A note on graphs without k-connected subgraphs

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Abstract

Given integers $k \geq 2$ and $n \geq k$, let $c(n,k)$ denote the maximum possible number of edges in an n-vertex graph which has no k-connected subgraph. It is immediate that $c(n,2) = n-1$. Mader [2] conjectured that for every $k \geq 2$, if n is sufficiently large then $c(n,k) \leq (1.5k-2)(n-k+1)$, where equality holds whenever $k-1$ divides n. In this note we prove that when n is sufficiently large then $c(n,k) \leq \frac{193}{120}(k-1)(n-k+1) < 1.61(k-1)(n-k+1)$, thereby coming rather close to the conjectured bound.

1 Introduction

All graphs considered here are finite, undirected and have no loops or multiple edges. For the standard terminology used the reader is referred to [1]. This paper is about a classical extremal problem in graph connectivity, raised by Mader in [3]. Let $k \geq 2$ be an integer. Recall that a graph with $n \geq k+1$ vertices is k-connected if the removal of any set of $k-1$ vertices from the graph results in a connected subgraph (graphs with $n \leq k$ vertices are considered non k-connected). For $n \geq k$, let $c(n,k)$ denote the maximum possible number of edges in an n-vertex graph which has no k-connected subgraph. It is easy to see that $c(n,2) = n-1$ since any tree does not have a 2-connected subgraph, and any n-vertex graph with n edges contains a cycle, which is a 2-connected subgraph. For the rest of this paper we shall assume $k \geq 3$, whenever necessary. Trivially, $c(k,k) = \binom{k}{2}$. Since the complete graph K_{k+1} is the only k-connected graph with $k+1$ vertices, one has $c(k+1,k) = \binom{k+1}{2} - 1$ where the unique extremal graph is K_{k+1}^- (the complete graph missing one edge).

In [2], Mader gave a construction of an n-vertex graph with no k-connected subgraph, and with a rather large number of edges. Let $G_{n,k}$ be defined as follows. Assume $n = (k-1)q + r$ where $1 \le r \le k-1$. The vertices of $G_{n,k}$ are arranged in $q+1$ classes $V_0, \ldots, V_q$, where each class contains exactly $k-1$ vertices, except for the final class V_q which contains r vertices. V_0 is an independent set, and V_i is a complete graph for $i = 1, \ldots, q$. Furthermore, there is an edge between each vertex of V_0 and each vertex of V_i for $i \ge 1$. Note that V_0 is a disconnecting set of size $k-1$. It is thus easy to check that $G_{n,k}$ has no k-connected subgraph. Let $e(n,k)$ denote the number of edges of $G_{n,k}$. We have:

$$e(n,k) = (q-1)\binom{k-1}{2} + \binom{r}{2} + (k-1)(n-k+1) \le (\frac{3}{2}k-2)(n-k+1), \quad (1)$$

and equality is obtained whenever n is a multiple of $k-1$. It follows that $c(n,k) \ge e(n,k)$. Mader [2] has conjectured the following:

Conjecture 1.1 (Mader [2]) *For n sufficiently large, $c(n,k) \le (\frac{3}{2}k - 2)(n-k+1)$. Consequently, if n is a multiple of $k-1$ then $c(n,k) = (\frac{3}{2}k-2)(n-k+1)$, and $G_{n,k}$ is an extremal graph.*

Mader [3] has proved Conjecture 1.1 for all $k \le 7$. The reason that n needs to be sufficiently large in Conjecture 1.1 follows from the fact that there exist n-vertex graphs with more than $(\frac{3}{2}k-2)(n-k+1)$ edges, and with no k-connected subgraph, for $n = \Theta(k^2)$.

A simple upper bound showing that $c(n,k) < (2k-3)(n-k+1)$ whenever $n \ge 2k-1$ is presented in [1], p. 45. Mader showed that for n sufficiently large, $c(n,k) < (1+\sqrt{2}/2)(k-1)(n-k+1)$. In this note we present a further improvement which is about halfway between Mader's bound and the bound in Conjecture 1.1:

Theorem 1.2 *For $k \ge 3$ and for $n \ge \frac{9}{4}(k-1)$, $c(n,k) \le \frac{193}{120}(k-1)(n-k+1)$.*

2 Proof of Theorem 1.2

An (S,A,B)-*partition* of a non k-connected graph G is a partition of the vertex set of G into three parts S, A and B, where $|S| = k-1$, $|A| \le |B|$ and there is no edge connecting a vertex of A and a vertex of B. Clearly, every non k-connected graph with at least $k+1$ vertices has an (S,A,B)-partition. Given an (S,A,B)-partition, let G_A and G_B denote the subgraphs of G induced by $S \cup A$ and $S \cup B$ respectively.

Proof of Theorem 1.2: Matula has proved [4] that

$$c(n,k) \le \binom{n}{2} - \frac{(n-k+1)^2-1}{3}. \quad (2)$$

We shall use this fact. For completeness, we reprove (2). This is done by induction on n. For $n = k$, (2) is obvious. For $n = k + 1$ we have $c(k + 1, k) = \binom{k+1}{2} - 1$, so (2) holds. Assume it holds for all $k \leq a < n$. Let G be an n-vertex graph without a k-connected subgraph. Consider an (S, A, B)-partition of G. Clearly, G misses at least the $|A||B|$ possible edges between A and B, and by the induction hypothesis, G_B, as a subgraph of G with $|B| + k - 1 < n$ vertices, misses at least $(|B|^2 - 1)/3$ additional edges. Hence, since $|A| \leq |B|$:

$$e(G) \leq \binom{n}{2} - |A||B| - (|B|^2 - 1)/3 \leq$$

$$\binom{n}{2} - \frac{(|A| + |B|)^2 - 1}{3} = \binom{n}{2} - \frac{(n - k + 1) - 1}{3}.$$

This proves (2) for all $n \geq k$.

