Packing cliques in graphs with independence number 2

Raphael Yuster *
Department of Mathematics
University of Haifa
Haifa 31905, Israel

Abstract

Let G be a graph with no three independent vertices. How many edges of G can be packed with edge-disjoint copies of K_k? More specifically, let $f_k(n,m)$ be the largest integer t such that for any graph with n vertices, m edges, and independence number 2, at least t edges can be packed with edge-disjoint copies of K_k. Turán's Theorem together with Wilson's Theorem assert that $f_k(n,m) = (1-o(1))\frac{n^2}{4}$ if $m \approx \frac{n^2}{4}$. A conjecture of Erdős states that $f_3(n,m) \geq (1-o(1))\frac{n^2}{4}$ for all plausible m. For any $\epsilon > 0$, this conjecture was still open even if $m \leq n^2(\frac{1}{4}+\epsilon)$. Generally, $f_k(n,m)$ may be significantly smaller than $\frac{n^2}{4}$. Already for $k=7$ it is easy to show that $f_7(n,m) \leq \frac{21}{90}n^2$ for $m \approx 0.3n^2$. Nevertheless, we prove the following result. For every $k \geq 3$ there exists $\gamma > 0$ so that if $m \leq n^2(\frac{1}{4}+\gamma)$ then $f_k(n,m) \geq (1-o(1))\frac{n^2}{4}$. In the special case $k=3$ we obtain the reasonable bound $\gamma \geq 10^{-4}$. In particular, the above conjecture of Erdős holds whenever G has less than $0.2501n^2$ edges.

1 Introduction

All graphs considered here are finite, undirected and simple. For standard graph-theoretic terminology see [1]. Let $g(n)$ be the largest integer t so that in any 2-coloring of the edges of K_n, at least t edges can be packed with edge-disjoint monochromatic triangles. Similarly, let $f(n)$ be the largest integer t so that in any n-vertex graph with independence number 2, at least t edges can be packed with edge-disjoint triangles. Clearly, $g(n) \leq f(n)$. On the other hand, $f(n) \leq n^2/4 - o(n^2)$ as can be seen by taking two vertex-disjoint cliques of sizes $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$. This led Erdős to conjecture (see, e.g., Problem 14 in [3])

$$g(n) = \frac{n^2}{4} - o(n^2).$$

This conjecture, studied recently in [2, 5], as well as the corresponding conjecture that $f(n) = \frac{n^2}{4} - o(n^2)$ (the latter raised explicitly in [5]), are still open. The best lower bound for $g(n)$, as well as for $f(n)$, is due to Keevash and Sudakov [5]. By examining the fractional version of the problem on small cases (with the assistance of a computer), together with a clever blow-up idea and a result of Haxell and Rödl, they obtain $g(n) \geq n^2/4.3 + o(n^2)$. All the extremal values in the small cases

*e-mail: raphy@research.haifa.ac.il

they examine also hold for graphs with independence number 2, and no lower bound for $f(n)$ is known to beat their lower bound for $g(n)$.

Generalizing the definition of $f(n)$, let $f_k(n,m)$ be the largest integer t so that in every graph with n vertices, m edges, and independence number 2, at least t edges can be packed with edge-disjoint K_k. Notice that the plausible values of m are at least $n^2/4 - n/2$ as Turán's Theorem guarantees that graphs with less edges have three independent vertices. Also note that $f(n) = \min_m f_3(n,m)$ and the conjecture of Erdős states that $f_3(n,m) \geq (1-o(1))\frac{n^2}{4}$. The simple example of two vertex-disjoint cliques shows that $f_k(n,m) \leq (1-o(1))\frac{n^2}{4}$ if $m = n^2(\frac{1}{4}+o(1))$. Wilson's Theorem [7] guarantees that large cliques almost completely decompose into copies of K_k and we therefore have that $f_k(n,m) = (1-o(1))\frac{n^2}{4}$ if $m = n^2(\frac{1}{4}+o(1))$ (namely, in graphs with density close to 1/2).

However, unlike the case of triangles, the analogue of the conjecture of Erdős does not hold for arbitrary large k. Namely, it is *not true* that $f_k(n,m) \geq (1-o(1))\frac{n^2}{4}$. The following proposition shows that already for $k = 7$ we have $f_7(n,m) \leq \frac{21}{90}n^2$ for $m = 0.3n^2(1+o(1))$.

Proposition 1.1

$$f_k\left(5n, \frac{5n(3n-1)}{2}\right) \leq \frac{5\binom{n}{2}\binom{k}{2}}{\lceil \frac{k}{2}(\frac{k}{2}-1)\rceil}.$$

In particular, $f_k(n, 0.3n^2(1+o(1)) \leq \frac{n^2\binom{k}{2}}{10\lceil \frac{k}{2}(\frac{k}{2}-1)\rceil}$.

It is therefore interesting to ask whether the analogue of the conjecture of Erdős *is true* for graphs whose density is greater than 1/2 (that is, m/n^2 is larger than 1/4). The main result of this paper gives an affirmative answer.

Theorem 1.2 *For every integer $k \geq 3$ there exists $\gamma > 0$ so that $f_k(n,m) \geq (1-o(1))\frac{n^2}{4}$ for $m \leq n^2(\frac{1}{4}+\gamma)$.*

Although we make no particular effort to optimize γ (and Proposition 1.1 shows that we cannot hope to have γ too large, at least when $k \geq 7$) we do make some effort in the case $k = 3$. In this case we can show that $\gamma \geq 10^{-4}$. In fact, we also prove, in the other end of the density scale, that $3n^2/8$ edges already guarantee $n^2/12 - o(n^2)$ edge-disjoint triangles.

