DISJOINT COLOR-AVOIDING TRIANGLES

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Abstract. A set of pairwise edge-disjoint triangles of an edge-colored K_n is *r-color avoiding* if it does not contain r monochromatic triangles, each having a different color. Let $f_r(n)$ be the maximum integer so that in every edge coloring of K_n with r colors, there is a set of $f_r(n)$ pairwise edge-disjoint triangles that is r-color avoiding. We prove that $0.1177n^2(1-o(1)) < f_2(n) < 0.1424n^2(1+o(1))$. The proof of the lower bound uses probabilistic arguments, fractional relaxation and some packing theorems. We also prove that $f_r(n)/n^2 < \frac{1}{6}(1-0.145^{r-1}) + o(1)$. In particular, for every r, if n is sufficiently large, there are edge colorings of K_n with r colors so that the removal of any $o(n^2)$ members from any Steiner triple system does not turn it r-color avoiding.

Key words. edge coloring, packing, triangles

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1. Introduction. All graphs considered here are finite, undirected, and simple. For standard graph-theoretic terminology the reader is referred to [2]. The study of properties of edge colorings of K_n is a central topic of research in Ramsey Theory and Extremal Graph Theory. In this paper a coloring always refers to an *edge* coloring.

A subgraph of a colored K_n is *monochromatic* if all of its edges are colored with the same color. A set of pairwise edge-disjoint subgraphs of a colored K_n is *r-color avoiding* if it does not contain r monochromatic elements, each having a different color. For an r-coloring C of K_n, and for an integer $k \ge 3$, let $f_{r,k}(C)$ be the maximum size of a set of pairwise edge-disjoint copies of K_k in K_n that is r-color avoiding. Let $f_{r,k}(n)$ be the minimum possible value of $f_{r,k}(C)$ where C ranges over all r-colorings of K_n. When $k = 3$, we denote $f_{r,k}(C) = f_r(C)$ and $f_{r,k}(n) = f_r(n)$. Thus, the value $f_2(n)$ guarantees that in *any* red-blue coloring of K_n we will always have a set of $f_2(n)$ edge-disjoint triangles that either does not contain a blue triangle or else does not contain a red one. The main result of this paper establishes nontrivial lower and upper bounds for $f_2(n)$.

THEOREM 1.1.

$$0.1177 - o(1) < \frac{f_2(n)}{n^2} < \frac{3\sqrt{5}-5}{12} + o(1).$$

Notice that $(3\sqrt{5}-5)/12 < 0.1424$. The term $o(1)$ denotes a quantity that tends to 0 as $n \to \infty$. The constant 0.1177 in the lower bound in Theorem 1.1 may be taken to be $(3\beta^2 - \beta^4)/12$, where $\beta = 0.7648...$ is the smallest root of $x^4 - 3x^3 + 1$. Multiplying the constants by 600, we obtain that, in terms of covering percentages, we can always cover more than 70% of the edges with a set of triangles that is 2-color avoiding, while we cannot, in general, expect to cover more than 86% of the edges with such a set. The main difficulty in the proof of Theorem 1.1 is in the lower bound. Our proof for it requires the use of some probabilistic arguments, some known packing theorems, the use of fractional relaxation and a connection between it and the integral problem. Closing the gap between the upper and lower bounds in Theorem 1.1 is currently beyond our reach.

The upper bound follows from a general construction. Notice that a $(\frac{1}{6} - o(1))n^2$ upper bound for $f_r(n)$ is trivial since every set of pairwise edge-disjoint triangles (we

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also use the expression *triangle packing*) has at most $n(n-1)/6$ elements. In fact, it is well-known that $f_r(n) = (\frac{1}{6} - o(1))n^2$ if r is sufficiently large as a function of n, as Kirkman [9] proved that there are triangle packings with $\frac{n^2}{6}(1-o(1))$ triangles. Our construction, however, shows that no finite number of colors suffices to guarantee an asymptotic optimal r-color avoiding triangle packing for *all* n.

THEOREM 1.2. *For all* $r \geq 2$, $\frac{f_r(n)}{n^2} < \frac{1}{6}(1-\zeta^{r-1}) + o(1)$ *where* $\zeta = \frac{7-3\sqrt{5}}{2} > 0.145$.

