Efficient covering designs of the complete graph

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Abstract

Let H be a graph. We show that there exists $n_0 = n_0(H)$ such that for *every* $n \geq n_0$, there is a covering of the edges of K_n with copies of H where every edge is covered at most twice and any two copies intersect in at most one edge. Furthermore, the covering we obtain is asymptotically optimal.

1 Introduction

All graphs considered here are finite, undirected and simple, unless otherwise noted. For the standard graph-theoretic notations the reader is referred to [5]. Let $H = (V_H, E_H)$ be a graph. An *H-covering design* of a graph $G = (V_G, E_G)$ is a set $L = \{G_1, \ldots G_s\}$ of subgraphs of G such that each G_i is isomorphic to H and every edge $e \in E_G$ appears in at least one member of L. The *H-covering number* of G, denoted by $cov(G, H)$, is the minimum number of members in an H-covering design of G. (If there is an edge of G which cannot be covered by a copy of H, we put $cov(G, H) = \infty$). Clearly, $cov(G, H) \geq |E_G|/|E_H|$. In case equality holds, the H-covering design is called an *H-decomposition* (or H-design) of G. Two trivial necessary conditions for a decomposition are that $|E_H|$ divides $|E_G|$ and that $gcd(H)$ divides $gcd(G)$ where the *gcd* of a graph

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is the greatest common divisor of the degrees of all the vertices. In case $G = K_n$, it was shown by Wilson in [17] that the two necessary conditions are also sufficient, provided $n \geq n_0(H)$, where $n_0(H)$ is a sufficiently large constant. If, however, the necessary conditions do not hold, the best one could hope for is an H-covering design of K_n where the following three properties hold:

1. **2-overlap**: Every edge is covered at most twice.
2. **1-intersection**: Any two copies of H intersect in at most one edge.
3. **Efficiency**: $s|E_H| < \binom{n}{2} + c(H) \cdot n$, where s is the number of members in the covering, and $c(H)$ is some constant depending only on H.

The papers of Mills and Mullin [12] and of Brouwer [4], provide an excellent survey of covering designs. Covering designs with the 2-overlap property were first introduced in statistical designs by [10] and are also mentioned in [2], [6] and [11]. Covering designs with the 1-intersection property (also called super-simple designs) are mentioned by Adams et. al. in [1], Teirlinck [15, 16], Fort and Hedlund [8], Brouwer [3] and Schreiber [14]. The existence of efficient Covering designs of *complete hypergraphs* was first proved by Rödl in [13].

Our main result is that H-covering designs of K_n, having these three properties, exist for every fixed graph H, and for all $n \geq n_0(H)$:

Theorem 1.1 *Let H be a fixed graph. There exists $n_0 = n_0(H)$ such that if $n \geq n_0$, K_n has an H-covering design with the 2-overlap, 1-intersection, and efficiency properties.*

2 Proof of the main result

We shall prove Theorem 1.1 whenever $H = K_h$ is a complete graph. This suffices, since if H is not a complete graph, it is known by Wilson's theorem [17] that there exists an $h_0 = h_0(H)$ such that K_{h_0} has an H-decomposition. By applying Theorem 1.1 to K_{h_0}, we shall obtain an $n_0 = n_0(h_0) = n_0(H)$, such that if $n \geq n_0$, K_n has a K_{h_0}-covering design with the 2-overlap and 1-intersection properties and such that $\binom{h_0}{2}s < \binom{n}{2} + h_0^3 \cdot n$, where s is the number of members in the covering. Thus, there is an H-covering design of K_n with the 2-overlap and 1-intersection properties, and with $s\frac{\binom{h_0}{2}}{|E_H|}$ elements, such that $s\frac{\binom{h_0}{2}}{|E_H|}|E_H| < \binom{n}{2} + h_0^3 \cdot n = \binom{n}{2} + c(H) \cdot n$.

Fix K_h, where $h \geq 3$ (for $h = 2$ the result is trivial), and let h_1 be the minimum positive integer such that whenever $n \geq h_1$ and $\binom{h}{2}$ divides $\binom{n}{2}$, and $h-1$ divides $n-1$, K_n has a K_h-decomposition. As mentioned before, the existence of h_1 is guaranteed by Wilson's Theorem [17]. Now let $n \geq \max\{h^8, h_1 + h(h-1)\}$. We will show that K_n has a K_h-covering design, as required in

Theorem 1.1. Let k be the minimum positive integer such that $\binom{h}{2}$ divides $\binom{n-k}{2}$ and $h-1$ divides $n-k-1$. It is easy to see that $0 \le k < h(h-1)$. If $k=0$ we are done, since in this case n satisfies the conditions in Wilson's Theorem, and there is a K_h-decomposition of K_n. Assume, therefore, that $1 \le k < h(h-1)$, and put $r = n-k$. Note that $r > h_1$. Partition the vertices of K_n into two subsets. The *big* subset has r vertices, namely $B = \{a_1, \ldots, a_r\}$. The *small* subset has k vertices, namely $S = \{b_1, \ldots, b_k\}$. We create the members of our efficient covering design in three stages.

