The number of orientations having no fixed tournament

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Abstract

Let T be a fixed tournament on k vertices. Let $D(n,T)$ denote the maximum number of orientations of an n-vertex graph that have no copy of T. We prove that $D(n,T) = 2^{t_{k-1}(n)}$ for all sufficiently (very) large n, where $t_{k-1}(n)$ is the maximum possible number of edges of a graph on n vertices with no K_k, (determined by Turán's Theorem). The proof is based on a directed version of Szemerédi's regularity lemma together with some additional ideas and tools from Extremal Graph Theory, and provides an example of a precise result proved by applying this lemma. For the two possible tournaments with three vertices we obtain separate proofs that avoid the use of the regularity lemma and therefore show that in these cases $D(n,T) = 2^{\lfloor n^2/4 \rfloor}$ already holds for (relatively) small values of n.

1 Introduction

All graphs considered here are finite and simple. For standard terminology on undirected and directed graphs the reader is referred to [5]. Let T be some fixed tournament. An orientation of an undirected graph $G = (V,E)$ is called *T-free* if it does not contain T as a subgraph. Let $D(G,T)$ denote the number of orientations of G that are T-free. Let $D(n,T)$ denote the maximum possible value of $D(G,T)$ where G is an n-vertex graph. In this paper we determine $D(n,T)$ precisely for every fixed tournament T and all sufficiently large n. Problems of counting orientations and directed subgraphs of a given type have been studied by several researchers. Examples of such results appear in [1, 7].

The problem of determining $D(n,T)$ even for three-vertex tournaments is already quite complicated (it is trivial for the unique two-vertex tournament). If G has no k-clique and T is a k-vertex tournament, then, clearly, $D(G,T) = 2^{e(G)}$ where $e(G)$ denotes the number of edges of G. Thus,

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for a k-vertex tournament T we obtain the following easy lower bound:

$$D(n,T) \geq 2^{t_{k-1}(n)} \tag{1}$$

where $t_{k-1}(n)$ is the maximum possible number of edges of a graph on n vertices with no K_k. Turán's Theorem shows that $t_{k-1}(n)$ is the number of edges of the unique complete $(k-1)$-partite graph with n vertices whose vertex classes are as equal as possible. In some cases, the lower bound in (1) is not the correct answer. For example, Let $T = C_3$ denote the directed triangle. For $n = 7$, the graph $G = K_7$ has 7! orientations that have no directed triangle (all the acyclic orientations). Hence $D(7) \geq 7! = 5040 > 2^{t_2(7)} = 2^{12} = 4096$. Similar examples are true for other tournaments with more than three vertices. However, all examples have n relatively small as a function of the number of vertices of the tournament. This suggests that possibly for every tournament T, and all n sufficiently large (as a function of T), the lower bound in (1) is the correct value. Our main theorem shows that this, indeed, is the case.

Theorem 1.1 *Let T be a fixed tournament on k vertices. There exists $n_0 = n_0(T)$ such that for all $n \geq n_0$,*

$$D(n,T) = 2^{t_{k-1}(n)}.$$

The proof of Theorem 1.1 is presented in the next two sections. It is based on the basic approach in [2] with some additional ideas, and uses several tools from Extremal Graph Theory, including a (somewhat uncommon) directed version of the regularity lemma of Szemerédi. It provides a rare example in which this lemma is used to prove results on directed graphs, and an even more rare example of a precise result obtained with the lemma.

Unfortunately, the use of the regularity lemma forces the constant n_0 appearing in Theorem 1.1 to be horribly large even for the case $k = 3$. In section 4 we outline a different proof for the special case $T = C_3$ that avoids using the regularity lemma, and obtain a moderate value for $n_0(C_3)$ (that can be optimized to less than 10000). Section 4 also contains a description of a simple reduction from the problem of counting the number of red-blue edge colorings of a graph G having no monochromatic K_k (solved in [10] for $k = 3$ and in [2] for $k > 3$) to the problem of counting the number of orientations of a graph G that do not contain the transitive tournament on k vertices, denoted T_k. Using this reduction we show, in particular, that $n_0(T_3) = 1$. The final section contains some concluding remarks and open problems.

In the rest of this paper, if x and y are vertices then xy refers to an edge between x and y in an undirected graph and (x, y) refers to a directed edge from x to y. If X and Y are disjoint subsets of vertices then $e(XY)$ denotes the number of edges between X and Y in an undirected graph, while $e(X, Y)$ denotes the number of edges from X to Y in a directed graph.

2 Graphs with many T-free orientations

Throughout the next two sections we assume that T is a fixed tournament on $k+1$ vertices and $k \geq 2$. Let G be an n-vertex graph with at least $2^{t_k(n)}$ distinct T-free orientations. Our aim in this section is to show that such graphs must be close to a k-partite graph. More precisely we prove the following.

Lemma 2.1 *For all $\delta > 0$ there exists $n_0 = n_0(k, \delta)$, such that if G is a graph of order $n \geq n_0$ which has at least $2^{t_k(n)}$ distinct T-free orientations then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_k$ such that $\sum_i e(V_i) < \delta n^2$.*

Our approach in the proof of Lemma 2.1 is similar to the one from [2] and [4], which is based on two important tools, the Simonovits stability theorem and the Szemerédi regularity lemma. However, we shall require a (somewhat uncommon) version of the regularity lemma for directed graphs and a few other additional ideas. We now introduce the necessary tools and lemmas needed for the proof of Lemma 2.1.

The stability theorem ([8], see also [5], p. 340) asserts that a K_{k+1}-free graph with almost as many edges as the Turán graph is essentially k-partite. The precise statement follows.