Theorem 1.3 *Every graph with n vertices, less than $0.2501n^2$ edges and independence number 2 has $n^2/12 - o(n^2)$ edge-disjoint triangles. Every graph with n vertices, more than $0.375n^2$ edges and independence number 2 has $n^2/12 - o(n^2)$ edge-disjoint triangles.*

Theorem 1.3 shows that the conjecture of Erdős for $f(n)$ holds when the density of the graph is at most 0.5002 or at least 0.75.

The proof of Theorem 1.2 is presented in Section 2. The proof of Theorem 1.3 requires a few additional ideas and is presented in Section 3. Two important ingredients in these proofs, both interesting on their own right, are Lemma 3.1 on packing induced paths of length two in bipartite graphs and Theorem 2.8 on packing edge-disjoint K_k in dense graphs. The final section contains some concluding remarks and the proof of Proposition 1.1.

2 Proof of the main result

We start with a sequence of three lemmas that can be viewed as a tailor-made sharpened version of the *stability theorem* of Simonovits [6] in the case of triangle-free graphs.

Lemma 2.1 *Every graph with m edges and n vertices has an edge so that the sum of the degrees of its endpoints is at least $4m/n$.*

Proof: Let $G = (V, E)$ be a graph with m edges and n vertices. For $e = (x, y) \in E$, let $w(e) = d(x) + d(y)$, where $d(v)$ is the degree of v. Clearly,

$$\sum_{e \in E} w(e) = \sum_{v \in V} d(v)^2 \geq n \left(\frac{2m}{n}\right)^2 .$$

It follows that for some $e \in E$, $w(e) \geq (n/m)(2m/n)^2 = 4m/n$. ■

Lemma 2.2 *Let $G = (V, E)$ be a triangle-free graph with n vertices and $\frac{1}{4}n^2(1-\rho^2)$ edges. Then (i) at most ρn vertices have degree less than $(1-\rho)(n/2)$,*
(ii) there exists a set of vertices U with $|U| \leq 2 + \rho^2 n$, such that $G - U$ is a bipartite graph.

Proof: (i) Assuming otherwise, let X be a set of ρn vertices, each $x \in X$ having $d(x) < (1-\rho)(n/2)$ (we ignore floors and ceilings here and anywhere else in this paper, where it does not affect the asymptotic nature of our results, and assume n is sufficiently large, whenever necessary). Consider the induced subgraph $G' = G[V \setminus X]$. Now,

$$e(G') > n^2(\frac{1}{4} - \frac{1}{4}\rho^2) - \rho(\frac{1}{2} - \frac{\rho}{2})n^2 \geq \frac{1}{4}n^2(1-\rho)^2 = \frac{1}{4}v(G')^2.$$

Hence, by Turán's Theorem, G' contains a triangle, a contradiction.
(ii) Let (x, y) be an edge as in Lemma 2.1. Then, $d(x) + d(y) \geq n(1-\rho^2)$. Let $U' \subset V$ be the set of vertices that are not connected to both x and y. Consider $W = V - U' - \{x, y\}$. Clearly, $W = W_x \cup W_y$ where each vertex of W_x (W_y) is connected to x (to y) and not to y (to x). Hence $|W| = d(x) + d(y) - 2 \geq n(1-\rho^2) - 2$, so $|U'| \leq n\rho^2$. Put $U = U' \cup \{x, y\}$. Now, $W = V - U = W_x \cup W_y$ induces a bipartite graph. ■

Lemma 2.3 *Let $G = (V, E)$ be a triangle-free graph with $\frac{1}{4}n^2(1-\rho^2)$ edges. Then, there exists a partition $V = V_1 \cup V_2$ such that $|V_i| \geq n/2 - n(\rho^2 + \rho/2 + 2/n)$ and the number of edges with both endpoints in the same part is at most $(\rho^3 + 1.5\rho^4)n^2$.*

Proof: By Lemma 2.2 there exist two disjoint independent sets V_1' and V_2' and $|V_1'| + |V_2'| \geq n(1-\rho^2)-2$. Furthermore, $|V_i'| \leq n/2+\rho n/2$, as if not, all vertices of V_i' would have degree less than $n/2-\rho n/2$ in G, contradicting the previous lemma. Thus, we also have $|V_i'| \geq n/2-n(\rho^2+\rho/2+2/n)$. Consider $U = V - (V_1' \cup V_2')$. We claim that no vertex of U has more than $3 + \rho n + \rho^2 n$ neighbors in each of V_1' and V_2'. Indeed, if $u \in U$ is such a vertex and $X_i \subset V_i'$ are $3 + \rho n + \rho^2 n$ neighbors of

u in V_i' then there are no edges between X_1 and X_2. Let $z \in X_1$ be a vertex with degree at least $(1-\rho)(n/2)$ in G. On the other hand, the degree of z in G is at most

$$n - |V_1'| - |X_2| \leq n - \left(\frac{n}{2} - n(\rho^2 + \frac{\rho}{2} + \frac{2}{n})\right) - (3 + \rho n + \rho^2 n) = (1-\rho)\frac{n}{2} - 1$$

a contradiction. We may therefore place u in one of V_1' or V_2', wherever it has less than $3+\rho n+\rho^2 n$ neighbors. After doing so for all $u \in U$ we obtain a partition of V to $V_i \supset V_i'$ for $i = 1, 2$ and no more than

$$(\rho^2 n + 2)(3 + \rho n + \rho^2 n) + \frac{1}{4}|U|^2 < (\rho^3 + 1.5\rho^4)n^2$$

edges with both endpoints in the same part. ■

Lemma 2.3 guarantees that in the complement G^c of a dense triangle-free graph G there are many edge-disjoint K_k, each of them having all k vertices in the same part of an appropriate partition. However, this is not enough, since we must compensate for the loss due to a non-negligible number of at most $(\rho^3 + 1.5\rho^4)n^2$ non-edges of G^c with both endpoints in the same part (and, possibly, some unpackable edges of G^c inside each part as well). The only way to do this is to pack sufficiently many edges of G^c with endpoints in both V_1 and V_2 into edge-disjoint K_k of G^c. An important step in this direction is established in the following lemma.

