# Nowhere 0 mod $p$ dominating sets in multigraphs

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**Abstract**

Let $G$ be a graph with integral edge weights. A function $d : V(G) \to Z_p$ is called a *nowhere* $0 \bmod p$ *domination function* if each $v \in V$ satisfies $\left(d(v) + \sum_{u \in N(v)} w(u,v)d(u)\right) \neq 0 \bmod p$, where $w(u,v)$ denotes the weight of the edge $(u,v)$ and $N(v)$ is the neighborhood of $v$. The subset of vertices with $d(v) \neq 0$ is called a *nowhere* $0 \bmod p$ *dominating set*. It is known that every graph has a nowhere $0 \bmod 2$ dominating set. It is known to be false for all other primes $p$. The problem is open for all odd $p$ in case all weights are one. In this paper we prove that every unicyclic graph (a graph containing at most one cycle) has a nowhere $0 \bmod p$ dominating set for all $p > 1$. In fact, for trees and cycles with any integral edge weights, or for any other unicyclic graph with no edge weight of $(-1) \bmod p$, there is a nowhere $0 \bmod p$ domination function $d$ taking only $0 - 1$ values. This is the first nontrivial infinite family of graphs for which this property is established. We also determine the minimal graphs for which there does not exist a $0 \bmod p$ dominating set for all $p > 1$ in both the general case and the $0 - 1$ case.

## 1 Introduction

All graphs and multigraphs considered here are finite, undirected and have no loops. For standard graph-theoretic terminology the reader is referred to [2]. A subset $D$ of vertices in a graph $G = (V, E)$ is a *dominating set* if every vertex not in $D$ has a neighbor in $D$. For an integer $p > 1$, $D$ is called a *nowhere* $0 \bmod p$ *dominating set* ($p$-NZDS for short) if $|D \cap N[v]| \neq 0 \bmod p$ for each $v \in V$ where $N[v]$ denotes the closed neighborhood of the vertex

$v$. If the edges of $G$ have integral weights then the following more general definition is used: A function $d : V(G) \to Z_p$ is called a *nowhere* $0 \bmod p$ *domination function* ($p$-NZDF for short) if each $v \in V$ satisfies $\left(d(v) + \sum_{u \in N(v)} w(u,v)d(u)\right) \neq 0 \bmod p$, where $w(u,v)$ denotes the weight of the edge $(u,v)$ and $N(v)$ is the open neighborhood of $v$. The subset of vertices with $d(v) \neq 0$ is the $p$-NZDS in this case. Notice that the analog definition where $d(v)$ is disregarded is trivially non-interesting as an isolated vertex cannot be satisfied (and there are small connected graph examples as well showing non-satisfiability). Also notice the analogy between multigraphs and integral edge-weighted graphs, where the multiplicity of an edge modulo $p$ is its weight (an edge with zero weight is the same as a nonexisting edge for this purpose).

Somewhat surprising, but simple to prove, is the fact observed by Sutner [6] (see also [4] for a generalization) that every graph has a 2-NZDS (edge weights play no role when $p = 2$). Unfortunately, Sutner's proof cannot be extended to other prime values of $p$, and in fact it has been shown by Alon and Lovász [1] that the analog theorem is not true:

**Proposition 1.1 (Alon and Lovász)** *For every odd prime $p$ there exists a symmetric matrix $A_p$ over $Z_p$, of order $2^{p-1} + p + 1$, with ones in the diagonal such that for every integral vector $x$, $Ax$ has a coordinate which is a multiple of $p$.* $\square$

Hence, if $A$ is interpreted as the adjacency matrix of an edge-weighted graph, the result follows. The matrix in proposition 1.1 is not a $0 - 1$ matrix. The following problem has been raised by Caro [3]:

