Monotone paths in edge-ordered sparse graphs

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Abstract

An *edge-ordered graph* is an ordered pair (G, f), where $G = G(V, E)$ is a graph and f is a bijective function, $f : E(G) \to \{1, 2, ..., |E(G)|\}$. f is called an *edge ordering* of G. A *monotone path of length* k in (G, f) is a simple path $P_{k+1} : v_1, v_2..., v_{k+1}$ in G such that either, $f((v_i, v_{i+1})) < f((v_{i+1}, v_{i+2}))$ or $f((v_i, v_{i+1})) > f((v_{i+1}, v_{i+2}))$ for $i = 1, 2, ..., k-1$. Given an undirected graph G, denote by $\alpha(G)$ the minimum over all edge orderings of the maximum length of a monotone path. In this paper we give bounds on $\alpha(G)$ for various families of sparse graphs, including trees, planar graphs and graphs with bounded arboricity.

1 Introduction

All graphs considered here are finite, undirected and simple, unless noted otherwise. For the standard graph-theoretic terminology the reader is referred to [3]. An *edge-ordered graph* is an ordered pair (G, f), where $G = G(V, E)$ is a graph and f is a bijective function, $f : E(G) \to \{1, 2, ..., |E(G)|\}$. f is called an *edge ordering* of G. A *monotone path of length* k in (G, f) is a simple path $P_{k+1} : v_1, v_2..., v_{k+1}$ in G such that either, $f((v_i, v_{i+1})) < f((v_{i+1}, v_{i+2}))$ or $f((v_i, v_{i+1})) > f((v_{i+1}, v_{i+2}))$ for $i = 1, 2, ..., k-1$. Given a graph G denote by $\alpha(G)$ the minimum over all edge orderings of the maximum length of a monotone path. Denote by $\alpha'(G)$ the minimum over all edge orderings of the maximum length of a monotone trail (in a trail vertices may appear more than once; a simple cycle is also considered a trail in our definition). Clearly, $\alpha(G) \leq \alpha'(G)$.

The problem of estimating $\alpha(K_n)$ was raised first by Chvátal and Komlós [5]. Graham and Kleitman [6] proved that:

$$\frac{1}{2}(\sqrt{4n-3}-1) < \alpha(K_n) < \frac{3}{4}n.$$

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The upper bound was improved by Calderbank, Chung and Sturtevant [4], showing,

$$\alpha(K_n) \leq (\frac{1}{2} + o(1))n.$$

They also conjectured that this is the right order of magnitude of $\alpha(K_n)$. However, no improvement upon the Graham-Kleitman lower bound is known.

There are very few results regarding $\alpha(G)$ for general graphs G. Bialostocki and Roditty [2] have characterized all the graphs G with $\alpha(G) \leq 2$. In fact, they showed that if $\alpha(G) \geq 3$ then either G is an odd cycle of length at least 5, or G contains as a subgraph one of six fixed graphs.

In this paper we give upper and lower bounds for $\alpha(G)$ and $\alpha'(G)$ for graphs G belonging to various well-known graph families. In order to describe our results we need to recall a few definitions. The *arboricity* of a graph G is the minimum number of subforests of G whose union covers all the edges of G. The *linear arboricity* $la(G)$ and *star arboricity* $st(G)$ are defined analogously, except that one requires that each forest contains only simple paths (in the linear arboricity case) or stars (in the star arboricity case). We assume the reader is familiar with the concepts of planar and bipartite planar graphs. Our main results can be summarized in the following two theorems:

Theorem 1.1 *Every planar graph G has $\alpha(G) \leq \alpha'(G) \leq 9$. There exist planar graphs with $\alpha(G) \geq 5$. Every bipartite planar graph G has $\alpha(G) \leq \alpha'(G) \leq 6$. There exist bipartite planar graphs with $\alpha(G) \geq 4$.*

Note the gap between the lower and upper bound in Theorem 1.1. It is rather difficult to find planar graphs where every edge ordering has a long monotone path. In fact, even the proof of the existence of such a graph with $\alpha(G) \geq 5$ is rather involved. The same difficulty holds for bipartite planar graphs. Thus, we have the following interesting open problems:

Problem 1: Determine the constant K_1 which is defined as $K_1 = \max \alpha(G)$ where the maximum is taken over all planar graphs.