Lemma 2.4 *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with n vertices and ηn^2 edges. Let G' be the graph obtained from G by adding all possible edges inside the vertex classes. Then, for n sufficiently large, there exists a set L of edge-disjoint K_k of G' so that $|L| \geq \frac{1}{40k^4}\eta^{4/3}n^2$. Furthermore, each element of L intersects both vertex classes.*

Proof: Let L be a maximum set of edge-disjoint copies of K_k in G', so that each $H \in L$ intersects both vertex classes. If $|L| \geq \frac{1}{40k^4}\eta^{4/3}n^2$ then we are done. Otherwise, let $M \subset E$ be the edges not belonging to elements of L. Let $G_M = (V_1 \cup V_2, M)$. For $v \in V_1 \cup V_2$ let t_v be the degree of v in G_M. We claim that $t_v \leq \eta^{2/3}n$ for all $v \in V_1 \cup V_2$. Indeed, assume otherwise and let v be with $t_v > \eta^{2/3}n$. Without loss of generality, assume $v \in V_1$ and let $T_v \subset V_2$ be the neighbors of v in G_M. The K_{t_v} induced by T_v in G' has at least ${t_v}^2/(2(k-2)) - t_v$ edges belonging to elements of L. Otherwise, this K_{t_v} would have had $\frac{t_v^2}{2} - \frac{t_v^2}{2(k-2)}$ edges not belonging to elements of L, but then Turán's Theorem guarantees a K_{k-1} inside this K_{t_v}, having no edge in elements of L. This K_{k-1}, together with v, contradicts the maximality of L. Now, if ${t_v}^2/(2(k-2)) - t_v$ edges belong to elements of L then, in particular,

$$\frac{1}{40k^4}\eta^{4/3}n^2 \geq |L| \geq \frac{{t_v}^2/(2(k-2)) - t_v}{\binom{k-1}{2}}.$$

This implies, in particular, that $t_v \leq \eta^{2/3}n$.

Let G_M' be the graph obtained from G_M by adding all the edges inside the vertex classes. In particular, G_M' is a spanning subgraph of G'. We shall produce the desired set L already in G_M'. Notice that $|M| \geq \eta n^2 - \binom{k}{2}|L| > \eta n^2/2$. Let $W \subset V$ be those vertices with $t_v \geq \eta n/2$. Hence,

$$\sum_{v \in W} t_v = 2|M| - \sum_{v \in V-W} t_v \geq \eta n^2 - n\frac{\eta n}{2} \geq \frac{\eta}{2}n^2.$$

For $v \in W$, we randomly choose $s_v = \lfloor \frac{1}{5k^4}\eta^{1/3}t_v \rfloor$ vertex-disjoint $(k-1)$-subsets of T_v. Let $S_v = \{S_v^1, \ldots, S_v^{s_v}\}$ be the set of chosen subsets. An edge of T_v has probability precisely

$$p_v = \frac{s_v\binom{k-1}{2}}{\binom{t_v}{2}} \le \frac{1}{5k^4}\eta^{1/3}t_v\frac{k^2}{t_v(t_v-1)} = \frac{1}{5k^2}\eta^{1/3}\frac{1}{t_v-1} \le \frac{1}{2k^2\eta^{2/3}n}$$

of belonging to some element of S_v. We say that S_v^i is *bad* if some edge with both endpoints in S_v^i also has both endpoints in some other S_u^j. Consider an edge (x,y) with $x,y \in S_v^i$. What is the probability that $x,y \in S_u^j$ for some other vertex $u \in W$? This probability is 0 if u is not a common neighbor of x,y and is precisely p_u if u is a common neighbor of x,y. Since the number of neighbors of, say, x, in the opposite vertex class is $t_x \le \eta^{2/3}n$, the probability that S_v^i is bad is at most

$$\binom{k-1}{2}\eta^{2/3}n\frac{1}{2k^2\eta^{2/3}n} \le \frac{1}{4}.$$

It follows that in $\cup_{v\in W}S_v$ the expected number of good subsets is at least

$$\frac{3}{4}\sum_{v\in W}s_v \ge \frac{1}{2}\cdot\frac{1}{5k^4}\eta^{1/3}\sum_{v\in W}t_v \ge \frac{1}{20k^4}\eta^{4/3}n^2.$$

In particular, there exists a choice of such subsets with this quantity. Let, therefore, L' be a family of good subsets with $|L'| \ge \frac{1}{20k^4}\eta^{4/3}n^2$ and let $S'_v \subset L'$ be those subsets belonging to T_v for $v \in W$. In particular, $\sum_{v\in W}|S'_v| = |L'|$ and hence, without loss of generality,

$$\sum_{v\in W\cap V_1}|S'_v| \ge \frac{|L'|}{2} \ge \frac{1}{40k^4}\eta^{4/3}n^2.$$

For each $v \in W \cap V_1$ and for each $(k-1)$-subset ${S'}_v^i \in S'_v$, notice that $v \cup {S'}_v^i$ is a copy of K_k in G'_M, and notice that all these copies are edge-disjoint. Hence, the result follows. ■

A function f from the set $F_k(G)$ of copies of K_k in a graph $G=(V,E)$ to $[0,1]$ is a *fractional K_k-packing* of G if $\sum_{e\in H\in F_k(G)} f(H) \le 1$ for each $e \in E$. The value of f is $|f| = \sum_{H\in F_k(G)} f(H)$. The *fractional K_k-packing number*, denoted $\nu_k^*(G)$, is the maximum possible value of a fractional K_k-packing. Clearly, $\nu_k^*(G) \ge \nu_k(G)$ where $\nu_k(G)$ is the maximum possible number of edge-disjoint copies of K_k in G. However, a result of Haxell and Rödl [4] shows that they do not differ by a lot (see also [9] for a shorter and more general proof).