**Problem 1.2** *Is it true that for every $p > 1$, every nonweighted graph has a $p$-NZDS ?*

The problem is open for all odd $p \geq 3$ (clearly a proof for $p$ is also a proof for all multiples of $p$, so for all even $p$ Sutner's result holds). Thus, it is interesting to find nontrivial pairs of families of graphs and values of $p$ for which the multigraph (or, at least, the simple graph) analog to Sutner's result holds. By "nontrivial" we mean that the existence of a $p$-NZDS is not implied directly from the degree sequence. For example, if a nonweighted graph has maximum degree $\Delta$ then it trivially has a nowhere 0 mod $p$ dominating set for each $p > \Delta + 1$, by simply taking all vertices into the dominating set. In fact, this also holds for $p = \Delta + 1$ by taking a maximal independent set (such a set is always a dominating set). As another trivial example consider a nonweighted graph whose degree sequence has gcd $= p$. Again, taking all vertices into the dominating set yields a nowhere 0 mod $p$ dominating set.

The first nontrivial family of graphs for which the answer to Problem 1.2 has been shown true is the family of trees. This has been proved by Caro and Jacobson [5] (they also determine when trees have a nowhere $z$ mod $p$ dominating set where $z$ is not necessarily zero). It turns out that proving the same result even for the slightly more general family of unicyclic graphs (graphs with at most one cycle) requires a considerable amount of additional effort (although Caro and Jacobson showed that an immediate consequence from the proof for trees guarantees that unicyclic graphs have a nowhere $0 \bmod p$ dominating set for $p \geq 4$, it does not cover the first odd prime, 3). Even more so, the Caro and Jacobson result cannot be generalized to unicyclic edge-weighted graphs. Our first theorem generalizes the Caro - Jacobson result in both aspects:

**Theorem 1.3** *For every $p > 1$, every edge-weighted unicyclic graph has a $p$-NZDF.*

In case the graph $G$ is any edge-weighted tree or cycle, or any edge-weighted unicyclic graph without the weight $(-1) \bmod p$ we can do even better:

**Theorem 1.4** *Let $p > 1$ and let $G$ be any edge-weighted tree or cycle, or any edge-weighted unicyclic graph with no edge weight of $(-1) \bmod p$. Then there exists a $p$-NZDF which takes only the values 0 or 1.*

By Theorem 1.3, every edge-weighted connected graph with $n$ vertices and at most $n$ edges has a $p$-NZDF. We cannot replace $n$ with $n+6$, and we cannot replace $n$ with $n+3$ if we also require a $0-1$ function, as we prove the following:

**Proposition 1.5** *The smallest edge-weighted graph that does not have a 3-NZDF has 6 vertices and 12 edges. As a multigraph it has 16 edges. The smallest edge-weighted graph that does not have a 3-NZDF taking only $0-1$ values has 6 vertices and 9 edges. As a multigraph it has 12 edges.*

An immediate corollary from Theorem 1.4 is the following:

**Corollary 1.6** *For every $p > 1$, every non-weighted unicyclic graph has a $p$-NZDS.*

The corollary follows since the case $p = 2$ is covered by Sutner's result, and for $p > 2$ we have that $1 \neq (-1) \bmod p$.

In the next section we prove Theorem 1.3 and Theorem 1.4. In the final section we describe the graphs yielding Proposition 1.5.

## 2 Proof of Theorems 1.3 and 1.4

In the remainder of this paper we assume that all edge weights (equivalently, all edge multiplicities) are between 1 and $p-1$, since otherwise we can always take the appropriate residue class mod $p$ and the results stay intact (if the multiplicity of an edge is a multiple of $p$ it is the same as if it did not exist at all). We shall also assume that $p \geq 3$ since the $p = 2$ case is handled in Sutner's result. This is convenient in order to avoid superfluous case analysis.

