A Note on Linear Coloring of Graphs

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Abstract

A proper vertex coloring of a graph is called *linear* if the subgraph induced by the vertices colored by any two colors is a set of vertex-disjoint paths. The *linear chromatic number* of a graph G, denoted by $lc(G)$, is the minimum number of colors in a linear coloring of G. Extending a result of Alon, McDiarmid and Reed concerning acyclic graph colorings, we show that if G has maximum degree d then $lc(G) = O(d^{3/2})$. We also construct explicit graphs with maximum degree d for which $lc(G) = \Omega(d^{3/2})$, thus showing that the result is optimal, up to an absolute constant factor.
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1 Introduction

All graphs considered here are finite, undirected and simple. For the standard graph-theoretic notations the reader is referred to [3]. A proper vertex coloring of a graph is called *linear* if the subgraph induced by the union of any two color classes is a linear forest, i.e. a set of vertex-disjoint paths. Let $lc(G)$ be the minimum number of colors in a linear coloring of the graph G. We call $lc(G)$ the *linear chromatic number* of G. A linear coloring is a special case of an *acyclic coloring*, where it is required that the subgraph induced by the union of any two color classes is a forest. Thus, $lc(G) \geq A(G)$, where $A(G)$ is the *acyclic chromatic number* of G. A related definition is that of *k-frugal colorings*, considered by Hind, Molloy and Reed [5], where a graph is k-frugal if it can be properly colored so that no color appears k times in a vertex neighborhood. Such a coloring is called a *k-frugal coloring*. Clearly, a linear coloring is 3-frugal, but the converse is not necessarily true. Erdös has conjectured in 1976 (cf. [6], problem 37) that if G has maximum degree d, then $A(G) = o(d^2)$. This conjecture has been proved by Alon, McDiarmid and Reed in [2], where it is shown that $A(G) = O(d^{4/3})$. They also give a probabilistic construction of a graph G with $A(G) = \Omega(d^{4/3}/(\log d)^{1/3})$. Thus, there is still a logarithmic-factor gap between the upper and lower bounds. Furthermore, there is no known explicit construction of a graph G which comes close to the probabilistic lower bound for $A(G)$. In this note we show that the conjecture of Erdös is still valid under much stricter conditions, namely that $lc(G) = o(d^2)$. In fact, we prove that $lc(G) = O(d^{3/2})$. Furthermore, unlike the case of acyclic coloring, we are able to supply a lower bound which matches the upper bound, up to a constant factor. That is, there are graphs G with $lc(G) = \Omega(d^{3/2})$. The lower bound is given by an *explicit* construction. Our proof of the upper bound is merely an extension of the result in [2], and is given is Section 2. The lower bound is constructed in Section 3. It is interesting to note that the probabilistic upper bound supplies the correct order of magnitude, and even the constant factor gap between the upper and lower bound is relatively small (less than $10 \cdot 11 = 110$).

2 The upper bound

Theorem 2.1 *If $\Delta(G) = d$ then $lc(G) \leq \lceil \max\{50d^{4/3}, 10d^{3/2}\} \rceil$.*

Proof: Our proof is an extension to the proof of Theorem 1.1 in [2]. Let $x = \lceil \max\{50d^{4/3}, 10d^{3/2}\} \rceil$, and let $f : V \rightarrow \{1, 2, \ldots, x\}$ be a random coloring of the vertices of G, where for $v \in V$, $f(v)$ is chosen randomly and uniformly from $\{1, 2, \ldots, x\}$. We consider events of five different types, and notice that if none of them occurs, then f is a linear coloring. The first four event types are described in [2] and are repeated here for the reader's convenience. The fifth type is a new set of events, which never occur in a linear coloring.

I. For each pair of adjacent vertices u and v of G, let $A_{\{u,v\}}$ be the event that $f(u) = f(v)$.

II. For each induced path of length four $v_0v_1v_2v_3v_4$ in G, let $B_{\{v_0,v_1,v_2,v_3,v_4\}}$ be the event that $f(v_0) = f(v_2) = f(v_4)$ and $f(v_1) = f(v_3)$.

III. For each induced 4-cycle $v_1v_2v_3v_4$ in G, in which v_1 and v_3 have at most $d^{2/3}$ common neighbors, and v_3 and v_4 have at most $d^{2/3}$ common neighbors, let $C_{\{v_1,v_2,v_3,v_4\}}$ be the event that $f(v_1) = f(v_3)$ and $f(v_2) = f(v_4)$.

IV. For each pair of non-adjacent vertices u, w with more than $d^{2/3}$ common neighbors, let $D_{\{u,w\}}$ be the event that $f(u) = f(w)$.

V. For each triple of vertices v_1, v_2, v_3 which have a common neighbor, let $E_{\{v_1,v_2,v_3\}}$ be the event that $f(v_1) = f(v_2) = f(v_3)$.

