Large induced subgraphs with equated maximum degree

Y. Caro * R. Yuster †

Abstract

For a graph G, denote by $f_k(G)$ the smallest number of vertices that must be deleted from G so that the remaining induced subgraph has its maximum degree shared by at least k vertices. It is not difficult to prove that there are graphs for which already $f_2(G) \geq \sqrt{n}(1-o(1))$, where n is the number of vertices of G. It is conjectured that $f_k(G) = \Theta(\sqrt{n})$ for *every* fixed k. We prove this for $k = 2, 3$. While the proof for the case $k = 2$ is easy, already the proof for the case $k = 3$ is considerably more difficult. The case $k = 4$ remains open.

A related parameter, $s_k(G)$, denotes the maximum integer m so that there are k vertex-disjoint subgraphs of G, each with m vertices, and with the *same* maximum degree. We prove that for every fixed k, $s_k(G) \geq n/k - o(n)$. The proof relies on probabilistic arguments.

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. Graph theory notation follows [2]. A vertex with maximum degree in a graph can be viewed, in many circumstances, as the "most influential" vertex in the graph. It may therefore be desired to distribute the influence among a few vertices of the graph. More formally, let $m(G)$ denote the number of vertices of the graph G that attain the maximum degree. Clearly, $m(G) = |V(G)|$ if and only if G is a regular graph, and, in the other obvious extreme, there are graphs in which $m(G) = 1$. Can we *always* guarantee that, after deleting just a few vertices of a given graph G, the remaining induced subgraph G' has $m(G') \geq k$? This is the focus of the first main result in this paper. We note that research concerning repetitions in the degree sequence of graphs (in this case, repetition of the maximum degree) has been treated by several researchers; see, e.g., [3, 4, 5, 7].

For a given positive integer k, let $f_k(G)$ denote the minimum number of vertices that we need to delete from G so that the remaining induced subgraph G' has $m(G') \geq k$. Denote by $f_k(n)$ the maximum of $f_k(G)$ where G ranges over all graphs with n vertices. Trivially, $f_1(n) = 0$, and clearly $f_2(n)$ is well defined for all $n \geq 2$. For completeness, we will set $f_k(n) = n$ whenever there is a graph G with n vertices for which no induced subgraph has $m(G') \geq k$. Thus, for example, $f_3(4) = 4$ as can be seen by taking the path with four vertices. It is easy to verify that $f_3(n) \leq n-3$

*Department of Mathematics and Physics, University of Haifa, Tivon 36006, Israel. email: yacaro@kvgeva.org.il
†Department of Mathematics, University Haifa, Haifa 31905, Israel. email: raphy@math.haifa.ac.il

for all $n \geq 5$. In fact, $f_k(n) \leq n-k$ for all n sufficiently large; in particular, if $n \geq R(k)$ where $R(k)$ is the k'th diagonal Ramsey number.

It is not difficult to prove that there are graphs with n vertices having the property that any induced subgraph with more than $n-\sqrt{n}+2$ vertices has a unique maximal vertex. This shows that $f_k(n) \geq \Theta(\sqrt{n})$ for all $k \geq 2$. We conjecture that $\Theta(\sqrt{n})$ is also the upper bound.

Conjecture 1.1 *For any fixed $k \geq 2$ we have $f_k(n) = \Theta(\sqrt{n})$.*

It is quite easy to prove Conjecture 1.1 for $k = 2$. However already the case $k = 3$ requires significant effort, as can be seen from our proof of the following theorem.

Theorem 1.2 *For $k = 2, 3$ we have $f_k(n) = \Theta(\sqrt{n})$.*

The case $k = 4$ of Conjecture 1.1 remains open. We are, however, able to verify Conjecture 1.1 for a large family of graphs. Recall that a graph is $K_{2,t}$-free if no two vertices have t common neighbors.

Proposition 1.3 *For fixed integers $k \geq 2$ and $t \geq 2$, if G is a $K_{2,t}$-free graph with n vertices then $f_k(G) = O(\sqrt{n})$.*

The proofs of Theorem 1.2 and Proposition 1.3 appear in Section 2.