Let $l \geq 3$ and let $C = \{r_1, \ldots, r_l\}$ be a cycle whose edges have nonzero weights from $Z_p$. A central argument in the proofs of Theorem 1.3 and Theorem 1.4 is the following lemma:

**Lemma 2.1** *Let $t : C \to \{-1, 0, 1\}$ be any function. Then, there exists a function $d : C \to \{0, 1\}$ such that the following holds (indices are modulo $l$):*

1. *If $t(r_i) = -1$ then $d(r_i) = 1$.*
2. *If $t(r_i) = 0$ then $d(r_i) = 0$.*
3. *If $t(r_i) = 1$ then $d(r_{i-1})w(r_{i-1}, r_i) + d(r_{i+1})w(r_i, r_{i+1}) + d(r_i) \neq 0 \bmod p$.*

**Proof:** We shall call $t(r_i)$ the *type* of $r_i$. Put

$$g(r_i) = d(r_{i-1})w(r_{i-1}, r_i) + d(r_{i+1})w(r_i, r_{i+1}) + d(r_i).$$

We must show how to define $d$ on vertices $r_i$ having positive type, such that $g(r_i) \neq 0 \bmod p$. We must handle two major cases: the case (a) where all vertices have positive type and the case (b) where there exists a vertex with negative or zero type. Notice that in the case (b) the set of vertices with positive type induce a (possibly empty) linear forest. Case (a) is divided into subcases according to the value of $l$ and case (b) is divided into subcases according to the types of the external neighbors of the endpoints of a path in the linear forest of positive vertices.

(a1) All vertices have positive type and $l = 0 \bmod 3$.
In this case we let $d(r_i) = 1$ if and only if $i = 0 \bmod 3$. Notice that $g(r_i) = 1$ for $i = 0 \bmod 3$, $g(r_i) = w(r_i, r_{i+1}) \neq 0 \bmod p$ for $i = 2 \bmod 3$ and $g(r_i) = w(r_{i-1}, r_i) \neq 0 \bmod p$ for $i = 1 \bmod 3$.

(a2) All vertices have positive type and $l = 1 \bmod 3$.
If all weights are equal to $p - 1$ then we may define $d(r_i) = 1$ for all $i$. In this case $g(r_i) = 2(p-1) + 1 = 2p - 1 \neq 0 \bmod p$. Otherwise, we may assume $w(r_1, r_2) = q < p - 1$.

We define $d(r_1) = 1$ and for $i \geq 2$ we define $d(r_i) = 1$ if and only if $i = 2 \bmod 3$. For $i \neq 1, 2$ we have $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ and hence $g(r_i) \neq 0 \bmod p$. For $i = 1, 2$ we have $g(r_i) = 1 + w(r_1, r_2) = 1 + q \neq 0 \bmod p$.

(a3) All vertices have positive type and $l = 2 \bmod 3$.
If for each $i = 1, \ldots, l$ we have $w(r_{i-1}, r_i) + w(r_i, r_{i+1}) = p$ then we define $d(r_i) = 1$ for all $i = 1, \ldots, l$. Notice that $g(r_i) = p + 1 \neq 0 \bmod p$ for all $i$ in this case. Thus, we may assume that $w(r_1, r_2) + w(r_2, r_3) \neq p$. We define $d(r_1) = 1$, $d(r_2) = 0$ and for $i = 3, \ldots, l$ we define $d(r_i) = 1$ if and only if $i = 0 \bmod 3$. Notice that for all $i \neq 2$ we have $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ and hence $g(r_i) \neq 0 \bmod p$. For $i = 2$ we have $g(r_2) = w(r_1, r_2) + w(r_2, r_3) \neq 0 \bmod p$.

