List Decomposition of Graphs

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Abstract

A family of graphs possesses the *common gcd property* if the greatest common divisor of the degree sequence of each graph in the family is the same. In particular, any family of trees has the common gcd property. Let $F = \{H_1, \ldots, H_r\}$ be a family of graphs having the common gcd property, and let d be the common gcd. It is proved that there exists a constant $N = N(F)$ such that for every $n > N$ for which d divides $n - 1$, and for every equality of the form $\alpha_1 e(H_1) + \ldots + \alpha_r e(H_r) = \binom{n}{2}$, where $\alpha_1, \ldots, \alpha_r$ are nonnegative integers, the complete graph K_n has a decomposition in which each H_i appears exactly α_i times. In case F is a family of trees the bound $N(F)$ is shown to be polynomial in the size of F, and, furthermore, a polynomial (in n) time algorithm which generates the required decomposition is presented.

1 Introduction

All graphs considered here are finite and undirected, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [4]. *List-decomposition* is a term frequently used [12, 10] to capture and unify several decomposition problems and conjectures having the following form: Given a complete multigraph λK_n and a multiset, or list, $L = \{H_1, \ldots, H_r\}$ of graphs such that $\sum_{i=1}^{r} e(H_i) = \lambda\binom{n}{2}$ and $gcd(L) \mid \lambda(n-1)$ (where $gcd(H_i)$ is the greatest common divisor of the degree sequence of H_i and $gcd(L) = gcd(gcd(H_1), \ldots, gcd(H_r)))$ is it then true that G has an L-decomposition, namely: $E(\lambda K_n)$ is the edge-disjoint union of the members of L such that each H_i appears exactly once in the decomposition. Note that since L is a list, there may be several graphs which are isomorphic in L. If H appears α times in L we say that H has *multiplicity* α in L.

Several particular cases of this general problem are already well known classical decomposition problems [6]. Here we mention a few of them to illustrate the generality of the concept of list-decomposition:

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1. *G-designs*: In this case L consists of a given graph G with multiplicity $\lambda\binom{n}{2}/e(G)$ and this is the classical G-design (G-decomposition) of λK_n solved asymptotically by Wilson [22].

2. The Gyárfás-Lehel conjecture [11]: In this case $L = \{T_1, \ldots, T_{n-1}\}$, where T_i denotes a tree having i edges. The conjecture is that there is an L-decomposition of K_n. This famous conjecture is mostly open (see e.g. [12, 5, 16]). No progress has been made on the weaker version in which each tree has multiplicity λ and the object is to decompose λK_n.

3. The Alspach conjecture [2, 14]: In this case L is any list of cycles of order at most n satisfying the necessary sum and divisibility conditions. For the case $\lambda = 1$, the Alspach conjecture is also stated for even values of n, where in this case the cycles should decompose K_n minus a one-factor. There are many recent developments, but only special cases of this conjecture are solved completely (see e.g. [1, 2, 3, 12, 13]). In particular, it has been solved for any set of two cycles whose length is at most 10 [17], and, for two cycles, the conjecture has been reduced to a finite problem [7].

4. Paths-list: In this case L is any list of paths of order at most n satisfying the necessary sum and divisibility conditions. The problem has been solved almost completely by Tarsi [19, 21] who showed that the necessary conditions are also sufficient provided all paths are of order at most $n - 3$ and any λ.

5. Stars-list: In this case L is any list of stars of order at most n satisfying the necessary sum and divisibility conditions. This problem has been solved recently in [15] who extended earlier results and ideas of Tarsi [18, 20].

6. *Designs with holes*: In this case L is usually a set of two kinds of complete graphs, say, K_p and K_q, $p \neq q$, such that K_q appears only once in L. There is rich literature on this issue and we refer the reader to [9].