Problem 1: Determine the constant K_2 which is defined as $K_2 = \max \alpha(G)$ where the maximum is taken over all bipartite planar graphs.

We currently have $5 \leq K_1 \leq 9$ and $4 \leq K_2 \leq 6$.

The next theorem supplies upper bounds for $\alpha(G)$ in terms of arboricity and maximum degree.

Theorem 1.2

1. *$\alpha(G) \leq \alpha'(G) \leq 3a(G)$. In particular, if G is a tree then $\alpha(G) \leq 3$.*
2. *$\alpha'(G) \leq 2st(G)$.*
3. *$\alpha'(G) \leq 2la(G)$.*

4. *If G has maximum degree $\Delta(G)$ then $\alpha'(G) \le 2a(G) + O(\log \Delta(G))$.*

5. *If G has maximum degree $\Delta(G)$ then $\alpha'(G) \le \Delta(G) + 1$.*

2 Proof of the main results

In this section we prove Theorem 1.1 and Theorem 1.2. We first prove a lemma which bounds $\alpha(G)$ in case G is a tree. A *caterpillar* is a tree $T(a_1, \ldots, a_t)$ which consists of a main path on t vertices, where a star with a_i edges is attached to vertex number i in the path. The vertices on the main path are the roots of the stars. Thus, T has $t + a_1 + \ldots + a_t$ vertices. Note that the family of caterpillars includes stars (the case $t = 1$), paths (the case where all $a_i = 0$), double stars (the case $t = 2$), etc.

Lemma 2.1 *If T is a tree then $\alpha(T) \le 3$. If T is a caterpillar with at least three vertices then $\alpha(T) = 2$.*

Proof: We first prove that $\alpha(T) \le 3$. Consider a rooted orientation of T. Hence, every directed edge (u, v) has a *layer* which is defined as the distance from v to the root. Let X denote the set of edges in odd layers and let $Y = E(T) \setminus X$ denote the set of edges in even layers. We define an edge ordering as follows. Arbitrarily assign the numbers $1, \ldots, |X|$ to the edges of X, and the numbers $|X| + 1, \ldots, e(T)$ to the edges of Y. Clearly, every monotone path contains edges from at most two consecutive layers. Since each layer induces a set of stars, each layer may contain at most two edges from the path, and if the path contains edges from two layers i and $i + 1$, then there is at most one edge from layer $i + 1$. Thus, any monotone path contains at most three edges.
Assume now that T is a caterpillar with at least three vertices. Since there are two adjacent edges, $\alpha(T) \ge 2$. We construct an edge ordering with no monotone path of length 3. If t is a star this is trivial. So we assume $T = T(a_1, \ldots, a_t)$ where $t \ge 2$. Let $m = t + a_1 + \ldots + a_t - 1$ denote the number of edges of T, and let $v_1, \ldots, v_t$ denote the main path of T. Partition the edges on the main path into two matchings X and Y where X consists of the edges (v_i, v_{i+1}) where i is odd, and Y consists of the edges (v_i, v_{i+1}) where i is even. Arbitrarily assign the numbers $1, \ldots, |X|$ to the edges of X, the numbers $m - |Y| + 1, \ldots, m$ to the edges of Y, and the numbers $|X| + 1, \ldots, m - |Y|$ to the other edges. It is immediate to check that there is no monotone path of length 3. □

The next lemma establishes a bound on $\alpha'(G)$ in terms of covering subgraphs.