Stage 1: Let B_0 be the subgraph induced by the vertices $\{a_1, \ldots, a_{r-1}\}$. Note that B_0 is a complete graph on $r-1$ vertices, and since $h-1$ divides $r-1$, there exists a K_{h-1}-factor in B_0. (Recall that an X-factor of a graph is a set of vertex-disjoint copies of X which cover all the vertices of the graph). Let F_1 be such a factor. We repeat the following process for $i = 2, \ldots, k$. Let B_{i-1} be the graph obtained from B_{i-2} after the edges of the members of F_{i-1} have been removed. Let F_i be a K_{h-1}-factor in B_{i-1}. In order to show that our process works, we need to show that a K_{h-1}-factor exists in B_{i-1}. We prove this by induction on i. For $i=1$, this is simply the factor F_1 defined above. Assume the claim holds for all $j < i$. This implies that B_{i-1} is regular of degree $(r-2)-(i-1)(h-2)$. According to the theorem of Hajnal and Szemerédi [9] if $(r-2)-(i-1)(h-2) \ge \frac{h-2}{h-1}(r-1)$ then B_{i-1} has a K_{h-1}-factor. Indeed,

$$(r-2)-(i-1)(h-2) \ge (r-2)-(k-1)(h-2) > (r-2)-h(h-1)(h-2) > r-h^3.$$

Since $r-\frac{r-1}{h-1} > \frac{h-2}{h-1}(r-1)$ it suffices to show that $r-h^3 \ge r-\frac{r-1}{h-1}$ and this holds since $r = n-k > h^4$. Having defined the K_{h-1}-factors $F_1, \ldots, F_k$, we now define a set L_1 of edge-disjoint copies of K_h in our K_n, which cover all the edges between S and $\{a_1, \ldots, a_{r-1}\}$. This is done by joining the vertex b_i to every member of F_i, for $i = 1, \ldots, k$. Note that whenever we join b_i to a member of F_i we obtain a copy of K_h. Note also that L_1 has exactly $k(r-1)/(h-1)$ members.

Stage 2: Since $r \ge h_1$, and since $h-1$ divides $r-1$ and $\binom{h}{2}$ divides $\binom{r}{2}$, we have by Wilson's Theorem that the subgraph induced by B (which is a K_r), has a K_h-decomposition. Fix a labeled K_h-decomposition D of this K_r. That is, D is a set of $\binom{r}{2}/\binom{h}{2}$ h-subsets of $\{a_1, \ldots, a_r\}$, where for each $1 \le i < j \le r$, the pair (a_i, a_j) appears in exactly one member of D. If π is any permutation of $\{1, \ldots, r\}$ then let D_π be the labeled K_h-decomposition obtained from D by replacing each appearance of a_i in any member of D with $\pi(a_i)$, for $i = 1, \ldots, r$. Our aim is to show that there exists a permutation π, and a set L^* of less than h^5 members of L_1 (recall that L_1 is constructed in stage 1), such that every member of D_π intersects every member of $L_1 \setminus L^*$ in at most one edge. In order to achieve this goal, we pick π randomly, where each of the $r!$ permutations is equally likely. Consider two distinct edges (a_i, a_j) and (a_k, a_l) which both appear in the same member of L_1 (note that when $h=3$, there is no such pair, since every member of L_1 contains only two vertices of B). We call such a pair of edges D_π-*bad* if they both appear in the same member of D_π. We shall compute the probability that two *fixed* edges (a_i, a_j) and (a_k, a_l) are D_π-bad. Consider first the case where (a_i, a_j) and (a_k, a_l) share an endpoint, say $a_k = a_i$. Since π is random, the probability that (a_i, a_j) and (a_i, a_l) appear in the same member of D_π is *exactly* $\frac{h-2}{r-2}$. To see this, fix $\pi(a_i)$ and $\pi(a_j)$, and let Q denote the unique member of D which contains both $\pi(a_i)$ and $\pi(a_j)$. There are $r-2$ possible choices for $\pi(a_l)$, where $h-2$ of them result in a member of Q. Thus, D_π is *bad* with probability $\frac{h-2}{r-2}$, given that $\pi(a_i)$ and $\pi(a_j)$ are known. Note, however, that the expression $\frac{h-2}{r-2}$ does not depend on the specific choices for $\pi(a_i)$ and $\pi(a_j)$. Now consider the case where (a_i, a_j) and (a_k, a_l) are two independent edges (this is possible only if $h-1 \geq 4$, since every member of L_1 contains only $h-1$ vertices from B). By a similar reasoning to the above, the probability that both these edges appear in the same member of D_π is exactly $\frac{h-2}{r-2}\frac{h-3}{r-3}$. There are $(h-1)(h-2)(h-3)/2$ pairs of adjacent edges of the form (a_i, a_j), (a_i, a_l) in every member of L_1. Thus, there are $k\frac{r-1}{h-1}(h-1)(h-2)(h-3)/2$ such pairs in all the members of L_1. There are $3\binom{h-1}{4}$ pairs of two independent edges of the form (a_i, a_j), (a_k, a_l) in every member of L_1. Thus there are $3k\frac{r-1}{h-1}\binom{h-1}{4}$ such pairs in all the members of L_1. Therefore, if μ is the expected number of D_π-bad pairs, then