We briefly mention three related parameters that have been investigated by several researchers. Erdős et al. [4] considered the function $N(n,k)$, which is the minimum number of pairwise edge-disjoint monochromatic K_k in any 2-coloring of K_n. Erdős conjectured that $N(n,3) = n^2/12 + o(n^2)$. This conjecture is still open. A lower bound of slightly more than $n^2/13$ is given in [8]. Similarly, let $N'(n,k)$ be the minimum number of pairwise edge-disjoint monochromatic K_k, all in the same color, in any 2-coloring of K_n. Jacobson (see, e.g., [4]) conjectured that $N'(n,3) = n^2/20 + o(n^2)$ (there is a simple example showing this would be best possible). Again, the result from [8] immediately implies a lower bound of slightly more than $n^2/26$. For a fixed graph H and a 2-coloring C of K_n, let $f_H(C)$ be the number of edges that do not belong to monochromatic copies of H. Now let $f(n,H) = \max_C f_H(C)$. It is shown in [7] that if H is a complete graph (or, in fact, any edge-color-critical graph) and n is sufficiently large, then $f(n,H)$ equals the Turán number $ex(n,H)$.

The rest of this paper is organized as follows. The proof of the lower bound in Theorem 1.1 is given in Section 2. The proof of the general upper bound yielding Theorem 1.2 is given in Section 3. Notice that the case $r = 2$ of Theorem 1.2 coincides with the upper bound in Theorem 1.1. In Section 4 we give some non-trivial proofs of the exact value of $f_2(n)$ for $n \leq 8$. The final section contains some concluding remarks and open problems.

2. A lower bound for $f_2(n)$. The proof of the lower bound in Theorem 1.1 is obtained by combining two different approaches; one approach (which we call the quadratic approach) is more suitable for colorings where no color is significantly more frequent than the other, and the second approach (the fractional approach) is more suitable when one color is significantly more frequent than the other.

For an integer $k \geq 3$, a Steiner system $S(2,k,n)$ is a set X of n points, and a collection of subsets of X of size k (called blocks), such that any 2 points of X are in exactly one of the blocks. In the case $k = 3$, we have a Steiner triple system, which exists if and only if $n \equiv 1,3 \bmod 6$. The case $k = 4$ is known to exist if and only if $n \equiv 1,4 \bmod 12$; see, e.g., [1].

In the proof of the lower bound for Theorem 1.1 we assume that C is a red-blue coloring of K_n with $\alpha\binom{n}{2}$ blue edges and $(1-\alpha)\binom{n}{2}$ red edges, and $1/2 \leq \alpha \leq 1$. We will also assume that $n \equiv 1 \bmod 12$ as this does not affect the asymptotic results. Each approach will yield a lower bound for $f_2(n)$ in terms of n and α. For each plausible α, one of these lower bounds will be at least as large as the claimed lower bound in Theorem 1.1.

2.1. The quadratic approach. For a red-blue coloring C of K_n, let $t(C)$ be the number of monochromatic triangles. Let $t(n,m)$ be the maximum value of $t(C)$ ranging over colorings with m blue edges. Clearly, $t(n,m) = t(n, \binom{n}{2} - m) = \Theta(n^3)$. Goodman [5] conjectured the value of $t(n,m)$. This conjecture has been proved by Olpp [10], who determined $t(n,m)$, and also determined at least one coloring with m blue edges having $t(n,m)$ monochromatic triangles.

Before we state Olpp's result we need to define two graphs. Let u and v be two integers which satisfy $m = \binom{v}{2} + u$ where $0 \le u \le v-1$. Note that for every $m \ge 0$, v and u are uniquely defined. Let $H_1(n,m)$ be the n-vertex graph which is composed of a clique on v vertices and, if $u > 0$, a unique vertex outside the clique, which is connected to exactly u vertices of that clique. (The remaining vertices, if there are any, are isolated). Note that $H_1(n,m)$ has exactly m edges. Let $H_2(n,m)$ be the complement of $H_1(n, \binom{n}{2} - m)$. Note that $H_2(n,m)$ has exactly m edges. Olpp has proved the following:

Lemma 2.1 (Olpp [10]). *Let C_1 be the coloring of K_n where the edges colored blue are defined by $H_1(n,m)$. Let C_2 be the coloring of K_n where the edges colored blue are defined by $H_2(n,m)$. Then $t(n,m) = \max\{t(C_1)\ ,\ t(C_2)\}$.* Note that Lemma 2.1 also supplies a formula for $t(n,m)$ since $t(C_1)$ and $t(C_2)$ can be explicitly computed.