Theorem 2.2 *For every $\alpha > 0$ there exists $\beta > 0$ (where $\beta \ll \alpha$), such that any K_{k+1}-free graph on m vertices with at least $t_k(m) - \beta m^2$ edges has a partition of the vertex set $V = V_1 \cup \cdots \cup V_k$ with $\sum_i e(V_i) < \alpha m^2$.* □

We also need the following lemma:

Lemma 2.3 *Let $\gamma > 0$ and let H be a k-partite graph with m vertices and with at least $t_k(m) - \gamma m^2$ edges. If we add to H at least $(2k+1)\gamma m^2$ new edges then the new graph contains a K_{k+1} with exactly one new edge connecting two vertices in the same vertex class of H.*

Proof: Let H' denote the new graph obtained from H by adding at least $(2k+1)\gamma m^2$ new edges. Since H is a k-partite graph, at least $(2k+1)\gamma m^2 - \gamma m^2 = 2k\gamma m^2$ new edges connect vertices in the same vertex class of H. Hence, some vertex class X contains at least $2\gamma m^2$ new edges. Since every graph contains a bipartite spanning subgraph with more than half the number of edges, we have that the induced subgraph of H' on X has a bipartite spanning subgraph with more than γm^2 edges. These edges, denoted F, together with the original edges of H define a subgraph of H' with more than $t_k(m)$ edges, which therefore contains a K_{k+1}. Such a K_{k+1} must contain exactly one edge of F and all other edges are original ones, as required. □

Next, we introduce the directed version of Szemerédi's regularity lemma. The proof, which is a relatively simple modification of the proof of the standard regularity lemma given in [9], can be found in [3]. For more details on the regularity lemma we refer the reader to the excellent survey of Komlós and Simonovits [6], which discusses various applications of this powerful result. We now give the definitions necessary in order to state the directed regularity lemma.

Let $G = (V, E)$ be a directed graph, and let A and B be two disjoint subsets of $V(G)$. If A and B are non-empty, define the *density of edges* from A to B as

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

For $\epsilon > 0$ the pair (A, B) is called *ϵ-regular* if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have

$$|d(X, Y) - d(A, B)| < \epsilon \qquad\qquad |d(Y, X) - d(B, A)| < \epsilon.$$

An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \ldots, V_m$ whose sizes are as equal as possible. An equitable partition of the set of vertices V of a directed graph G into the classes $V_1, \ldots, V_m$ is called *ϵ-regular* if $|V_i| \leq \epsilon|V|$ for every i and all but at most $\epsilon\binom{m}{2}$ of the pairs (V_i, V_j) are ϵ-regular.

The directed regularity lemma states the following:

Lemma 2.4 *For every $\epsilon > 0$, there is an integer $M(\epsilon) > 0$ such that for every directed graph G of order $n > M$ there is an ϵ-regular partition of the vertex set of G into m classes, for some $1/\epsilon \leq m \leq M$.* □

A useful notion associated with an ϵ-regular partition is that of a *cluster graph*. Suppose that G is a directed graph with an ϵ-regular partition $V = V_1 \cup \cdots \cup V_m$, and $\eta > 0$ is some fixed constant (to be thought of as small, but much larger than ϵ). The *undirected* cluster graph $C(\eta)$ is defined on the vertex set $\{1, \ldots, m\}$ by declaring ij to be an edge if (V_i, V_j) is an ϵ-regular pair with $d(V_i, V_j) \geq \eta$ and also $d(V_j, V_i) \geq \eta$. From the definition, one might expect that if a cluster graph contains a copy of K_{k+1} then the original directed graph contains T (assuming ϵ was chosen small enough with respect to η and k). This is indeed the case, as established in the following slightly more general lemma whose proof is similar to an analogous lemma for the undirected case (see [6]).

Lemma 2.5 *Let $\eta > 0$ and suppose that $\epsilon < (\eta/2)^k/k$. Let G be a directed graph with an ϵ-regular partition $V = V_1 \cup \cdots \cup V_m$ and let $C(\eta)$ be the cluster graph of the partition.*

1. *If $C(\eta)$ contains a copy of K_{k+1} then G contains a copy of T.*

2. *If $C(\eta)$ does not have a copy of K_{k+1} and (V_s, V_t) is an ϵ-regular pair with $d(V_s, V_t) \geq \eta$ but $st \notin C(\eta)$, and the addition of st to $C(\eta)$ forms a K_{k+1}, then G contains a copy of T.*

Proof: It clearly suffices to prove the second statement. Without loss of generality assume $s = 1$ and $t = 2$. Label the vertices of T with $\{1, \ldots, k+1\}$ such that $(1, 2) \in T$ (namely, there is an edge directed from 1 to 2). We may assume that the addition of $(1,2)$ to $C(\eta)$ forms a K_{k+1} whose vertices are $1,\ldots,k+1$. We will find a copy of T in G where vertex i of T corresponds to a vertex of G belonging to V_i, for $i=1,\ldots,k+1$.

We prove that for every p, $0 \le p \le k+1$ there are subsets $B_i \subset V_i$, $1 \le i \le k+1$, and a set of vertices $\{a_1,\ldots,a_p\}$ where $a_i \in B_i$ with the following properties.
(i) $|B_i| \ge (\frac{\eta}{2})^{i-1}|V_i|$ for all $1 \le i \le p$ and $|B_i| \ge (\frac{\eta}{2})^p|V_i|$ for all $p < i \le k+1$.
(ii) For all $i=1,\ldots,p$ and for all $i < j \le k+1$, if $(i,j) \in T$ then $(a_i,v) \in G$ for all $v \in B_j$ and if $(j,i) \in T$ then $(v,a_i) \in G$ for all $v \in B_j$.

The assertion of the lemma clearly follows from the above statement for $p = k+1$ since the vertices $\{a_1,\ldots,a_{k+1}\}$ induce T in G.

