}$. Obviously, $H(k,r)$ has a K_r-decomposition, and $gcd(H(k,r)) = r-1$.

A graph G is called *d-consistent* if the degrees of all its vertices are the same, modulo d. Obviously, every regular graph (and thus, also, the complete graph) is d-consistent. Our next lemma is taken from [2]. In fact, we only cite here a very special case of the lemma which we need for the proof of Theorem 1.1.

Lemma 3.3 *Let H be a nonempty graph, $gcd(H) = d$, and let s be a positive integer. There exists $N_0 = N_0(H,s)$ such that if $G = (V,E)$ is a d-consistent graph with $n > N_0$ vertices and $\delta(G) \geq n-s$, then:*

$$P(H,G) = \lfloor \frac{\sum_{v \in V} \alpha_v}{2e(H)} \rfloor,$$

where α_v is the degree of vertex v, rounded down *to the closest multiple of d. The right hand side of this formula should be* **reduced** *by 1 if d divides $gcd(G)$ and $0 < |E| \bmod e(H) \leq d^2/2$.*

Using Lemma 2.1, the properties of $H(k,r)$ and Lemma 3.3, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 We first define the constant N appearing in the theorem:

$$N = \max\{N_0(H(k,r),s)\ ,\ N_0(K_r,1)\ ,\ 2s-1+\frac{2s}{1-(1-k^{-2})^{1/(2s)}}\ ,\ 2k^2s^{18}\},$$

where N_0 is the constant defined in Lemma 3.3. Clearly, $N = N(k,r)$ is only a function of k and r. Let $n > N$, we need to show that K_n has a k-orthogonal optimal K_r-packing. We begin by defining the following integers:

- $a = n-1 \bmod r-1$ where $0 \leq a < r-1$.
- $b = n(n-1-a) \bmod r(r-1)$ where $0 \leq b < r(r-1)$.
- $h = \binom{s}{2} + \binom{s-r+1}{2}$. Note that h is the number of edges of $H(k,r)$, and that h is a rather large multiple of $\binom{r}{2}$, since $H(k,r)$ is K_r-decomposable, and each of the two cliques comprising $H(k,r)$ is $6(k-1)$-evasive.
- $c = n(n-1-a) \bmod 2h$ where $0 \leq c < 2h$.
- $x = \frac{c-b}{r(r-1)}$ Note that x is a nonnegative integer since $2h$ divides $r(r-1)$.

- $y=-1$ if $a=0$ and $x>0$ and $0<\binom{n}{2} \bmod \binom{r}{2} \le (r-1)^2/2$. $y=h/\binom{r}{2}-1$ if $a=0$ and $x=0$ and $0<\binom{n}{2} \bmod \binom{r}{2} \le (r-1)^2/2$. Otherwise, $y=0$.

Claim 1: $(x+y)\cdot\binom{r}{2}<h$.
Proof of Claim 1: If $x>0$ then y is not positive, so

$$(x+y)\binom{r}{2} \le x\binom{r}{2} = \frac{c-b}{2} \le \frac{c}{2} < h.$$

If $x=0$ then either $y=h/\binom{r}{2}-1$ or $y=0$. In any case,

$$(x+y)\binom{r}{2} = y\binom{r}{2} \le h-\binom{r}{2} < h.$$

□

Our first task is to delete from K_n a set L_0 of $x+y$ edge-disjoint copies of K_r. This can be easily achieved since a single copy of $H(k,r)$ in K_n already contains $h/\binom{r}{2}$ copies of K_r and by Claim 1, we may pick $x+y<h/\binom{r}{2}$ of them. Denote the spanning subgraph of K_n after the deletion of the elements of L_0 by G. Note that G is still $r-1$-consistent, since the degree of every vertex of G, modulo $r-1$, is still a. G has $\binom{n}{2}-(x+y)\binom{r}{2}$ edges and $\delta(G) \ge n-1-\Delta(H(k,r)) = n-1-(s-1)=n-s$. Let L_1 be an optimal $H(k,r)$-packing of G. Lemma 3.3 enables us to compute the number of elements of L_1. We can apply Lemma 3.3 to $H(k,r)$ and G, since G is d-consistent, $n>N \ge N_0(H(k,r),s)$ and $\delta(G) \ge n-s$. The formula stated in Lemma 3.3 gives:

$$P(H(k,r),G) = \lfloor \frac{n(n-1-a)-2(x+y)\binom{r}{2}}{2h} \rfloor = \frac{n(n-1-a)-c}{2h} + \lfloor \frac{b-2y\binom{r}{2}}{2h} \rfloor$$

unless $a=0$ and $0<(\binom{n}{2}-(x+y)\binom{r}{2}) \bmod h \le (r-1)^2/2$ in which case the last formula should be reduced by 1.
Claim 2: The condition $a=0$ and $0<(\binom{n}{2}-(x+y)\binom{r}{2}) \bmod h \le (r-1)^2/2$ does not happen.
Proof of Claim 2: If $a \ne 0$ we are done. Assume, therefore, that $a=0$. Thus, $y=-1$ or $y=h/\binom{r}{2}-1$. Consider first the case $y=-1$. In this case we have that $0<\binom{n}{2} \bmod \binom{r}{2} \le (r-1)^2/2$. We must show that $(\binom{n}{2}-(x-1)\binom{r}{2}) \bmod h > (r-1)^2/2$. Indeed, $b=n(n-1) \bmod r(r-1)$ and $c=n(n-1) \bmod 2h$. Thus, both b and c are even integers and therefore $b/2=\binom{n}{2} \bmod \binom{r}{2} \le (r-1)^2/2$ and $c/2=\binom{n}{2} \bmod h$. Now

$$(\binom{n}{2}-(x-1)\binom{r}{2}) \bmod h = (\binom{n}{2}-c/2+b/2+\binom{r}{2}) \bmod h = (b/2+\binom{r}{2}) \bmod h.$$

It remains to show that $(b/2+\binom{r}{2}) \bmod h > (r-1)^2/2$. It suffices to show that $b/2+\binom{r}{2}<h$, (since, trivially $b/2+\binom{r}{2}>(r-1)^2/2$). Indeed, this holds since $b/2 \le (r-1)^2/2 < \binom{r}{2} \le h/2$. The case where $y = h/\binom{r}{2} - 1$ which happens only when $x = 0$ is proved similarly, and is, in fact, easier. This completes the proof of the claim. □

We now have that in any case,

$$P(H(k,r),G) = \frac{n(n-1-a)-c}{2h} + \lfloor \frac{b-2y\binom{r}{2}}{2h} \rfloor. \tag{1}$$

Consider L_0 and L_1. We claim that one can obtain an optimal K_r-packing of K_n using them. This is done as follows: All the elements of L_1 are edge-disjoint copies of $H(k,r)$, and all the $x+y$ elements of L_0 are edge-disjoint copies of K_r. Furthermore, the elements of L_0 are pairwise edge-disjoint from the elements of L_1. Every element of L_1 is K_r-decomposable, so one can obtain a K_r-packing of K_n by performing a K_r-decomposition of each element of L_1, and, finally, adding the elements of L_0 to the packing. We now show that any K_r-packing obtained in this way is an optimal K_r-packing. By (1) the number of elements of any K_r-packing obtained in this way is

$$Q(L_0,L_1) = \frac{h}{\binom{r}{2}}|L_1| + |L_0| = \frac{h}{\binom{r}{2}}(\frac{n(n-1-a)-c}{2h} + \lfloor \frac{b-2y\binom{r}{2}}{2h} \rfloor) + (x+y) =$$

$$\frac{n(n-1-a)-b}{r(r-1)} + \frac{h}{\binom{r}{2}} \lfloor \frac{b-2y\binom{r}{2}}{2h} \rfloor + y.$$