$$\mu = k\frac{r-1}{h-1}\frac{(h-1)(h-2)(h-3)}{2}\frac{h-2}{r-2} + k\frac{r-1}{h-1}3\binom{h-1}{4}\frac{h-2}{r-2}\frac{h-3}{r-3} <$$

$$\frac{h^5}{2} + \frac{3}{24}h^7\frac{r-1}{(r-2)(r-3)} < h^5.$$

Thus, there exists a permutation π such that the number of D_π-bad pairs is less than h^5. Fix such a permutation, and let $L_2 = D_\pi$. Let L^* be the set of all members of L_1 which contain a D_π-bad pair. Clearly, $|L^*| < h^5$. Thus, every member of L_2 intersects every member of $L_1 \setminus L^*$ in at most one edge. Put $L_3 = L_2 \cup (L_1 \setminus L^*)$.

Stage 3: Every edge of K_n appears in at most two members of L_3 and any two members of L_3 intersect in at most one edge. However, there may still be uncovered edges. In fact, all the $\binom{k}{2}$ edges connecting two members of S are not covered, and all the k edges of the form (b_i, a_r), for $i = 1, \ldots, k$, are not covered. Furthermore, each member of L^* covers $h-1$ edges connecting some $b_i \in S$ to a subset of $h-1$ vertices of $\{a_1, \ldots, a_{r-1}\}$, and these edges are uncovered in L_3. Thus there are $|L^*|(h-1)$ uncovered edges of this form. Hence, if M denotes the set of uncovered edges, we have that

$$|M| = \binom{k}{2} + k + |L^*|(h-1) < h^6.$$

The crucial point is that the number of uncovered edges is bounded by a constant depending only on h. We shall show how to sequentially create a set L_4 of copies of K_h, beginning with $L_4 = \emptyset$, where at each stage, a new copy of K_h containing at least one non-covered edge by members of $L_3 \cup L_4$, is added to L_4 (thus $|L_4| < h^6$) and such that the following three invariants are maintained:

1. Every edge is covered at most twice by members of $L_3 \cup L_4$.
2. Any two members of $L_3 \cup L_4$ intersect in at most one edge.
3. If L_4 already contains j members, then any vertex of $B \cup S$ is adjacent to at most $jh + h^3$ edges which are covered twice by members of $L_3 \cup L_4$.

Note that at the beginning of the process, when $L_4 = \emptyset$, the first two invariants hold, since they hold for L_3. We must show that the third invariant holds initially, when $j = 0$. Indeed, in L_3, all the edges adjacent to a vertex of S are either non-covered, or covered once in L_1. Now consider a vertex $a_i \in B$. If $i < r$, a_i is adjacent to exactly $(h-2)k$ edges which are covered twice by members of $L_1 \cup L_2$ (recall that a_r is not adjacent to any edge which is covered in L_1). Since $L_3 \subset L_1 \cup L_2$, we have that any vertex in $B \cup S$ is adjacent to at most $(h-2)k < h^3$ edges which are covered twice by members of L_3.