Given a family of nonempty graphs $F = \{H_1, \ldots, H_r\}$ we say that F is *totally list-decomposable* if there exists $N = N(F)$ such that for *every* $n > N$ for which $gcd(F)$ divides $n - 1$, and for *every* equality of the form $\alpha_1 e(H_1) + \ldots + \alpha_r e(H_r) = \binom{n}{2}$, where $\alpha_1, \ldots, \alpha_r$ are nonnegative integers, the complete graph K_n has a decomposition in which each H_i appears exactly α_i times. It is not difficult to construct examples of families of graphs which are not totally list decomposable. In the final section we give such an example. The common phenomena of all these examples is that there are at least two graphs in the family with different gcd. We say that a family of nonempty graphs $F = \{H_1, \ldots, H_r\}$ has the *common gcd property* if $gcd(H_i) = gcd(H_j)$ for any pair $1 \leq i < j \leq r$. In particular, note that any family of trees has the common gcd property (the gcd of a tree is

1). Also, any family of d-regular graphs has the common gcd property, and there are many other examples. Our main result in this paper is the following.

Theorem 1.1 *Every finite family of graphs which possesses the common gcd property is totally list-decomposable.*

The proof of Theorem 1.1 appears in Section 2. It should be pointed out that the constant $N(F)$ in the definition of total list decomposability, which is computed in Theorem 1.1, is very large. In fact, it is exponential in the product of the sizes of the graphs appearing in F. This is not surprising as even the best known lower bounds in Wilson's Theorem mentioned above (which is clearly a special case of Theorem 1.1, where the set F consists of a single graph) are exponential [8]. Furthermore, the proof of Theorem 1.1 is an existence proof. It is not algorithmic. In case the family of graphs consists only of trees we can overcome both of these disadvantages using a different proof.

Theorem 1.2 *Every finite family $F = \{H_1, \ldots, H_r\}$ of trees is totally list-decomposable. In fact, $N(F) \leq (6h)^{26}$, where $h = \sum_{i=1}^{r} e(H_i)$. Furthermore, given any equality of the form $\alpha_1 e(H_1) + \ldots + \alpha_r e(H_r) = \binom{n}{2}$, where $\alpha_1, \ldots, \alpha_r$ are nonnegative integers and $n > N(F)$, we can produce a decomposition of K_n into α_i copies of H_i for $i = 1, \ldots, r$ in polynomial (in n) time.*

The proof of Theorem 1.2 appears in Section 3. The final section contains some concluding remarks and open problems.

2 Proof of the main result

Before we prove Theorem 1.1 we need two important lemmas. The first one is a theorem of Gustavsson [10] which says that for every fixed graph H, if G is a large enough graph, which is also very dense (as a function of H), then G has an H-decomposition, provided, of course, that the necessary conditions hold, namely, $gcd(H)$ divides $gcd(G)$ and $e(H)$ divides $e(G)$.

Lemma 2.1 [Gustavsson [10]] *Let H be a fixed nonempty graph. There exists a positive integer $n_0 = n_0(H)$, and a small positive constant $\gamma = \gamma(H)$, such that if G is a graph with $n > n_0$ vertices, and $\delta(G) \geq (1-\gamma)n$, and G satisfies the necessary conditions for an H-decomposition, then G has an H-decomposition.*

We note here that the constant $\gamma(H)$ used in Gustavsson's proof is very small. In fact, even for the case where H is a triangle, Gustavsson's proof uses $\gamma = 10^{-24}$. Thus, the graph G is very dense. We also note that Gustavsson's proof is an *existence* proof, and is non-constructive. Namely, it does not provide a polynomial time algorithm which generates the guaranteed decomposition.

The proof of the next lemma uses the special case of Lemma 2.1, where the graph G is K_n. This special case is the famous theorem of Wilson [22] which states that for every fixed graph H, there exists $n_0 = n_0(H)$ such that if $n > n_0(H)$ and $e(H)$ divides $\binom{n}{2}$ and $gcd(H)$ divides $n-1$, then K_n has an H-decomposition. Using Wilson's theorem one can prove the next result.