Lemma 2.5 *[Haxell and Rödl [4]] Let G be a graph with n vertices. Then, $\nu_k^*(G) \le \nu_k(G) + o(n^2)$.*

Yuster [8] proved that if a graph has sufficiently large minimum degree, then it has a *fractional K_k-decomposition*, namely, a fractional K_k-packing of value $2|E|/(k(k-1))$. The exact statement of his result follows.

Lemma 2.6 *[Yuster [8]] Let $k \ge 3$ be an integer. For n sufficiently large, every graph with n vertices and minimum degree at least $n(1-1/9k^{10})$ has a fractional K_k-decomposition.*

Corollary 2.7 *Let $k \ge 3$ be an integer. For r sufficiently large, if G is obtained from K_r by deleting at most $r-k$ edges sharing an endpoint then G has a fractional K_k-decomposition.*

Proof: Let v be the vertex in common to all the deleted edges. The degree of v in G is at least $k-1$ and the degree of each other vertex of G is at least $r-2$. Partition the neighbors of v into parts $P_1, \ldots, P_t$ where $k-1 \leq |P_i| \leq 2k-3$. Each $v \cup P_i$ induces a complete graph with at least k vertices and at most $2k-2$ vertices, and, in particular, $v \cup P_i$ has a trivial fractional K_k-decomposition. Deleting all these t complete graphs we obtain a graph with $r-1$ vertices (v is now isolated and can be ignored) and minimum degree at least $(r-2)-(2k-4) = r-2k+2$. By Lemma 2.6, if r is sufficiently large, this remaining graph has a fractional K_k-decomposition, and hence so does G. ■

Theorem 2.8 *For a positive integer k, there exists $\epsilon_0 > 0$ so that for all $\epsilon < \epsilon_0$ the following holds. If G is a graph with n vertices and at least $n^2(\frac{1}{2}-\epsilon)$ edges then G has a packing with edge-disjoint copies of K_k so that at most $(2k-3)\epsilon^{6/5}n^2 + o(n^2)$ edges are unpacked.*

Proof: Let $r_0 = r_0(k)$ be a sufficiently large integer to be chosen later, and let $\epsilon_0 = r_0^{-5}$. Given $\epsilon < \epsilon_0$ let $r = \epsilon^{-1/5}$ (we may and will assume that r is an integer). We may also assume that $n \equiv 1 \bmod r$ is sufficiently large as this does not affect the asymptotic nature of the result. Thus, by Wilson's Theorem [7], G has a decomposition L into $n(n-1)/(r(r-1))$ *induced* r-graphs. Let π be a permutation of $V(G) = \{1, \ldots, n\}$ and let L_π be the decomposition obtained by taking each element of L with vertices $\{v_1, \ldots, v_r\}$ and mapping it to the subgraph of G induced by $\{\pi(v_1), \ldots, \pi(v_r)\}$. Now choose π uniformly at random. Let T be the set of non-edges of G and notice that $|T| \leq \epsilon n^2$. For $t_1, t_2 \in T$ sharing no endpoint, the probability that they are in the same element of L_π is precisely

$$\frac{\binom{n-4}{r-4}}{\binom{n-2}{r-2}} < \frac{r^2}{n^2}.$$

Thus, the expected number of elements of L_π having two elements of T sharing no endpoint is less than

$$\frac{r^2}{n^2}\binom{|T|}{2} < \frac{r^2}{n^2}\epsilon^2 n^4 < \epsilon^2 r^2 n^2.$$

It follows that there *exists* an L where less than $\epsilon^2 r^2 n^2$ elements of L have two independent non-edges. Let, therefore, $L = L_1 \cup L_2 \cup L_3 \cup L_4$ where L_1 consists of all induced r-graphs with two or more independent non-edges, L_2 consists of all induced r-graphs isomorphic to $K_r \setminus K_3$, L_3 consists of all induced r-graphs with one vertex with degree at most $k-2$ and all other vertices with degree at least $r-2$, and L_4 consists of all induced r-graphs with one vertex with degree at least $k-1$ and all other vertices with degree at least $r-2$. Since $|L_1| < \epsilon^2 r^2 n^2$, the overall number of edges in the elements of L_1 is less than $\epsilon^2 r^4 n^2$. The total number of elements in L_3 is at most $|T|/(r-k+1) < 2\epsilon n^2/r$. Each element of L_3 contains a K_{r-1} and has a trivial fractional K_k-packing of value $(r-1)(r-2)/(k(k-1))$. By Corollary 2.7, if r_0 is sufficiently large, all the elements in L_4 have a fractional K_k-decomposition. By Lemma 2.6, if r_0 is sufficiently large, all the elements in L_2 have a fractional K_k-decomposition. It follows that

$$\nu_k^*(G) \geq \frac{|E| - \epsilon^2 r^4 n^2 - 2(k-2)\epsilon n^2/r}{\binom{k}{2}}.$$