Lemma 2.2. *If C is a red-blue coloring with $m = \alpha\binom{n}{2}$ blue edges and $\alpha \ge 0.5$, then*

$$f_2(C) \ge \frac{n^2}{12}(1 + 3\alpha(1-\sqrt{\alpha})) - o(n^2).$$

Proof. Let C_1 and C_2 be the colorings in Lemma 2.1 where $m = \alpha\binom{n}{2}$. By examining the graphs $H_1(n,m)$ and $H_2(n,m)$ it is easy to verify that

$$t(C_1) = \binom{n}{3}(1 - 3\alpha(1-\sqrt{\alpha})) - o(n^3),$$
$$t(C_2) = \binom{n}{3}(1 - 3(1-\alpha)(1-\sqrt{1-\alpha})) - o(n^3).$$

Since $\alpha \ge 0.5$, we have $t(C_1) \ge t(C_2)$. Thus, by Lemma 2.1,

$$t(C) \le t(n,m) = \binom{n}{3}(1 - 3\alpha(1-\sqrt{\alpha})) - o(n^3). \tag{2.1}$$

Fix a Steiner triple system $S(2,3,n)$. A random permutation π of $[n]$ that maps the vertices of K_n to the elements of $S(2,3,n)$ corresponds to a random triangle packing L_π of K_n of order $n(n-1)/6$. Every triangle is equally likely to appear in L_π, each with probability $1/(n-2)$. The expected number of monochromatic triangles in L_π is, therefore, equal to $t(C)/(n-2)$. Fix a π for which L_π contains at most $t(C)/(n-2)$ monochromatic triangles. Thus, there is a packing $M \subset L_\pi$, of size at least $|L_\pi| - t(C)/(2n-4)$ which is 2-color avoiding. By (2.1),

$$\begin{aligned} f_2(C) &\ge \frac{n(n-1)}{6} - \frac{t(C)}{2n-4} \\ &\ge \frac{n(n-1)}{6} - \frac{1}{2n-4}\left(\binom{n}{3}(1 - 3\alpha(1-\sqrt{\alpha})) - o(n^3)\right) \\ &\ge \frac{n^2}{12}(1 + 3\alpha(1-\sqrt{\alpha})) - o(n^2). \end{aligned}$$

□

2.2. The fractional approach. We start with the definition of our fractional relaxation. For a red-blue coloring C of K_n, Let $\mathcal{T}_r$ be the set of triangles that contain a red edge and let $\mathcal{T}_b$ be the set of triangles that contain a blue edge. A *fractional* blue-avoiding packing is a function $\nu : \mathcal{T}_r \to [0,1]$ satisfying, for each edge e, $\sum_{e \in T \in \mathcal{T}_r} \nu(T) \le 1$. Similarly, a fractional red-avoiding packing $\nu : \mathcal{T}_b \to [0,1]$ satisfies, for each edge e, $\sum_{e \in T \in \mathcal{T}_b} \nu(T) \le 1$. The *value* of ν is $|\nu| = \sum_{T \in \mathcal{T}_c} \nu(T)$ where $c = r$ or $c = b$, depending on whether ν is blue-avoiding or red-avoiding. Let $r^*(C)$ (resp. $b^*(C)$) be the maximum possible value of a fractional blue-avoiding (resp. red-avoiding) packing. Let $f_2^*(C) = \max\{r^*(C)\ ,\ b^*(C)\}$. Finally, let $f_2^*(n)$ be the minimum of $f_2^*(C)$ ranging over all red-blue colorings of K_n.

It is easy to see that $f_2^*(n) \ge f_2(n)$, by considering only functions ν that take values 0 and 1. It is also not difficult to construct examples showing strict inequality. For example, we trivially have $f_2(4) = 1$, while $f_2^*(4) = 2$. It is interesting, however, and far from trivial, that the gap between $f_2^*(n)$ and $f_2(n)$ cannot be too large. Haxell and Rödl showed in [6] that the gap between a fractional and an integral triangle packing is $o(n^2)$. This, however, is not sufficient since our graphs are colored. In other words, our packings are *not allowed* to assign positive values to certain triangles. In [11] the author has extended the result from [6] to packings whose elements are taken from any given family of graphs, using a different (probabilistic) approach. In fact, the same proof from [11] also holds for *induced* packings. More formally, let $\mathcal{F}$ be any given family of graphs. An *induced* $\mathcal{F}$-packing of a graph G is a set of induced subgraphs of G, each of them isomorphic to an element of $\mathcal{F}$, and any two of them intersecting in at most one vertex. Let $\nu_{\mathcal{F}}(G)$ be the maximum cardinality of an induced $\mathcal{F}$-packing. Similarly, a *fractional* induced $\mathcal{F}$-packing is a function that assigns weights from $[0,1]$ to the induced subgraphs of G that are isomorphic to elements of $\mathcal{F}$, so that for each pair of vertices x, y, the sum of the weights of the subgraphs containing both x and y is at most one. Let $\nu^*_{\mathcal{F}}(G)$ be the maximum value of a fractional induced $\mathcal{F}$-packing.