The packing number $P(K_r,K_n)$ can be computed by using Lemma 3.3. We can use Lemma 3.3 since K_n is $r-1$-consistent and since $n > N \geq N_0(K_r,1)$. We therefore have

$$P(K_r,K_n) = \lfloor \frac{n(n-1-a)}{r(r-1)} \rfloor = \frac{n(n-1-a)-b}{r(r-1)},$$

unless $a = 0$ and $0 < \binom{n}{2} \bmod \binom{r}{2} \leq (r-1)^2/2$, in which case the last formula should be reduced by 1. Note that the last condition happens if and only if $y \neq 0$. We need to show that in any case, $Q(L_0,L_1) = P(H,K_n)$. Consider first the case $y = 0$. In this case $\lfloor \frac{b-2y\binom{r}{2}}{2h} \rfloor = 0$ so $Q(L_0,L_1) = P(H,K_n)$. If $y = -1$ then, since $b \leq r(r-1)-1$ and since h is at least twice $\binom{r}{2}$, we have that $\lfloor \frac{b-2y\binom{r}{2}}{2h} \rfloor = 0$. Thus, once again, we have $Q = P(H,K_n)$. If $y = h/\binom{r}{2} - 1$ then, by a similar argument, $\lfloor \frac{b-2y\binom{r}{2}}{2h} \rfloor = -1$. Thus,

$$Q(L_0,L_1) = \frac{n(n-1-a)-b}{r(r-1)} - \frac{h}{\binom{r}{2}} + \frac{h}{\binom{r}{2}} - 1 = \frac{n(n-1-a)-b}{r(r-1)} - 1 = P(H,K_n).$$

Put $L = L_0 \cup L_1$. We have shown how to obtain an optimal K_r-packing using L. We may view L as a set of edge-disjoint cliques whose sizes are either r, s or $s-r+1$. We now show how to get a family of k-orthogonal optimal K_r-packings. This will be shown by using Lemma 2.1, together with the fact that K_s and K_{s-r+1} are K_r-decomposable and $6(k-1)$-evasive. Label the vertices of K_n with the numbers $1,\ldots,n$, and let π_i for $i=1,\ldots,k$ be a set of k permutations of $\{1,\ldots,n\}$, each chosen randomly with uniform distribution, and each chosen *independently*. Let L^{π_i} be defined as in Section 2. Note that $L^{\pi_j} = (L^{\pi_i})^{\pi_j \circ \pi_i^{-1}}$. The reasoning behind the last notation is to emphasize that π_j is a completely random permutation with respect to π_i (they were chosen independently).

Claim 3: With probability greater than 0.5, for all $1 \le i < j \le k$, every element of $L_0^{\pi_i}$ is vertex-disjoint with every element of $L_0^{\pi_j}$.

Proof of Claim 3: It suffices to show that for every fixed pair of indices i and j, every element of $L_0^{\pi_i}$ is vertex-disjoint with every element of $L_0^{\pi_j}$ with probability greater than $1 - 1/k^2$. Indeed, recall that all the $x + y$ elements of L_0 are taken from a single copy of $H(k, r)$ in K_n. Thus, there are at most $s + (s - r + 1) \le 2s$ vertices in all the elements of L_0 together. The probability that *all* the numbers from a set S of $2s$ numbers of $\{1, \dots n\}$ are mapped by a random permutation to numbers outside S is exactly:

$$\frac{(n-2s)}{n}\frac{n-2s-1}{n-1}\cdots\frac{n-4s+1}{n-2s+1} \ge (1 - \frac{2s}{n-2s+1})^{2s} > 1 - \frac{1}{k^2},$$

where the last inequality follows from the fact that $n > N \ge 2s - 1 + \frac{2s}{1-(1-k^{-2})^{1/(2s)}}$. □

