Orthogonal Colorings of Graphs

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Abstract

An *orthogonal coloring* of a graph G is a pair $\{c_1, c_2\}$ of proper colorings of G, having the property that if two vertices are colored with the same color in c_1, then they must have distinct colors in c_2. The notion of orthogonal colorings is strongly related to the notion of orthogonal Latin squares. The *orthogonal chromatic number* of G, denoted by $O\chi(G)$, is the minimum possible number of colors used in an orthogonal coloring of G. If G has n vertices, then the definition implies that $\lceil\sqrt{n}\,\rceil \leq O\chi(G) \leq n$. G is said to have an *optimal* orthogonal coloring if $O\chi(G) = \lceil\sqrt{n}\,\rceil$. If, in addition, n is an integer square, then we say that G has a *perfect* orthogonal coloring, since for any two colors x and y, there is exactly one vertex colored by x in c_1 and by y in c_2.

The purpose of this paper is to study the parameter $O\chi(G)$ and supply upper bounds to it which depend on other graph parameters such as the maximum degree and the chromatic number. We also study the structure of graphs having an optimal or perfect orthogonal coloring, and show that several classes of graphs always have an optimal or perfect orthogonal coloring. We also consider the *strong* version of orthogonal colorings, where no vertex may receive the same color in both colorings.

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1 Introduction

All graphs considered here are finite, undirected, and have no loops or multiple edges. For the standard graph-theoretic and design-theoretic notations the reader is referred to [4] and to [6]. A *k-orthogonal coloring* of a graph G, is a set $\{c_1, \ldots, c_k\}$ of proper colorings of G, with the additional property that if u and v are two distinct vertices having the same color in some coloring c_i, then they must have distinct colors in all the other colorings c_j where $j \neq i$. A 2-orthogonal coloring is simply called an *orthogonal coloring*. The *k-orthogonal chromatic number* of G, denoted by $O\chi_k(G)$, is the minimum possible number of colors used in a k-orthogonal coloring of G. When $k = 2$ we simply define $O\chi(G) = O\chi_2(G)$, to be the *orthogonal chromatic number* of G. Clearly, we may take c_1 to be a coloring which colors every vertex by a distinct color, and $c_i = c_1$, for $i = 2, \ldots, k$. This shows that $O\chi_k(G) \leq n$, where n is the number of vertices of G. (Note that, trivially, $O\chi_k(G) \leq O\chi_{k+1}(G)$). On the other hand, the definition implies that $O\chi(G) \geq \chi(G)$, and also that $O\chi(G) \geq \lceil\sqrt{n}\,\rceil$, since otherwise, there are less than n possible color pairs. We can therefore summarize:

$$\max\{\chi(G), \lceil\sqrt{n}\,\rceil\} \leq O\chi(G) \leq O\chi_3(G) \leq \ldots \leq n. \tag{1}$$

There are many graphs which satisfy $O\chi(G) = \lceil\sqrt{n}\,\rceil$. For example, $O\chi(C_5) = 3$ as we may color the cycle once by the colors $(1, 2, 3, 1, 2)$ and then by the colors $(3, 1, 3, 1, 2)$. Note that we have $O\chi(C_5) = \chi(C_5)$. These observations naturally raise the following definitions:

1. G is said to have an *optimal* k-orthogonal coloring (k-OOC) for short) if $O\chi_k(G) = \lceil\sqrt{n}\,\rceil$. A 2-OOC is simply called an OOC.

2. If n is an integer square and G has a k-OOC we say that G has a *perfect* k-orthogonal coloring (k-POC for short), as, in this case, each ordered color pair appears in each ordered pair of colorings in exactly one vertex. A 2-POC is simply called a POC.

An example of a graph having a POC is C_9, since we may color the cycle first by $(1, 2, 3, 1, 2, 3, 1, 2, 3)$ and then by $(1, 2, 1, 2, 3, 2, 3, 1, 3)$.

The notion of k-orthogonal colorings is strongly related to the notion of orthogonal Latin squares. Recall that two Latin squares L_1, L_2 of order r are orthogonal if for any ordered pair (s, t) where $1 \leq s \leq r$ and $1 \leq t \leq r$, there is exactly one position (i, j) for which $L_1(i, j) = s$ and $L_2(i, j) = t$. It is well-known that orthogonal Latin squares exist for every $r \notin \{2, 6\}$ (cf. [6]). A family of k-orthogonal Latin squares of order r, is a set of k Latin squares every two of which are orthogonal. It is well-known that for every k, there exists $L(k)$, such that for every $r \geq L(k)$, there exists a family of k-orthogonal Latin squares of order r (cf. [6], and [3] who showed that $L(k) = O(k^{14.8})$).

Given a family $F = \{L_1, \ldots, L_k\}$ of k-orthogonal Latin squares of order r, we define the graph $U(F)$ as follows: G has r^2 vertices, which are denoted by the ordered pairs (i, j) for $i = 1, \ldots r$, $j = 1, \ldots, r$. A vertex (i_1, j_1) is joined to a vertex (i_2, j_2) if for every $p = 1, \ldots, k$, $L_p(i_1, j_1) \neq L_p(i_2, j_2)$. Note that $U(F)$ is regular of degree $r^2 - k(r-1) - 1$. The crucial fact is that $O\chi_k(U(F)) = r$, since we can define the colorings $\{c_1, \ldots, c_k\}$ in the obvious way: $c_p((i, j)) = L_p(i, j)$. The pairwise-orthogonality of the members of F, and the definition of $U(F)$ show that this is a k-orthogonal coloring of $U(F)$. Since the coloring only uses r colors, and since the number of vertices is r^2, we have that $O\chi_k(U(F)) = r$, and that $U(F)$ has a k-POC. This discussion yields the following fact:

FACT 1: Let k and r be positive integers with $r \geq L(k)$. Let G be a subgraph of every graph X with r^2 vertices where X is $r^2 - k(r-1) - 1$-regular, then $O\chi_k(G) \leq r$. If, in addition, G has r^2 vertices, then G has a k-POC.