To prove (i) and (ii) we use induction on p. For $p=0$ simply take $B_i = V_i$ for all i. Given the sets B_i and $\{a_1,\ldots,a_{p-1}\}$ satisfying (i), (ii) for $p-1$ we show how to modify them to hold for p. Observe that by assumption the cardinality of each B_j, for $p < j \le k+1$, is bigger than $(\eta/2)^k|V_j| \ge \epsilon|V_j|$. For each such j if $(p,j) \in T$ ($(j,p) \in T$) let B_p^j denote the set of all vertices in B_p that have outdegree (indegree) less than $(\eta-\epsilon)|B_j|$ into (from) B_j. We claim that $|B_p^j| \le \epsilon|V_p|$ for each j. This is because otherwise the two sets $X = B_p^j$ and $Y = B_j$ would contradict the ϵ-regularity of the pair (V_p,V_j), since $d(B_p^j,B_j) < \eta-\epsilon$, whereas $d(V_p,V_j) \ge \eta$, by assumption. Therefore, the cardinality of the set $B_p \setminus (B_p^{p+1} \cup \ldots \cup B_p^{k+1})$ is at least

$$|B_p| - (k+1-p)\epsilon|V_p| \ge \left(\frac{\eta}{2}\right)^{p-1}|V_p| - k\epsilon|V_p| > 0.$$

We can now choose arbitrarily a vertex a_p in $B_p \setminus (B_p^{p+1} \cup \cdots \cup B_p^{k+1})$ and replace each B_j for $p < j \le k+1$ by the set of outgoing (resp. incoming) neighbors of a_p in B_j. Since $\eta - \epsilon > \eta/2$ this will not decrease the cardinality of each B_j by more than a factor of $\eta/2$ and it is easily seen that the new sets B_i, and the set $\{a_1,\ldots,a_p\}$ defined in this manner satisfy the conditions (i), (ii) for p. □

Proof of Lemma 2.1. Let $\delta > 0$ and let $\alpha < \delta/(4k+7)$. Whenever necessary we shall assume n is sufficiently large as a function of δ and k. Let $\beta = \beta(\alpha,k)$ be chosen as in Theorem 2.2. Recall that $\beta < \alpha$. Let $\eta < \beta$ be a positive constant to be chosen later. Let $\epsilon < (\eta/2)^k/k$ and notice that η and ϵ satisfy the conditions of Lemma 2.5. Let $M = M(\epsilon)$ be as in Lemma 2.4.

Let $G = (V,E)$ be an undirected graph with n vertices and at least $2^{t_k(n)}$ distinct T-free orientations.

Let $\vec{G}$ be a T-free orientation of G. By applying Lemma 2.4 to $\vec{G}$ we get a partition $V = V_1 \cup \cdots \cup V_m$ satisfying the conditions of the lemma. In particular, $1/\epsilon \le m \le M$. Let $C = C(\eta)$ be the corresponding cluster graph on the vertex set $\{1,\ldots,m\}$. By Lemma 2.5, $C(\eta)$ is K_{k+1}-free and thus by Turán's theorem $C(\eta)$ has at most $t_k(m)$ edges.

Our first goal is to show that for some orientation of G the resulting cluster graph has *more than* $t_k(m) - \beta m^2$ edges. Assume this is false. In order to derive a contradiction we first bound the number of orientations of G that could give rise to a particular partition and a particular cluster graph $C = C(\eta)$. We therefore fix the partition (that is, the vertex sets $V_1, \ldots, V_m$ and the non regular pairs) and a cluster graph agreeing with the partition.

Note that by definition, there are at most $m\binom{\lceil n/m \rceil}{2} < \epsilon n^2$ edges of G with both endpoints in the same part of the partition. Hence, there are at most $2^{\epsilon n^2}$ ways to orient such edges. Similarly, there are at most $\epsilon\binom{m}{2} \cdot (\lceil n/m \rceil)^2 < \epsilon n^2$ edges of G that belong to non ϵ-regular pairs. There are at most $2^{\epsilon n^2}$ ways to orient such edges.

Next, consider an ϵ-regular pair (V_i, V_j) such that $ij \notin C(\eta)$. Thus, either $e(V_i, V_j) \leq |V_i||V_j|\eta$ or else $e(V_j, V_i) \leq |V_i||V_j|\eta$. In either case, if $e(V_i V_j)$ is the number of undirected edges of G between V_i and V_j then there are at most

$$2\left(\sum_{q=0}^{\lfloor |V_i||V_j|\eta \rfloor} \binom{e(V_i V_j)}{q}\right) < 2\frac{n^2}{m^2}\eta \cdot 2^{H(\eta)n^2/m^2} \ll 2^{H(2\eta)n^2/m^2}$$

orientations of the edges of G belonging to this pair. Here we use the well known estimate $\binom{a}{xa} \leq 2^{H(x)a}$ for $0 < x < 1$, where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the entropy function.

Finally, consider a pair corresponding to an edge of $C(\eta)$. Trivially there are at most $2^{(\lceil n/m \rceil)^2}$ possible orientations of the edges belonging to this pair.

Altogether, the total number of orientations of G giving rise to a fixed partition and a fixed cluster graph with $r \leq t_k(m) - \beta m^2$ edges is at most

$$2^{\epsilon n^2} \cdot 2^{\epsilon n^2} \cdot 2^{H(2\eta)(n^2/m^2)\binom{m}{2}} \cdot 2^{(\lceil n/m \rceil)^2 r} < 2^{2\epsilon n^2} 2^{H(2\eta)n^2} 2^{(n^2/m^2)(t_k(m)-\beta m^2)} 2^{nm} <$$

$$2^{2\epsilon n^2} 2^{H(2\eta)n^2} 2^{(t_k(n)-\beta n^2)} 2^{nM} 2^k$$

where the last inequality follows from the well known fact that for every x,

$$\frac{k-1}{k}\frac{x^2}{2} - k < t_k(x) \leq \frac{k-1}{k}\frac{x^2}{2}.$$

Note that M is a constant and there are at most M^n partitions of the vertex set of G into at most M parts. Also, for every such partition there are at most $2^{M^2/2}$ choices for the cluster graph $C(\eta)$ and (significantly) less than $2^{M^2/2}$ choices for the non-regular pairs.