Fixing i and j, we have by Lemma 2.1, that with probability at least $q = 1 - 1/(2k^2)$, every element of L^{π_i} has at most 6 bad edges with respect to L^{π_j} (or, using the notations of Section 2, $\pi_j \circ \pi_i^{-1}$ is $(6, L^{\pi_i})$ semi-orthogonal). Note that the conditions of Lemma 2.1 are met, since $n > N \ge 2k^2 s^{18} = s^{18}/(1 - q)$, and every element of L^{π_i} has at most s vertices. Thus, with probability at least $1 - (k - 1)/(2k^2)$, for *every* $j \ne i$, every element of L^{π_i} has at most 6 bad edges with respect to L^{π_j}. Therefore, with probability at least $1 - k(k - 1)/(2k^2) > 0.5$, for every ordered pair i and j, every element of L^{π_i} has at most 6 bad edges with respect to L^{π_j}. Using this observation, together with Claim 3 we can immediately prove the following claim:

Claim 4: There exist k permutations π_i, $i = 1, \dots, k$ of $\{1, \dots, n\}$ such that for every ordered pair i and j every element of $L_0^{\pi_i}$ is vertex-disjoint from every element of $L_0^{\pi_j}$, and every element of L^{π_i} has at most 6 bad edges with respect to L^{π_j}.

Proof of Claim 4: Immediate from the obvious fact that two events with probability greater than 0.5 simultaneously hold with positive probability. □

Let π_i for $i = 1, \dots, k$ be permutations satisfying Claim 4. We may now use L^{π_i} for $i = 1, \dots, k$, to create a set of k-orthogonal optimal K_r-packings. This is done as follows: Let X be a K_s or a K_{s-r+1} element of L^{π_i}, and recall that X is $6(k - 1)$-evasive. Let $T(X)$ be the set of edges of X which are bad with respect to some L^{π_j}, for $j \ne i$. By Claim 4, $|T(X)| \le 6(k - 1)$. Since X is $6(k - 1)$-evasive, we may decompose X to copies of K_r such that each edge of $T(X)$ appears in a distinct copy of K_r. We do these K_r-decompositions for each $X \in L^{\pi_i}$ which is a K_s or a K_{s-r+1} and by taking the union of all these decompositions, together with the elements of $L_0^{\pi_i}$, we obtain an optimal K_r-packing of K_n, denoted by $L_2^{\pi_i}$.

Claim 5: $L_2^{\pi_i}$ for $i = 1, \dots, k$ is a k-orthogonal optimal K_r-packing of K_n.

Proof of Claim 5: Let $U_1 \in L_2^{\pi_i}$ and $U_2 \in L_2^{\pi_j}$. We need to show that they share at most one edge. If $U_1 \in L_0^{\pi_i}$ and $U_2 \in L_0^{\pi_j}$ then they are vertex-disjoint, and we are done. Thus, we may assume that U_1 belongs to some K_r-decomposition of some $X \in L^{\pi_i}$, where X is either a K_s or a K_{s-r+1}. We cannot have two edges e_1 and e_2 of U_1 both in U_2, since if this were the case, then both e_1 and e_2 are in $T(X)$, but in the K_r-decomposition of X, in which U_1 is one of the elements, every edge of $T(X)$ appears in a different copy of K_r. □

The final claim completes the proof of Theorem 1.1 □

4 The hardness of orthogonal star decompositions

In this section we prove the following theorem:

Theorem 4.1 *For every fixed integer $r \geq 3$ and for every fixed integer $k \geq 1$, It is NP-Complete to decide whether an input graph G has a k-orthogonal $K_{1,r}$-decomposition.*