We may now assume that some $r_i$ has $t(r_i) \neq 1$, and hence it suffices to show how to define $d$ on each consecutive maximal path of vertices with positive type. Assume without loss of generality that $r_1, \ldots, r_k$ is such a path, and $k \geq 1$. Let $r_0$ be the outside neighbor of $r_1$ and $r_{k+1}$ be the outside neighbor of $r_k$. Notice that it is possible that $r_0 = r_{k+1}$ in case only one vertex has non-positive type. Since $r_0$ and $r_{k+1}$ have non-positive type, the values $d(r_0)$ and $d(r_{k+1})$ are determined. By symmetry we may assume $d(r_0) \geq d(r_{k+1})$.

b(1) $d(r_0) = d(r_{k+1}) = 0$.
If $k \neq 0 \bmod 3$ we define, for $i = 1, \ldots, k$, $d(r_i) = 1$ if and only if $i = 1 \bmod 3$. If $k = 0 \bmod 3$ we define, for $i = 1, \ldots, k$, $d(r_i) = 1$ if and only if $i = 2 \bmod 3$. Notice that in any case we have $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ for all $i = 1, \ldots, k$ and hence $g(r_i) \neq 0 \bmod p$.

b(2) $d(r_0) = 1$ and $d(r_{k+1}) = 0$.
If $k \neq 2 \bmod 3$ we define, for $i = 1, \ldots, k$, $d(r_i) = 1$ if and only if $i = 0 \bmod 3$. In this case we have $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ for all $i = 1, \ldots, k$ and hence $g(r_i) \neq 0 \bmod p$. We may therefore assume $k = 2 \bmod 3$. If $w(r_0, r_1) + w(r_1, r_2) \neq 0 \bmod p$ we define, for $i = 1, \ldots, k$, $d(r_i) = 1$ if and only if $i = 2 \bmod 3$. In this case $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ for all $i = 2, \ldots, k$ and hence $g(r_i) \neq 0 \bmod p$. For $i = 1$ we have $g(r_1) = w(r_0, r_1) + w(r_1, r_2) \neq 0 \bmod p$. We remain with the case $w(r_0, r_1) + w(r_1, r_2) = p$. If $w(r_0, r_1) \neq p - 1$ we put $d(r_i) = 1$ if and only if $i = 1 \bmod 3$. Again, $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ for all $i = 2, \ldots, k$ and hence $g(r_i) \neq 0 \bmod p$, and for $i = 1$ we have $g(r_1) = 1 + w(r_0, r_1) \neq 0 \bmod p$. Finally, we remain with the case $w(r_0, r_1) = p - 1$ and $w(r_1, r_2) = 1$. We put $d(r_1) = 1$, and for $i \geq 2$, $d(r_i) = 1$ if and only if $i = 2 \bmod 3$. In this case $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ for all $i = 3, \ldots, k$ and hence $g(r_i) \neq 0 \bmod p$. Also, $g(r_1) = w(r_0, r_1) + 1 + w(r_1, r_2) = p + 1 \neq 0 \bmod p$, and $g(r_2) = w(r_1, r_2) + 1 = 2 \neq 0 \bmod p$ (here we use our assumption $p > 2$).

b(3) $d(r_0) = d(r_{k+1}) = 1$.
If $k = 2 \bmod 3$ we define, for $i = 1, \ldots, k$, $d(r_i) = 1$ if and only