$U(F)$ is a graph which can be obtained from the complete graph K_{r^2} be deleting k edge-disjoint K_r-factors, since each Latin square $L_i \in F$ eliminates one K_r-factor from K_{r^2}, where every K_r in this factor corresponds to r cells having the same symbol in L_i. The fact that the distinct K_r-factors are edge-disjoint follows from the pairwise-orthogonality of the members of F. It is interesting to note that in case $k = 2$, the graph $U(F)$ (considered as an unlabeled graph) is independent of the actual Latin squares $\{L_1, L_2\}$. This is because whenever we delete two edge-disjoint K_r-factors from K_{r^2}, we always get the same graph, which we denote by U_r. We therefore call U_r the *universal orthogonal graph of order* r. Note that U_r exists for every $r \geq 1$, although for $r = 2, 6$ there is no corresponding pair of orthogonal Latin squares. For example, $U_2 = 2K_2$, since by deleting two independent edges from K_4 we get C_4, and then deleting another pair of independent edges we get $2K_2$. Now put $U_r = K_{r^2} \setminus \{F_1, F_2\}$ where F_1 is the first K_r-factor deleted from K_{r^2} and F_2 is the second K_r factor deleted from $K_{r^2} \setminus \{F_1\}$. We can associate each vertex v of U_r with an ordered pair of integers (i, j), $1 \leq i \leq r$ and $1 \leq j \leq r$, where i is the serial number of the clique K_r in F_1 containing v, and j is the serial number of the clique K_r in F_2 containing v. This association shows that for all r, $O\chi(U_r) = r$ and U_r has a POC. Another obvious result of this association is that every graph G with $O\chi(G) \leq r$ is isomorphic to a subgraph of U_r, since we may map $v \in G$ colored with (i, j) to the vertex of U_r associated with the pair (i, j). We can summarize this discussion in the following statement:

FACT 2: Let r be a positive integer. A graph G is isomorphic to a subgraph of U_r if and only if $O\chi(G) \leq r$. If, in addition, G has r^2 vertices, then G has a POC.

A related concept to orthogonal coloring is the notion of *orthogonal edge coloring*, introduced in [2]. In this case, one requires two proper edge colorings with the property that any two edges which receive the same color in the first coloring, receive distinct colors in the second coloring. For a survey of the results on orthogonal edge colorings the reader is referred to [1]. The results on orthogonal edge coloring naturally translate to results on orthogonal vertex coloring when one considers line graphs. In this paper we study the parameter $O\chi_k(G)$, with our main attention on $O\chi(G)$. In section 2 we supply several upper bounds to $O\chi(G)$ and $O\chi_k(G)$ which depend on other graph parameters such as the maximum degree and the chromatic number. In some cases we are able to obtain exact results. In particular, we prove the following theorems:

Theorem 1.1 *Let G be a graph with n vertices, and with maximum degree Δ. Then*

$$O\chi(G) \le \left\lceil \frac{n}{\Delta+1} \right\rceil + \Delta.$$

Furthermore, if $n > \Delta(\Delta+1)$ then the r.h.s. can be reduced by 1. For $k \ge 2$ the following upper bound holds:

$$O\chi_k(G) \le \min\{2\sqrt{k-1}\max\{\Delta, \sqrt{n}\}\ ,\ (k-1)\left\lceil \frac{n}{\Delta+1} \right\rceil + \Delta\}.$$

Theorem 1.2 *Let G be a graph with n vertices, and with $\chi = \chi(G)$. Then*

$$O\chi(G) \le O\chi_3(G) \le \chi + \sqrt{\chi}\sqrt{n}.$$

For $k \ge 4$ we have that

$$O\chi_k(G) < \chi L(k-2) + \chi + \sqrt{\chi}\sqrt{n}.$$

Furthermore, for every χ and k there exists $N = N(\chi, k)$ such that if $n > N$ then $O\chi_k(G) \le \chi + \sqrt{\chi}\sqrt{n}$.

Theorem 1.3 *Let G be a complete t-partite graph with vertex classes of sizes $s_1, \ldots, s_t$. Then,*

$$O\chi(G) = \sum_{i=1}^{t} \lceil \sqrt{s_i} \rceil - \lfloor m/2 \rfloor$$

where m is the number of vertex classes whose size s_i satisfies $\lceil \sqrt{s_i} \rceil \lfloor \sqrt{s_i} \rfloor \ge s_i$ but is not an integer square.

Recall that a graph G is *d-degenerate* if we may order the vertices of G such that every vertex has at most d neighbors preceding it in the ordering. Such an ordering is called a *d-degenerate ordering.* For example, trees are 1-degenerate and planar graphs are 5-degenerate. Obviously, d-degenerate graphs have a greedy coloring with $d+1$ colors. The next theorem bounds the k-orthogonal chromatic number of d-degenerate graphs.

Theorem 1.4 *Let G be a d-degenerate graph with n vertices. If t satisfies*

$$(t-d)^k > \binom{k}{2}(n-d-1)t^{k-2}$$

then $O\chi_k(G) \le t$. Consequently, for $k = 2$ we get

$$O\chi(G) \le d + \left\lceil \sqrt{n-d} \right\rceil .$$

In Section 3 we consider graphs having an OOC or a POC. We prove several extensions of Facts 1 and 2, and, in particular, we show that every graph with maximum degree which is not too large has a k-OOC:

Theorem 1.5 *If G is an n-vertex graph satisfying $n \ge L(k-2)^2$, and $\Delta(G) \le (\sqrt{n}-1)/(2k)$, then G has a k-OOC. In particular, if n is a perfect square, then G has a k-POC.*

(Note that for $k = 2, 3$ the condition $n \ge L(k-2)^2$ is vacuous, so in these cases, Theorem 1.5 applies to every n). In section 4 we consider strong orthogonal colorings in which no vertex is allowed to receive the same color in both colorings. We will show the existence of a non-trivial family of graphs which are perfect w.r.t strong orthogonality. The final section contains some concluding remarks and open problems.