By Lemma 2.5, $\nu_k(G) \geq \nu_k^*(G) - o(n^2)$. Recalling that $\epsilon = r^{-5}$, the number of edges not packed by an optimal K_k-packing is

$$|E| - \binom{k}{2}\nu_k(G) \leq \epsilon^2 r^4 n^2 + 2(k-2)\epsilon n^2/r + o(n^2) \leq (2k-3)\epsilon^{6/5}n^2 + o(n^2)$$

as required. ■

Proof of Theorem 1.2: Fix $k \geq 3$, and let $\gamma_0 = \gamma_0(k)$ be a sufficiently small constant to be chosen later. We will assume, whenever necessary, that n is sufficiently large. For $\gamma < \gamma_0$, let $G = (V, E)$ be a graph with n vertices and $m = n^2(\frac{1}{4} + \gamma)$ edges and with independence number 2. Hence, $G^c = (V, E^c)$, the complement of G, is triangle-free. We must prove that $\nu_k(G) \geq (1 - o(1))\frac{n^2}{2k(k-1)}$. Let $\rho = \sqrt{8\gamma}$. Thus,

$$|E^c| = \binom{n}{2} - m \geq n^2(\frac{1}{4} - 2\gamma) = \frac{1}{4}n^2(1 - \rho^2).$$

By Lemma 2.3 applied to G^c, there exists a partition $V = V_1 \cup V_2$ such that $|V_i| \geq n/2 - n(\rho^2 + \rho/2 + 2/n)$ and the number of edges of G^c with both endpoints in the same part is at most $(\rho^3 + 1.5\rho^4)n^2$.

Let E_i and E_i^c be, respectively, the set of edges of G and G^c inside V_i, for $i = 1, 2$. Let E_{12} and E_{12}^c be, respectively, the set of edges of G and G^c between V_1 and V_2. Putting $|E_{12}| = \eta n^2$, we consider two cases.

If $\eta < \gamma/2$ then

$$|E_1| + |E_2| = m - \eta n^2 > n^2(\frac{1}{4} + \gamma - \frac{\gamma}{2}) = n^2(\frac{1}{4} + \frac{\gamma}{2}).$$

Consider the graph induced by E_i. It has $|V_i| > n/3$ vertices and at most $|E_i^c| \leq (\rho^3 + 1.5\rho^4)n^2 < 9(\rho^3 + 1.5\rho^4)|V_i|^2$ non-edges. Let $\epsilon_0 = \epsilon_0(k)$ be the parameter from Theorem 2.8. By choosing $\gamma_0 = \gamma_0(k)$ sufficiently small we can guarantee that $\rho = \sqrt{8\gamma}$ satisfies $9(\rho^3 + 1.5\rho^4) < \epsilon_0$. Thus, by Theorem 2.8, the graph induced by E_i has a K_k-packing in which at most $(2k-3)(9(\rho^3 + 1.5\rho^4))^{6/5}n^2 + o(n^2)$ edges are unpacked. By considering V_1 and V_2 together, we obtain a K_k-packing of G so that at least

$$|E_1| + |E_2| - 2(2k-3)(9(\rho^3 + 1.5\rho^4))^{6/5}n^2 - o(n^2) \geq n^2(\frac{1}{4} + \frac{\gamma}{2} - (4k-6)(9(\rho^3 + 1.5\rho^4))^{6/5}) - o(n^2)$$

edges are packed. By setting $\gamma_0 = \gamma_0(k)$ sufficiently small we can make sure that $\frac{\gamma}{2} - (4k-6)(9(\rho^3 + 1.5\rho^4))^{6/5} > 0$. Thus, $\binom{k}{2}\nu_k(G) \geq \frac{1}{4}n^2 - o(n^2)$, as required.

If $\eta \geq \gamma/2 = \rho^2/16$ we apply Lemma 2.4. Let $G' = (V, E_1 \cup E_1^c \cup E_2 \cup E_2^c \cup E_{12})$. By Lemma 2.4, there exists a set L of edge-disjoint K_k of G' so that $|L| \geq \frac{1}{40k^4}\eta^{4/3}n^2$. Furthermore, each element of L intersects both V_1 and V_2. Not every K_k in L is also a K_k in G, as elements of L may contain edges of $E_1^c \cup E_2^c$. If $L' \subset L$ is the set of elements of L that are also a K_k in G, then

$$|L'| \geq |L| - |E_1^c \cup E_2^c| \geq \frac{1}{40k^4}\eta^{4/3}n^2 - (\rho^3 + 1.5\rho^4)n^2 \geq n^2(\frac{\rho^{8/3}}{1613k^2} - \rho^3 - 1.5\rho^4).$$

Thus, by setting $\gamma_0 = \gamma_0(k)$ sufficiently small we can make sure that $|L'| \geq \rho^{17/6}n^2$. In fact, we will assume that $|L'| = \rho^{17/6}n^2$ (otherwise we will take only a subset).

Let $F_i \subset E_i$ consist of all the edges of E_i not belonging to elements of L'. Consider the graph induced by F_i. It has $|V_i| > n/3$ vertices and at most $(\rho^3 + 1.5\rho^4)n^2 + k^2\rho^{17/6}n^2 < 9(\rho^3 + 1.5\rho^4 + k^2\rho^{17/6})|V_i|^2$ non-edges. By choosing $\gamma_0 = \gamma_0(k)$ sufficiently small we can guarantee that $\rho = \sqrt{8\gamma}$ satisfies $9(\rho^3 + 1.5\rho^4 + k^2\rho^{17/6}) < \epsilon_0$. Thus, by Theorem 2.8, the graph induced by F_i has a K_k-packing in which at most $(2k-3)(9(\rho^3 + 1.5\rho^4 + k^2\rho^{17/6}))^{6/5}n^2 + o(n^2)$ edges are unpacked. By considering both F_1, F_2 and L' we obtain a packing of G with at least

$$|L'|\binom{k}{2} + |F_1| + |F_2| - (4k-6)(9(\rho^3 + 1.5\rho^4 + k^2\rho^{17/6}))^{6/5}n^2 - o(n^2)$$

packed edges. Notice also that since each element of L' has vertices in both vertex classes,