	I	II	III	IV	V
I	$2d$	$6d^4$	$2d^{8/3}$	$2d^{4/3}$	d^3
II	$5d$	$15d^4$	$5d^{8/3}$	$5d^{4/3}$	$2.5d^3$
III	$4d$	$12d^4$	$4d^{8/3}$	$4d^{4/3}$	$2d^3$
IV	$2d$	$6d^4$	$2d^{8/3}$	$2d^{4/3}$	d^3
V	$3d$	$9d^4$	$3d^{8/3}$	$3d^{4/3}$	$1.5d^3$

Table 1: Upper bounds in the dependency graph H

It is proved in [2] that if none of the events of the first four types occurs, then f is an acyclic coloring. Thus, if none of the events of the five types above occurs, then f is a linear coloring. It remains to show that with positive probability none of these events happen. For this purpose, we construct H, a dependency graph on the events. The nodes of H are the events of all the five different types, and two nodes X_S and Y_T (where $X, Y \in \{A, B, C, D, E\}$) are adjacent if and only if $S \cap T \neq \emptyset$. In order to apply the Lovász Local Lemma (cf. [4], see also [1]), we need to estimate the number of nodes of each type in H which are adjacent to any given node. This estimate is given in Table 1. In this table, the (i, j) entry is an upper bound on the number of nodes of type j (i.e. nodes which correspond to events of type j) which are adjacent in H to a node of type i. The values in the first four rows and first four columns are proved correct in Lemmas 2.4 and 2.5 of [2]. We need to prove the correctness of the bounds in the last row and column. Consider a vertex $w \in G$. Since $\Delta(G) = d$, there are at most $d \cdot \binom{d-1}{2} < d^3/2$ events of type V in which w participates. Thus, an event of type I, denoted by $A_{\{u,v\}}$, is adjacent in H to at most $2 \cdot d^3/2 = d^3$ events of type V. This validates entry (1,5) in Table 1. Similarly, an event of type II, denoted by $B_{\{v_0,v_1,v_2,v_3,v_4\}}$ is adjacent in H to at most $5 \cdot d^3/2$ events of type V, validating entry (2,5). The other entries in the last column are verified analogously. The entries in the last row are proved correct by the fact that events of type V are formed by sets of three vertices, and by Lemma 2.4 of [2], which bounds the number of occurrences of a vertex v in an event of type i, for $i = 1, 2, 3, 4$ (for $i = 1$ this bound is d, for $i = 2$ it is $3d^4$, for $i = 3$ it is $d^{8/3}$ and for $i = 4$ it is $d^{4/3}$).

Since the coloring f is random, we obviously have that $\text{Prob}[E] = \frac{1}{x^2}$, where E is an event of type V. We add this fact to fact 2.6 in [2] which states that $\text{Prob}[A] = \frac{1}{x}$, $\text{Prob}[B] = \frac{1}{x^3}$, $\text{Prob}[C] = \frac{1}{x^2}$, and $\text{Prob}[D] = \frac{1}{x}$, where A,B,C,D are events of types I, II, III, IV respectively. Finally, in order to apply the Local Lemma we define the weight of each event to be twice its probability. Hence, in order to prove that with positive probability none of the forbidden events hold, we need to show that the following four inequalities hold.

$$\frac{1}{x} \le \frac{2}{x}(1-\frac{2}{x})^{2d+2d^{4/3}}(1-\frac{2}{x^3})^{6d^4}(1-\frac{2}{x^2})^{2d^{8/3}+d^3} \tag{1}$$

$$\frac{1}{x^3} \le \frac{2}{x^3}(1-\frac{2}{x})^{5d+5d^{4/3}}(1-\frac{2}{x^3})^{15d^4}(1-\frac{2}{x^2})^{5d^{8/3}+2.5d^3} \tag{2}$$

$$\frac{1}{x^2} \le \frac{2}{x^2}(1-\frac{2}{x})^{4d+4d^{4/3}}(1-\frac{2}{x^3})^{12d^4}(1-\frac{2}{x^2})^{4d^{8/3}+2d^3} \tag{3}$$

$$\frac{1}{x^2} \le \frac{2}{x^2}(1-\frac{2}{x})^{3d+3d^{4/3}}(1-\frac{2}{x^3})^{9d^4}(1-\frac{2}{x^2})^{3d^{8/3}+1.5d^3}. \tag{4}$$

Inequality (1) corresponds to events of type I and IV, inequality (2) to events of type II, inequality (3) to events of type III and inequality (4) to events of type V. Note that the validity of inequality (2) implies that of the other three, and inequality (2) is valid since

$$(1-\frac{2}{x})^{5d+5d^{4/3}}(1-\frac{2}{x^3})^{15d^4}(1-\frac{2}{x^2})^{5d^{8/3}+2.5d^3} \ge$$

$$(1-\frac{20d^{4/3}}{x})(1-\frac{30d^4}{x^3})(1-\frac{15d^3}{x^2}) \ge (1-\frac{20}{50})(1-\frac{30}{50^3})(1-\frac{15}{100}) > \frac{1}{2}.$$

□

3 The lower bound

We give an explicit construction of a graph $G_r = (V_r, E_r)$ with r^3 vertices, with $\Delta(G_r) = \Theta(r^2)$ and with $lc(G_r) \ge r^3/2$. G_r is defined as follows: $V_r = A * A * A$ where $A = \{1, \ldots, r\}$. A vertex (a_1, a_2, a_3) is connected to (b_1, b_2, b_3) if and only if they coincide in at least one coordinate, i.e, $a_i = b_i$ for some i. Note that G_r has r^3 vertices, and the degree of every vertex is $3r(r-1)$. We show that $lc(G_r) \ge r^3/2$. It suffices to show that in every linear coloring of G_r, no color appears in more than two vertices. Indeed, assume that (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) are all colored with the same color. Since the coloring is proper, this means that for $i = 1, 2, 3$, the set $\{a_i, b_i, c_i\}$ contains three distinct values. Thus, the vertex (a_1, b_2, c_3) is distinct from all of these three vertices, but it is a neighbor of each of them. This cannot happen in a linear coloring.

By putting $d = 3r^2$ we obtain a graph G_r with maximum degree less than d, and with $lc(G_r) \ge d^{3/2}/(6\sqrt{3}) > d^{3/2}/11$. By taking vertex-disjoint components of G_r, we obtain that for every $n > 0$, there exists a graph G with more than n vertices, and maximum degree less than d, and $lc(G) > d^{3/2}/11$.

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