A related problem, along a line recently suggested in [6], asks for the deletion of as few as possible vertices so that the remaining vertices can be split into k *equal* parts, each inducing a subgraph with the *same* maximum degree. Formally, denote by $s_k(G)$ the maximum integer m so that there are k vertex-disjoint subgraphs of G, each with m vertices, and with the *same* maximum degree. Trivially, $s_k(G) \geq \lfloor n/k \rfloor$. However, it is easy to see that already $s_2(G)$ may be smaller than $\lfloor n/2 \rfloor$. Indeed, consider the star $K_{1,n-1}$ when n is even. Clearly, $s_2(K_{1,n-1}) = n/2 - 1$. Nevertheless, we prove that $s_k(G)$ is very close to n/k for every fixed k.

Theorem 1.4 *For any fixed k, and for any graph G with n vertices, $s_k(G) \geq n/k - o(n)$.*

The proof of Theorem 1.4 that appears in Section 3 also shows that k can, actually, be more than a constant. In fact, for sufficiently small ϵ, Theorem 1.4 also holds for $k = n^{\epsilon}$ where $o(n)$ is replaced with $o(n^{1-\epsilon})$.

2 Large induced subgraph with several maximum degree vertices

We begin this section by observing that $f_2(n) > \sqrt{n}(1-o(1))$. We construct a graph G with $n = p^2$ vertices having the property that any induced subgraph with more than $n-p+2$ vertices has a unique maximal vertex. This shows that $f_2(G) \geq p-2$. The vertex set of G will be denoted by $V = \{v_1, \ldots, v_n\}$. The edge set of G is constructed as follows. For $i = 0, \ldots, p-3$, vertex v_{n-i} will have degree $n-(i+1)p$ by arbitrarily choosing this amount of neighbors from $\{v_1, \ldots, v_{n-p+2}\}$. Notice that the remaining $n-p+2$ vertices have degree at most $p-2$. Now, any subgraph that contains more than $n-p+2$ vertices must contain at least one vertex from $\{v_n, \ldots, v_{n-p+3}\}$. The vertex v_{n-k} of this subgraph, where k is the smallest possible, is the unique vertex with maximum degree in it, as its degree is between $n-(k+1)p$ and greater than $n-(k+2)p$, and no other vertex of the subgraph has degree in this range.

Proof of Theorem 1.2: By the construction above, we only need to prove that $f_3(G) \le C\sqrt{n}$. Let us first prove the much easier assertion that $f_2(G) \le \sqrt{8n}$. Assume that the vertices of G are ordered by $v_1, \ldots, v_n$ where $d(v_i) \ge d(v_{i+1})$ for $i = 1, \ldots, n-1$. Clearly, there must be and index $i < \sqrt{n/2}$ so that $d(v_i) < d(v_{i+1}) + \sqrt{2n}$. Let, therefore, i be the smallest such index. Delete from G the vertices $v_1, \ldots, v_{i-1}$. In the remaining induced subgraph G', the degree of every vertex has decreased from its original degree in G by at most $i-1$. Since both v_i and v_{i+1} are in G' we have that the difference between the largest degree in G' and the second largest one is less than $\sqrt{2n} + \sqrt{n/2}$ (notice that we do not claim that v_i or v_{i+1} are the largest or second largest vertices in G'). Let x be the largest vertex in G' with degree $d'(x)$ and let y be the second largest vertex with degree $d'(y)$. We can delete $d'(x) - d'(y)$ neighbors of x in G' that are not neighbors of y, thereby equating their degrees. The total number of deleted vertices is less than $2\sqrt{n/2} + \sqrt{2n} = \sqrt{8n}$, as required.

We now proceed to the more difficult case, $k = 3$. Again, we assume that the vertices of G are ordered by $v_1, \ldots, v_n$ where $d(v_i) \ge d(v_{i+1})$ for $i = 1, \ldots, n-1$. By a similar argument to the above, there is an index $i < \sqrt{n}$ such that $d(v_i) < d(v_{i+5}) + 5\sqrt{n}$. Letting i be the smallest such index and deleting from G the vertices $v_1, \ldots, v_{i-1}$ we obtain an induced subgraph G' where the degrees of the first six largest vertices differ by less than $6\sqrt{n}$. We denote by t_i the i'th largest degree in G' where t_1 is the maximum degree of G'. Hence, $t_1 - t_6 < 6\sqrt{n}$.