if $i = 0 \bmod 3$. In this case we have $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ for all $i = 1, \ldots, k$ and hence $g(r_i) \neq 0 \bmod p$. If $k = 0 \bmod 3$ and $w(r_0, r_1) \neq p-1$ we define $d(r_i) = 1$ if and only if $i = 1 \bmod 3$. In this case $d(r_{i-1})+d(r_i)+d(r_{i+1}) = 1$ for all $i = 2, \ldots, k$ and hence $g(r_i) \neq 0 \bmod p$, and for $i = 1$ we have $g(r_1) = 1 + w(r_0, r_1) \neq 0 \bmod p$. If $k = 0 \bmod 3$ and $w(r_0, r_1) = p - 1$ we may assume by symmetry that also $w(r_k, r_{k+1}) = p - 1$. In this case we define $d(r_1) = 1$ if and only if $w(r_1, r_2) = 1$. Similarly we define $d(r_k) = 1$ if and only if $w(r_{k-1}, r_k) = 1$. For $i = 2, \ldots, k - 1$ we define $d(r_i) = 1$ if and only if $i = 2 \bmod 3$. Notice that $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ for all $i = 3, \ldots, k - 2$ and hence $g(r_i) \neq 0 \bmod p$. For $i = 1$ and $i = k$ we have $d(r_i) \in \{p-1, p+1\} \neq 0 \bmod p$. For $i = 2$ and $i = k-1$ we have $d(r_i) \in \{1, 2\} \neq 0 \bmod p$. Finally we may assume $k = 1 \bmod 3$. If $w(r_1, r_2) \neq p - 1$ or $w(r_0, r_1) \neq 1$ we define for $i = 2, \ldots, k$, $d(r_i) = 1$ if and only if $i = 2 \bmod 3$. We put $d(r_1) = 1$ if and only if $w(r_0, r_1) + w(r_1, r_2) = p$. In this case we have $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ for all $i = 3, \ldots, k$ and hence $g(r_i) \neq 0 \bmod p$. For $i = 1$ we have $g(r_1) = p + 1$ if $w(r_0, r_1) + w(r_1, r_2) = p$ and $g(r_1) = w(r_0, r_1) + w(r_1, r_2)$ if $w(r_0, r_1) + w(r_1, r_2) \neq p$. In any case, $g(r_1) \neq 0 \bmod p$. Also, $g(r_2) = 1$ if $d(r_1) = 0$, and if $d(r_1) = 1$ we must have $w(r_0, r_1) + w(r_1, r_2) = p$ and hence we cannot have $w(r_1, r_2) = p - 1$ since otherwise this forces $w(r_0, r_1) = 1$ and we assume this is not the case. In this case $g(r_2) = 1 + w(r_1, r_2) \neq 0 \bmod p$ as well. Finally, we may assume $k = 1 \bmod 3$, $w(r_1, r_2) = p - 1$, $w(r_0, r_1) = 1$ and by symmetry also $w(r_{k-1}, r_k) = p - 1$ and $w(r_k, r_{k+1}) = 1$ (this forces $k \geq 4$). We put, for $i = 1, \ldots, k$, $d(r_i) = 1$ if and only if $i = 1 \bmod 3$. In this case $d(r_{i-1}) + d(r_i) + d(r_{i+1}) = 1$ for all $i = 2, \ldots, k - 1$. For $i = 1, k$ we have $d(r_i) = 1 + 1 + (p - 1) = p + 1 \neq 0 \bmod p$. $\square$

If $C$ is an edge-weighted cycle then by applying Lemma 2.1 to $C$ and the constant function $t = 1$ we immediately obtain the following Corollary:

**Corollary 2.2** *For all $p > 1$, every edge-weighted cycle has a $p$-NZDF which takes only $0 - 1$ values.*

Let $T$ be a rooted weighted tree. Our next goal is to define a *type assignment* on the vertices of $T$, that assigns to each vertex $v$ of $T$ one of the three types, 0, 1 and $-1$, denoted $t(v)$ using the following postorder recursive definition (this type assignment is similar to the nonweighted type assignment appearing in [5]):

1. $t(v) = 0$ if and only if $v$ has at least one child with a negative type.
2. $t(v) = 1$ if and only if it has no negative child and the sum of the weights of the edges connecting $v$ to its positive children is $0 \bmod p$. Notice that, in particular, all leaves have positive type.

3. $t(v) = -1$ if and only if it has no negative child and the sum of the weights of the edges connecting $v$ to its positive children is not $0 \bmod p$.

Clearly, by traversing from the leaves upwards (postorder traversal) we get the unique type assignment of the rooted tree.