Lemma 2.2 *Let $H = \{H_1, \ldots, H_r\}$ be any family of nonempty graphs. Then for every $M > 0$ there exists $m > M$ such that K_m has an H_i-decomposition for each $i = 1, \ldots, r$.*

Proof: Let $N = \max_{i=1}^{r} n_0(H_i)$, where $n_0(H_i)$ is the constant appearing in Wilson's Theorem. Now let $M > 0$ be any number. Let $m > \max\{N, M\}$ be the smallest integer such that $(m-1)/2$ is a multiple of all the $2r$ numbers $e(H_1), \ldots, e(H_r), gcd(H_1), \ldots, gcd(H_r)$. Then, $\binom{m}{2}$ is a multiple of $e(H_i)$ for each $i = 1, \ldots, r$ and $m-1$ is a multiple of $gcd(H_i)$ for each $i = 1, \ldots, r$. Since $m > n_0(H_i)$ for each $i = 1, \ldots, r$ it follows from Wilson's Theorem that K_m has an H_i-decomposition for each $i = 1, \ldots, r$. $\square$

Proof of Theorem 1.1 Let $F = \{H_1, \ldots, H_r\}$ be a set with the common gcd property, and let $d = gcd(F)$ denote the common gcd. We need to define a number of constants before we can proceed. Let $h_i = e(H_i)$ for $i = 1, \ldots, r$. Let $k = \max_{i=1}^{r} v(H_i)$. Let m be the smallest positive integer such that $\binom{m}{2} \geq \binom{k}{2}r$ and such that K_m has an H_i-decomposition for each $i = 1, \ldots, r$. According to Lemma 2.2, such an m exists. Now, for each $i = 1, \ldots, r$ define the graph $F_i = K_m \cup H_i$, namely, F_i is the vertex-disjoint union of K_m and H_i. Note that $gcd(F_i) = d$. Now define $\gamma_i = \gamma(F_i)$ and $n_i = n_0(F_i)$ as in Lemma 2.1. Put $\gamma = \min_{i=1}^{r} \gamma_i$. Finally put

$$N = \max\{n_1, \ldots, n_r \ , \ \frac{k}{\gamma} \ , \ kr\binom{m}{2}\}.$$

Note that $N = N(F)$. Now let $n > N$, where d divides $n-1$, and assume that $\alpha_1, \ldots, \alpha_r$ are nonnegative integers satisfying $\alpha_1 h_1 + \ldots + \alpha_r h_r = \binom{n}{2}$. We must show that K_n has a decomposition with α_i copies of H_i for each $i = 1, \ldots, r$.

We claim that there exists some j such that $\alpha_j \geq \binom{n}{2}/\binom{m}{2}$. To see this, note that by averaging we have that there exists some j such that $\alpha_j h_j \geq \binom{n}{2}/r$. Now, since $h_j \leq \binom{k}{2}$ and since $\binom{m}{2} \geq \binom{k}{2}r$ the claim holds. For the remainder of the proof we fix a j having the property

$$\alpha_j \geq \frac{\binom{n}{2}}{\binom{m}{2}}.$$

For each $i = 1, \ldots, r$ except for $i = j$, we perform the integer division of α_i by the integer $\binom{m}{2}/h_i$ and define the quotient q_i and the remainder t_i in the obvious manner:

$$\alpha_i = q_i \cdot \frac{\binom{m}{2}}{h_i} + t_i \ , \ 0 \leq t_i \leq \frac{\binom{m}{2}}{h_i} - 1.$$

Let $q = q_1 + \ldots + q_{j-1} + q_{j+1} + \ldots + q_r$. We claim that $\alpha_j > q$. Indeed,

$$q = q_1 + \ldots + q_{j-1} + q_{j+1} + \ldots + q_r < \sum_{i=1}^{r} \frac{\alpha_i h_i}{\binom{m}{2}} = \frac{\binom{n}{2}}{\binom{m}{2}} \leq \alpha_j.$$