Theorem 2.3. *[Yuster [11], induced version] Let $\mathcal{F}$ be a family of graphs. If G is a graph with n vertices then $\nu^*_{\mathcal{F}}(G) - \nu_{\mathcal{F}}(G) = o(n^2)$.* From Theorem 2.3 it is easy to show that $f_2^*(n)$ and $f_2(n)$ are close.

Corollary 2.4. $f_2^*(n) - f_2(n) = o(n^2)$.

Proof. Consider a red-blue coloring C of K_n. Let $r(C)$ be the maximum cardinality of a blue-avoiding triangle packing and let $b(C)$ be the maximum cardinality of a red-avoiding triangle packing. It suffices to show that $r^*(C) - r(C) = o(n^2)$ and that $b^*(C) - b(C) = o(n^2)$. Let G be the n-vertex graph obtained by taking only the edges colored red. Consider the family $\mathcal{F} = \{K_3, K_{1,2}, \overline{K_{1,2}}\}$. Clearly, $r(C) = \nu_{\mathcal{F}}(G)$ and $r^*(C) = \nu^*_{\mathcal{F}}(G)$. The result now follows from Theorem 2.3. Similarly $b^*(C) - b(C) = o(n^2)$ by considering the complement of G. □

By Corollary 2.4, in order to prove the lower bound claimed for $f_2(n)$ in Theorem 1.1, it suffices to prove the same lower bound for $f_2^*(n)$.

Let $\mathcal{F}_r$ be the set of non-isomorphic graphs on r vertices. We note that each element of $\mathcal{F}_r$ corresponds to a red-blue coloring of K_r by coloring the edges blue and the non-edges red. It is easy to verify that $\mathcal{F}_4$ consists of 11 graphs, each being one of $\{K_4, K_4^-, Q, C_4, P_4, K_{1,3}\}$ or a complement of one of these (the complement of P_4 is P_4; Q is the graph with four edges that contains a triangle). For a graph H let $b^*(H) = b^*(C)$ where C is the red-blue coloring corresponding to H. It is easy to verify that $b^*(K_4) = 2$, $b^*(K_4^-) = 2$, $b^*(Q) = 2$, $b^*(C_4) = 2$, $b^*(P_4) = 2$, $b^*(K_{1,3}) = 1.5$, $b^*(\overline{K_{1,3}}) = 2$, $b^*(\overline{C_4}) = 2$, $b^*(\overline{Q}) = 1.5$, $b^*(\overline{K_4^-}) = 1$, $b^*(\overline{K_4}) = 0$.

Lemma 2.5. *If C is a red-blue coloring with $m = \alpha\binom{n}{2}$ blue edges and $\alpha \ge 0.5$, then*

$$f_2(C) \ge \frac{n^2}{12}(3\alpha - \alpha^2) - o(n^2).$$

Proof. By corollary 2.4 it suffices to prove the claimed lower bound for $f_2^*(C)$. In fact, we shall prove a stronger statement:

$$b^*(C) \geq \frac{n^2}{12}(3\alpha - \alpha^2) - o(n^2). \tag{2.2}$$

Fix a Steiner system $T = S(2,4,n)$ on the set $X = \{1,\ldots,n\}$. We shall also fix, for each block $B = \{i,j,k,\ell\}$ of T, a matching $M(B) = \{\{i,j\},\{k,\ell\}\}$. Let π be a permutation of $[n]$ selected uniformly at random from S_n. The permutation π defines a decomposition of the edges of K_n into a set L_π of $n(n-1)/12$ pairwise edge-disjoint red-blue colored K_4. Indeed, assume that the set of vertices of K_n is $V = \{v_1,\ldots,v_n\}$ and use π to map the blocks of T to pairwise edge-disjoint red-blue colored K_4. A block $B = \{i,j,k,\ell\}$ is mapped to the element of L_π which is the subgraph induced by $\{\pi(i),\pi(j),\pi(k),\pi(\ell)\}$. As noted earlier, each element of L_π corresponds to an element of $\mathcal{F}_4$. Now let

$$f_\pi = \sum_{H \in L_\pi} b^*(H) \leq b^*(C).$$

We will prove that the expectation of the random variable f_π is at least $n^2(3\alpha - \alpha^2)/12 - o(n^2)$, which implies (2.2).