Suppose L_4 already contains j members, and there still exists an uncovered edge $e = (q_1, q_2)$ in M. We shall find a set $Q = \{q_3, \ldots, q_h\}$ of $h-2$ vertices in $B \cup S$, and add the complete graph K_h induced by $\{q_1, q_2, \ldots, q_h\}$ to L_4, while maintaining our three invariants. We select the elements of Q sequentially. The first element, q_3, needs to have the property that (q_1, q_3) is not covered twice, and (q_2, q_3) is not covered twice. Indeed there are at most $2(jh + h^3)$ vertices of $(B \cup S) \setminus \{q_1, q_2\}$ which are ruled out as candidates for q_3. Since

$$2(jh + h^3) < 2(h^7 + h^3) \leq h^8 - 2 \leq n - 2$$

we can find the desired q_3. It is important to note that there does not exist any member of $L_3 \cup L_4$ which contains *both* (q_1, q_3) and (q_2, q_3), since this would require it to contain (q_1, q_2) which we assume to be uncovered. Therefore, invariants 1 and 2 still hold. Suppose we have already found appropriate vertices $q_3, \ldots, q_i$, where $i < h$, and we wish to find q_{i+1}. Our requirements of q_{i+1} are as follows: All the edges (q_t, q_{i+1}) for $t = 1, \ldots, i$ should each be covered at most once, and for each once-covered edge (q_t, q_p) where $1 \leq t < p \leq i$, q_{i+1} does not appear in the unique copy of $L_3 \cup L_4$ containing (q_t, q_p). These requirements rule out at most

$$i \cdot (jh + h^3) + \binom{i}{2}(h-2)$$

possible candidates for q_{i+1} from $(B \cup S) \setminus \{q_1, \ldots, q_i\}$. In order to show that q_{i+1} can be selected we need to show that

$$n - i > i(jh + h^3) + \binom{i}{2}(h-2).$$

Indeed,

$$i(jh+h^3)+\binom{i}{2}(h-2)\le (h-1)(h^7+h^3)+\binom{h-1}{2}(h-2)<h^8-(h-1)\le n-i.$$

Our construction of Q shows that after adding the K_h subgraph induced by $\{q_1,\ldots,q_h\}$ as the $j+1$'th element to L_4, invariants 1 and 2 still hold. Note also that invariant 3 holds as any vertex may only have at most $h-1$ edges which are now covered twice, and which were not covered twice prior to this stage. (The only vertices for which this may happen are $q_1,\ldots,q_h$).

In order to complete our proof we only need to show that if $L=L_3\cup L_4$ contains s elements then $s\binom{h}{2}<\binom{n}{2}+h^3n$. Clearly, it suffices to show that

$$sh(h-1)<n(n-1)+h^3(n-1). \tag{1}$$

L_4 contains less than h^6 members. L_1 contains exactly $k(r-1)/(h-1)$ members, and L_2 contains exactly $\binom{r}{2}/\binom{h}{2}$ members. Thus,

$$s<h^6+k\frac{r-1}{h-1}+\frac{\binom{r}{2}}{\binom{h}{2}}. \tag{2}$$

We shall prove (1) using (2) and using the facts that $k<h(h-1)$, $r=n-k$ and $n\ge h^8$. Indeed

$$sh(h-1)<h^7(h-1)+hk(r-1)+r(r-1)=h^8-h^7+hkn-hk^2-hk+n^2-2kn+k^2-n+k<$$

$$h^8-h^3+hkn+n^2-2kn-n<n(n-1)+h^3(n-1).$$

□

3 Concluding remarks and an open problem

When $H=K_h$, the constant $n_0(H)$ in Theorem 1.1 is shown in the proof to be no larger than $\max\{h^8,h_1+h(h-1)\}$, where $h_1=h_1(h)$ is the corresponding constant in Wilson's Theorem. However, the best known bound for h_1 (and, consequently, for $n_0(H)$), is rather large, and highly exponential in h [7]. It is plausible, however, that the statement of Theorem 1.1 is still valid for $n_0(H)$ which is much smaller. In fact, we conjecture the following:

Conjecture 3.1 *There exists a positive constant C such that for all $h\ge 2$, if $n\ge Ch^2$ then K_n has a K_h covering design where each edge is covered at most twice and any two copies intersect in at most one edge.*

Note that a positive answer to Conjecture 3.1 requires a proof which does not use Wilson's Theorem, as improving Wilson's constant to $O(h^2)$ is unlikely. The h^2 factor in Conjecture 3.1 cannot be reduced since we have the following simple $0.25h^2$ lower bound: Assume that $h \geq 10$. If $n = \lfloor 0.25h^2 \rfloor$ then any K_h-covering of K_n contains $\binom{n}{2}/\binom{h}{2} > h/2$ members. However, the union of t K_h-subgraphs with the 1-intersection property contains at least $h + (h-2) + \ldots + (h-2t+2)$ vertices. For $t = \lceil h/2 \rceil$ this sum is greater than $0.25h^2 \geq n$. Thus, any K_h-covering of K_n does not have the 1-intersection property.

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