We will now modify G' by considering several cases. In each case we do one of the following. We either delete a few vertices from G', so as to directly obtain an induced subgraph in which the maximum degree is attained by at least three vertices, or else we reduce the case to another case (namely, we modify G' so that its structure applies to another case). We shall denote the vertices of the current G' by $x_1, x_2, x_3, \ldots$ and their current degrees by $d_1, d_2, d_3, \ldots$ where $d_i \ge d_{i+1}$. When we reduce once case to another we shall use the notation $x_i := x_j$ to signify that vertex x_j in the current case becomes vertex x_i in the case that we reduce to. We shall also use the notation $d_i := d_j - q$ to signify that the i'th largest degree in the case that we reduce to equals the j'th largest degree in the current case, minus q. Similarly we will denote $d_i :\ge d_j - q$ to signify that the i'th largest degree in the case that we reduce to is at least as large as the j'th largest degree in the current case, minus q.

We denote by Z the set of vertices of the current G', not including x_1, x_2, and that have degree at least d_4. In particular, $Z \supset \{x_3, x_4\}$ but also contains other vertices with the same degree as x_4, if there are any. We call a vertex of Z *good* if it is adjacent to x_2 or if it is not adjacent to x_1. The following are the cases we consider.

Case 1: $(x_1, x_2) \notin E$ and there exists a good vertex. We modify G' by deleting from it a set of $d_1 - d_2 \ge 0$ neighbors of x_1 that are not neighbors of x_2. Notice that a good vertex is not deleted.

After the deletion, we have that x_1 and x_2 both have the same maximum degree which is d_2 and that the third largest vertex, denoted by u, has degree denoted by ℓ, and $\ell \geq d_4 - d_1 + d_2$. If x_1 and x_2 now have $d_2 - \ell$ common neighbors that are not neighbors of u then we can delete them and we are done, as x_1, x_2, u all have the same maximum degree ℓ. Otherwise, if there are only $p < d_2 - \ell$ such common neighbors we first delete them. Then, we delete $d_2 - \ell - p$ neighbors of x_1 that are not neighbors of u (none of these are neighbors of x_2) and delete $d_2 - \ell - p$ neighbors of x_2 that are not neighbors of u (none of these are neighbors of x_1). Again, we are done. Notice that we have used here the fact that x_2 is not adjacent to x_1. Altogether, the number of vertices we have deleted from G' is at most

$$d_1 - d_2 + 2(d_2 - \ell) \leq d_1 - d_2 + 2(d_2 - (d_4 - d_1 + d_2)) \leq 3(d_1 - d_4) \ . \tag{1}$$

Case 2: $(x_1, x_2) \notin E$ and there is no good vertex. In this case we delete x_1 from G'. Notice that x_1 is connected to all the vertices of Z and that x_2 is not connected to any of the vertices of Z. After the deletion, the largest vertex is x_2 and its degree is d_2. The second largest vertex is x_3 and its degree is $d_3 - 1$. The third largest vertex is x_4 and its degree is $d_4 - 1$. Since the (now) first largest vertex (x_2) is not connected to the (now) second largest vertex (x_3) and since x_2 is also not connected to x_4, we reduce to Case 1 and notice that $x_1 := x_2$, $x_2 := x_3$, $x_3 := x_4$, $d_1 := d_2$, $d_2 := d_3 - 1$ and $d_3 := d_4 - 1$. We notice also that when applying Case 1, the new x_3 (which is the old x_4) is a good vertex. Thus, in Case 1, the value of ℓ is at least $d_3 - d_1 + d_2$ (in terms of the new d_i's) and hence, by (1), at most $3(d_1 - d_3)$ (in terms of the new d_i's) vertices are deleted to settle Case 1. In terms of the current d_i's we have that Case 2 can be settled directly by deleting at most

$$1 + 3(d_2 - d_4 + 1) = 4 + 3(d_2 - d_4) \tag{2}$$

vertices.