Thus, the total number of T-free orientations of G is at most

$$M^n 2^{M^2} 2^{2\epsilon n^2} 2^{H(2\eta)n^2} 2^{nM} 2^k 2^{-\beta n^2} 2^{t_k(n)}.$$

Since $\epsilon < \eta$ and since $H(2\eta)$ tends to zero with η we have that for η sufficiently small as a function of β, the number of T-free orientations of G is less than $2^{t_k(n)}$, a contradiction.

Fix an orientation $\vec{G}$ of G for which $C = C(\eta)$ has at least $t_k(m) - \beta m^2$ edges. Let $V_1, \ldots, V_m$ denote the parts in the ϵ-regular partition. According to Theorem 2.2, C has a vertex partition

$W = W_1 \cup \cdots \cup W_k$ with $\sum_i e(W_i) < \alpha m^2$. Thus, let C^* be the spanning subgraph of C from which the edges with both endpoints in W_i have been removed, for $i = 1, \ldots, k$. Notice that C^* is a k-partite graph with at least $t_k(m) - (\beta + \alpha)m^2 = t_k(m) - \gamma m^2$ edges where $\gamma = \alpha + \beta$. We call a pair (V_i, V_j) a *one-sided dense pair* if it is an ϵ-regular pair and ij is not an edge of C but either $d(V_i, V_j) > \eta$ or $d(V_j, V_i) > \eta$. We claim that there are at most $(2k+1)\gamma m^2$ one-sided dense pairs. Assume this is false, adding to C^* the edges corresponding to one-sided dense pairs we get, by Lemma 2.3, that there are $k+1$ vertices of C (w.l.o.g. assume they are $\{1, \ldots, k+1\}$) such that $ij \in C$ for all $1 \le i < j \le k+1$ except for the edge 1 2 which is not in C but corresponds to the one-sided dense pair (V_1, V_2) where $d(V_1, V_2) > \eta$. By Lemma 2.5, $\vec{G}$ has T, yielding the contradiction.

We now delete from G the following edges:

1. The edges with both endpoints in V_i for $i = 1, \ldots, m$. We have shown that there are at most ϵn^2 such edges.

2. The edges belonging to non ϵ-regular pairs. We have shown that there are at most ϵn^2 such edges.

3. The edges belonging to non-dense pairs or one-sided dense pairs. There are at most $(2\eta + (2k+1)\gamma)n^2$ such edges.

4. The edges belonging to pairs (V_i, V_j) such that $ij \in W_s$ for $s = 1, \ldots, k$. Since there are at most αm^2 such pairs, there are at most αn^2 such edges.

In other words, we keep only edges belonging to pairs (V_i, V_j) such that $ij \in C^*$. Denote this subgraph of G by G'. Then, G' is k-partite and, recalling that $\epsilon < \eta < \beta < \alpha$ and $\gamma = \alpha + \beta$, the number of edges deleted from G is at most

$$(\alpha + 2\eta + (2k+1)\gamma + 2\epsilon)n^2 < (4\eta + (4k+3)\alpha)n^2 \le (4k+7)\alpha n^2 < \delta n^2.$$

This concludes the proof of Lemma 2.1. $\square$

3 Proof of Theorem 1.1

In this section we complete the proof of our main theorem. The proof follows along the lines of [2] with several essential modifications required to deal with directed graphs. We start by recalling some notation and facts. $T_k(n)$ denotes the Turán graph, which is a complete k-partite graph on n vertices with class sizes as equal as possible, and, as denoted earlier, $t_k(n)$ is the number of edges in $T_k(n)$. Let $\delta_k(n)$ denote the minimum degree of $T_k(n)$. The following equalities are well known simple observations.

$$t_k(n) = t_k(n-1) + \delta_k(n), \qquad \delta_k(n) = n - \lceil n/k \rceil, \qquad \frac{k-1}{k}n^2/2 - k < t_k(n) \le \frac{k-1}{k}n^2/2. \quad (2)$$

We also need one additional easy lemma, before we present the proof of Theorem 1.1.

Lemma 3.1 *Let S be a tournament with the vertices $\{1, \dots, k\}$. Let G be a directed graph and let $W_1, \dots, W_k$ be subsets of vertices of G such that for every $i \neq j$ and every pair of subsets $X_i \subseteq W_i, |X_i| \geq 10^{-k}|W_i|$ and $X_j \subseteq W_j, |X_j| \geq 10^{-k}|W_j|$ there are at least $\frac{1}{10}|X_i||X_j|$ edges of G from X_i to X_j if $(i,j) \in S$ or at least $\frac{1}{10}|X_i||X_j|$ edges of G from X_j to X_i if $(j,i) \in S$. Then G contains a copy of S where the vertex playing the role of $i \in S$ belongs to W_i.*

Proof. We use induction on k. For $k = 1$ and $k = 2$ the statement is obviously true. Suppose it is true for $k-1$ and let $W_1, \dots, W_k$ be the subsets of vertices of G which satisfy the conditions of the lemma for the fixed tournament S.