Let $G$ be a unicyclic graph. Let $C = (r_1, \ldots, r_l)$ be the unique cycle of $G$. If $G$ is a tree we pick an arbitrary vertex $r_1$ and consider the degenerate cycle $C = (r_1)$. For each $v \in V$, we define $c(v) = i$ if $r_i$ is the vertex closest to $v$ on the cycle. Notice that $c(v)$ is well-defined for each $v \in V$ and that, trivially, $c(r_i) = i$. Now let $V_i = \{v \in V \mid c(v) = i\}$. Clearly, for $i \neq j$, $V_i \cap V_j = \emptyset$. Also note that the subgraph of $G$ induced by $V_i$ is a tree, which we denote by $T_i$, and consider $T_i$ as a tree rooted at $r_i$. Thus, for each $v \in V$, $t(v)$ is defined as the type of $v$ determined by the type assignment of $T_{c(v)}$.

We now show how to construct a $p$-NZDF for $G$, denoted $d$. In case $G$ is a tree this function will be a $0-1$-function. Also, in case $G$ is not a tree and there is no edge weight equal to $(p-1) \bmod p$, this function will be a $0-1$-function. This (together with Corollary 2.2 in case $G$ is a simple cycle) will simultaneously prove Theorem 1.3 and Theorem 1.4.

Vertices with *type* $= 0$ will always have $d = 0$. We begin by determining the value of $d$ on the vertices of $C$. This is done according to Lemma 2.1. (Since the lemma assumes the cycle is nontrivial, we just note that in case $C = (r_1)$ is the trivial cycle we simply put $d(r_1) = 0$ if and only if $t(r_1) = 0$).

We now determine the value of $d$ on the children of $r_i$, $i = 1, \ldots, l$. Let $g(r_i) = d(r_{i-1})w(r_{i-1}, r_i) + d(r_{i+1})w(r_i, r_{i+1}) + d(r_i)$ (indices modulo $l$, and in case the cycle is trivial define $g(r_1) = d(r_1)$) as in the proof of Lemma 2.1.

If $t(r_i) = 1$ and $d(r_i) = 1$ then all the children of $r_i$ get $d = 0$. If $t(r_i) = 1$ and $d(r_i) = 0$ all positive children of $r_i$ get $d = 1$. If $t(r_i) = 0$, let $s_i$ denote the sum of the weights from $r_i$ to its positive children. All positive children of $r_i$ get $d = 1$. If $s_i + g(r_i) = 0 \bmod p$ then we let exactly one negative child of $r_i$ receive $d = 1$, all other negative children get $d = 0$. Otherwise, all negative children get $d = 0$. If $t(r_i) = -1$ then if $g(r_i) = 0 \bmod p$ we let exactly one positive child $u$ of $r_i$ get $d(u) = 1$, unless $w(r_i, u) = p - 1$ in which case we let $d(u) = p - 1$ (this is the only case where $d$ is not $0 - 1$, and notice that this case cannot happen in case $G$ is a tree since in this case a negative root has $g(r_1) = 1$). All other positive children get $d = 0$. Otherwise, all positive children get $d = 0$. It is immediate to verify that for each $i = 1, \ldots, l$, $\left(d(r_i) + \sum_{u \in N(r_i)} w(u, r_i)d(u)\right) \neq 0 \bmod p$.

We now determine the value of $d$ on all other vertices (i.e. vertices that are not on $C$ nor connected to vertices of $C$). Whenever we determine the value of such a vertex $v$ we assume that the value of the parent of $v$ and possibly some subset of siblings of $v$ has already been determined (i.e. we perform a preorder traversal). If $t(v) = 0$ then $d(v) = 0$. If $t(v) = 1$ then $d(v) = 1$ if and only if its parent has $d = 0$. If $t(v) = -1$ then its parent has type zero and thus the parent has $d = 0$. Let $q = 0$ if $v$'s grandfather has $d = 0$ and otherwise let $q$ be the weight of the edge connecting $v$'s father to $v$'s grandfather. Let $s$ denote the sum of the weights of the edges connecting $v$'s father to its positive children (these are also the positive siblings of $v$). We put $d(v) = 1$ if and only if $s + q = 0 \bmod p$ and $v$ is the first among its negative siblings for which the value of $d$ is determined.