$$|F_1| + |F_2| \geq |E_1| + |E_2| - |L'|\binom{k-1}{2} \geq \binom{|V_1|}{2} + \binom{|V_2|}{2} - (\rho^3 + 1.5\rho^4)n^2 - |L'|\binom{k-1}{2} \geq$$

$$n^2(\frac{1}{4} - \rho^3 - 1.5\rho^4) - |L'|\binom{k-1}{2} - o(n^2).$$

Hence, the packing above consists of at least

$$(k-1)|L'| + n^2(\frac{1}{4} - \rho^3 - 1.5\rho^4) - (4k-6)(9(\rho^3 + 1.5\rho^4 + k^2\rho^{17/6}))^{6/5}n^2 - o(n^2) \geq$$

$$n^2(\frac{1}{4} + (k-1)\rho^{17/6} - \rho^3 - 1.5\rho^4 - (4k-6)(9(\rho^3 + 1.5\rho^4 + k^2\rho^{17/6}))^{6/5}) - o(n^2)$$

packed edges. By setting $\gamma_0 = \gamma_0(k)$ sufficiently small we can make sure that $(k-1)\rho^{17/6} - \rho^3 - 1.5\rho^4 - (4k-6)(9(\rho^3 + 1.5\rho^4 + k^2\rho^{17/6}))^{6/5} > 0$. Thus, $\binom{k}{2}\nu_k(G) \geq \frac{1}{4}n^2 - o(n^2)$, as required. ■

3 Packing triangles in graphs with independence number 2

The following lemma, that is interesting in its own right, considerably strengthens Lemma 2.4 in the case $k = 3$. Recall that P_3 denotes a path of length 2.

Lemma 3.1 *Let $G = (V, E)$ be a bipartite graph with n vertices. G has a set L of copies of P_3, such that any two elements of L intersect in at most one vertex, and $|E| - 2|L| < n$.*

Proof: By induction on n. The case $n = 1$ trivially holds. Now let G have n vertices, and consider a bipartition (A, B). Let v have minimum degree in G, and assume, with no loss of generality, that $v \in A$. Let $d(v) = \delta(G) = k$. If $k = 0$ or $k = 1$ we are done by applying the induction hypothesis to $G - v$. Otherwise, let $N(v) \subset B$ be v's neighbors. Let $N(v) = \{w_1, \ldots, w_k\}$. For $i = 1, ..., k-1$ greedily pick $a_i \in A$ where $(a_i, w_i) \in E$ and $a_i \notin \{v, a_1, \ldots, a_{i-1}\}$. We can clearly do this since $d(w_i) \geq k$. Now consider the set $M = \{(v, w_1, a_1), (v, w_2, a_2), \ldots, (v, w_{k-1}, a_{k-1})\}$ of copies of P_3 in G. The elements of M cover all edges incident with v but at most one. Now delete v from the graph, and also delete the $k-1$ edges (w_i, a_i) and apply induction. The obtained subgraph of G, denoted G', has $n-1$ vertices and $|E| - k - (k-1)$ edges. By the induction hypothesis we can find a set L' of copies of P_3 in G' such that $|E| - 2k + 1 - 2|L'| < n - 1$ and such that any two elements of L' intersect in at most one vertex. Now, let $L = L' \cup M$. Clearly, any two elements of L intersect in at most one vertex and $|L| = |L'| + k - 1$. Therefore, $|E| - 2|L| = |E| - 2(|L'| + k - 1) = |E| - 2k - 2|L'| + 2 < n$, as required. ■

Lemma 3.2 *Every graph with n vertices and $\binom{n}{2} - t$ edges has a fractional triangle packing whose value is at least $(\binom{n}{2} - 7t/3)/3$. Furthermore, if the graph has no independent set of size 3 then it has a fractional triangle packing whose value is at least $(\binom{n}{2} - 2t)/3$.*

Proof: We first claim that a graph on 7 vertices and $21 - t$ edges has a fractional triangle packing whose value is at least $(21 - 7t/3)/3$. This is trivially true for $t \geq 9$.

If $t = 4, 5$ then H has 16 or 17 edges. It is not difficult to check that a graph with 7 vertices and 16 edges has 4 edge-disjoint triangles. Thus, the claim holds for $t = 4, 5$.

For $t = 6, 7, 8$ it follows from the case $t = 5$ that H has $9 - t$ edge-disjoint triangles. Thus, the claim holds for $t = 6, 7, 8$.

If $t = 2$ there are two cases. If both missing edges share an endpoint, then there are six edge-disjoint triangles in H. In this case the claim holds. Otherwise, let (a, b) and (c, d) be the two missing edges and let (x, y, z) be the remaining three vertices. There are three types of edges. Type I edges consists of the four edges in $\{a, b, c, d\}$. Type II edges consists of the 12 edges with a single endpoint in $\{x, y, z\}$ and type III edges are the three edges with both endpoints in $\{x, y, z\}$. There are also three types of triangles. Type I triangles contain precisely one edge of type I and two edges of type II. There are 12 such triangles. Type II triangles contain two edges of type II and one edge of type III. There are 12 such triangles. Type III triangles contain only edges of type III. There is one such triangles. Assign to each triangle of type I the value 1/3. Assign to each triangle of type II the value 1/6. Assign to the triangle of type III the value 1/3. It is easy to verify that this is a legal fractional triangle packing (in fact, it is a fractional triangle decomposition) of total value 19/3. Hence, the claim holds for $t = 2$.

If $t = 3$ then either there are two missing edges sharing an endpoint, in which case there are at least five edge-disjoint triangles in G, and the claim holds. Otherwise, there are two independent missing edges (a, b) and (c, d). Putting back the third missing edge, we have, by the previous case of $t = 2$ a fractional triangle decomposition of value 19/3. Hence, without the third missing edge this corresponds to a fractional triangle packing of value $19/3 - 1 = 16/3$. Hence, the claim holds for $t = 3$.

