Mean Ramsey-Turán numbers

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Abstract

A ρ-mean coloring of a graph is a coloring of the edges such that the average number of colors incident with each vertex is at most ρ. For a graph H and for $\rho \geq 1$, the *mean Ramsey-Turán number* $RT(n, H, \rho - mean)$ is the maximum number of edges a ρ-mean colored graph with n vertices can have under the condition it does not have a monochromatic copy of H. It is conjectured that $RT(n, K_m, 2 - mean) = RT(n, K_m, 2)$ where $RT(n, H, k)$ is the maximum number of edges a k edge-colored graph with n vertices can have under the condition it does not have a monochromatic copy of H. We prove the conjecture holds for K_3. We also prove that $RT(n, H, \rho - mean) \leq RT(n, K_{\chi(H)}, \rho - mean) + o(n^2)$. This result is tight for graphs H whose clique number equals their chromatic number. In particular we get that if H is a 3-chromatic graph having a triangle then $RT(n, H, 2 - mean) = RT(n, K_3, 2 - mean) + o(n^2) = RT(n, K_3, 2) + o(n^2) = 0.4n^2(1 + o(1))$.

1 Introduction

All graphs considered are finite, undirected and simple. For standard graph-theoretic terminology see [1]. Ramsey and Turán type problems are central problems in extremal graph theory. These two topics intersect in *Ramsey-Turán Theory* which is now a wide field of research with many interesting results and open problems. The survey of Simonovits and Sós [11] is an excellent reference for Ramsey-Turán Theory.

The *Ramsey number* $R(H, k)$ is the minimum integer n such that in any k-coloring of the edges of K_n there is a monochromatic H. An edge coloring is called *k-local* if every vertex is incident with at most k colors. The *local Ramsey number* $R(H, k - loc)$ is the minimum integer n such that in any k-local coloring of the edges of K_n there is a monochromatic H. An edge coloring is called *ρ-mean* if the average number of colors incident with each every vertex is at most ρ. The *mean Ramsey number* $R(H, \rho - mean)$ is the minimum integer n such that in any ρ-mean coloring of the edges of K_n

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there is a monochromatic H. Clearly, $R(H, k) \leq R(H, k-loc) \leq R(H, k-mean)$. The relationship between these three parameters has been studied by various researchers. See, e.g., [2, 4, 7, 10]. In particular, Gyárfás et. al. [7] proved that $R(K_m, 2) = R(K_m, 2-loc)$. Caro and Tuza proved that $R(K_m, 2-loc) = R(K_m, 2-mean)$ and Schelp [10] proved that $R(K_m, k-loc) = R(K_m, k-mean)$.

The *Ramsey-Turán number* $RT(n, H, k)$ is the maximum number of edges a k-colored graph with n vertices can have under the condition it does not have a monochromatic copy of H. We analogously define the local and mean Ramsey-Turán numbers, denoted $RT(n, H, k-loc)$ and $RT(n, H, \rho-mean)$ respectively, to be the maximum number of edges a k-local (resp. ρ-mean) colored graph with n vertices can have under the condition it does not have a monochromatic copy of H. Clearly, $RT(n, H, k) \leq RT(n, H, k-loc) \leq RT(n, H, k-mean)$.

The relationship between $RT(n, H, k)$, Ramsey numbers and Turán numbers is well-known. The Turán graph $T(n, k)$ is the complete k-partite graph with n vertices whose vertex classes are as equal as possible. Let $t(n, k)$ be the number of edges of $T(n, k)$. Burr, Erdős and Lovász [3] introduced the Ramsey function $r(H, k)$ which is the smallest integer r for which there exists a complete r-partite graph having the property that any k edge-coloring of it has a monochromatic H. For example, $r(K_m, k) = R(K_m, k)$ and $r(C_5, 2) = 5$. Clearly, $RT(n, K_m, k) = t(n, R(K_m, k) - 1)$. As shown in Theorem 13 in [11], it follows from the Erdős-Stone Theorem [6] that

$$RT(n, H, k) = \left(1 - \frac{1}{r(H, k) - 1}\right)\binom{n}{2} + o(n^2).$$

Clearly, a similar relationship holds between $RT(n, H, k-loc)$ and the analogous Ramsey function $r(H, k-loc)$. However, no such relationship is known for $RT(n, H, k-mean)$. We conjecture that such a relationship holds.