For $H \in \mathcal{F}_4$, let $t_\pi(H)$ denote the number of elements of L_π corresponding to H. Clearly,

$$\sum_{H \in \mathcal{F}_4} t_\pi(H) = \frac{n(n-1)}{12}.$$

We may therefore rewrite f_π as

$$f_\pi = \sum_{H \in \mathcal{F}_4} t_\pi(H) b^*(H). \tag{2.3}$$

We need to estimate the expectation $E[t_\pi(H)]$ for various H.

Our first observation is that $E[t_\pi(K_4)] \leq \frac{\alpha^2}{12} n(n-1)(1-o(1))$. Indeed, consider a block B of T, and consider its preassigned matching $M(B) = \{\{i,j\},\{k,\ell\}\}$. The probability that $(\pi(i),\pi(j))$ is blue is precisely α. The probability that $(\pi(k),\pi(\ell))$ is blue given that we are *told* that $(\pi(i),\pi(j))$ is blue (and even told its identity) is $\alpha(1-o(1))$. Since there are $n(n-1)/12$ blocks we have that $E[t_\pi(K_4)] \leq \frac{\alpha^2}{12} n(n-1)(1-o(1))$. Similarly, $E[t_\pi(\overline{K_4})] \leq \frac{(1-\alpha)^2}{12} n(n-1)(1-o(1))$. However, we can do much better.

LEMMA 2.6.

$$E\left[t_\pi(K_4) + \frac{2}{3}t_\pi(K_4^-) + \frac{1}{3}t_\pi(Q) + \frac{2}{3}t_\pi(C_4) + \frac{1}{3}t_\pi(P_4) + \frac{1}{3}t_\pi(\overline{C_4})\right]$$

$$= \frac{\alpha^2}{12} n(n-1)(1-o(1)).$$

$$E\left[t_\pi(\overline{K_4}) + \frac{2}{3}t_\pi(\overline{K_4^-}) + \frac{1}{3}t_\pi(\overline{Q}) + \frac{2}{3}t_\pi(\overline{C_4}) + \frac{1}{3}t_\pi(P_4) + \frac{1}{3}t_\pi(C_4)\right]$$

$$= \frac{(1-\alpha)^2}{12} n(n-1)(1-o(1)).$$

Proof. For each element $H \in L_\pi$, let $m(H)$ be the number of blue perfect matchings it contains, and let $g_\pi = \sum_{H \in L_\pi} m(H)$. Clearly,

$$g_\pi = 3t_\pi(K_4) + 2t_\pi(K_4^-) + t_\pi(Q) + 2t_\pi(C_4) + t_\pi(P_4) + t_\pi(\overline{C_4}).$$

Since K_4 has precisely three perfect matchings, the expected number of blocks B for which $M(B)$ is mapped to two blue edges is $\frac{1}{3}E[g_\pi]$. On the other hand, the expected number of blocks B for which $M(B)$ is mapped to two blue edges is $\frac{\alpha^2}{12}n(n-1)(1-o(1))$ as noted in the paragraph preceding the lemma. Thus, the first equality in the statement of the lemma follows. The second equality follows analogously. □

To simplify notation, consider the following eleven variables: $x_1 = E[t_\pi(K_4)]$, $x_2 = E[t_\pi(K_4^-)]$, $x_3 = E[t_\pi(Q)]$, $x_4 = E[t_\pi(C_4)]$, $x_5 = E[t_\pi(P_4)]$, $x_6 = E[t_\pi(K_{1,3})]$, $x_7 = E[t_\pi(\overline{K_{1,3}})]$, $x_8 = E[t_\pi(\overline{C_4})]$, $x_9 = E[t_\pi(\overline{Q})]$, $x_{10} = E[t_\pi(\overline{K_4^-})]$, $x_{11} = E[t_\pi(\overline{K_4})]$. With these variables, placing expectations on both sides of (2.3) we obtain

$$E[f_\pi] = 2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + 1.5x_6 + 2x_7 + 2x_8 + 1.5x_9 + x_{10}.$$