We may now define q_j and t_j by the integer division of $\alpha_j - q$ by the integer $1 + \binom{m}{2}/h_j$ namely:

$$\alpha_j - q = q_j \cdot \frac{\binom{m}{2} + h_j}{h_j} + t_j \quad 0 \leq t_j \leq \frac{\binom{m}{2}}{h_j}.$$

Consider the graph X composed of t_i vertex-disjoint copies of H_i for each $i = 1, \ldots, r$. The maximum degree of X is, obviously, at most $k-1$. X has at most $k(t_1 + \ldots + t_r) \leq kr\binom{m}{2}$ vertices. Also, trivially, $gcd(X) = d$. Since $n > N \geq kr\binom{m}{2}$ it follows that X is a subgraph of K_n. Let $G = K_n \setminus X$ denote the graph obtained from K_n by deleting a copy of X. We claim that G satisfies the conditions of Lemma 2.1 for the graph $H = F_j$. First note that G has $n > N \geq n_j = n_0(F_j)$ vertices. Next, note that since $N \geq k/\gamma$, we have that the minimum degree of G satisfies:

$$\delta(G) \geq (n-1) - (k-1) = n - k \geq n(1-\gamma) \geq n(1-\gamma_j).$$

Since d divides $n-1$ and since $gcd(X) = d$ we have that $gcd(G) = d = gcd(F_j)$. Finally, the number of edges of G satisfies:

$$e(G) = \binom{n}{2} - e(X) = \binom{n}{2} - \sum_{i=1}^{r} t_i h_i = (\binom{m}{2} + h_j)(q_1 + \ldots + q_r) = e(F_j)(q_1 + \ldots + q_r).$$

Hence, by Lemma 2.1, G has an F_j-decomposition into $q_1 + \ldots + q_r$ copies of F_j. Since $F_j = K_m \cup H_j$ we also have a decomposition of G into $q_1 + \ldots + q_r$ copies of K_m and $q_1 + \ldots + q_r$ copies of H_j. For each $i = 1, \ldots, r$ and $i \neq j$ we can obtain α_i edge-disjoint copies of H_i in K_n as follows: We take the t_i copies of H_i from X, and take q_i copies of K_m from the decomposition of G, which have not yet been used, and decompose each of these copies of K_m to H_i. This results in an additional $q_i\binom{m}{2}/h_i$ copies of H_i. Together we have $t_i + q_i\binom{m}{2}/h_i = \alpha_i$ edge-disjoint copies of H_i. We can continue taking non-used copies of K_m in the decomposition of G since there are $q_1 + \ldots + q_r$ such copies. Finally, we remain with q_j copies of K_m, the $q_1 + \ldots + q_r$ copies of H_j in the decomposition of G, and with the t_j copies of H_j in X. Decomposing each of the remaining K_m's to H_j this amounts to:

$$q_j \frac{\binom{m}{2}}{h_j} + (q_1 + \ldots + q_r) + t_j = q_j \frac{\binom{m}{2}}{h_j} + q + q_j + t_j = \alpha_j$$

edge-disjoint copies of H_j. Thus, we obtained a decomposition of K_n into α_i copies of H_i for each $i = 1, \ldots, r$. $\square$

3 An algorithmic proof for trees

Before we prove Theorem 1.2, we need the following lemma, whose proof appears in [23]:

Lemma 3.1 [Yuster [23]] *If H is a tree and G is an n-vertex graph where $e(H)$ divides $e(G)$, and $\delta(G) \ge n/2 + 10v(H)^4\sqrt{n \log n}$ then G has an H-decomposition.* □

Proof of Theorem 1.2 Let $F = \{H_1, \ldots, H_r\}$ be a family of trees, and let $h_i = e(H_i)$ denote the number of edges of H_i. Put $h = h_1 + \ldots + h_r$. Clearly, we can assume $h \ge 3$, otherwise there is nothing to prove. Let

$$N = (6h)^{26}.$$

We must show that if $n > N$, and if $\alpha_1, \ldots, \alpha_r$ are nonnegative integers satisfying $\alpha_1 h_1 + \ldots + \alpha_r h_r = \binom{n}{2}$, then K_n has a decomposition in which there are exactly α_i copies of H_i for $i = 1, \ldots, r$.