Conjecture 1.1

$$RT(n, H, k-mean) = \left(1 - \frac{1}{r(H, k-mean) - 1}\right)\binom{n}{2} + o(n^2).$$

Combining this with the fact that $R(K_m, 2) = R(K_m, 2-loc) = R(K_m, 2-mean)$ we have the following stronger conjecture for complete graphs and $k = 2$.

Conjecture 1.2

$$RT(n, K_m, 2-mean) = RT(n, K_m, 2) = t(n, R(K_m, 2) - 1).$$

For non-integral values of ρ is is not even clear what the right conjecture for $RT(n, H, \rho-mean)$ should be.

The first result of this paper shows that Conjecture 1.2 holds for K_3.

Theorem 1.3 $RT(n, K_3, 2-mean) = RT(n, K_3, 2) = t(n, R(K_3, 2) - 1) = t(n, 5) = \lfloor 0.4n^2 \rfloor$.

The second result of this paper asserts that $RT(n,H,\rho-mean)$ is bounded by a function of the chromatic number of H. In fact, for graphs whose clique number equals their chromatic number, $RT(n,H,\rho-mean)$ is essentially determined by the chromatic number of H.

Theorem 1.4 *For all $\rho \geq 1$ and for all graphs H, $RT(n,H,\rho-mean) \leq RT(n,K_{\chi(H)},\rho-mean)+o(n^2)$. In particular, if the chromatic number of H equals its clique number then $RT(n,H,\rho-mean) = RT(n,K_{\chi(H)},\rho-mean)+o(n^2)$.*

The proof of Theorem 1.4 uses a colored version of Szemerédi's Regularity Lemma together with several additional ideas. Notice that the trivial case $\rho = 1$ in Theorem 1.4 is equivalent to the Erdős-Stone Theorem. Combining Theorem 1.3 with Theorem 1.4 we obtain:

Corollary 1.5 *Let H be a 3-chromatic graph. Then, $RT(n,H,2-mean) \leq 0.4n^2(1+o(1))$. If H contains a triangle then $RT(n,H,2-mean) = 0.4n^2(1+o(1))$.* ■

The next section contains the proof of Theorem 1.3. Section 3 contains the proof of Theorem 1.4.

2 Proof of Theorem 1.3

We need to prove that $RT(n,K_3,2-mean) = t(n,5)$. Since K_5 has a 2-coloring with no monochromatic triangle, so does $T(n,5)$. Hence, $RT(n,K_3,2-mean) \geq t(n,5)$. We will show that $RT(n,K_3,2-mean) \leq t(n,5)$. Clearly, the result is trivially true for $n < 6$, so we assume $n \geq 6$. Our proof proceeds by induction on n. Let G have $n \geq 6$ vertices and more than $t(n,5)$ edges. Clearly we may assume that G has precisely $t(n,5)+1$ edges. Consider any given 2-mean coloring of G. If $n = 6$ then $G = K_6$. Recall from the introduction that $R(K_3,2-mean) = R(K_3,2) = 6$. As a 2-mean coloring of K_6 contains a monochromatic triangle this base case of the induction holds. If $n = 7$ then G is K_7^-. Again, it is trivial to check that any 2-mean coloring of K_7^- contains a monochromatic triangle. Similarly, if $n = 8$ then G is a K_8 missing two edges and it is straightforward to verify that any 2-mean coloring of such a G contains a monochromatic triangle.

Assume the theorem holds for all $6 \leq n' < n$ and $n \geq 9$. For a vertex v, let $c(v)$ denote the number of colors incident with v and let $d(v)$ denote the degree of v.