Case 3: (x_1, x_2, x_3) form a triangle. We first delete $d_1 - d_2$ neighbors of x_1 that are not neighbors of x_2. After the deletion, both x_1 and x_2 have maximum degree d_2. The degree of x_3 is ℓ and we have $\ell \geq d_3 - d_1 + d_2$. We now perform the following operation for at most $d_2 - \ell$ times, as long as there are no three vertices with the same maximum degree. If there is a neighbor common to x_1 and x_2 that is not a neighbor of x_3 we delete it. Otherwise, we delete a neighbor of x_1 that is not a neighbor of x_3 and a neighbor of x_2 that is not a neighbor of x_3. Notice that in either case, the maximum degree is reduced by 1 and is still shared by x_1 and x_2, while x_3 remained with degree ℓ. After performing this operation at most $d_2 - \ell$ times we must have three vertices sharing the maximum degree. The overall number of vertices deleted in Case 3 is at most

$$d_1 - d_2 + 2(d_2 - \ell) \leq d_1 - d_2 + 2(d_1 - d_3) \leq 3(d_1 - d_3) \ . \tag{3}$$

Case 4: $(x_1, x_2) \in E$, $(x_1, x_3) \notin E$, $(x_2, x_3) \notin E$. We delete x_2 from G'. After the deletion, x_1 is still the largest vertex with degree $d_1 - 1$, and x_3 becomes the second largest vertex with degree d_3. Since x_1 is not connected to x_3 we can now reduce to either Case 1 or Case 2. We notice that

$x_1 := x_1$, $x_2 := x_3$, $d_1 := d_1 - 1$, $d_2 := d_3$, $d_3 :\geq d_4 - 1$ and $d_4 :\geq d_5 - 1$. Notice that if we reduce to Case 1 then, by (1), we can settle Case 4 directly by deleting at most $1 + 3(d_1 - d_5)$ vertices (in terms of the current d_i's) and if we reduce to Case 2 then, by (2), we can settle Case 4 directly by deleting at most $8 + 3(d_3 - d_5)$ vertices (in terms of the current d_i's). In any case, the number of vertices we need to delete in order to settle Case 4 directly is at most

$$8 + 3(d_1 - d_5) \ . \tag{4}$$

Case 5: $(x_1, x_2) \in E$, $d_3 > d_4$, and x_3 is connected to precisely one of x_1 or x_2. If $(x_1, x_3) \in E$ we delete x_1. We reduce to either Case 1 or Case 2 with $x_1 := x_2$, $x_2 := x_3$, $d_1 := d_2 - 1$, $d_2 := d_3 - 1$, $d_3 :\geq d_4 - 1$ and $d_4 :\geq d_5 - 1$. Similarly, if $(x_2, x_3) \in E$ we delete x_2. Again we reduce to either Case 1 or Case 2 with $x_1 := x_1$, $x_2 := x_3$, $d_1 := d_1 - 1$, $d_2 := d_3 - 1$, $d_3 :\geq d_4 - 1$ and $d_4 :\geq d_5 - 1$. Again, by (1) and (2) we see that in terms of the current d_i's, the number of vertices that we need to delete in order to resolve Case 5 directly is at most

$$5 + 3(d_1 - d_5) \ . \tag{5}$$

Case 6: $(x_1, x_2) \in E$, $d_3 = d_4 = \cdots = d_k$ where $k \geq 5$, and each of $x_3, \ldots, x_k$ is adjacent to precisely one of x_1 or x_2. We delete x_1 and x_2. After the deletion all the vertices $x_3, \ldots, x_k$ have the same maximum degree $d_3 - 1$ and we are done. We deleted only two vertices.