Lemma 2.2 *Let G be a graph, and let $H_1, \ldots, H_k$ be subgraphs of G such that each edge of G appears in at least one of the H_i. Then $\alpha'(G) \le \sum_{i=1}^{k} \alpha'(H_i)$.*

Proof:, Since α' is a monotone increasing graph parameter (adding edges cannot decrease α') we may assume each edge of G appears in *exactly* one of the H_i. Let h_i denote the number of edges of H_i. Thus, $h_1 + \ldots + h_k = e(G)$. Let f_i be an edge ordering of H_i in which every monotone trail is of length at most $\alpha'(H_i)$ for $i = 1, \ldots, k$. We define an edge ordering f of G as follows. For each $e \in G$, if $e \in H_i$ then $f(e) = f_i(e) + h_1 + \ldots + h_{i-1}$. Clearly, f is a bijection, and thus an edge ordering. Furthermore, if both e and e' belong to H_i then if e'' satisfies $f(e) < f(e'') < f(e')$ then we must have $e'' \in H_i$. Thus, in every monotone trail in (G, f) all the edges belonging to H_i appear consecutively in the trail. Hence, there are at most $\alpha'(H_i)$ edges from H_i in the trail. Consequently, the trail contains at most $\sum_{i=1}^{k} \alpha'(H_i)$ edges. □

Proof of Theorem 1.2: The upper bounds appearing in Theorem 1.2 follow by applying Lemma 2.1 and Lemma 2.2 to several powerful results in graph theory. We consider the various items in Theorem 1.2:

1. A graph G with arboricity $k = a(G)$ can be decomposed into k forests $H_1, \ldots, H_k$. By Lemma 2.1, $\alpha(H_i) = \alpha'(H_i) \le 3$. Hence, by Lemma 2.2, $\alpha(G) \le \alpha'(G) \le 3k$.

2. A graph G with star arboricity $k = st(G)$ can be decomposed into k stars $H_1, \ldots, H_k$. By Lemma 2.1, $\alpha(H_i) = \alpha'(H_i) \le 2$. Hence, by Lemma 2.2, $\alpha(G) \le \alpha'(G) \le 2k$.

3. A graph G with linear arboricity $k = la(G)$ can be decomposed into k paths $H_1, \ldots, H_k$. By Lemma 2.1, $\alpha(H_i) = \alpha'(H_i) \le 2$. Hence, by Lemma 2.2, $\alpha(G) \le \alpha'(G) \le 2k$. Note that the last two items could be united and extended by defining the *caterpillar arboricity* of a graph in the obvious manner, thus achieving a more powerful bound.

4. Alon, McDiarmid and Reed [1] proved the very powerful statement that:

$$st(G) \le a(G) + O(log\ \Delta(G)).$$

   Since $\alpha'(G) \le 2stG)$ it follows that $\alpha'(G) \le 2a(G) + O(\log \Delta(G))$. Note that this bound is superior to the bound $\alpha'(G) \le 3a(G)$ in case the logarithm of the maximum degree is significantly smaller than the arboricity.

5. By Vizing's Theorem (cf. [3]), every graph G with maximum degree $\Delta(G)$ can be decomposed into $k \le \Delta(G) + 1$ matchings $H_1, \ldots, H_k$. Since, trivially, $\alpha'(H_i) = 1$ we have by Lemma 2.2 that $\alpha'(G) \le \Delta(G) + 1$.

□

Proof of Theorem 1.1: Nash-Williams has shown in [7] that

$$a(G) = \max_{H \subseteq G} \lceil \frac{e(H)}{|V(H)|-1} \rceil.$$

(we assume in the last equality that H contains at least one edge). Since every n-vertex planar graph contains at most $3n-6$ edges, and since subgraphs of planar graphs are planar, it follows that $a(G) \leq 3$ for a planar graph G. Thus, we have that for every planar graph, $\alpha(G) \leq \alpha'(G) \leq 9$. Similarly, since every n-vertex bipartite planar graph contains at most $2n-4$ edges, and since subgraphs of bipartite planar graphs are bipartite and planar, it follows that $a(G) \leq 2$ for a bipartite planar graph G. Thus, we have that for every bipartite planar graph, $\alpha(G) \leq \alpha'(G) \leq 6$. We now turn to the construction of lower bounds for planar and bipartite planar graphs. We begin with the planar case. For $n \geq 3$, define the planar graph G_n as follows: Draw a cycle with n vertices on the plane, and denote the vertices by the numbers $1, \ldots, n$. Now add a new vertex, denoted by a, inside the cycle, and connect a to each vertex on the cycle. Then, add a new vertex b outside the cycle, and connect b to each vertex on the cycle. Note that G_n is, indeed, planar, has $n+2$ vertices and $3n = 3(n+2)-6$ edges. We claim that for n sufficiently large, $\alpha(G_n) \geq 5$. This is proved in the following lemma:

Lemma 2.3 *For each $n > 98$, $\alpha(G_n) \geq 5$.*

Proof: Let f be an arbitrary edge ordering of G_n. For simplicity we shall use the notation $e' > e$ whenever $f(e') > f(e)$ for two edges e, e' of G_n. There are exactly $2n$ stars with three edges in G_n which contain as the center a vertex i of the cycle, and the edges (i,a), (i,b) and (i,j) where $j = i-1$ or $j = i+1$ (if $i = n$ then we use the convention $n+1 = 1$). Let L be the set of these $2n$ stars. We distinguish three cases:

1. L contains at least 17 elements in which the edge (i,j) is smaller than both (i,a) and (i,b). By majority, we may assume w.l.o.g. that in at least 9 of these elements $(i,j) < (i,a) < (i,b)$. Once again, by majority we may assume w.l.o.g. that in at least 5 of these elements $j = i+1$. Since each i is a root of exactly one element of L which contains the edge $(i, i+1)$, we have that there exist three such elements of L, whose roots are i_1, i_2, i_3 such that $i_1 + 1 < i_2$ and $i_2 + 1 < i_3$. Consider the largest edge of all the 9 edges of these three stars. Such an edge must be one of the edges (i_j, b) for $j = 1, 2, 3$. Assume w.l.o.g. that this is the edge (i_3, b). Now, if $(i_1, a) < (i_2, a)$ then we have the following monotone simple path of length 5: $(i_1+1, i_1, a, i_2, b, i_3)$ since $(i_1, i_1+1) < (i_1, a) < (i_2, a) < (i_2, b) < (i_3, b)$. Otherwise, $(i_2, a) < (i_i, a)$ and then we have the monotone simple path $(i_2+1, i_2, a, i_1, b, i_3)$ since $(i_2, i_2+1) < (i_2, a) < (i_1, a) < (i_1, b) < (i_3, b)$.

2. L contains at least 17 elements in which the edge (i, j) is larger than both (i, a) and (i, b). This case is analogous to the previous one.

3. The remaining case is where there is a subset L' of at least $2n - 33$ elements of L in which the edge (i, j) is in the between (i, a) and (i, b). Let (a, s_1) be the smallest edge adjacent to a, and let (a, t_1) be the largest edge adjacent to a. Similarly, let (b, s_2) be the smallest edge adjacent to b and let (b, t_2) be the largest edge adjacent to b (note that $s_1 \neq t_1$ and $s_2 \neq t_2$ but it may be that $s_1 = s_2$ or $s_1 = t_2$ or $s_2 = t_1$ or $t_1 = t_2$). Since each vertex from the cycle $(1, \ldots, n)$ appears in exactly 4 elements of L, there is a subset L'' of L' of at least $2n - 49$ elements which do not contain any vertex of $\{s_1, t_1, s_2, t_2\}$. Since each edge of the cycle $(1, \ldots, n)$ appears in exactly two elements of L there are at least $n - 49$ edges of the cycle $(1, \ldots, n)$ which have the property that both copies of L containing the edge appear in L''. Since $n - 49 > n/2$, there are at least two consecutive edges in the cycle $(1, \ldots, n)$, denote them by $(i, i+1)$ and $(i+1, i+2)$, such that both copies of L containing $(i, i+1)$ appear in L'' and both copies of L containing $(i+1, i+2)$ appear in L''. In other words, $(i, i+1)$ is between (i, a) and (i, b), and also between $(i+1, a)$ and $(i+1, b)$ and, similarly, $(i+1, i+2)$ is between $(i+1, a)$ and $(i+1, b)$ and also between $(i+2, a)$ and $(i+2, b)$. Assume, w.l.o.g. that $(i+1, a) < (i+1, b)$ and that $(i, i+1) < (i+1, i+2)$. Now, if $(i, b) > (i, i+1)$ we are done since we can take the simple monotone path $(s_1, a, i+1, i, b, t_2)$. Otherwise, if $(i+2, a) < (i+1, i+2)$ we are done since we can take the simple monotone path $(s_1, a, i+2, i+1, b, t_2)$. Otherwise we have that both $(i, b) < (i, i+1)$ and $(i+2, a) > (i+1, i+2)$ and in this case we can take the simple monotone path $(s_2, b, i, i+1, i+2, a, t_1)$ (note that this path is of length 6).