If some v has $c(v) \geq 2$ and $d(v) \leq 4n/5$ then $G - v$ is also 2-mean colored and has more than $t(n-1,5)$ edges. Hence, by the induction hypothesis, $G - v$ has a monochromatic triangle.

Otherwise, if some v has $c(v) = 1$ and $d(v) \leq 3n/5$ then let w be a vertex with maximum $c(w)$. Then, $G - \{v,w\}$ is also 2-mean colored and has more than $t(n-2,5)$ edges. Hence, by the induction hypothesis, $G - \{v,w\}$ has a monochromatic triangle.

Otherwise, if v is an isolated vertex of G then let u and w be two distinct vertices having maximum $c(u)+c(w)$. Then, $G - \{v,u,w\}$ is 2-mean colored and has more than $t(n-3,5)$ edges. Hence, by the induction hypothesis, $G - \{v,u,w\}$ has a monochromatic triangle.

We are left with the case where $\delta(G) > 3n/5$ and whenever $c(v) \geq 2$ then also $d(v) > 4n/5$. Let v be with $c(v) = 1$ (if no such v exists then the graph is 2-local colored and hence contains a monochromatic triangle as, trivially, $RT(n, K_3, 2 - loc) = t(n, 5)$). We may assume that $3n/5 < d(v) \leq 4n/5$, since otherwise we would have $\delta(G) > 4n/5$ which is impossible for a graph with $t(n, 5) + 1$ edges. Consider the neighborhood of v, denoted $N(v)$. Clearly, if $w \in N(v)$ then $c(w) > 1$ otherwise (because $d(w) > 3n/5$) there must be some $w' \in N(v)$ for which (v, w, w') is a monochromatic triangle and we are done. Thus, the minimum degree of $G[N(v)]$ is greater than $d(v) - n/5$. Since $d(v) > 3n/5$ it follows that $G[N(v)]$ has minimum degree greater than $2|N(v)|/3$. If $|N(v)|$ is divisible by 3 then the theorem of Corrádi and Hajnal [5] implies that $G[N(v)]$ has a triangle factor. If $|N(v)| - 1$ is divisible by 3 then the theorem of Hajnal and Szemerédi [8] implies that $G[N(v)]$ has a factor into $(|N(v)| - 4)/3$ triangles and one K_4. If $|N(v)| - 2$ is divisible by 3 then, similarly, $G[N(v)]$ has a factor into $(|N(v)| - 8)/3$ triangles and two K_4 or $(|N(v)| - 5)/3$ triangles and one K_5. Assume that G has no monochromatic triangle. The sum of colors incident with the vertices of any non-monochromatic triangle is at least $5 = 3 \cdot (5/3)$. The sum of colors incident with the vertices of any K_4 having no monochromatic triangle is at least $8 > 4 \cdot (5/3)$. The sum of colors incident with the vertices of any K_5 having no monochromatic triangle is at least $10 > 5 \cdot (5/3)$. Thus,

$$2n \geq \sum_{v \in V} c(v) \geq n + \frac{5}{3} d(v) > n + \frac{5}{3} \cdot \frac{3}{5} n = 2n$$

a contradiction. ■

3 Proof of Theorem 1.4

Before we prove Theorem 1.4 we need several to establish several lemmas.

Lemma 3.1 *For every $\epsilon > 0$ there exists $\alpha = \alpha(\epsilon) > 0$ such that for all m sufficiently large, if a graph has m vertices and more than $RT(m, K_s, \rho - mean) + \epsilon m^2/4$ edges and is $(\rho + \alpha)$-mean colored, then it has a monochromatic K_s.*

Proof: Pick α such that $\epsilon m^2/4 > (\alpha m + 1)(m - 1)$ for all sufficiently large m. Given a graph G with m vertices and more than $RT(m, K_s, \rho - mean) + \epsilon m^2/4$ edges, consider a $(\rho + \alpha)$-mean coloring of G. By picking $\lceil \alpha m \rceil$ non-isolated vertices of G and deleting all edges incident with them we obtain a spanning subgraph of G with m vertices, more than $RT(m, K_s, \rho - mean) + \epsilon m^2/4 - (\alpha n + 1)(n - 1) \geq RT(m, K_s, \rho - mean)$ edges, and which is ρ-mean colored. By definition, it has a monochromatic K_s. ■