For every $1 \leq i \leq k-1$ denote by W_k^i the subset of vertices in W_k defined as follows. If $(i,k) \in S$ then $v \in W_k^i$ if v has less than $|W_i|/10$ incoming edges from W_i. If $(k,i) \in S$ then $v \in W_k^i$ if v has less than $|W_i|/10$ outgoing edges to W_i. By definition, if $(i,k) \in S$, we have $e(W_i, W_k^i) < |W_k^i||W_i|/10$ and if $(k,i) \in S$, we have $e(W_k^i, W_i) < |W_k^i||W_i|/10$ and therefore, in any case, $|W_k^i| < 10^{-k}|W_k|$. Thus we deduce that $\left|\bigcup_{i=1}^{k-1} W_k^i\right| < (k-1)10^{-k}|W_k| < |W_k|/2$. So in particular there exists a vertex v in W_k which does not belong to $\bigcup_{i=1}^{k-1} W_k^i$. For every $1 \leq i \leq k-1$ if $(i,k) \in S$ let W_i' be the set of incoming neighbors of v in W_i, and if $(k,i) \in S$ let W_i' be the set of outgoing neighbors of v in W_i. By definition, W_i' has size at least $|W_i|/10$. Note that for every pair of subsets $X_i \subseteq W_i'$ and $X_j \subseteq W_j'$ with sizes $|X_i| \geq 10^{-(k-1)}|W_i'| \geq 10^{-k}|W_i|$ and $|X_j| \geq 10^{-(k-1)}|W_j'| \geq 10^{-k}|W_j|$, G contains at least $\frac{1}{10}|X_i||X_j|$ edges between X_i and X_j in the appropriate direction. By the induction hypothesis there exists a copy of $S - k$ with one vertex in each W_i', playing the role of $i \in S$ in this copy, for $1 \leq i \leq k-1$. This copy, induced together with the vertex v, forms a copy of S where v plays the role of k. □

Proof of Theorem 1.1. Let n_0 be large enough to guarantee that the assertion of Lemma 2.1 holds for $\delta = 10^{-8k}$. Suppose that G is a graph on $n > n_0^2$ vertices with at least $2^{t_k(n)+m}$ distinct T-free orientations, for some $m \geq 0$. Our argument is by induction with an improvement at every step. More precisely, we will show that if G is not the corresponding Turán graph then it contains a vertex x such that $G - x$ has at least $2^{t_k(n-1)+m+1}$ distinct T-free orientations. Iterating, we obtain a graph on n_0 vertices with at least $2^{t_k(n_0)+m+n-n_0} > 2^{n_0^2}$ distinct T-free orientations. But a graph on n_0 vertices has at most $n_0^2/2$ edges and hence at most $2^{n_0^2/2}$ orientations. This contradiction will prove the theorem for $n > n_0^2$.

Recall from (2) that $\delta_k(n)$ denotes the minimum degree of $T_k(n)$, and $t_k(n) = t_k(n-1) + \delta_k(n)$. If G contains a vertex x of degree less than $\delta_k(n)$, then the edges incident with x can have, together, at most $2^{\delta_k(n)-1}$ orientations. Thus $G - x$ should have at least $2^{t_k(n-1)+m+1}$ distinct T-free orientations and we are done. Hence we may and will assume that all the vertices of G have degree at least $\delta_k(n)$.

Consider a partition $V_1 \cup \cdots \cup V_k$ of the vertex set of G which minimizes $\sum_i e(V_i)$. By our choice of n_0 in Lemma 2.1, we have that $\sum_i e(V_i) < 10^{-8k}n^2$. Note that if $|V_i| > \big(1/k + 10^{-6k}\big)n$, for some i, then every vertex in V_i has at least $\delta_k(n) - \big(\frac{k-1}{k}n - 10^{-6k}n\big) \geq 10^{-6k}n - 1$ neighbors in V_i. Thus $\sum_i e(V_i) > (10^{-6k}n - 1)(1/k + 10^{-6k})n/2 > 10^{-8k}n^2$, a contradiction. Therefore, $|V_i| - n/k \leq 10^{-6k}n$ for every i and also $|V_i| = n - \sum_{j \neq i} |V_j| \geq n/k - (k-1)10^{-6k}n$. So for every i we have $\big||V_i| - n/k\big| < 10^{-5k}n$. Let $\mathcal{D}$ denote the set of all possible T-free orientations of G.

First consider the case when there is some vertex with many neighbors in its own class of the partition, say $x \in V_1$ with $|N(x) \cap V_1| > n/(400k)$. Our choice of partition guarantees that in this case $|N(x) \cap V_i| > n/(400k)$ also for all $2 \leq i \leq k$, or by moving x to another part we could reduce $\sum_i e(V_i)$. Consider a permutation σ of $\{1, \ldots, k+1\}$. Let $\mathcal{D}_\sigma \subset \mathcal{D}$ be a subset of orientations defined as follows: An orientation belongs to $\mathcal{D}_\sigma$ if for all $i = 1, \ldots, k$ there exist $W_i \subset V_i$ with $|W_i| \geq n/(900k)$ such that if $(\sigma(i), \sigma(k+1)) \in T$ then x has an incoming edge from each $v \in W_i$ and if $(\sigma(k+1), \sigma(i)) \in T$ then x has an outgoing edge to each $v \in W_i$. Let $\mathcal{D}^* = \mathcal{D} \setminus \big(\cup_{\sigma \in S(k+1)} \mathcal{D}_\sigma\big)$.

Consider an orientation of G belonging to $\mathcal{D}_\sigma$. Since the orientation is T-free we have by Lemma 3.1 that there is some ordered pair (i,j) (corresponding to $(\sigma(i), \sigma(j)) \in T$) and subsets $X_i \subset W_i$, $X_j \subset W_j$ with $|X_i| \geq 10^{-k}|W_i|$ and $|X_j| \geq 10^{-k}|W_j|$ with at most $\frac{1}{10}|X_i||X_j|$ edges from X_i to X_j. There are at most $\binom{k}{2}2^{|V_i|}2^{|V_j|} < 2^{2n}$ ways to choose such an ordered pair (i,j) and to choose X_i and X_j and at most

$$\frac{1}{10}|X_i||X_j|\binom{|X_i||X_j|}{\lfloor |X_i||X_j|/10 \rfloor} < \frac{1}{10}|X_i||X_j|2^{H(0.1)|X_i||X_j|} < 2^{H(0.11)|X_i||X_j|}$$

ways to orient at most $\frac{1}{10}|X_i||X_j|$ edges from X_i to X_j. In addition, from the structure of G we know that there are at most $t_k(n) + 10^{-8k}n^2 - |X_i||X_j|$ other edges in this graph, so the number of orientations in $\mathcal{D}_\sigma$ can be bounded as follows