□

We now turn to the bipartite planar case which is simpler. Recall the graph $K_{2,5}$ which is bipartite planar. This graph does the job:

Lemma 2.4 $\alpha(K_{2,5}) = \alpha'(K_{2,5}) = 4$.

Proof: The fact that $\alpha'(K_{2,5}) \leq 4$ follows from the fact that $st(K_{2,5}) = 2$ and Theorem 1.2. It remains to show that $\alpha(K_{2,5}) \geq 4$. Let f be an arbitrary edge-ordering of $K_{2,5}$. Let the vertices of degree 5 be denoted by the letters x, y, and let the vertices of degree 2 be denoted by the numbers $1, 2, 3, 4, 5$. Hence, for each $i = 1, 2, ..., 5$ the sequence (x, i, y) is a simple path having two edges. For each i there are two options, either$f((x, i)) < f((i, y))$ or$f((x, i)) > f((i, y))$. Since $i = 1, 2, ..., 5$ we can assume without loss of generality that at least for three distinct values of i it happens that $f((x, i)) < f((i, y))$, and once again we may assume that it happens for $i = 1, 2, 3$. Thus, we may assume$f((x, 1)) < f((1, y))$ and $f((x, 2)) < f((2, y))$ and $f((x, 3)) < f((3, y))$. Look

at the maximum of all these six edges, and assume w.l.o.g. that it is $f((3, y))$ (of course it has to be some edge adjacent to y). Now, if $f((x, 1)) < f((x, 2))$ then the path $(1, x, 2, y, 3)$ is a monotone simple path of length 4 since $f((x, 1)) < f((x, 2)) < f((2, y)) < f((3, y))$. Otherwise $f((x, 2)) < f((x, 1))$ and then the path $(2, x, 1, y, 3)$ is a monotone simple path of length 4 since $f((x, 2)) < f((x, 1)) < f((1, y)) < f((3, y))$. $\square$
This completes the proof of Theorem 1.1. $\square$

We end this section with a conjecture. Let $ca(G)$ denote the caterpillar arboricity of a graph G.

Conjecture 2.5 *Every planar graph G has $ca(G) \leq 4$.*

The motivation of this conjecture is the following: As mentioned above, every planar graph has arboricity $a(G) \leq 3$, where every planar graph with more than $2n - 2$ edges has $a(G) = 3$. It has recently been proved that the star arboricity of every planar graph is $st(G) \leq 5$, and equality is known for some planar graphs. A forest of caterpillars is more general than a forest of stars, and less general then an arbitrary forest. Thus, $ca(G) \leq 5$ and $ca(G) \geq 3$. Furthermore It is easy to construct triangulated planar graphs (i.e. graphs with $3n - 6$ edges) which do not contain a caterpillar as a spanning subgraph. These graphs obviously cannot be covered by three caterpillar forests. Thus, there exist planar graphs with $ca(G) \geq 4$. Note that the assertion of Conjecture 2.5 would imply, using Lemma 2.1 and Lemma 2.2, that $\alpha'(G) \leq 8$ for every planar graph G.

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