Lemma 3.2 *If n is a multiple of m then $RT(n, K_s, \rho - mean) \geq RT(m, K_s, \rho - mean) n^2/m^2$.*

Proof: Let G be a graph with m vertices and $RT(m, K_s, \rho - mean)$ edges having a ρ-mean coloring without a monochromatic K_s. Let G' be obtained from G by replacing each vertex v with an independent set X_v of size n/m. For $u \neq v$, we connect a vertex from X_u with a vertex from X_v if and only if uv is an edge of G, and we color this edge with the same color of uv. Clearly, G' has $RT(m, K_s, \rho - mean)n^2/m^2$ edges, the corresponding coloring is also ρ-mean, and there is no monochromatic K_s in G'. As G' has n vertices we have that $RT(n, K_s, \rho - mean) \geq RT(m, K_s, \rho - mean)n^2/m^2$. ■

As mentioned in the introduction, our main tool in proving Theorem 1.4 is a colored version of Szemerédi's Regularity Lemma. We now give the necessary definitions and the statement of the lemma.

Let $G = (V, E)$ be a graph, and let A and B be two disjoint subsets of V. If A and B are non-empty, let $e(A, B)$ denote the number of edges with one endpoint in A and another endpoint in B and define the *density of edges* between A and B by

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

For $\gamma > 0$ the pair (A, B) is called *γ-regular* if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \gamma|A|$ and $|Y| > \gamma|B|$ we have

$$|d(X, Y) - d(A, B)| < \gamma.$$

An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \ldots, V_m$ of almost equal size, i.e., $\big||V_i| - |V_j|\big| \leq 1$ for all i, j. An equitable partition of the set of vertices V of G into the classes $V_1, \ldots, V_m$ is called *γ-regular* if $|V_i| < \gamma|V|$ for every i and all but at most $\gamma\binom{m}{2}$ of the pairs (V_i, V_j) are γ-regular. Szemerédi [12] proved the following.

Lemma 3.3 *For every $\gamma > 0$, there is an integer $M(\gamma) > 0$ such that for every graph G of order $n > M$ there is a γ-regular partition of the vertex set of G into m classes, for some $1/\gamma < m < M$.*

To prove Theorem 1.4 we will need a colored version of the Regularity Lemma. Its proof is a straightforward modification of the proof of the original result (see, e.g., [9] for details).

Lemma 3.4 *For every $\gamma > 0$ and integer r, there exists an $M(\gamma, r)$ such that if the edges of a graph G of order $n > M$ are r-colored $E(G) = E_1 \cup \cdots \cup E_r$, then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_m$, with $1/\gamma < m < M$, which is γ-regular simultaneously with respect to all graphs $G_i = (V, E_i)$ for $1 \leq i \leq r$.*

A useful notion associated with a γ-regular partition is that of a *cluster graph.* Suppose that G is a graph with a γ-regular partition $V = V_1 \cup \cdots \cup V_m$, and $\eta > 0$ is some fixed constant (to be thought of as small, but much larger than γ.) The cluster graph $C(\eta)$ is defined on the vertex set $\{1,\ldots,m\}$ by declaring ij to be an edge if (V_i,V_j) is a γ-regular pair with edge density at least η. From the definition, one might expect that if a cluster graph contains a copy of a fixed clique then so does the original graph. This is indeed the case, as established in the following well-known lemma (see [9]), which says more generally that if the cluster graph contains a K_s then, for any fixed t, the original graph contains the Turán graph $T(st,s)$.