Case 7: $(x_1, x_2) \in E$, $d_3 = d_4 > d_5$, $(x_3, x_4) \notin E$ and each of x_3, x_4 is adjacent to precisely one of x_1 or x_2. We delete x_1 and x_2. After the deletion x_3 and x_4 have maximum degree $d_3 - 1$. Since they are not connected we reduce to either Case 1 or Case 2 with $x_1 := x_3$, $x_2 := x_4$, $d_1 := d_3 - 1$, $d_2 := d_3 - 1$, $d_3 :\geq d_5 - 2$ and $d_4 :\geq d_6 - 2$. Again, by (1) and (2) we see that in terms of the current d_i's, the number of vertices that we need to delete in order to resolve Case 7 directly is at most

$$9 + 3(d_3 - d_6) \ . \tag{6}$$

Case 8: $(x_1, x_2) \in E$, $d_3 = d_4 > d_5$, $(x_3, x_4) \in E$ and each of x_3, x_4 is adjacent to precisely one of x_1 or x_2. There are several sub-cases to consider.

If both x_3, x_4 are adjacent to x_1, we delete x_1 and reduce to Case 1 or Case 2 with $x_1 := x_2$, $x_2 := x_3$, $x_3 := x_4$, $d_1 := d_2 - 1$, $d_2 := d_3 - 1$, $d_3 := d_3 - 1$ and $d_4 :\geq d_5 - 1$. In terms of the current d_i's, the number of vertices that we need to delete in order to resolve this subcase of Case 8 directly is at most $5 + 3(d_2 - d_5)$.

Similarly, if both x_3, x_4 are adjacent to x_2, we delete x_2 and reduce to Case 1 or Case 2 with $x_1 := x_1$, $x_2 := x_3$, $x_3 := x_4$, $d_1 := d_1 - 1$, $d_2 := d_3 - 1$, $d_3 := d_3 - 1$ and $d_4 :\geq d_5 - 1$. In terms of the current d_i's, the number of vertices that we need to delete in order to resolve this subcase of Case 8 directly is at most $5 + 3(d_1 - d_5)$.

Finally, we can assume that $(x_1, x_3) \in E$ and $(x_2, x_4) \in E$ and hence (x_1, x_2, x_3, x_4) induce a 4-cycle. We delete a set of $d_2 - d_3$ neighbors of x_2 that are not neighbors of x_4. After this deletion, the degree of x_2 and x_4 is d_3, while the degree of x_1 is at least $d_1 - d_2 + d_3$ and the degree of x_3 is at least $2d_3 - d_2$. Thus, we can reduce to Case 1 or Case 2 with $x_1 := x_1$, $x_2 := x_4$, $x_3 := x_2$, $x_4 := x_3$ and $d_1 :\geq d_1 - d_2 + d_3$, $d_2 := d_3$, $d_3 := d_3$, $d_4 :\geq 2d_3 - d_2$. If we reduce to Case 1, then by (1), we can settle this subcase of Case 8 directly by deleting at most $d_2 - d_3 + 3(d_1 - 2d_3 + d_2) \leq 7(d_1 - d_3)$ (all in terms of the current d_i's). If we reduce to Case 2, then by (2), we can settle this subcase of Case 8 directly by deleting at most $d_2 - d_3 + 4 + 3(d_3 - 2d_3 + d_2) = 4 + 4(d_2 - d_3)$ (all in terms of the current d_i's).

Considering all the above subcases of Case 8, we see that in terms of the current d_i's, the number of vertices that we need to delete in order to resolve Case 8 directly is at most

$$\max\{5 + 3(d_2 - d_5)\ ,\ 5 + 3(d_1 - d_5)\ ,\ 7(d_1 - d_3)\ ,\ 4 + 4(d_2 - d_3)\}\ . \tag{7}$$

We have concluded the description of all cases. Notice that these cases cover all possibilities for the original G'. Indeed Cases 1 and 2 cover the possibility that $(x_1, x_2) \notin E$ in the original G'. Case 3 covers the possibility that x_1 and x_2 form a triangle with any vertex with degree d_3. Case 4 covers the possibility that x_1 and x_2 are both not connected to a vertex with degree d_3. We remain with the possibility that each vertex with degree d_3 is connected to exactly one of x_1 or x_2. Cases 5,6,7,8 cover this possibility.