Proof: The problem is in NP since given k families of subgraphs of G we can verify in polynomial time if each family is a $K_{1,r}$-decomposition and if they are pairwise orthogonal. We will prove the NP-Completeness by reducing from the corresponding non-orthogonal $K_{1,r}$-decomposition problem (i.e. the case $k = 1$), which is known to be NP-Complete for every fixed $r \geq 3$ [19]. Suppose we are given an instance $G = (V, E)$ for the non-orthogonal $K_{1,r}$-decomposition problem. We create a graph G' from G by adding to each vertex v a set $S(v)$ of $r(kr-k+1)$ new neighbors, each connected only to v. G' can clearly be constructed in polynomial time, and has $(r(kr-k+1)+1)|V|$ vertices. We claim that G' has a k-orthogonal $K_{1,r}$-decomposition if and only if G has a $K_{1,r}$-decomposition. Assume that G' has a k-orthogonal $K_{1,r}$-decomposition. In particular, G' has a $K_{1,r}$-decomposition denoted by L. For each vertex v and for each $i = 1, \dots, r$ let $s(i, v)$ be the number of elements of L rooted at v (the root of $K_{1,r}$ is the vertex with degree r), and having exactly i leaves in $S(v)$. Clearly, $\sum_{i=1}^{r} i \cdot s(i, v) = r(kr - k + 1)$. Thus, the number of elements rooted at v and having a leaf in $V(G)$ is exactly $q(v) = \sum_{i=1}^{r}(r-i)s(i, v) = 0 \bmod r$. Thus, this set of $q(v)$ edges connecting v to vertices of $V(G)$ can be regrouped into $q(v)/r$ copies of $K_{1,r}$, all entirely within G. By doing this for each $v \in V$ we get a $K_{1,r}$-decomposition of G.

Assume now that G has a $K_{1,r}$-decomposition L. We need to create k distinct $K_{1,r}$-decompositions of G' which are pairwise orthogonal. let $Q(v)$ be the set of vertices of G, adjacent to v, which belong to elements of L rooted at v. Putting $q(v) = |Q(v)|$ we obviously have $q(v) = 0 \bmod r$. It thus suffices to show that the star whose root is v, and whose leaves are $Q(v) \cup S(v)$ has a

k-orthogonal $K_{1,r}$-decomposition. Consider a star with $x = q(v) + r(kr - k + 1)$ vertices. The line graph of this star is K_x. It suffices to show that K_x has k distinct K_r-factors, where each two factors are edge-disjoint (a K_r-factor is a set of x/r vertex-disjoint subgraphs isomorphic to K_r). This can be deduced from the Theorem of Hajnal and Szemerédi [26], stating that if r divides x, and X is a graph with x vertices, $\delta(X) \geq (1 - 1/r)x$, then X has a K_r-factor. Thus, one may take K_x and delete from it t edge-disjoint K_r-factors, obtaining a regular spanning subgraph with degree $x - 1 - t(r - 1)$ as long as $x - 1 - t(r - 1) \geq (1 - 1/r)x$. Thus, we need only to show that $x - 1 - k(r - 1) \geq (1 - 1/r)x$. This, in turn, is true since $x \geq r(kr - k + 1)$. □

It is interesting to note that the same NP-Completeness proof applies not only when k is fixed, but even when $k = \lfloor n^\alpha \rfloor$ for any fixed $\alpha < 1$, where n denotes the number of vertices of the graph G. One cannot expect to have $\alpha > 1$, since, by a simple counting argument, the number of pairwise-orthogonal $K_{1,r}$-decompositions is always $O(n)$.

In closing this paper we would like to add a few comments:

- Corollary 1.2 shows that for every fixed positive integer k, there exists a k-orthogonal K_r-decomposition of K_n (a $CRSS(n, r, k)$ design) provided that n is large enough, and that n satisfies the trivial necessary divisibility conditions. An easy counting argument shows that one cannot have more than $n - 2$ pairwise-orthogonal K_r-decompositions. It would be interesting to determine tight upper and lower bounds for the maximum possible value of k (as a function of r and n), for which a k-orthogonal K_r-decomposition (or, equivalently, a $CRSS(n, r, k)$ design) still exists.

- It is possible to extend Theorem 1.1 to the case of arbitrary fixed graphs instead of complete graphs. Namely, a k-orthogonal optimal H-packing of K_n.

- Although we are able to prove NP-Completeness for orthogonal star decompositions, it would be interesting to prove a full orthogonal analog to the Dor-Tarsi result:

  Conjecture 4.2 *For every fixed graph H having at least three edges in some connected component, and for every fixed positive integer k, it is NP-Complete to decide if a given input graph G has a k-orthogonal H-decomposition.*

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