Lemma 3.5 *For every $\eta > 0$ and positive integers s,t there exist a positive $\gamma = \gamma(\eta,s,t)$ and a positive integer $n_0 = n_0(\eta,s,t)$ with the following property. Suppose that G is a graph of order $n > n_0$ with a γ-regular partition $V = V_1 \cup \cdots \cup V_m$. Let $C(\eta)$ be the cluster graph of the partition. If $C(\eta)$ contains a K_s then G contains a $T(st,s)$.*

Proof of Theorem 1.4: Fix an s-chromatic graph H and fix a real $\rho \geq 1$. We may assume $s \geq 3$ as the theorem is trivially true (and meaningless) for bipartite graphs. Let $\epsilon > 0$. We prove that there exists $N = N(H,\rho,\epsilon)$ such that for all $n > N$, if G is a graph with n vertices and more than $RT(n,K_s,\rho-mean)+\epsilon n^2$ edges then any ρ-mean coloring of G contains a monochromatic copy of H.

We shall use the following parameters. Let t be the smallest integer for which $T(st,s)$ contains H. Let $r = \lceil 18\rho^2/\epsilon^2 \rceil$. In the proof we shall choose η to be sufficiently small as a function of ϵ alone. Let $\alpha = \alpha(\epsilon)$ be as in lemma 3.1. Let γ be chosen such that (i) $\gamma < \eta/r$, (ii) $\rho/(1-\gamma r) < \rho + \alpha$, (iii) $1/\gamma$ is larger than the minimal m for which Lemma 3.1 holds. (iv) $\gamma < \gamma(\eta,s,t)$ where $\gamma(\eta,s,t)$ is the function from Lemma 3.5. In the proof we shall assume, whenever necessary, that n is sufficiently large w.r.t. all of these constants, and hence $N = N(H,\rho,\epsilon)$ exists. In particular, $N > n_0(\eta,s,t)$ where $n_0(\eta,s,t)$ is the function from Lemma 3.5 and also $N > M(\gamma,r)$ where $M(\gamma,r)$ is the function from Lemma 3.4.

Let $G=(V,E)$ be a graph with n vertices and with $|E| > RT(n,K_s,\rho-mean)+\epsilon n^2$. Notice that since $s \geq 3$ and since $RT(n,K_s,\rho-mean) \geq RT(n,K_3,1) = t(n,2) = \lfloor n^2/4 \rfloor$ we have that $n^2/2 > |E| > n^2/4$. Fix a ρ-mean coloring of G. Assume the colors are $\{1,\ldots,q\}$ for some q and let c_i denote the number of edges colored with i. Without loss of generality we assume that $c_i \geq c_{i+1}$. We first show that the first r colors already satisfy $c_1+c_2+\cdots+c_r \geq |E|-\epsilon n^2/2$. Indeed, assume otherwise. Since, trivially, $c_{r+1} \leq |E|/r$, let us partition the colors $\{r+1,\ldots,q\}$ into parts such that for each part (except, perhaps, the last part) the total number of edges colored with a color belonging to the part is between $|E|/r$ and $2|E|/r$. The number of edges colored by a color from the last part is at most $2|E|/r$. The number of parts is, therefore, at least

$$\frac{\frac{\epsilon}{2}n^2}{\frac{2|E|}{r}} > \frac{\epsilon}{2}r.$$

Since any set of z edges is incident with at least $\sqrt{2z}$ vertices we have that the total number of vertices incident with colors $r+1$ and higher is at least

$$\left(\frac{\epsilon}{2}r-1\right)\sqrt{2|E|/r} > \frac{\epsilon}{3}r\frac{n}{\sqrt{2r}} = \frac{\epsilon\sqrt{r}}{\sqrt{18}}n > \rho n,$$

a contradiction to the fact that G is ρ-mean colored.

Let E_i be the set of edges colored i, let $G_i = (V, E_i)$, let $E' = E_1 \cup \cdots \cup E_r$ and let $G' = (V, E')$. By the argument above, $|E'| > RT(n, K_s, \rho - mean) + \epsilon n^2/2$ and G' is ρ-mean colored. It suffices to show that G' has a copy of H.