We claim that the function $d$ is a $p$-NZDF. Notice also that $d$ is $0-1$ on all vertices except, maybe, a positive child of some negative cycle vertex $r_i$ that is connected to $r_i$ with an edge whose weight is $p-1$ and $g(r_i) = 0 \bmod p$ (this cannot happen if $G$ is a tree). We have already shown that $d$ is a legitimate $p$-NZDF on the vertices of $C$. Let $v$ be any vertex not on $C$, and let $y$ be its parent. Let $g(v) = d(v) + \sum_{u \in N(v)} w(u,v)d(u)$.

- If $t(v) = 1$ and $d(v) \neq 0$ then all children of $v$ have $d = 0$. If $d(y) = 0$ then $g(v) = d(v) \neq 0 \bmod p$. Otherwise we must have $d(y) = 1$, $y$ is on the cycle, and $t(y) = -1$. In this case $g(v) = d(v) + w(v,y)$. If $w(v,y) = p-1$ then recall that we defined $d(v) = p-1$ in this case so $g(v) = 2p - 2 \neq 0 \bmod p$. If $w(v,y) \neq p-1$ then we defined $d(v) = 1$ so $g(v) = 1 + w(v,y) \neq 0 \bmod p$.

- If $t(v) = 1$ and $d(v) = 0$ then we must have $d(y) \neq 0$ and all positive children of $v$ have $d = 1$, but since the sum of the weights from $v$ to its positive children is $0 \bmod p$, they contribute nothing to $g(v) \bmod p$. Thus, $g(v) = d(y)w(u,y) \bmod p$. Since $d(y) \in \{1, p-1\}$ we have $g(v) \neq 0 \bmod p$.

- If $t(v) = 0$ then recall that we have intentionally selected either zero or one of the negative children of $v$ to have $d = 1$ (and all other negative children to have $d = 0$) so as to guarantee that $g(v) \neq 0 \bmod p$.

- If $t(v) = -1$ then $t(y) = 0$, and hence $d(y) = 0$. Now, if $d(v) = 1$ then all children of $v$ have $d = 0$ and hence $g(v) = 1 \neq 0 \bmod p$. Otherwise we must have $d(v) = 0$ and then all (and nothing else but) its positive children have $d = 1$. Since the sum of the weights from $v$ to its positive children is not $0 \bmod p$ we have $g(v) \neq 0 \bmod p$ in this case. $\square$

# 3 Proof of Proposition 1.5

Clearly the minimal counterexamples must be connected graphs, and they cannot have a vertex of degree $n-1$, where $n$ is the number of vertices.

Figure 1: The minimal counterexamples

By Theorem 1.4 they also cannot be a cycle or a tree. Hence $n \geq 5$. If there exists a counterexample with 5 vertices it must have 5, 6 or 7 edges (in fact, a minimal counterexample for the general non $0-1$ case cannot have 5 edges by Theorem 1.3). There is one possible graph with 7 edges, five with 6 edges and three with 5 edges. Each can be separately verified (even manually) to always have a $0-1$ $p$-NZDF for all $p > 1$. Hence, $n \geq 6$. The smallest counterexample of a graph having no $0-1$ 3-NZDF appears in the left side of Figure 1. It has 6 vertices, 9 edges, three of which have weight 2. Hence, as a multigraph it has 12 edges. (It is not too difficult to prove manually that graphs with 6 vertices and 6,7 or 8 edges always have a $0,1$ $p$-NZDF for all $p > 1$). The smallest counterexample of a graph having no general 3-NZDF appears in the right side of Figure 1. It also has 6 vertices, 12 edges, four of which have weight 2. Hence, as a multigraph it has 16 edges. The latter graph was verified as minimal with the aid of a computer. Both minimal counterexamples are unique up to isomorphism. Notice the somewhat amusing fact that both counterexamples are planar, showing that one cannot replace "unicyclic" with "planar" in the statement of Theorem 1.3.

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