Let $y_i = x_i/n(n-1)$ for $i = 1, \ldots, 11$. Using Lemma 2.6, a lower bound for $E[f_\pi]$ is obtained by solving the following linear program:

$$\begin{aligned}
&\min\ 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 1.5y_6 + 2y_7 + 2y_8 + 1.5y_9 + y_{10} \\
&s.t.\ \sum_{i=1}^{11} y_i = \frac{1}{12} \\
&\quad y_1 + \frac{2}{3}y_2 + \frac{1}{3}y_3 + \frac{2}{3}y_4 + \frac{1}{3}y_5 + \frac{1}{3}y_8 = \frac{\alpha^2}{12} - o(1) \\
&\quad y_{11} + \frac{2}{3}y_{10} + \frac{1}{3}y_9 + \frac{2}{3}y_8 + \frac{1}{3}y_5 + \frac{1}{3}y_4 = \frac{(1-\alpha)^2}{12} - o(1) \\
&\quad y_i \geq 0 \quad \text{for } i = 1, \ldots, 11.
\end{aligned}$$

In order to derive an optimal solution for this linear program, we exhibit matching solutions both for it and for its dual. The dual program is:

$$\begin{aligned}
&\max \quad \frac{1}{12}z_1 + (\frac{\alpha^2}{12} - o(1))z_2 + (\frac{(1-\alpha)^2}{12} - o(1))z_3 \\
&s.t. \\
&\quad \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix} \\
&\quad \leq \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 1.5 & 2 & 2 & 1.5 & 1 & 0 \end{pmatrix}.
\end{aligned}$$

(In the argument below and, in fact, throughout Lemma 2.6, we could write all expressions explicitly, instead of writing $o(1)$ terms. However, this would be somewhat cumbersome and, moreover, the reader will be able to check that this is not necessary.) A feasible solution for the dual is $z_1 = 3/2$, $z_2 = 1/2$ and $z_3 = -3/2$ (notice that the constraint set of the dual does not involve $o(1)$ terms). The value this solution attains is $(3\alpha - \alpha^2)/12 - o(1)$. To prove that this is, in fact, an asymptotically optimal solution, we exhibit a feasible solution for the primal problem whose value is also $(3\alpha - \alpha^2)/12 - o(1)$. Indeed, consider the solution $y_1 = \alpha^2/12 - o(1)$, $y_{11} = (1-\alpha)^2/12 - o(1)$ and $y_6 = (\alpha - \alpha^2)/6 + o(1)$ and all the other eight variable are zero, so that all constraints are satisfied. Indeed this solution attains the value $(3\alpha - \alpha^2)/12 - o(1)$, as required. It follows that $E[f_\pi] \geq n^2(3\alpha - \alpha^2)/12 - o(n^2)$, as required. □

2.3. Combining the results. Given Lemma 2.2 and Lemma 2.5, we see that if $\alpha \geq 0.5$ is close to 0.5 then the bound in Lemma 2.2 is larger than the bound in Lemma 2.5. On the other hand, when α approaches 1, the bound in Lemma 2.5 approaches the optimal packing of size $n^2/6 - o(n^2)$. By equating $1 + 3\alpha(1 - \sqrt{\alpha})$ with $3\alpha - \alpha^2$ we get that the point of equilibrium is the square of the smallest root of $x^4 - 3x^3 + 1$. If $\beta = 0.7648...$ denotes this root we clearly have

$$f_2(n) \geq \frac{3\beta^2 - \beta^4}{12} n^2 - o(n^2) \ ,$$

proving the lower bound in Theorem 1.1. □

3. An upper bound for $f_r(n)$. We start this section with a construction of a red-blue coloring of K_n that cannot avoid a monochromatic red triangle and a monochromatic blue triangle in any large triangle packing.