We apply Lemma 3.4 to G' and obtain a partition of V into m classes $V_1 \cup \cdots \cup V_m$ where $1/\gamma < m < M$ which is γ-regular simultaneously with respect to all graphs $G_i = (V, E_i)$ for $1 \le i \le r$. Consider the cluster graph $C(\eta)$. By choosing η sufficiently small as a function of ϵ we are guaranteed that $C(\eta)$ has at least $RT(m, K_s, \rho - mean) + \epsilon m^2/4$ edges. To see this, notice that if $C(\eta)$ had less edges then, by Lemma 3.2, by the definition of γ-regularity and by the definition of $C(\eta)$, the number of edges of G' would have been at most

$$(RT(m, K_s, \rho - mean) + \frac{\epsilon}{4}m^2)\frac{n^2}{m^2} + \eta\frac{n^2}{m^2}\binom{m}{2} + \gamma\binom{m}{2}\frac{n^2}{m^2} + \binom{n/m}{2}m$$

$$< RT(m, K_s, \rho - mean)\frac{n^2}{m^2} + \frac{\epsilon}{2}n^2 \le RT(n, K_s, \rho - mean) + \frac{\epsilon}{2}n^2$$

contradicting the cardinality of $|E'|$. In the last inequality we assume each color class has size n/m precisely. This may clearly be assumed since floors and ceilings may be dropped due to the asymptotic nature of our result.

We define a coloring of the edges of $C(\eta)$ as follows. The edge ij is colored by the color whose frequency in $E'(V_i, V_j)$ is maximal. Notice that this frequency is at least $(n^2/m^2)\eta/r$. Let ρ^* be the average number of colors incident with each vertex in this coloring of $C(\eta)$. We will show that $\rho^* \le \rho + \alpha$. For $i = 1, \ldots, m$ let $c(j)$ denote the number of colors incident with vertex j in our coloring of $C(\eta)$. Clearly, $c(1) + \cdots + c(m) = \rho^* m$. For $v \in V$, let $c(v)$ denote the number of colors incident with vertex v in the coloring of G'. Clearly, $\sum_{v \in V} c(v) \le \rho n$. We will show that almost all vertices $v \in V_j$ have $c(v) \ge c(j)$. Assume that color i appears in vertex j of $C(\eta)$. Let $V_{j,i} \subset V_j$ be the set of vertices of V_j incident with color i in G'. We claim that $|V_j - V_{j,i}| < \gamma n/m$. Indeed, if this was not the case then by letting $Y = V_j - V_{j,i}$ and letting $X = V_{j'}$ where j' is any class for which jj' is colored i we have that $d(X, Y) = 0$ with respect to color i, while $d(V_j, V_{j'}) \ge \eta/r$ with respect to color i. Since $\eta/r > \gamma$ this contradicts the γ-regularity of the pair $(V_j, V_{j'})$ with respect to color i. Now, let $W_j = \{v \in V_j \ : \ c(v) \ge c(j)\}$. We have therefore shown that $|W_j| \ge |V_j| - \gamma rn/m$. Hence,

$$\rho n \ge \sum_{v \in V} c(v) \ge \sum_{j=1}^{m} \sum_{v \in W_j} c(v) \ge \sum_{j=1}^{m} c(j)\frac{n}{m}(1 - \gamma r) = \rho^* n(1 - \gamma r).$$

It follows that

$$\rho^* \leq \frac{\rho}{1-\gamma r} \leq \rho + \alpha.$$

We may now apply Lemma 3.1 to $C(\eta)$ and obtain that $C(\eta)$ has a monochromatic K_s, say with color j. By Lemma 3.5 (applied to the spanning subgraph of $C(\eta)$ induced by the edges colored j) this implies that $G_j = (V, E_j)$ contains a copy of $T(st, s)$. In particular, there is a monochromatic copy of H in G. We have therefore proved that $RT(n, H, \rho - mean) \leq RT(n, K_s, \rho - mean) + \epsilon n^2$. Now, if H contains a K_s then we also trivially have $RT(n, H, \rho - mean) \geq RT(n, K_s, \rho - mean)$. This completes the proof of Theorem 1.4. ■

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