Since in the original G' we have $d_i = t_i$ for $i = 1, 2, 3, 4, 5, 6$ we see from (1),(2),(3),(4),(5),(6),(7) that all of the cases can be resolved completely by deleting from the original G' at most $7(t_1 - t_6)$ vertices, assuming $t_1 - t_6 \geq 3$, or otherwise at most 15 vertices if $t_1 - t_6 \leq 2$. In any case, this is less than $42\sqrt{n}$ vertices. Recalling the (less than) $\sqrt{n}$ vertices that were deleted from G to obtain G' we have that we can always delete from G less than $43\sqrt{n}$ vertices and obtain a subgraph in which the maximum degree is attained by at least three vertices. ■

Proof of Proposition 1.3: Let G be a $K_{2,t}$-free graph on n vertices, and suppose that $n \geq t^2\binom{k}{2}^2$. We will prove that $f_k(G) \leq (3k-3)n^{1/2}$.

We perform the following process. If there are k vertices with degrees in $(n, n - \sqrt{n}]$ we halt. Otherwise, we delete all (less than k) vertices with degrees in $(n, n-\sqrt{n}]$ from G. If in the remaining graph there are k vertices with degrees in $(n - \sqrt{n}, n - 2\sqrt{n}]$ we halt. Otherwise, we delete all vertices with degrees in this interval. Assume that halt after $j \geq 0$ deletion steps. Hence, we have a subgraph G' with at least $n - (k-1)j$ vertices where the k vertices with highest degrees all have degrees in $(n - j\sqrt{n}, n - (j+1)\sqrt{n}]$. Notice that we must have $j \leq \sqrt{n}$.

Assume first that the maximum degree of G' is less than $2\sqrt{n}$. This case is resolved by performing the following process on G'. As long as there are less than k vertices that share the maximum degree, delete the vertices that share the maximum degree, and continue. Notice that after at most $2\sqrt{n}$ deletion steps we obtain a graph G'' with $m(G'') \geq k$. We obtained G'' by deleting at most $2(k-1)\sqrt{n}$ vertices of G'. We obtained G' by deleting at most $(k-1)j \leq (k-1)\sqrt{n}$ vertices of G. Thus. $f_k(G) \leq (3k-3)\sqrt{n}$.

Assume next that the maximum degree of G' is at least $2\sqrt{n}$. Let A be the set consisting of the k vertices with largest degree in G'. Let B be the set of vertices of G' that are not in A but have at least two neighbors in A. Since G' is $K_{2,t}$-free, we have that $|B| \leq (t-1)\binom{k}{2}$. Since a vertex $v \in A$ has at most $k-1$ neighbors in A and since its degree in G' is at least $2\sqrt{n}$ we have that v has at least $2\sqrt{n} - (k-1) - (t-1)\binom{k}{2} \geq \sqrt{n}$ neighbors in G' that are not in A and that are not neighbors of any other vertex of A. Since the difference of the degrees of two vertices of A is less than $\sqrt{n}$ we see that we can equate the degrees of all the vertices of A by deleting less than $\sqrt{n}k$ vertices (for each $v \in A$ we only delete the correct amount of the unique neighbors of v). Altogether we have deleted at most $j(k-1) + \sqrt{n}k$ vertices of G. Hence $f_k(G) \leq (3k-3)\sqrt{n}$ in this case as well. ∎

3 Disjoint induced subgraphs with equal order and maximum degree

Proof of Theorem 1.4: For the rest of the proof, we assume that $k \geq 2$ is fixed and that G is an n-vertex graph. Whenever necessary we shall assume that n is sufficiently large, as this does not affect the asymptotic result.