Let $0 < \alpha < 1$ be a parameter, let A be a set of αn vertices and B a set of $n(1-\alpha)$ vertices. The vertices of A induce a monochromatic red clique, and all other edges are colored blue. Suppose there is a K_3-packing of size x with no monochromatic red K_3. Then, each element of this packing either contains two edges from the cut (A, B), or has all its three vertices from B. Thus,

$$x < \frac{\alpha(1-\alpha)}{2} n^2 + \frac{(1-\alpha)^2}{6} n^2 + o(n^2). \tag{3.1}$$

Suppose there is a packing of size y with no monochromatic blue K_3. Then we cannot use edges with both endpoints in B at all. Thus,

$$y < \frac{\alpha^2/2 + \alpha(1-\alpha)}{3} n^2 + o(n^2). \tag{3.2}$$

Now, let $z = \max\{x, y\}$. By equating (3.1) and (3.2) we get that for $\alpha = (\sqrt{5} - 1)/2$ we have

$$z < \frac{3\sqrt{5} - 5}{12} n^2 + o(n^2) \approx 0.1424 n^2 (1 + o(1)).$$

In particular, this proves the upper bound in Theorem 1.1.

The construction for $r > 2$ generalizes the construction above. Suppose the set of vertices V of K_n is partitioned into vertex classes $V_1, \ldots, V_r$. The edges with both endpoints in V_i are colored with color i, and an edge between V_i and V_j for $i < j$ is colored with color j. The idea is to choose the sizes of the vertex classes so that a sufficiently large K_3-packing must contain an i-monochromatic K_3 for each color i. Fix $0 < \alpha < 1$, and assume that $|V_i| = \alpha(1-\alpha)^{i-1} n$ for $i = 1, \ldots, r-1$ and $|V_r| = (1-\alpha)^{r-1} n$ (we ignore floors and ceilings as these have no effect on the asymptotic result).

Suppose there is a K_3-packing L_i of size x_i with no i-monochromatic K_3. An upper bound for x_1 is identical to the upper bound for x in (3.1):

$$x_1 < \frac{\alpha(1-\alpha)}{2} n^2 + \frac{(1-\alpha)^2}{6} n^2 + o(n^2). \tag{3.3}$$

For $i = 2, \ldots, r-1$, we notice that no two edges inside V_i appear together in a non i-monochromatic K_3. Since the third vertex of a non i-monochromatic K_3 having two vertices in V_i must belong to some V_j with $j > i$ we have

$$x_i < \frac{1}{6} n^2 - \frac{\alpha^2(1-\alpha)^{2i-2}}{6} n^2 + \frac{\alpha(1-\alpha)^{i-1}(1-\alpha)^i}{6} n^2 + o(n^2). \tag{3.4}$$

For $i = r$, L_r cannot cover edges with both endpoints in V_r at all. Thus, similarly to (3.2) we get

$$x_r < \frac{(1-(1-\alpha)^{r-1})^2/2 + (1-\alpha)^{r-1}(1-(1-\alpha)^{r-1})}{3} n^2 + o(n^2). \tag{3.5}$$

Simplifying (3.3),(3.4), and (3.5) we get that

$$\frac{6x_i}{n^2} - o(1) \le 1 - \alpha(2\alpha-1)(1-\alpha)^{2i-2} \quad i = 1,\ldots,r-1. \tag{3.6}$$

$$\frac{6x_r}{n^2} - o(1) \le 1 - (1-\alpha)^{2r-2}. \tag{3.7}$$

Now, let $z = \max\{x_1,\ldots,x_r\}$, and notice that, in fact, it suffices to consider $z = \max\{x_{r-1}, x_r\}$. By equating the case $i = r-1$ in (3.6) with (3.7) we get that for $\alpha = (\sqrt{5}-1)/2$ we have:

$$z < \frac{1 - (\frac{3-\sqrt{5}}{2})^{2r-2}}{6} n^2 + o(n^2).$$

It follows that $\frac{f_r(n)}{n^2} < \frac{1}{6}(1-\zeta^{r-1}) + o(1)$ where $\zeta = \frac{7-3\sqrt{5}}{2}$. This completes the proof of Theorem 1.2. □

4. Determining $f_2(n)$ for small n. Clearly, $f_2(3) = f_2(4) = 1$. For $n = 5$ we notice that there are 15 distinct pairs of edge-disjoint triangles. Each of the 10 triangles appears in three of these pairs. If each pair contains a red triangle and a blue triangle we must have five red triangles and five blue triangles. Suppose, w.l.o.g., that there are at most five red edges. Notice that five edges cannot induce five triangles. Thus, $f_2(5) = 2$.