$$\begin{aligned}
|\mathcal{D}_\sigma| &\leq 2^{t_k(n)+10^{-8k}n^2-|X_i||X_j|}\, 2^{2n} 2^{H(0.11)|X_i||X_j|} \\
&\leq 2^{t_k(n)+10^{-8k}n^2}\, 2^{2n}\, (\sqrt{2}/2)^{|X_i||X_j|} \leq 2^{t_k(n)+10^{-8k}n^2}\, 2^{2n}\, (\sqrt{2}/2)^{10^{-2k-6}k^{-2}n^2} \\
&< 2^{t_k(n)+10^{-8k}n^2}\, 2^{2n} \left(2^{-0.01}\right)^{10^{-2k-6}k^{-2}n^2} = 2^{t_k(n)}\, 2^{2n}\, 2^{-(10^{-2k-8}k^{-2}-10^{-8k})n^2} \\
&\ll \frac{2^{t_k(n)}}{2(k+1)!}.
\end{aligned}$$

In this estimate we used the facts that $H(0.11) < 1/2$, $|X_i|, |X_j| \geq n/(k10^{k+3})$, $\sqrt{2}/2 < 2^{-0.01}$ and that $10^{-2k-8}k^{-2} - 10^{-8k} > 0$ for all $k \geq 2$.

By the above discussion, $|\mathcal{D}^*|$ contains at least $2^{t_k(n)+m} - 2^{t_k(n)}/2 \geq 2^{t_k(n)+m-1}$ distinct T-free orientations of G. Let $\vec{G}$ be one of them. Since $\vec{G} \notin \mathcal{D}_\sigma$ for no $\sigma \in S(k+1)$ we must have some i such that there are at most $n/(900k)$ edges from x to V_i or at most $n/(900k)$ edges from V_i to x. Assume w.l.o.g. that there are at most $n/(900k)$ edges from x to V_i. Thus, there are at least

$n/(400k) - n/(900k) > n/(900k)$ edges from V_i to x. Let $\sigma \in S(k+1)$ be a permutation for which $(\sigma(i), \sigma(k+1)) \in T$. Since $\vec{G} \notin \mathcal{D}_\sigma$ we must have some $j \neq i$ for which there are at most $n/(900k)$ edges from x to V_j or at most $n/(900k)$ edges from V_j to x.

We have shown that for every element of $\mathcal{D}^*$ there are (at least) two distinct indices i, j such that there are at most $n/(900k)$ edges connecting x to V_i in at least one of the two possible directions and the same hold for V_j (although not necessarily in the same direction). We call the direction with less than $n/(900k)$ edges the *sparse* direction.

Since the size of V_i is at most $(1/k + 10^{-5k})n$, we obtain that the number of orientations of edges between x and V_i, given the sparse direction, is bounded by

$$\frac{n}{900k}\binom{\lfloor (1/k + 10^{-5k})n \rfloor}{\lfloor n/(900k) \rfloor} \leq 2^{H(0.002)(1/k+10^{-5k})n} \leq 2^{0.03(1/k+10^{-5k})n}, \tag{3}$$

since $H(0.002) < 0.03$. Clearly, this estimate is also valid for the number of orientations of edges between x and V_j, given the sparse direction between them. Note that in addition x is incident to at most $n - |V_i| - |V_j| \leq (\frac{k-2}{k} + 2 \cdot 10^{-5k})n$ other edges, which can have two possible directions. Using the above inequalities together with the facts that there are $\binom{k}{2}$ possible pairs i, j and four possible choices for the sparse directions between x and V_i and between x and V_j we obtain that the number of orientations of the edges incident with x is at most

$$4\binom{k}{2}\left(2^{0.03(1/k+10^{-5k})n}\right)^2 2^{\left(\frac{k-2}{k}+2\cdot 10^{-5k}\right)n} < 2^{\left(\frac{k-1}{k}-\frac{1}{100k}\right)n}.$$

But we had that $|\mathcal{D}^*| \geq 2^{t_k(n)+m-1}$. Hence the number of T-free orientations of $G - x$ is at least

$$2^{t_k(n)+m-1-(\frac{k-1}{k}-\frac{1}{100k})n} \gg 2^{t_k(n-1)+m+1}.$$

This completes the induction step in the first case.

Now we may assume that every vertex has degree at most $n/(400k)$ in its own class. We may suppose that G is not k-partite, or else by Turán's theorem $e(G) \leq t_k(n)$ and therefore $|\mathcal{D}| \leq 2^{t_k(n)}$ with equality only for $G = T_k(n)$. So, without loss of generality, we suppose that G contains an edge xy with $x, y \in V_k$. For $\sigma \in S(k+1)$, let $\mathcal{D}_\sigma$ denote the set of all T-free orientations $\vec{G}$ of G in which $(x, y) \in \vec{G}$ if and only if $(\sigma(k), \sigma(k+1)) \in T$ and there are sets $W_i \subset V_i, |W_i| \geq n/(900k)$ for every $1 \leq i \leq k-1$ such that all the edges from x to W_i exist and are oriented from x to W_i if $(\sigma(k), \sigma(i)) \in T$ or oriented from W_i to x if $(\sigma(i), \sigma(k)) \in T$, and also all the edges from y to W_i exist and are oriented from y to W_i if $(\sigma(k+1), \sigma(i)) \in T$ or oriented from W_i to y if $(\sigma(i), \sigma(k+1)) \in T$. Let $\mathcal{D}^* = \mathcal{D} \setminus \left(\cup_{\sigma \in S(k+1)} \mathcal{D}_\sigma\right)$ denote the remaining orientations.