2 Upper bounds

In this section we prove Theorems 1.1-1.4 which all give upper bounds to $O\chi(G)$ and $O\chi_k(G)$. Depending on the graph, each theorem may give a different estimate. The first theorem supplies a useful upper bound for graphs with a rather large chromatic number.

Proof of Theorem 1.1: We shall use the result of Hajnal and Szemerédi [7], which states that every graph has a proper vertex coloring with $\Delta + 1$ colors, in which every color class contains at most $\lceil n/(\Delta+1) \rceil$ vertices and at least $\lfloor n/(\Delta+1) \rfloor$ vertices. Let c_1 be such a coloring of G. Now add to G edges between each two vertices colored the same by c_1. The resulting graph, denoted by G_1 has maximum degree

$$\Delta(G_1) \le \Delta + \left\lceil \frac{n}{\Delta+1} \right\rceil - 1.$$

Let c_2 be a greedy coloring of G_1 with $\Delta(G_1) + 1$ colors. The definition of G_1 implies that c_1 and c_2 are orthogonal. Since the number of colors used by c_1 is $\Delta + 1 \le \Delta + \lceil n/(\Delta+1) \rceil$, it follows that

$$O\chi(G) \le \Delta + \left\lceil \frac{n}{\Delta+1} \right\rceil .$$

We can improve this bound in case $n > \Delta(\Delta+1)$. We will show that in this case, G_1 satisfies the conditions of the theorem of Brooks [4]. Put $x = \Delta + \lceil n/(\Delta+1) \rceil$. We first show that G_1 does not have a clique of order x. Assume X is any set of x vertices in G_1. There are at most $x \cdot \Delta/2$ edges of G with both endpoints in X. Each vertex is adjacent in G_1 to at most $\lceil n/(\Delta+1) \rceil - 1$ vertices to which it was not adjacent in G. Thus, there are at most $x \cdot (\lceil n/(\Delta+1) \rceil - 1)/2$ such edges with both endpoints in X. Summing up, there are at most $x(x-1)/2$ edges in G_1 with both endpoints in X, where the only way to achieve this number is if X is a union of $y = x/\lceil n/(\Delta+1) \rceil$ vertex classes of c_1 with size $\lceil n/(\Delta+1) \rceil$ each. Namely, if $(\Delta + \lceil n/(\Delta+1) \rceil)/\lceil n/(\Delta+1) \rceil$ is an integer, which imposes that Δ be a multiple of $\lceil n/(\Delta+1) \rceil$. This, however, is impossible, since $n > \Delta(\Delta+1)$. Thus, X is not a clique. Consequently, G_1 does not have a clique of order x. Also, note that if $\Delta > 1$ then $x > 3$, and if $\Delta = 1$ the claim holds trivially, so in any case, the Theorem of Brooks applies to G_1, and G_1 has a coloring c_2 with $x-1$ colors. As before, c_1 and c_2 are orthogonal, and c_1 uses only $\Delta+1$ colors, which is not greater than $x-1$. Thus,

$$O\chi(G) \leq x - 1 = \Delta + \left\lceil \frac{n}{\Delta+1} \right\rceil - 1.$$

For $k \geq 2$ we may use a recursive application of the Hajnal and Szemerédi Theorem. Instead of coloring G_1 using a greedy coloring, we can color it once again using $\Delta(G_1)+1$ colors using the Hajnal and Szemerédi result. Denote this coloring by c_2. We now define G_2 by adding to G_1 edges between two vertices having the same color in c_2. Clearly, $\Delta(G_2) \leq \lceil n/(\Delta(G_1)+1) \rceil + \Delta(G_1) - 1$. After $k-1$ applications we obtain a graph G_{k-1} with $\Delta(G_{k-1}) \leq \lceil n/(\Delta(G_{k-2})+1) \rceil + \Delta(G_{k-2}) - 1$. We may color G_{k-1} greedily using, say, $\Delta(G_{k-1})+1$ colors, and denote the final coloring by c_k. The construction shows that $\{c_1, \ldots, c_k\}$ is a family of k-orthogonal colorings of G. The recurrence equation $\Delta(G_p) \leq \lceil n/(\Delta(G_{p-1})+1) \rceil + \Delta(G_{p-1}) - 1$ for $p = 1, \ldots, k-1$ (define $G = G_0$) is dominated by both $2\sqrt{p}\Delta(G_0) = 2\sqrt{p}\Delta$, assuming $\Delta \geq \sqrt{n}$, and by $p\lceil n/(\Delta+1) \rceil + \Delta - 1$. Thus,

$$O\chi_k(G) \leq \min\{2\sqrt{k-1}\max\{\Delta, \sqrt{n}\}\ ,\ (k-1)\left\lceil \frac{n}{\Delta+1} \right\rceil + \Delta\}.$$

Note that whenever $\Delta \geq k^{1/4}\sqrt{n}$ the estimate $(k-1)\lceil n/(\Delta+1) \rceil + \Delta$ is better than the estimate $2\sqrt{k-1}\Delta$. □