If $t = 1$, let (a, b) denote the missing edge. Assign to every triangle that contains a or b the value 1/4 (there are 20 such triangles). Assign to every triangle that contains neither a nor b the value 1/6 (there are 10 such triangles). It is easy to verify that this is a legal fractional triangle packing (in fact, it is a fractional triangle decomposition) of total value 20/3. Hence, the claim holds for $t = 1$.

The claim trivially holds for $t = 0$, as K_7 has 7 edge-disjoint triangles.

Now, let G be a graph with n vertices and $\binom{n}{2} - t$ edges. Let P be the set of $\binom{n}{7}$ induced 7-vertex subgraphs of G. For $H \in P$ with $21 - t_H$ edges, let f_H be a fractional triangle packing of H with $|f_H| \geq (21 - 7t_H/3)/3$, and set $f_H(T) = 0$ for $T \notin H$. Let

$$f = \frac{1}{\binom{n-2}{5}} \sum_{H \in P} f_H.$$

Then, clearly, $|f| \geq (\binom{n}{2} - 7t/3)/3$, as required.

Now, the second part of the lemma can clearly be proved analogously by showing that every graph with 7 vertices, $21-t$ edges, and no independent set of size 3, has a fractional triangle packing whose value is at least $(21-2t)/3$. Indeed it is shown in [5] that a graph with seven vertices and no independent set of size 3 always has a fractional triangle packing whose value is at least 2. Thus, the claim holds for $t \geq 8$. Recall from the first part of the lemma that a graph with 7 vertices and 16 edges has 4 edge-disjoint triangles. Thus, the claim holds for $t = 5$ and also for $t = 6$ (as in this case the graph has 3 edge-disjoint triangles). Recall from the previous lemma that a graph with 7 vertices and 18 edges either has 5 edge-disjoint triangles or a fractional triangle packing whose value is at least $16/3$. This proves the case $t = 3$. In fact it also proves the case $t = 4$ because if the graph has four missing edges we can always add one missing edge so that the remaining 3 missing edges are not all independent. In this case there is a fractional packing of value at least $16/3$, and hence of value $13/3$ in the original graph. The case $t = 2$ holds since, as in the first part of the lemma, either there are six edge-disjoint triangles, or a fractional triangle decomposition. The case $t = 1$ holds because K_7^- has a fractional triangle decomposition and the case $t = 0$ holds because K_7 has seven vertex-disjoint triangles. The only remaining case is $t = 7$. It can be easily verified that a graph with 7 vertices, 14 edges and no independent set of size 3 has a fractional triangle packing whose value is at least 3. ■

Proof of Theorem 1.3: For the first part of the theorem, let $G = (V, E)$ be a triangle-free graph with n vertices and $\frac{1}{4}n^2(1-\rho^2)$ edges, where $\rho^2/4 < 0.0001$. Notice that $\rho < 0.02$. It will be convenient to color the edges of G blue and the non-edges by red. We must prove that there is a set of $(1-o(1))n^2/12$ red triangles. By Lemma 2.3, there exists a partition $V = V_1 \cup V_2$ such that the number of blue edges inside the parts is at most $(\rho^3 + 1.5\rho^4)n^2$, and such that $|V_i| \geq n/2 - n(\rho^2 + \rho/2 + 2/n)$. Let B_i and R_i be, respectively, the set of blue edges and red edges inside V_i and let $b_i = |B_i|$ and $r_i = |R_i|$. Let B_{12} and R_{12} be, respectively, the set of blue edges and red edges with one endpoint in V_1 and the other in V_2. By Lemma 3.1, the edges of $R_{1,2}$ can be packed with a set L of paths of length 2, such that any two paths intersect in at most one vertex, and the number of unpacked edges of $R_{1,2}$ in at most n (negligible). For $P \in L$ let P_c be the triangle induced by P. Notice that P_c is either a completely red triangle or it has precisely one blue edge belonging to $B_1 \cup B_2$. Let $T = \{P_c \ : \ P_c \text{ is red}\}$. Notice that T is a set of *edge-disjoint* red triangles. Let $Z \subset R_{1,2}$ be the red edges not covered by elements of T. Clearly,

$$|Z| \leq n + 2(b_1 + b_2) \leq n + 2(\rho^3 + 1.5\rho^4)n^2. \tag{1}$$

Let $F_i \subset R_i$ be the set of red edges inside V_i that are covered by the elements of T, and let $f_i = |F_i|$. Clearly,

$$f_1 + f_2 = |T| \leq \frac{1}{2}|R_{12}| = \frac{1}{2}(|V_1||V_2| - \frac{1}{4}n^2(1-\rho^2) + b_1 + b_2) \leq \frac{1}{2}\left(\frac{n^2\rho^2}{4} + (\rho^3 + 1.5\rho^4)n^2\right). \tag{2}$$

Let G_i be the subgraph induced by the vertex set V_i and the red edges $R_i \setminus F_i$. It has $r_i - f_i = \binom{|V_i|}{2} - (b_i + f_i)$ edges. By Lemma 3.2 and Lemma 2.5

$$\nu_3(G_i) \geq \frac{1}{3}\left(\binom{|V_i|}{2} - \frac{7}{3}(b_i + f_i)\right) - o(n^2).$$