We will partition F into two parts F_1, F_2 as follows. If $\alpha_i < \binom{n}{2}/(2h^2)$ then $H_i \in F_1$, otherwise $H_i \in F_2$. Note that it is possible that $F_1 = \emptyset$, but, obviously, we must always have $F_2 \neq \emptyset$. Put $|F_1| = s$ and $|F_2| = r - s$, and assume, w.l.o.g. that $F_1 = \{H_1, \ldots, H_s\}$.

Our first goal is to show that there exists an n-vertex graph G, with $\Delta(G) \le n/h$, which has a decomposition in which there are exactly α_i copies of H_i for $i = 1, \ldots, s$. This can be done using a greedy algorithm as follows. Let $\alpha = \alpha_1 + \ldots + \alpha_s$. We shall construct graphs G_j for $j = 0, \ldots, \alpha$, where $\Delta(G_j) \le n/h$ and G_j has a decomposition into α_i copies of H_i for $i = 1, \ldots, k-1$ and $j - \alpha_1 - \ldots - \alpha_{k-1}$ copies of H_k, where $1 \le j - \alpha_1 - \ldots - \alpha_{k-1} \le \alpha_k$. Thus, $G_\alpha = G$ is the required graph. We begin with G_0 which is the empty graph on n vertices. Assume that we have already constructed G_{j-1} and we wish to construct G_j. We wish to add to G_{j-1} a copy of H_k by adding an appropriate set of h_k new edges, such that the resulting graph G_j still has $\Delta(G_j) \le n/h$. The number of edges in G_{j-1} satisfies

$$e(G_{j-1}) < \alpha_1 h_1 + \ldots + \alpha_s h_s < \frac{\binom{n}{2}}{2h^2}(h_1 + \ldots + h_s) < \frac{\binom{n}{2}}{2h}.$$

Thus, there are at least $n/4$ vertices of G_{j-1} with degree at most $(2/3)n/h$. Consider such a set of $\lceil n/4 \rceil$ vertices of G_{j-1}. The subgraph G' of G_{j-1} induced by them has $\Delta(G') \le (2/3)n/h$. Thus, the complement of G', denoted by G^* has $\delta(G^*) \ge \lceil n/4 \rceil - (2/3)n/h - 1 \ge n/36 - 1 \ge h - 1$. Thus, G^* contains every tree with $h-1$ edges, and, in particular, it contains H_k. Adding the edges of such a copy of H_k into G_{j-1} we obtain G_j and, clearly,

$$\Delta(G_j) \le \max\{\Delta(G_{j-1})\ ,\ (2/3)n/h + \Delta(H_k)\} \le \max\{n/h\ ,\ (2/3)n/h + h - 1\} = n/h.$$

Having obtained the graph G described above, we now consider $M = K_n \setminus G$. We need to show that M has a decomposition into α_i copies of H_i for $i = s+1, \ldots, r$. We note that $\delta(M) =$

$n-1-\Delta(G) \geq n-1-n/h$. For $i=s+1,\dots,r$ we define

$$t_i = \left\lfloor 40h^2 \frac{\alpha_i}{\alpha_{s+1}+\ldots+\alpha_r} \right\rfloor.$$