If the chromatic number of G is large (say, greater than $\sqrt{n}$), and close to the maximum degree, then the estimate in Theorem 1.1 is very good. For example, consider a graph with $\chi(G) = n^\alpha$ and $\Delta(G) = n^{\alpha+\epsilon}$ where $\alpha > 0.5$ and $\epsilon \geq 0$ is small. By (1) and by Theorem 1.1 we have that

$$n^\alpha \leq O\chi(G) \leq n^{\alpha+\epsilon} + n^{1-\alpha-\epsilon} + 1 = n^{\alpha+\epsilon}(1 + o(1)).$$

Theorem 1.2 supplies a useful bound for graphs with a rather small chromatic number. Before proving it, we need the following lemma:

Lemma 2.1 *Let I_t denote the independent set of size t. Then, $O\chi(I_t) = O\chi_3(I_t) = \lceil \sqrt{t} \rceil$, and if $k \geq 4$ then $O\chi_k(I_t) \leq \max\{\lceil \sqrt{t} \rceil\ ,\ L(k-2)\}$.*

Proof: Let p be a positive integer, and let $k \geq 3$. Suppose there exist $k-2$ orthogonal Latin squares of order p. We claim that I_{p^2} has a k-POC. Let $L_1, \ldots, L_{k-2}$ be $k-2$ orthogonal Latin squares or order p. We first assign to every vertex v of I_{p^2} a distinct pair of indices (i,j) where $1 \leq i \leq p$ and $1 \leq j \leq p$. We now define the k orthogonal colorings $c_1, \ldots, c_k$. Assume that v is assigned the pair (i,j). Then we define $c_1(v) = i$, $c_2(v) = j$ and $c_s(v) = L_{s-2}(i,j)$ for $s = 3, \ldots, k$. It is easily verified that $c_1, \ldots, c_k$ are pairwise orthogonal.
Trivially, $L(1) = 1$, since there exists a Latin square of every positive order. In any case, if $\lceil \sqrt{t} \rceil \geq L(k-2)$, then by the proof above, $O\chi_k(I_t) = \lceil \sqrt{t} \rceil$, and therefore, $O\chi_k(I_t) \leq \max\{\lceil \sqrt{t} \rceil\ ,\ L(k-2)\}$, for every $k \geq 3$. Since $L(1) = 1$ and since $\lceil \sqrt{t} \rceil \leq O\chi(I_t) \leq O\chi_3(I_t)$ we also have $O\chi(I_t) = O\chi_3(I_t) = \lceil \sqrt{t} \rceil$. □

Proof of Theorem 1.2: We partition the vertices of G into χ independent sets, denoted by $C_1, \ldots, C_\chi$. By using disjoint color sets for each C_i, $i = 1, \ldots, \chi$, and by applying Lemma 2.1 to each C_i we obtain that for $k = 2, 3$

$$O\chi_k(G) \leq \sum_{i=1}^{\chi} \left\lceil \sqrt{|C_i|} \right\rceil,$$

and for $k \geq 4$,

$$O\chi_k(G) \leq \sum_{i=1}^{\chi} \max\{\left\lceil \sqrt{|C_i|} \right\rceil\ ,\ L(k-2)\}.$$

Since $|C_1| + \ldots + |C_\chi| = n$, it follows by an elementary convexity argument that the last two inequalities are maximized when all the sets have equal size. Thus, for $k = 2, 3$

$$O\chi_k(G) \leq \sum_{i=1}^{\chi} \left\lceil \sqrt{\frac{n}{\chi}} \right\rceil \leq \chi + \sqrt{\chi}\sqrt{n},$$

and for $k \geq 4$, if s denotes the number of vertex classes whose size is less than $L(k-2)^2$ then

$$O\chi_k(G) \leq sL(k-2) + (\chi - s)\left\lceil \sqrt{\frac{n}{\chi - s}} \right\rceil \leq sL(k-2) + (\chi - s) + \sqrt{\chi - s}\sqrt{n} < \chi L(k-2) + \chi + \sqrt{\chi}\sqrt{n}.$$

If n is sufficiently large then $sL(k-2) \leq \sqrt{n}(\sqrt{\chi} - \sqrt{\chi - s})$, and thus

$$O\chi_k(G) \leq \chi + \sqrt{\chi}\sqrt{n}.$$

□

Theorem 1.3 shows that the orthogonal chromatic number of complete partite graphs can be computed exactly.

Proof of Theorem 1.3: Let $S_1, \ldots, S_t$ denote the vertex classes of G, where $|S_i| = s_i$, and the sizes of the first m classes have the property that s_i is not an integer square and $\lceil \sqrt{s_i} \rceil \lfloor \sqrt{s_i} \rfloor \geq s_i$. We first create an orthogonal coloring $\{c_1, c_2\}$ with the required number of colors. For $i = m+1, \ldots, t$ we use $\lceil \sqrt{s_i} \rceil$ distinct colors to color the vertices of S_i in both c_1 and c_2, while maintaining orthogonality. This can be done since S_i is an independent set. If m is odd, then S_m is also colored with $\lceil \sqrt{s_m} \rceil$ distinct colors in both c_1 and c_2. We now consider the $\lfloor m/2 \rfloor$ pairs of classes $\{S_1, S_2\}, \ldots, \{S_{2\lfloor m/2 \rfloor -1}, S_{2\lfloor m/2 \rfloor}\}$. In coloring the vertices of each of these pairs we proceed as follows. Assume the pair is $\{S_i, S_{i+1}\}$, and let $\{w_1, \ldots, w_z\}$ be a set of z distinct colors where $z = \lceil \sqrt{s_i} \rceil + \lceil \sqrt{s_{i+1}} \rceil - 1$. Consider all the ordered pairs of colors of the form (w_p, w_q) where $1 \leq p \leq \lfloor \sqrt{s_i} \rfloor$ and $1 \leq q \leq \lceil \sqrt{s_i} \rceil$. There are at least s_i such pairs, so we may color the vertices of S_i with these pairs, where the first coordinate is the color in c_1 and the second is the color in c_2. Now consider all the ordered pairs of colors of the form (w_p, w_q) where $\lfloor \sqrt{s_i} \rfloor + 1 \leq p \leq z$ and $\lceil \sqrt{s_i} \rceil + 1 \leq q \leq z$. There are at least s_{i+1} such pairs, so we may color the vertices of S_{i+1} with these pairs where the first coordinate is the color in c_1 and the second is the color in c_2. Note that no vertex of S_i receives the same color as a vertex of S_{i+1} in neither c_1 nor c_2. Summing up over all the distinct sets of colors we have used at most $\sum_{i=1}^{t} \lceil \sqrt{s_i} \rceil - \lfloor m/2 \rfloor$ colors. Thus

$$O\chi(G) \leq \sum_{i=1}^{t} \lceil \sqrt{s_i} \rceil - \lfloor m/2 \rfloor .$$