Consider an orientation $\vec{G} \in \mathcal{D}_\sigma$. Let T_σ denote the sub-tournament of T obtained by deleting the vertices $\sigma(k)$ and $\sigma(k+1)$. Since there is no T in $\vec{G}$, there is also no copy of T_σ in which the role of vertex $\sigma(i)$ is played by a vertex from W_i for $i = 1, \ldots, k-1$. Thus, by Lemma 3.1, there is a pair (i, j) and subsets $X_i \subset W_i$, $X_j \subset W_j$ with $|X_i| \geq 10^{-(k-1)}|W_i|$ and $|X_j| \geq 10^{-(k-1)}|W_j|$ with at most $\frac{1}{10}|X_i||X_j|$ edges from X_i to X_j if $(\sigma(i),\sigma(j)) \in T$ or at most $\frac{1}{10}|X_i||X_j|$ edges from X_j to X_i if $(\sigma(j),\sigma(i)) \in T$. Arguing exactly as before in the first case we can prove that $|\mathcal{D}_\sigma| < \frac{2^{t_k(n)}}{2(k+1)!}$ and thus $|\mathcal{D}^*| \geq 2^{t_k(n)+m-1}$.

Next consider an orientation $\vec{G}$ of G from $\mathcal{D}^*$ and suppose, without loss of generality, that $(x,y) \in \vec{G}$. Let $\sigma \in S(k+1)$ be such that $(\sigma(k),\sigma(k+1)) \in T$. Since $\vec{G} \notin \mathcal{D}_\sigma$ there is some class V_i, $i \leq 1 \leq k-1$, in which x and y have at most $n/(900k)$ "common neighbors" in the sense that x has an outgoing edge to all these common neighbors in case $(\sigma(k),\sigma(i)) \in T$ or else x has an incoming edge from all these common neighbors in case $(\sigma(i),\sigma(k)) \in T$ and also y has an outgoing edge to all these common neighbors in case $(\sigma(k+1),\sigma(i)) \in T$ or else y has an incoming edge from all these common neighbors in case $(\sigma(i),\sigma(k+1)) \in T$. Note that for any other vertex z in V_i which is not such a common neighbor, we can only have at most three possible simultaneous orientations of the two edges xz and yz of G (assuming they exist). Since there are at most $(1/k + 10^{-5k})n$ vertices in V_i we have at most $3^{(1/k+10^{-5k})n}$ ways to orient such edges and, as in (3), at most

$$\frac{n}{900k}\binom{\lfloor (1/k + 10^{-5k})n \rfloor}{\lfloor n/(900k) \rfloor} \leq 2^{H(0.002)(1/k+10^{-5k})n} \leq 2^{0.03(1/k+10^{-5k})n}$$

possibilities to choose a set of common neighbors of x and y in V_i. Thus, there are at most

$$2^{0.03\left(1/k+10^{-5k}\right)n}\, 3^{\left(1/k+10^{-5k}\right)n} < 2^{1.7\left(1/k+10^{-5k}\right)n}$$

ways to orient edges from x,y to V_i. Note that, since the degree of x and y in V_k is at most $n/(400k)$ we have that the number of edges from x, y to $\bigcup_{j \neq i} V_j$ is bounded by $n(2(\frac{k-2}{k}+2\cdot 10^{-5k})+2/(400k))$. Even if all these edges can be oriented arbitrarily, since we have $k-1$ choices for the index i, and four possible combinations for the direction between x and y to their common neighbors in V_i, we can bound the number of orientations of the edges incident at x and y by

$$4(k-1)\, 2^{1.7\left(1/k+10^{-5k}\right)n}\, 2^{2\left(\frac{k-2}{k}+\frac{1}{400k}+2\cdot 10^{-5k}\right)n} < 2^{2\left(\frac{k-1}{k}-\frac{1}{100k}\right)n}.$$

But we know that $|\mathcal{D}^*| \geq 2^{t_k(n)+m-1}$. Thus the number of T-free orientations of $G - \{x,y\}$ is at least

$$2^{t_k(n)+m-1-2(\frac{k-1}{k}-\frac{1}{100k})n} \gg 2^{t_k(n-2)+m+2}.$$

This completes two induction steps for the second case and proves the theorem. □

4 Directed triangles and transitive tournaments

In this section we consider two special cases of Theorem 1.1. We first show an easy proof of Theorem 1.1 in case $T = T_k$ is the transitive tournament with k vertices. Next, we consider the smallest non-transitive tournament, namely $T = C_3$ and outline a proof for C_3 that avoids using the regularity lemma. Indeed, the proof for C_3 is more complicated than the proof for T_3. The proof for T_3 follows rather easily from a result of the second author in [10] concerning the number of red-blue edge colorings of a graph that avoid monochromatic triangles and the result for T_k ($k > 3$) follows from a recent result of [2] that generalizes the result of [10] to larger cliques. The proof for C_3 does not follow from these coloring results and requires an ad-hoc proof (although some arguments are similar to those appearing in the proof of [10]). To see the difficulty consider the following argument. Let $F(G)$ denote the number of red-blue edge colorings of G with no monochromatic triangle and let $D(G)$ denote the number of orientations of G with no C_3. Since the Ramsey number $R(3) = 6$, we have $F(G) = 0$ whenever G has a K_6. In particular, $F(K_n) = 0$ for $n \geq 6$. On the other hand, $D(K_n) = n!$, and $D(G) > 0$ always. Thus, it is more difficult to show that dense graphs have a relatively small $D(G)$ than it is to show that dense graphs have a relatively small $F(G)$. In fact, our proof for $T = C_3$ uses some powerful decomposition results that are not needed in the coloring case.

4.1 Orientations with no transitive tournaments

Let $F(G, k)$ denote the number of red-blue edge colorings of a graph G that have no monochromatic K_k. Let $F(n, k)$ denote the maximum possible value of $F(G, k)$ where G has n vertices. The following result is proved in [10] for $k = 3$ and in [2] for all $k > 3$ (the result in [2] also considers colorings with more than two colors).