Consider first the case where $\Delta(G)$ (the maximum degree of G) is at most $n^{1/4}$. We perform the following process. If $m(G) > k\Delta(G)^2$, then we can clearly select k vertices of G with maximum degree $\Delta(G)$ with the property that no two of them are neighbors or have a common neighbor. Thus if $v_1, \ldots, v_k$ are such vertices, we can define k pairwise vertex-disjoint subgraphs of G, each with $\lfloor n/k \rfloor$ vertices, as follows. G_i will contain v_i and all of its neighbors, for $i = 1, \ldots, k$, and the other vertices of G are assigned arbitrarily to the G_i. Notice that each G_i has maximum degree $\Delta(G)$, as required, and hence $s_k(G) = \lfloor n/k \rfloor$ in this case. Otherwise, we can assume that $m(G) < k\Delta(G)^2$. So we modify G by deleting from it all the $m(G)$ vertices with maximum degree. We have deleted less than $k\Delta(G)^2 \leq kn^{1/2}$ vertices and the new graph has maximum degree at most $n^{1/4} - 1$. We now repeat the same process as before. Notice that the process consists of at most $n^{1/4}$ steps (as each step decreases the maximum degree) and that in each step we delete less than $kn^{1/2}$ vertices. Thus, once we halt we have only deleted at most $kn^{3/4}$ vertices and hence $s_k(G) \geq \lfloor (n - kn^{3/4})/k \rfloor = n/k - o(n)$, and we are done.

We may now assume that $\Delta(G) > n^{1/4}$. We denote the vertices of G by $v_1, \ldots, v_n$ where $d(v_i) \geq d(v_{i+1})$ for $i = 1, \ldots, n-1$. We perform the following process. If $d(v_1) - d(v_k) < n^{0.1}$ we halt. Otherwise we delete $v_1, \ldots, v_k$ from G and repeat the same argument for the new graph which now has maximum degree less than $n - n^{0.1}$. After t repetitions we have deleted kt vertices and have a subgraph of G with maximum degree less than $n - tn^{0.1}$. Hence, by deleting at most $kn^{0.9} = o(n)$ vertices from G we can assume that the degrees of the first k vertices with highest degree differ by at most $n^{0.1}$. We therefore have a subgraph G' of G with $s = n - o(n)$ vertices that we denote $v_1, \ldots, v_s$ and $d(v_1) - d(v_k) < n^{0.1}$. Notice that if $d(v_1) \leq s^{1/4}$ we are done by the previous case. We may therefore assume that $d(v_1) = t > s^{1/4}$. Another easy case is when $t \geq s - s^{3/4}$. In this case, we delete from the graph all the vertices other than $v_1, \ldots, v_k$ that are not common neighbors of all the $v_1, \ldots, v_k$. This amounts to deleting less than $k(s^{3/4} + n^{0.1}) = o(n)$

vertices. We can now partition the remaining vertices to equal parts $V_1, \ldots, V_k$ (after throwing away at most $k-1$ vertices because of divisibility considerations) where we place v_i in part V_i for $i = 1, \ldots, k$. Notice that in each V_i the vertex v_i is adjacent to all other vertices, and we are done. Thus, we can now assume that $s - s^{3/4} > t > s^{1/4}$.

We now partition the set of vertices of G' into k parts $V_1, \ldots, V_k$ as follows. We place v_i in V_i for $i = 1, \ldots, k$. Each of the remaining vertices $v_{k+1}, \ldots, v_s$ is placed in one of the parts $V_1 \ldots, V_k$ by selecting the part uniformly at random. All $s - k$ choices are independent.

Notice that the expected cardinality of V_i is s/k. By a Chernoff large deviation estimate (see [1]), we have that with probability exponentially small in n, V_i deviates from its mean s/k by more than $s^{2/3}$. Thus, with extremely high probability, for all $i = 1, \ldots, k$,

$$| \, |V_i| - s/k \, | < s^{2/3} \; . \tag{8}$$

Since no vertex of G' has degree greater than t, we have that for each vertex, its expected degree in its part is at most t/k. Again, by the same Chernoff large deviation estimate, with very high probability, no vertex deviates from its expected degree by more than $t^{2/3}$. Now consider vertex v_i in V_i. We know, in particular, that with very high probability the degree of v_i in $G'[V_i]$ is at least $d(v_i)/k - t^{2/3} > (t - n^{0.1})/k - t^{2/3} > t/k - 2t^{2/3}$ (we use here the fact that $t^{2/3} > (s^{1/4})^{2/3} > n^{0.1}$). Notice the we do not claim that v_i has maximum degree in $G'[V_i]$, but we have shown that with very high probability, for all $i = 1, \ldots, k$,