We now need to show that any orthogonal coloring requires at least this number of colors. Let $\{c_1, c_2\}$ be an orthogonal coloring of G. The colors used by c_1 in S_i cannot be used in any S_j for $j \neq i$ since c_1 is proper. The same holds for c_2. Let a_i and b_i denote the number of colors used in S_i by c_1 and c_2, respectively. Thus, c_1 uses $a_1 + \ldots + a_t$ colors and c_2 uses $b_1 + \ldots + b_t$ colors. Since c_1 and c_2 are orthogonal, we know that $a_i b_i \geq s_i$ for $i = 1, \ldots, t$. The overall number of colors used by the pair $\{c_1, c_2\}$ is

$$\max\{\sum_{i=1}^{t} a_i, \sum_{i=1}^{t} b_i\} \geq \left\lceil \frac{\sum_{i=1}^{t}(a_i + b_i)}{2} \right\rceil .$$

Since $a_i b_i \geq s_i$, the r.h.s. of the last inequality is minimized when $a_i + b_i = 2\lceil \sqrt{s_i} \rceil$ if $i > m$, and when $a_i + b_i = \lceil \sqrt{s_i} \rceil + \lfloor \sqrt{s_i} \rfloor = 2\lceil \sqrt{s_i} \rceil - 1$ if $i \leq m$. Thus,

$$O\chi(G) \geq \left\lceil \frac{\sum_{i=1}^{t} 2\lceil \sqrt{s_i} \rceil - m}{2} \right\rceil = \sum_{i=1}^{t} \lceil \sqrt{s_i} \rceil - \lfloor m/2 \rfloor .$$

□

As an example, we have that $O\chi(K_{6,6}) = 5$ since 6 is not an integer square and $2 \cdot 3 \geq 6$, so $m = 2$ in this case. Note that the same reasoning yields $O\chi(K_{5,5}) = 5$ and $O\chi(K_{5,4}) = 5$.

Proof of Theorem 1.4: Consider a d-degenerate ordering $\{v_1, \ldots, v_n\}$ of the vertices of G. We need to create a set $\{c_1, \ldots, c_k\}$ of k-orthogonal colorings. We prove the theorem by coloring the vertices one by one while maintaining orthogonality. Coloring $v_1, \ldots, v_d$ is trivial, since we may define, say, $c_j(v_i) = i$ for all $j = 1, \ldots, k$ and $i = 1, \ldots, d$. Note that $t \geq d$ so we are still within bounds. Assume we have successfully colored $v_1, \ldots, v_{i-1}$ $(i > d)$ by using no more than t colors. We now wish to color v_i. Let R be the set of neighbors of v_i in G which have already been colored. Clearly, $|R| \leq d$. Without loss of generality, we may assume $|R| = d$. There are at most d colors used in R in the coloring c_j. Thus, there are at least $t - d$ ways to extend c_j to v_i while still maintaining that c_j is a proper coloring. Overall, there are at least $(t - d)^k$ ways to extend all the colorings to v_i, and still have that all the colorings are proper. We still need to show that at least one of these extensions maintains orthogonality. Consider a vertex v_j where $j < i$ and $j \notin R$. Any extension of the colorings to v_i which satisfies that for some distinct colorings c_x and c_y $c_x(v_j) = c_x(v_i)$ and $c_y(v_j) = c_y(v_i)$ is illegal. This eliminates at most t^{k-2} extensions of the colorings to v_i. Since this holds for every pair of distinct colorings and for all the $i - d - 1$ vertices v_j where $j < i$ and $j \notin R$, there are at most $\binom{k}{2}(i - d - 1)t^{k-2}$ illegal extensions. This still leaves at least one legal extension since $(t - d)^k > \binom{k}{2}(i - d - 1)t^{k-2}$.

For $k = 2$ we can solve this inequality explicitly and obtain that if $t > d + \sqrt{n - d - 1}$ then $O\chi(G) \leq t$. In particular, $O\chi(G) \leq d + 1 + \left\lfloor \sqrt{n - d - 1} \right\rfloor = d + \left\lceil \sqrt{n - d} \right\rceil$. $\Box$

Theorem 1.4 is rather tight since $O\chi(G) \geq \lceil \sqrt{n} \rceil$ always. In fact, for every d, we can show that there exist d-degenerate graphs G for which $O\chi(G) = d + \left\lceil \sqrt{n - d} \right\rceil$. Consider the graph $G_{n,d} = I_{n-d} * K_d$ which is defined by taking an independent set of order $n - d$ and a clique of order d and joining every vertex of the clique with every vertex of the independent set. It is easy to see that $G_{n,d}$ is d-degenerate and that $\chi(G_{n,d}) = d + 1$. We claim that we cannot color $G_{n,d}$ orthogonally with less than $d + \left\lceil \sqrt{n - d} \right\rceil$ colors. Consider two orthogonal colorings c_1 and c_2 of $G_{n,d}$. Since they are orthogonal on I_{n-d}, at least one of them uses $\left\lceil \sqrt{n - d} \right\rceil$ colors on I_{n-d}, and, obviously, an additional set of d colors on K_d.