For $n = 6$, notice that K_6 has 15 distinct perfect matchings. Each perfect matching uniquely defines two sets of four pairwise edge-disjoint triangles (by considering the $K_{2,2,2}$ obtained by deleting the matching). All together, there are 30 distinct triangle packings of size 4. Totally, they contain 120 triangles, but since K_6 has 20 triangles, each triangle appears in precisely 6 such packings. Suppose each packing has a red and a blue monochromatic triangle. Then there are at least 5 monochromatic red triangles and at least 5 monochromatic blue triangles. Assume, w.l.o.g., that there are at most seven red edges. Notice that seven edges cannot induce five triangles. It follows that $f_2(6) = 4$ (K_6 does not have five pairwise edge-disjoint triangles).

For $n = 7$, we first notice that $f_2(7) \le 6$ (although K_7 does have a Steiner triple system with 7 edge-disjoint triangles). Indeed, take a red K_5 and color the remaining 11 edges blue. In a packing that has no red triangle there are at least two blue edges in each triangle, and hence its size is at most 5. In a packing that has no blue triangle the unique blue edge that is not incident with any red edge does not appear. Hence, the packing contains at most 6 triangles. In fact, it is easy to verify that this coloring indeed contains 6 edge-disjoint triangles, non of which is entirely blue. For the other direction, K_7 contains precisely 30 distinct Steiner triple systems. Totally, they contain 210 triangles, but since K_7 has 35 triangles, each triangle appears in precisely 6 such systems. If each system contains two blue triangles and two red triangles, then there are 10 red triangles and 10 blue triangles. Assume, w.l.o.g., that there are at most 10 red edges. The only way 10 edges can induce 10 triangles is if they form a K_5, and this is precisely the construction we examined earlier. Thus, $f_2(7) = 6$.

For $n = 8$, notice that K_8 has 105 distinct perfect matchings. Each perfect matching uniquely defines 8 sets of 8 pairwise edge-disjoint triangles (by considering the $K_{2,2,2,2}$ obtained by deleting the matching). All together, there are 840 distinct triangle packings of size 8. Totally, they contain 6720 triangles, but since K_8 has 56 triangles, each triangle appears in precisely 120 such packings. Suppose each packing has two red and two blue monochromatic triangles. Then there are at least 14 monochromatic red triangles and at least 14 monochromatic blue triangles. Assume, w.l.o.g., that there are at most 14 red edges. If there are 14 red edges then, by Lemma 2.1 $t(8, 14) = 24$ so we cannot have 28 monochromatic triangles. If there are only 13 red edges or less, they cannot induce 14 triangles. It follows that $f_2(8) \geq 7$. To see that $f_2(8) = 7$ consider a red K_5 and color the remaining 18 edges blue. In any packing of 8 pairwise edge-disjoint triangles, this coloring has both a red and a blue triangle.

5. Concluding remarks. The most obvious open problem is to determine the true asymptotic behavior of $f_2(n)$. We conjecture that the upper bound construction is the right (asymptotic) answer. Namely, $f_2(n) = \frac{3\sqrt{5}-5}{12}n^2 - o(n^2)$. The fractional approach yielding Lemma 2.5 uses the Steiner system $S(2, 4, n)$. At the price of significantly complicating the proof we can use higher order system such as $S(2, k, n)$ (Wilson's Theorem guarantees the existence of an $S(2, k, n)$ when n is any sufficiently large integer satisfying $n \equiv 1 \bmod k(k-1)$). This, however, requires the analysis of all possible colorings of K_k and their expected frequencies, which is already a daunting task for $k = 6$, and which will not lead to a significant improvement in the lower bound.

A *Steiner packing* of K_n is a triangle packing of maximum cardinality. As already mentioned, if $n \equiv 1, 3 \bmod 6$, there is a Steiner triple system, which, by definition, is a Steiner packing that covers every edge and hence consists of $n(n-1)/6$ elements. For other moduli, the cardinality of a Steiner packing is also well-known [3]. It is $\lfloor n(n-2)/6 \rfloor$ if n is even and $\lfloor n(n-1)/6 - 1 \rfloor$ if $n \equiv -1 \bmod 6$. Let $g(r)$ be the maximum integer n so that in every r-edge coloring of K_n there is a Steiner packing that is r-color avoiding. The arguments in Section 4 show that $g(2) = 6$, since already for $n = 7$ the Steiner packing has 7 elements while $f_2(7) = 6$. It would be interesting to determine the behavior of $g(r)$ as a function of r. The proof of Theorem 1.2 shows that $g(r)$ is at most exponential in r (the base being at most roughly 2.7).

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