We have $t_i \geq 20$ since $\alpha_i \geq \binom{n}{2}/(2h^2)$ and since, clearly, $\alpha_{s+1}+\ldots+\alpha_r < \binom{n}{2}$. Thus,

$$20 \leq t_i \leq 40h^2. \tag{1}$$

Given any set of trees, we can *concatenate* them into one tree by choosing one vertex from each tree, and identifying all the chosen vertices. The concatenated tree is, by definition, decomposable into its original constituents. Let H denote the tree obtained by concatenating t_i copies of H_i for each $i=s+1,\dots,r$. Note that H has exactly $t = t_{s+1}h_{s+1}+\ldots+t_rh_r$ edges. By (1), $t \leq 40h^3$. Now define

$$q = \left\lfloor 0.95\frac{e(M)}{t} \right\rfloor.$$

Claim: $t_iq \leq \alpha_i$ for $i=s+1,\dots,r$.
Proof: It suffices to prove that

$$0.95\frac{e(M)}{t} \cdot 40h^2 \frac{\alpha_i}{\alpha_{s+1}+\ldots+\alpha_r} \leq \alpha_i.$$

Since $e(M) = \alpha_{s+1}h_{s+1}+\ldots+\alpha_rh_r$ it suffices to show that

$$0.95\frac{\alpha_{s+1}h_{s+1}+\ldots+\alpha_rh_r}{t}40h^2 \leq \alpha_{s+1}+\ldots+\alpha_r. \tag{2}$$

We will use the fact that

$$t_i = \left\lfloor 40h^2 \frac{\alpha_i}{\alpha_{s+1}+\ldots+\alpha_r} \right\rfloor \geq 40h^2 \frac{\alpha_i}{\alpha_{s+1}+\ldots+\alpha_r} - 1 \geq 38h^2 \frac{\alpha_i}{\alpha_{s+1}+\ldots+\alpha_r}.$$

We therefore have

$$t = t_{s+1}h_{s+1}+\ldots+t_rh_r \geq 38h^2 \frac{\alpha_{s+1}h_{s+1}+\ldots+\alpha_rh_r}{\alpha_{s+1}+\ldots+\alpha_r},$$

and, therefore, (2) holds. This completes the proof of the claim. According to the last claim, we can define $b_i = \alpha_i - t_iq$ for $i=s+1,\dots,r$ and we are guaranteed that the b_i are nonnegative integers. Our next goal is to find in M a spanning subgraph M' with the property that M' has a decomposition in which there are exactly b_i copies of H_i for each $i=s+1,\dots,r$, and $\Delta(M') \leq n/9$. This is done in a similar way as when creating G. However, we must now be more careful, since M' must be a spanning subgraph of M (unlike G which had no such restriction). We use the greedy procedure once again. Assume that we have already found a subgraph M'' of M with $\Delta(M'') \leq n/9$ and which contains a decomposition into b_i copies of each H_i, $i=s+1,\dots,r-1$ and b_r-1 copies of

H_r (completing the last element is, clearly, the most difficult situation in the greedy construction, as we may assume h_r is the largest tree in F_2, and the following arguments work at any earlier stage of the process). We wish to add a copy of H_r to M'' such that the edges of H_r are taken from $e(M) \setminus e(M'')$, and such that the resulting graph M' has $\Delta(M') \leq n/9$. We first estimate the number of edges in M'':

$$e(M'') < h_{s+1}b_{s+1} + \ldots + h_r b_r = \sum_{i=s+1}^{r} h_i(\alpha_i - t_i q) = e(M) - qt \leq$$

$$e(M) - t(0.95\frac{e(M)}{t} - 1) = 0.05e(M) + t \leq 0.05e(M) + 40h^3.$$

It follows that M'' has at least $\lceil n/2 \rceil$ vertices whose degrees do not exceed $(0.2e(M) + 160h^3)/n$. Let X be such a set of $\lceil n/2 \rceil$ vertices. Consider the graph induced by the vertices of X and the edges of $M \setminus M''$. We denote this graph by X as well. Clearly,

$$\delta(X) \geq \delta(M) - \lfloor n/2 \rfloor - \frac{0.2e(M) + 160h^3}{n}.$$

Recalling the facts that $\delta(M) \geq n - 1 - n/h$, $h \geq 3$, $e(M) \leq \binom{n}{2}$ and $n \geq (6h)^{26}$ we get that

$$\delta(X) \geq \frac{2n}{3} - 1 - \frac{n}{2} - 0.1n - 160\frac{h^3}{n} \geq 0.05n.$$