3 Optimal and perfect orthogonal colorings

In this section we focus on graphs having a k-OOC or a k-POC. Clearly, if $\chi(G) > \lceil \sqrt{n} \rceil$ then G does not have a k-OOC. Hence, there exist graphs G with $\Delta(G) = \lceil \sqrt{n} \rceil$ which do not have a k-OOC (e.g. any graph G on n vertices with $\Delta(G) = \lceil \sqrt{n} \rceil$ having a clique on $\lceil \sqrt{n} \rceil + 1$ vertices as a connected component). It turns out that graphs with a somewhat lower maximum degree, but still with $\Delta(G) = \Omega(\sqrt{n})$, always have a k-OOC. This is shown in Theorem 1.5:

Proof of Theorem 1.5: Consider the set V of the vertices of G, as a set of n isolated vertices. Since $n \geq L(k-2)^2$, we have, by Lemma 2.1, that V has k-OOC. Let $c_1, \ldots, c_k$ be such a k-OOC. We will now add to V edges of G, one by one, until we obtain G. Every time we add a new edge, we will modify the colorings $c_1, \ldots, c_k$ so that they will remain proper and pairwise orthogonal. Thus, at the end, we will have a k-OOC of G. Assume that we have already added some edges of G to V, and we now wish to add the next edge $e = (u, v)$. Denote the graph after the addition of e by G^*. Note that G^* is a spanning subgraph of G, and we assume that $c_1, \ldots, c_k$ is a k-OOC of $G^* \setminus \{e\}$. If $c_i(v) \neq c_i(u)$ for each $i = 1, \ldots, k$, then $c_1, \ldots, c_k$ form a k-OOC of G^*. Otherwise, we will show how to find a vertex x, such that by interchanging the k colors of x with the corresponding k colors of v, we still have that every coloring is proper, and hence this modification constitutes a k-OOC of G^*. Consider the set Z of the neighbors of v in G^*. Clearly, $|Z| \leq \Delta = \Delta(G)$. Let $W \subset V$ be the set of vertices w having, for some i, and some $z \in Z$, $c_i(w) = c_i(z)$. (We allow $z = w$, so $Z \cup \{v\} \subset W$). Since the colorings form a k-OOC in $G^* \setminus \{e\}$, we know that each color appears at most $\lceil \sqrt{n} \rceil$ times in each coloring. Thus, $|W| \leq k|Z| \lceil \sqrt{n} \rceil$. Now let $Y \subset V$ denote the set of vertices y, other than v, which have $c_i(y) = c_i(v)$ for some $i = 1, \ldots, k$. Clearly, $|Y| \leq k(\lceil \sqrt{n} \rceil - 1)$. Now consider the set Y^* of all the vertices of G^* which have a neighbor in Y. $|Y^*| \leq |Y|\Delta$. Finally, let $X = V \setminus (W \cup Y^*)$. We first show that X is not empty. This is true since

$$|X| \geq n - |W| - |Y^*| \geq n - k|Z| \lceil \sqrt{n} \rceil - \Delta|Y| \geq n - k\Delta \lceil \sqrt{n} \rceil - k\Delta(\lceil \sqrt{n} \rceil - 1) > 0$$

where the last inequality follows from the fact that $\Delta \leq (\sqrt{n} - 1)/(2k) < n/(k(2\lceil \sqrt{n} \rceil - 1))$. Now let $x \in X$. We can interchange the k colors given to x with the k corresponding colors given to v, and remain with a proper coloring. This is because after the interchange, the colorings in the neighborhood of v are proper since $x \notin W$, and the colorings in the neighborhood of x are proper since $x \notin Y^*$, and so it has no neighbor which shares the same color with the original color of v, in any of the k colorings. □

Theorem 1.4 and (1) show that for every n-vertex tree T, $\lceil \sqrt{n} \rceil \leq O\chi(T) \leq 1 + \left\lceil \sqrt{n-1} \right\rceil$. Thus, $O\chi(T)$ is one of two consecutive possible values, and if $n-1$ is an integer square, those upper and lower bounds coincide, so in this case, T has an OOC. In case $n-1$ is not an integer square, the example after Theorem 1.4 shows that the star on n vertices, $K_{1,n-1}$, has $O\chi(K_{1,n-1}) = 1 + \left\lceil \sqrt{n-1} \right\rceil$, and thus, $K_{1,n-1}$ does not have an OOC. However, stars are not the only examples of trees which do not have an OOC. In fact, every tree with n vertices, having a vertex of degree $(\left\lfloor \sqrt{n-1} \right\rfloor)^2 + 1$ does not have an OOC. For example, all the trees with $18 \leq n \leq 25$ vertices which have a vertex of degree 17 do not have an OOC since they contain $K_{1,17}$ and $O\chi(K_{1,17}) = 6$. It is, however, an easy exercise to establish that when $n-2$ is an integer square, and T is a tree with n vertices which is not a star, then T has an OOC.

There are trees with a much lower maximal degree which do not have an OOC. Let n be an integer square, and assume (although this is not necessary) that n is even. Let T be the double star obtained by joining two $K_{1,n/2-1}$ at the roots. T has maximum degree $n/2$, and we claim that T does not have an OOC whenever $(\lceil\sqrt{n}\,\rceil)(\lceil\sqrt{n}-1\rceil) < n$. Let c_1 and c_2 be two orthogonal colorings using x colors. The roots must have distinct colors in c_1, denote these colors by 1 and 2. At most $x-2$ leaves may have color 1 (otherwise c_2 must use $x+1$ colors if it is to be proper), and similarly, at most $x-2$ leaves may have color 2. Consequently, there are at least $n-2-2(x-2) = n+2-2x$ leaves that have colors other than 1 or 2 in c_1. No other color may appear x times at a leaf, so the number of other colors is at least $(n+2-2x)/(x-1)$ So we must have $(n+2-2x)/(x-1)+2 \le x$. Consequently, $x(x-1) \ge n$. Thus, whenever $(\lceil\sqrt{n}\,\rceil)(\lceil\sqrt{n}-1\rceil) < n$, we must have $x > \lceil\sqrt{n}\,\rceil$.