$$|\Delta(G'[V_i]) - t/k| < 2t^{2/3}. \tag{9}$$

Let us therefore *fix* a partition $V_1, \ldots, V_k$ so that (8) and (9) hold for $i = 1, \ldots, k$. Let u_i be a vertex of maximum degree $\Delta(G'[V_i])$ in V_i. We next select a random subset $W_i \subset V_i - \{u_i\}$ of cardinality $10k|V_i|/t^{1/3}$ for $i = 1, \ldots, k$. Notice that for each vertex $v \in V_i$, the expected number of neighbors of v in W_i is asymptotically a $10kt^{-1/3}$ fraction of its degree in $G'[V_i]$, and by (9) this expectation is much larger than $9t^{2/3}$. From standard large deviation estimates we obtain that with very high probability, v has at least $8t^{2/3}$ neighbors in W_i and this holds for each $v \in V_i$ and for all $i = 1, \ldots, k$. Let us therefore *fix* the W_i for $i = 1, \ldots, k$, having this property.

Consider the following process that equates the maximum degrees. Define

$$y = \min_{i=1}^{k} \Delta(G'[V_i]) \; .$$

We will show that it is possible to delete relatively few vertices from each V_i until the maximum degree becomes y. So, lets pick V_i and recall that by (9), $\Delta(G'[V_i]) - y \le 4t^{2/3}$ and that u_i has degree $\Delta(G'[V_i])$ in V_i. Also recall that $u_i \notin W_i$. We start deleting from V_i the vertices of W_i one by one, until the first time the maximum degree becomes y. Indeed, this will eventually happen at some point since after deleting all of W_i each vertex has lost at least $8t^{2/3}$ neighbors, and hence all vertices have degree smaller than y. Notice that once the maximum degree becomes y we have obtained a subset V_i' with at least $|V_i| - 10k|V_i|/t^{1/3}$ vertices.

After equating all of the maximum degrees to y it may still be the case that not all the V_i' have the same cardinality. By (8), however, we know the difference in cardinalities is at most $2s^{2/3} + 10k(s/k + s^{2/3})/t^{1/3} = o(s)$. Let z_i be a vertex with degree y in V_i'. Recall that the degree of z_i in V_i is at most $t/k + 2t^{2/3}$. Hence, z_i has at least $|V_i| - t/k - 2t^{2/3}$ non-neighbors in V_i. By (8), this is at least

$$\frac{s}{k} - s^{2/3} - \frac{t}{k} - 2t^{2/3}$$

non-neighbors. Thus, z_i has at least

$$\frac{s}{k} - s^{2/3} - \frac{t}{k} - 2t^{2/3} - 10k\frac{s/k + s^{2/3}}{t^{1/3}}$$

non-neighbors in V_i'. It therefore remains to show that

$$2s^{2/3} + 10k\frac{s/k + s^{2/3}}{t^{1/3}} < \frac{s}{k} - s^{2/3} - \frac{t}{k} - 2t^{2/3} - 10k\frac{s/k + s^{2/3}}{t^{1/3}} .$$

Since $t < s$ it suffices to prove that

$$21\frac{s}{t^{1/3}} < \frac{s}{k} - \frac{t}{k} - 5s^{2/3}.$$

The last inequality clearly holds since $s^{1/4} < t < s - s^{3/4}$. ■

4 Concluding remarks

By taking graph complements it is easy to see that both Theorem 1.2 and Theorem 1.4 hold for the analogous problems for minimum degrees. Proposition 1.3 holds, in particular, for trees as they are $K_{2,2}$-free. However, for trees a somewhat better bound can be obtained. For every n-vertex tree T we have $f_k(T) \le (2k-1)n^{1/3}$. In fact, it is not difficult to construct trees for which $f_k(T) > c(k)n^{1/3}$. We omit the relatively easy proof.

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