Since $0.05n > h > h_r$ we can find in X a copy of H_r. Joining the edges of a copy of H_r in X to M'' we obtain the graph M' which, by construction, is a subgraph of M and, furthermore,

$$\Delta(M') \leq \max\{\Delta(M'') \,,\; \frac{0.2e(M) + 160h^3}{n} + \Delta(H_r)\} \leq$$

$$\max\{\frac{n}{9} \,,\; 0.1n + \frac{160h^3}{n} + h\} \leq \max\{\frac{n}{9} \,,\; \frac{n}{9}\} = \frac{n}{9}$$

(in the last inequality we used the fact that $n > N = (6h)^{26}$). Having constructed the graph M' we now come to the final stage of the proof. Denote by M^* the spanning subgraph of M obtained by deleting the edges of M'. We claim that M^* has an H-decomposition, and the number of elements in this decomposition in q. We prove this using Lemma 3.1. First, we must show that $e(M^*) = q \cdot e(H) = qt$. This is true since

$$e(M^*) = e(M) - e(M') = \sum_{i=s+1}^{r} \alpha_i h_i - \sum_{i=s+1}^{r} b_i h_i = q \sum_{i=s+1}^{r} h_i t_i = qt.$$

Next, we show that M^* and H satisfy the other condition of Lemma 3.1, namely:

$$\delta(M^*) \geq n/2 + 10(t+1)^4\sqrt{n \log n}.$$

We can estimate $\delta(M^*)$ by

$$\delta(M^*) \geq \delta(M) - \Delta(M') \geq n - 1 - \frac{n}{h} - \frac{n}{9} \geq 0.555n.$$

Now, since $t \leq 40h^3$ it suffices to show that

$$0.555n \geq n/2 + 10(40h^3 + 1)^4 \sqrt{n \log n}$$

and this holds since $n \geq (6h)^{26}$. We thus have by Lemma 3.1 that M^* has an H-decomposition into q copies of H. Since every copy of H is decomposable into t_i copies of H_i for each $i = s+1, \ldots, r$, we have that M^* has a decomposition into qt_i copies of H_i for each $i = s+1, \ldots, r$. It is now easy to see that the decompositions of G, M' and M^* together supply the required decomposition of K_n. First note that by our construction, G, M' and M^* are edge-disjoint and their edge union is K_n. Now consider H_i. If $i \leq s$ then the decomposition of G constructed above contains exactly α_i copies of H_i. If $i \geq s+1$ then the decomposition of M' has b_i copies of H_i and the decomposition of M^* has qt_i copies of H_i. Together, this gives $b_i + qt_i = \alpha_i$ copies of H_i.