4 Strong orthogonal colorings

A k-strong orthogonal coloring, is a k-orthogonal coloring $c_1,\ldots,c_k$, where for each vertex v and for every pair of distinct colorings c_i and c_j, $c_i(v) \ne c_j(v)$. The analogous definitions of $SO\chi_k(G)$ and $SO\chi(G)$ are obvious. A graph G is said to have a *strong perfect orthogonal coloring* (SPOC for short) if it has $r(r-1)$ vertices and $SO\chi(G) = r$. A graph G with r^2 vertices is said to have a *strong orthogonal scheme* if it has a strong orthogonal coloring with $r+1$ colors, where the first coloring only uses r colors (thus, all possible r^2 pairs of colors, under this restriction, are used).

If G has n vertices, then, clearly, $SO\chi(G) \ge (1+\sqrt{1+4n})/2$, and $O\chi(G) \ge \sqrt{n}$. It is therefore plausible to conjecture that $O\chi(G) \le SO\chi(G) \le O\chi(G)+1$. This, however, is far from being true, as $O\chi(U_4) = 4$ while it is not difficult to check that $SO\chi(U_4) = 6$. In fact, it can be shown that $SO\chi(U_r) - O\chi(U_r)$ grows linearly with r. It is possible, however, to prove that $SO\chi(G) \le O\chi(G) + \lceil\chi(G)/2\rceil$. To see this, assume c_1, c_2 is an orthogonal coloring of G with $O\chi(G)$ colors (assume the colors are $1,\ldots,O\chi(G)$). Let X be the set of vertices of G colored the same in c_1 and c_2. X, being a subgraph of G, can be partitioned into at most $\chi(G)$ independent sets $X_1,\ldots,X_{\chi(G)}$. Now, for every $v \in X_i$ with $i \le \chi(G)/2$ we redefine $c_1(v) = O\chi(G)+i$. For every $v \in X_i$ with $i > \chi(G)/2$ we redefine $c_2(v) = O\chi(G)+i-\lfloor\chi(G)/2\rfloor$. After these modifications, c_1, c_2 is a strong orthogonal coloring of G using at most $O\chi(G)+\lceil\chi(G)/2\rceil$ colors. Note that, in particular, every bipartite graph has $SO\chi(G) \le O\chi(G)+1$, and, by Theorem 1.4, every d-degenerate graph has $SO\chi(G) \le \lceil(3d+1)/2\rceil + \left\lceil\sqrt{n-d}\right\rceil$.

There exist universal graphs w.r.t. strong orthogonality and strong orthogonal schemes. In the first case, one may take the set of all $r(r-1)$ ordered pairs (i,j) where $i \ne j$ and $1 \le i \le r$, $1 \le j \le r$ as the vertices, and connect a pair (i,j) to a pair (k,l) with an edge if and only if $i \ne k$

and $j \neq l$. The resulting graph, denoted by W_r, has $SO\chi(W_r) = r$, and therefore has a SPOC. Also, any graph G having $SO\chi(G) \leq r$ is isomorphic to a subgraph of W_r, as can be seen from the obvious isomorphism, where $v \in G$ is mapped to the vertex $(c_1(v), c_2(v))$ of W_r, c_1, c_2 being a strong orthogonal coloring of G with at most r colors. The universal graph w.r.t strong orthogonal schemes, denoted by X_r. is defined analogously, where now the set of vertices consists of all the ordered pairs (i, j) where $1 \leq i \leq r$ and $1 \leq j \leq r+1$, and $i \neq j$.

All the theorems proved in Section 2 and Section 3 have analogous versions when strong orthogonality is required, where only minor modifications are needed. We will therefore not prove them here. We will, however, show that an interesting family of graphs, namely the family of the complement graphs of U_r, denoted by U_r^c, has a strong orthogonal scheme, with the exception of $r \in \{2, 3, 6\}$. In other words, U_r^c is a spanning subgraph of X_r, unless $r \in \{2, 3, 6\}$. In fact we will show something more general:
Consider the grid graph $G_{a,b}$, where $a \geq b$ are positive integers. The vertices of $G_{a,b}$ are defined by all pairs of ordered integers (i, j) where $1 \leq i \leq a$ and $1 \leq j \leq b$, and a vertex (i, j) is connected to a vertex (k, l) if $i = k$ or $j = l$. Clearly, $G_{a,b}$ has ab vertices and is regular of degree $a + b - 2$. Also, $G_{a,b}$ has a maximum clique of order a. Clearly, $G_{r,r} = U_r^c$.