Lemma 4.1 *Let $k \geq 3$. There exists $n_0 = n_0(k)$ such that for all $n \geq n_0$, $F(n, k) = 2^{t_{k-1}(n)}$.* □

In fact, in [10] it is shown that $n_0(3) = 6$ (and this is tight) while the $n_0(k)$ obtained in [2] is a huge number already for $k = 4$, as their proof uses the regularity lemma.
Lemma 4.1 and (1) enable us to prove the following:

Proposition 4.2 *Let $k \geq 3$. Then, $F(n, k) \geq D(n, T_k)$. Consequently, $D(n, T_k) = 2^{t_{k-1}(n)}$ for all $n \geq n_0(k)$ where $n_0(k)$ is the constant appearing in Lemma 4.1.*

Proof: Consider a graph G on n vertices. Label its vertices with the numbers $1, \ldots, n$. There is a bijection between red-blue edge colorings of G and orientations of G as follows: An edge is colored blue if and only if in the associated orientation the edge is oriented from the smaller vertex to the larger. Now assume that G has an orientation with no T_k. We show that the associated coloring has no monochromatic K_k. Consider a K_k of G. It must contain a directed cycle in the orientation. In the associated coloring, we cannot have all the edges of such a cycle colored with the same color. We have shown that $F(G, k) \geq D(G, T_k)$. Hence, $F(n, k) \geq D(n, T_k)$. □.

Notice that although $n_0(3) = 6$ (in fact, $F(5, 3) = 82 > 2^6$) it is easy to check that $D(n, T_3) = 2^{\lfloor n^2/4 \rfloor}$ for all $n \geq 1$ (one needs to check only $n = 1, \ldots, 5$). For $k = 4$, however, we have $D(4, T_4) = 2^6 - 4! = 40 > 2^{t_3(4)} = 32$.

4.2 Orientations with no directed triangles

Theorem 4.3 *For all $n \geq 600000$, $D(n, C_3) = 2^{\lfloor n^2/4 \rfloor}$.*

Outline of proof: Let H be a graph, and let $H + x$ denote the graph obtained from H by adding a new vertex x and connecting it to all vertices of H. For a C_3-free orientation $\vec{H}$ of H, let $ext(\vec{H})$ denote the number of C_3-free orientations of $H + x$ that are extensions of $\vec{H}$. Let $ext(H)$ denote the maximum possible value of $ext(\vec{H})$ taken over all C_3-free orientations of H. It is not difficult to see that $ext(K_k) = k + 1$ and $ext(S_k) = 2^{k-1} + 1$, where S_k is the star with k vertices. Slightly more complicated arguments show that for all $k \geq 3$, $ext(P_k) = ext(P_{k-1}) + ext(P_{k-2})$ where P_k is the path with k vertices. In particular, for all $k \geq 1$, $ext(P_k) = z_k$ where z_k is the $k + 2$ element of the Fibonacci sequence. If $H_1, \ldots, H_k$ are the components of a spanning subgraph of H then $ext(H) \leq \prod_{i=1}^{k} ext(H_i)$.

Theorem 4.3 follows from the following lemma by a simple inductive argument.

Lemma 4.4 *If $n \geq 320$, and G is a graph with n vertices, then at least one of the following must hold:*

1. $D(G) \leq 2^{\lfloor n^2/4 \rfloor}$.
2. *There exists a vertex x of minimum degree such that if H is the subgraph of G induced by the neighbors of x then $ext(H) \leq 0.94 \cdot 2^{\lfloor n/2 \rfloor}$. Thus, $D(G) \leq 0.94 \cdot 2^{\lfloor n/2 \rfloor} D(G - x)$.*
3. *$\delta(G) = \lfloor n/2 \rfloor$, there exist two vertices x and y such that $D(G) \leq 2^{\lfloor n/2 \rfloor} D(G - x)$ and $D(G - x) \leq 0.94 \cdot 2^{\lfloor (n-1)/2 \rfloor} D(G - \{x, y\})$.*

The proof of Lemma 4.4 is quite involved and consists of a detailed case analysis which corresponds to the structure of G. We use several lemmas that provide upper bounds for $ext(H)$ for various types of graphs H. The details are available upon request from the authors.

5 Concluding remarks and open problems

- Another interesting problem is to determine $D(n, m, T)$, that is, the maximum possible number of T-free orientations of a graph with n vertices and m edges. By (1) we trivially have $D(n, m, T) = 2^m$ whenever $m \leq t_{k-1}(n)$, where k is the number of vertices of T. The problem becomes considerably more difficult for $m > t_{k-1}(n)$. Even for $T = C_3$ the exact values for all (n, m) pairs are unknown. Using the fact that every non-transitive tournament contains a triangle we trivially have $D(n, \binom{n}{2}, C_3) = n!$. It is also not difficult to prove the following proposition

  Proposition 5.1

1. $D(n, \binom{n}{2} - 1, C_3) = (n-1)!(n-1)$ *for* $n \geq 2$.
2. $D(n, \binom{n}{2} - 2, C_3) = n! - 2(n-1)! + (n-2)! + 2(n-3)!$ *for* $n \geq 4$. □

- It is of some interest to determine $D(n, C_3)$ for all n. Using a computer program we have $D(n, C_3) = n!$ for $n = 1, \ldots, 7$. The same program yields $D(8, C_3) = 2^{16}$. The case $n = 9$ is too large for a straightforward computer verification. We conjecture that the following holds for all $n \geq 1$
$$D(n, C_3) = \max\{2^{\lfloor n^2/4 \rfloor}, n!\}.$$
In particular, it is conjectured that $n_0(C_3) = 8$ in the statement of Theorem 1.1.

- It would be interesting to generalize Theorem 1.1 to the situation of finding the number of H-free orientations, where H is any directed graph, not necessarily a tournament. In fact, it is not difficult to generalize Lemma 2.1 to apply also for $H = T(t)$, where t is any positive integer and $T(t)$ is the directed graph obtained from the k-vertex tournament T by replacing each vertex with an independent set of size t. In particular, this shows that an asymptotic version of Theorem 1.1 holds for $T(t)$.

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