Now let $n \geq \frac{9}{4}(k-1)$, and let G be an n-vertex graph without a k-connected subgraph. Put $n = \gamma(k - 1)$ and assume first that $\gamma \leq \frac{17}{5}$. According to (1) we have:

$$e(G) \leq \frac{\gamma^2(k-1)^2 - \gamma(k-1)}{2} - \frac{(\gamma-1)^2(k-1)^2 - 1}{3} \leq$$

$$(k-1)^2(\frac{\gamma^2}{2} - \frac{(\gamma-1)^2}{3}) \leq \frac{193}{120}(\gamma-1)(k-1)^2 = \frac{193}{120}(k-1)(n-k+1).$$

Now assume that $\gamma > \frac{17}{5}$. We use induction once again, and assume the theorem holds for each value smaller than n. Consider an (S, A, B)-partition of G, put $a = |A|$ and $b = |B|$, and recall that $a \leq b$. Let α and β be defined by $a = \alpha(k - 1)$ and $b = \beta(k - 1)$. Notice that $a + b + k - 1 = n$ and so $\alpha + \beta = \gamma - 1$. Consider first the case $\alpha \leq 1$. In this case, $\beta + 1 \geq \frac{12}{5}$, so the induction hypothesis holds for G_B. Hence, the number of edges of G is at most

$$\frac{a(a-1)}{2} + a(k-1) + \frac{193}{120}(k-1)b < 1.5(k-1)a + \frac{193}{120}(k-1)b <$$

$$\frac{193}{120}(k-1)(a+b) = \frac{193}{120}(k-1)(n-k+1).$$

Now consider the case where $\alpha \geq \frac{5}{4}$. Since $\beta \geq \alpha$ we also have $\beta \geq \frac{5}{4}$. In this case, both G_A and G_B have at least $\frac{9}{4}(k - 1)$ edges, and since $e(G) \leq e(G_A) + e(G_B)$ we have by the induction hypothesis that:

$$e(G) \leq \frac{193}{120}(k-1)a + \frac{193}{120}(k-1)b = \frac{193}{120}(k-1)(n-k+1).$$

We remain with the case where $1 < \alpha < \frac{5}{4}$. A useful observation is the following: For every $1 < \alpha < \frac{5}{4}$:

$$\frac{\alpha^2}{2} - \frac{(\alpha-1)^2}{3} + \alpha - \frac{193}{120}\alpha \le 0. \tag{3}$$

Furthermore, the l.h.s. of (3) is monotone increasing in the range $[1, \frac{5}{4}]$.
Since there are at most $a(k-1)$ edges between S and A we have that $e(G) \le e(A) + a(k-1) + e(G_B)$. If $\beta \ge \frac{5}{4}$ then, according to (2) applied to $e(A)$ and the induction hypothesis applied to $e(G_B)$, and using (3) we have:

$$e(G) \le \binom{a}{2} - \frac{(a-k+1)^2-1}{3} + a(k-1) + \frac{193}{120}b(k-1) =$$

$$\frac{\alpha(k-1)(\alpha(k-1)-1)}{2} - \frac{(\alpha-1)^2(k-1)^2-1}{3} + \alpha(k-1)^2 + \frac{193}{120}\beta(k-1)^2 \le$$

$$(k-1)^2\left(\frac{\alpha^2}{2} - \frac{(\alpha-1)^2}{3} + \alpha + \frac{193}{120}\beta\right) \le (k-1)^2\frac{193}{120}(\alpha+\beta) =$$

$$\frac{193}{120}(k-1)(n-k+1).$$

Finally, if $\beta < \frac{5}{4}$ then we can use (2) also for $e(G_B)$ and obtain:

$$e(G) \le (k-1)^2\left(\frac{\alpha^2}{2} - \frac{(\alpha-1)^2}{3} + \alpha\right) + e(G_B) \le$$

$$(k-1)^2\left(\frac{\alpha^2}{2} - \frac{(\alpha-1)^2}{3} + \alpha + \frac{(\beta+1)^2}{2} - \frac{\beta^2}{3}\right).$$

We therefore need to show that:

$$\frac{\alpha^2}{2} - \frac{(\alpha-1)^2}{3} + \alpha + \frac{(\beta+1)^2}{2} - \frac{\beta^2}{3} \le \frac{193}{120}(\alpha+\beta).$$

Since the l.h.s. of (3) is monotone increasing in the selected range, and since $\alpha \le \beta$, the worst case in the last inequality occurs when $\alpha = \beta$. It therefore suffices to show that:

$$\frac{\alpha^2}{3} + \frac{8}{3}\alpha + \frac{1}{6} \le \frac{193}{60}\alpha$$

which, in turn, is true for $1 < \alpha < \frac{5}{4}$. □

References

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