We still need to show how to implement the proof of Theorem 1.2 as a polynomial time algorithm. Fix a family $F = \{H_1, \ldots, H_r\}$ of trees. The algorithm receives as its input a set of r nonnegative integers $\alpha_1, \ldots, \alpha_r$ which satisfy $\alpha_1 e(H_1) + \ldots + \alpha_r e(H_r) = \binom{n}{2}$ for some integer n which satisfies $n > N(F)$. The algorithm must output a list-decomposition of K_n which consists of α_i copies of each H_i. Reviewing the proof of Theorem 1.2, this is done as follows: The sets F_1 and F_2 are easily created by checking for each i if $\alpha_i < \binom{n}{2}/(2h^2)$. Now, the graph G consisting of α_i edge-disjoint copies of H_i for each $H_i \in F_1$ is created in polynomial time since the embedding of the graphs is done by a greedy method. The numbers t_i and the concatenated graph H are constructed in constant time, as they only depend on the fixed family F and the fixed set of $r - s$ numbers $\alpha_{s+1}, \ldots, \alpha_r$. Hence, the number q and the nonnegative numbers b_i are also computed in constant time. The graph $M = K_n \setminus G$ is generated in polynomial time since we have already generated G. The spanning subgraph M' of M with b_i edge-disjoint copies of H_i for each $H_i \in F_2$ is created in polynomial time since, as in the creation of G, the embeddings of the graphs composing M' are done by the greedy method. Now, $M^* = M \setminus M'$ is, obviously, generated in polynomial time since both M and M' are already given. Since H is a fixed graph which has already been generated, we can now generate the H-decomposition of M^* in polynomial time using the algorithmic version of Lemma 3.1 [23]. This completes the algorithm and the proof of Theorem 1.2. $\square$

4 Concluding remarks

1. We demonstrate the existence of simple families which are not totally list-decomposable. Consider the family $F = \{K_3, K_4\}$. Let N be any positive integer. We will find $n > N$ and positive integers α_1, α_2 which satisfy $3\alpha_1 + 6\alpha_2 = \binom{n}{2}$ while K_n *does not* have the corresponding list-decomposition with α_1 copies of K_3 and α_2 copies of K_4. Indeed, Let $n > N$ be a number satisfying $n \equiv 0 \pmod{12}$. Choose $\alpha_1 = 2$ and $\alpha_2 = \binom{n}{2}/6 - 1$. Clearly, these numbers are integers, and
$$3\alpha_1 + 6\alpha_2 = \binom{n}{2}.$$
We will prove that K_n does not have a decomposition into two copies of K_3 and $\binom{n}{2}/6 - 1$ copies of K_4. Assume the contrary, then the two copies of K_3 contain at most 6 vertices. Thus, there is some vertex which does not appear in any K_3, so it must appear in exactly $(n-1)/3$ copies of K_4, but $(n-1)/3$ is *not* an integer, so this is impossible.

2. It would be interesting to find an algorithmic proof of Theorem 1.1. This may be plausible since the major non-algorithmic part is Gustavsson's Theorem, namely Lemma 2.1. However, note that in the proof we only use a very weak form of this theorem, since the graph G on which we apply Lemma 2.1 is very close to being complete, since its complement (the graph X in the proof) has bounded degree k. Thus, a weaker form of Gustavsson's theorem replacing γn with any function $w(n)$, where $w(n) \to \infty$ arbitrarily slowly suffices. Such a weaker form may be easier to prove and implement as an algorithm.

3. Reviewing the proof of Theorem 1.1 it is obvious that the decomposed graph does not have to be K_n, and it suffices that the graph should be very dense, as in Lemma 2.1. Thus, we can state the following theorem.

   Theorem 4.1 *Let $F = \{H_1, \ldots, H_r\}$ be a set of graphs having the common gcd property. Then, there exists a positive integer $N = N(F)$, and a positive constant $\gamma = \gamma(F)$, such that for every graph G with $n > N$ vertices, $\delta(G) \geq n(1-\gamma)$ for which gcd(F) divides gcd(G), and for every linear combination $\alpha_1 e(H_1) + \ldots + \alpha_r e(H_r) = e(G)$, where $\alpha_1, \ldots, \alpha_r$ are nonnegative integers, there exists a decomposition of G in which there are α_i copies of H_i for $i = 1, \ldots, r$.*

   Note that, in particular, Theorem 4.1 solves the Alspach conjecture mentioned in the introduction, for any set of fixed cycles, and for every n sufficiently large. Note also that in the proof of Theorem 4.1 we need the full strength of Gustavsson's Theorem.

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