Theorem 4.1

1. *Every subgraph G of $G_{r,r-1}$ has $SO\chi(G) \leq r$. In particular, every spanning subgraph of $G_{r,r-1}$ has a SPOC.*

2. *For every positive integer $r \notin \{2, 3, 6\}$, every spanning subgraph of $G_{r,r}$ has a strong orthogonal scheme.*

Proof: We begin with the proof of the first part. It suffices to show that $G_{r,r-1}$ has a SPOC. We define two colorings c_1 and c_2 of $G_{r,r-1}$ as follows. Both colorings will only use the colors $0, \ldots, r-1$. Consider first the case where r is odd. We define $c_1((i, j)) = (i + j - 2) \bmod r$. Next, we define $c_2((i, j)) = (c_1(i, j) + j) \bmod r$. Now, c_1 is a proper coloring since $c_1((i, j)) \neq c_1((i, k))$ when $k \neq j$, and $c_1((i, j)) \neq c_1((k, j))$ when $k \neq i$. Similar reasoning shows that c_2 is proper (we use the fact that r is odd). No vertex uses the same color in both c_1 and c_2, since $1 \leq j \leq r-1$. Finally, both colorings are orthogonal since if, for two distinct vertices, $c_1((i, j)) = c_1((k, l))$ then we must have $j \neq l$, and therefore $c_2((i, j)) \neq c_2((k, l))$. Now assume that r is even. We define $c_1((i, j)) = (i + j - 2) \bmod r$ for all $1 \leq i \leq r$ and for all $1 \leq j \leq r/2$, and we define $c_1((i, j)) = (i + j - 1) \bmod r$ for all $1 \leq i \leq r$ and $r/2 + 1 \leq j \leq r - 1$. We define $c_2((i, j)) = (c_1(i, j) + j) \bmod r$ for all i and j. Once again, it is easy to check that c_1 and c_2 are both proper, orthogonal, and no vertex has the same color in both c_1 and c_2.

For the second part of the proof, it suffices to show that $G_{r,r}$ has a strong orthogonal scheme. The proof of this relies on the existence of self-orthogonal Latin squares for every order $r \notin 2, 3, 6$ [5]. A Latin square L is called self-orthogonal if the Latin square L^t (the transpose of L) is orthogonal to L. Let, therefore, L be a self-orthogonal Latin square of order r. We define a strong orthogonal scheme c_1, c_2 of $G_{r,r}$ as follows. $c_1((i,j)) = L(i,j)$ for every $1 \le i \le r$ and $1 \le j \le r$. If $i \ne j$ we define $c_2((i,j)) = L^t(i,j) = L(j,i)$. Finally, we define $c_2((i,i)) = r+1$, for $i = 1, \ldots, r$. Clearly, c_1 is a proper coloring of $G_{r,r}$ since L is a Latin square. Also, c_2 is a proper coloring of $G_{r,r}$ since L^t is a Latin square and since $\{(1,1), \ldots, (r,r)\}$ are an independent set in $G_{r,r}$. Also, the colorings are orthogonal since L and L^t are orthogonal and since any two vertices of the set $\{(1,1), \ldots, (r,r)\}$ do not have the same color in c_1 as the diagonal of L and L^t is the same, and they are orthogonal, which implies that no symbol may appear twice in the diagonal. Finally, c_1 and c_2 are strong orthogonal since no vertex has the same color in both c_1 and c_2. Since c_1 uses only r colors, the result follows. □

In the cases $r = 2$ or $r = 3$ it is easy to check that $G_{2,2}$ and $G_{3,3}$ do not have a strong orthogonal scheme. The fact that there is no strong orthogonal scheme for $G_{6,6}$ is less trivial, and relies on the fact that there are no two orthogonal Latin squares for $r = 6$. (It is also possible to check this by computer since one only needs to check that W_6 does not contain $G_{6,6}$. A very small fraction of the 36! possible mappings need to be checked since there are many equivalences and restrictions, which result from the large automorphism group of $G_{6,6}$) □

5 Concluding remarks and open problems

1. Theorem 1.5 shows, in particular, that if $\Delta(G) \le (\sqrt{n} - 1)/4$, then G has an OOC. It is interesting to find out if the bound $(\sqrt{n} - 1)/4$ can be improved to a bound of the form $\sqrt{n}/c$ where $c < 4$. Clearly we cannot expect $c \le 1$, as shown in the discussion prior to the proof of Theorem 1.5. The obvious generalization for $k > 2$ (i.e. replacing the denominator $2k$ with something smaller) may also be considered.

2. Determining if a graph G has a POC is equivalent, according to FACT 2, to a spanning subgraph isomorphism problem, namely, is a given graph G with r^2 vertices isomorphic to a spanning subgraph of U_r. Since general spanning subgraph isomorphism problems are known to be NP-Complete, it is safe to conjecture that determining if an input graph G has a POC is NP-Complete. Determining $O\chi(T)$ for a given tree T with n vertices is, on the other hand, a much more interesting problem, since, as shown in Section 3, this number is either $\lceil \sqrt{n} \rceil$ or $1 + \left\lceil \sqrt{n-1} \right\rceil$. For some trees the answer is known (see section 3), while for a general input

tree T, we do not yet have an algorithm that determines, in polynomial time, which of the two values is the right one for T.

3. Is it true that every tree with maximum degree smaller than $n/2$ has an OOC? In section 3 we have shown that there are trees with degree $n/2$ which do not have an OOC.

4. It would be interesting to determine the exact relationship between $O\chi(G)$ and $SO\chi(G)$.

5. A graph G is called $[n,k,r]$*-partite* if the vertex set of G can be partitioned into n independent sets (called vertex classes), each of size k exactly, and there are exactly r independent edges between any two vertex classes. (See [8] which focuses on these graphs). For example, K_n is $[n,1,1]$-partite, and C_9 is $[3,3,3]$-partite. An *independent covering* of an $[n,k,r]$-partite graph is a set of k vertex-disjoint independent sets of size n, each containing exactly one vertex from each vertex class. It is shown in [8] that every $[k,k,2]$-partite graph has an independent covering. This implies that every $[k,k,2]$-partite graph has a POC, since one can define the first coloring according to the vertex classes, and the second coloring according to the independent coverings. It is not even known whether every $[k,k,3]$-graph has an independent covering, although much more is conjectured, namely that every $[k,k,k]$-partite graph has an independent covering ([8] Conjecture 1.5). A seemingly easier conjecture is the following: "Does every $[k,k,k]$-partite graph have a POC?"

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