In particular, there is a packing of G_i with red triangles in which at most $(4/3)(b_i+f_i)+o(n^2)$ red edges are uncovered. Recalling that the edges of Z are also unpacked in red triangles, we obtain, together with T, a set of red edge-disjoint triangles that cover all but at most $|Z|+(4/3)(b_1+f_1+b_2+f_2)+o(n^2)$ red edges. Therefore, together with (1) and (2),

$$\nu_3(G) \geq \frac{1}{3}\left(\binom{n}{2} - \frac{1}{4}n^2(1-\rho^2) - \left(|Z| + \frac{4}{3}(b_1+f_1+b_2+f_2)+o(n^2)\right)\right)$$

$$\geq n^2\left(\frac{1}{12} + \frac{\rho^2}{12} - \frac{1}{3}(2\rho^3+3\rho^4) - \frac{4}{9}(\rho^3+1.5\rho^4+\frac{\rho^2}{8}+\frac{\rho^3}{2}+\frac{3\rho^4}{4})\right) - o(n^2)$$

$$= n^2\left(\frac{1}{12} + \frac{1}{36}\rho^2 - \frac{4}{3}\rho^3 - 2\rho^4\right) - o(n^2).$$

In order to prove that $\nu_3(G) \geq n^2/12 - o(n^2)$ it remains to show that $\frac{1}{36}\rho^2 - \frac{4}{3}\rho^3 - 2\rho^4 \geq 0$. Indeed, this holds for all $\rho \leq 0.02$, as required.

For the second part of the theorem, let G be a graph with $n^2/2 - \alpha n^2$ edges. By letting $t = \alpha n^2 - o(n^2)$ in Lemma 3.2 we have, together with Lemma 2.5 that

$$\nu_3(G) \geq \nu_3^*(G) - o(n^2) \geq \frac{1}{3}\left(\binom{n}{2} - \frac{7}{3}t\right) - o(n^2) = \left(\frac{1}{6} - \frac{7}{9}\alpha\right)n^2 - o(n^2).$$

Similarly, by the second part of Lemma 3.2, if G does not have three independent vertices then $\nu_3(G) \geq (1/6 - 2\alpha/3)n^2 - o(n^2)$. By using $\alpha = 1/8$ the second part of the theorem follows. ∎

4 Concluding remarks

- Let $f_k(n)$ be the minimum possible value of $f_k(n,m)$ ranging over all plausible values of m. Let $\alpha_k = \liminf f_k(n)/n^2$. Thus, the conjecture of Erdős states that $\alpha_3 = 1/4$, the example using two vertex-disjoint cliques shows that $\alpha_k \leq \frac{1}{4}$, and Proposition 1.1 shows that $\alpha_k \leq \frac{\binom{k}{2}}{10\lceil \frac{k}{2}(\frac{k}{2}-1)\rceil}$. It would be very interesting to determine α_k for all k.

- **Proof of Proposition 1.1:** We construct a graph $G=(V,E)$ with $5n$ vertices where V consists of five vertex-disjoint sets $V_1,\ldots,V_5$ of size n each. Each V_i induces a complete graph. Furthermore, E contains all n^2 edges between V_i and V_{i+1} (indices modulo 5). Thus, $|E| = 5\binom{n}{2} + 5n^2 = \frac{5n(3n-1)}{2}$. Notice that G does not have three independent vertices. Now, any copy of K_k in G cannot have vertices in three distinct vertex classes. Thus, at least $\lceil \frac{k}{2}(\frac{k}{2}-1)\rceil$ edges of such a copy have both endpoints in the same vertex class. As there are only $5\binom{n}{2}$ edges with both endpoints in the same vertex class the proposition follows. ∎

- We have shown that if G has independence number 2 and $\alpha n^2 + o(n^2)$ edges where $\alpha \in [0.25, 0.2501]$ or $\alpha \in [0.375, 0.5]$ then $\nu_3(G) \geq n^2/12 - o(n^2)$. Both density intervals can be slightly extended at the price of complicating the proof. It would be interesting to *significantly* extend both intervals so that they eventually intersect, thereby proving the conjecture for $f(n)$.

- Lemma 3.2 is based on the analysis of the possible fractional triangle packings of a graph with seven vertices. One can certainly improve the constant 7/3 appearing in the statement of the lemma by examining larger constant size graphs. Using a computer we have found out that a graph with 9 vertices and $36 - t$ edges has a fractional triangle packing whose value is at least $(36 - 9t/4)/3$, thereby showing that the constant 7/3 can be improved to 9/4. It is plausible to assume that a constant closer to 2 can be obtained by examining larger cases. Such an improvement, however, would only have a minor effect on the constants in Theorem 1.3.
- Lemma 3.1 shows that the edges of an n-vertex bipartite graph can be packed with paths of length 2, any two of them intersecting in at most one vertex, so that less than n edges are unpacked. It is plausible that an even sharper estimate holds. For example, we propose the following problem, that we verified for small values of n. Is it true that if $G \subset K_{n/2,n/2}$ then its edges can be packed with paths of length 2, any two of them intersecting in at most one vertex, so that at most $n/2$ edges are unpacked? If true, this would be best possible by considering a perfect matching with $n/2$ edges.

Acknowledgment

I wish to thank Noga Alon for insightful suggestions.

References

[1] B. Bollobás, *Modern Graph Theory*, Springer-Verlag, 1998.

[2] P. Erdős, R. J. Faudree, R. J. Gould, M. S. Jacobson and J. Lehel, *Edge disjoint monochromatic triangles in 2-colored graphs*, Discrete Math. 231 (2001), 135-141.

[3] P. Erdős, *Some recent problems and results in graph theory*, Discrete Math. 164 (1997), 81-85.

[4] P. E. Haxell and V. Rödl, *Integer and fractional packings in dense graphs*, Combinatorica 21 (2001), 13–38.

[5] P. Keevash and B. Sudakov, *Packing Triangles in a graph and its complement*, J. Graph Theory 47 (2004), 203–216.

[6] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: *Theory of Graphs (Proc. Colloq. Tihany, 1966)*, Academic Press, New York, and Akad. Kiadó, Budapest, 1968, 279–319.

[7] R. M. Wilson, *Decomposition of complete graphs into subgraphs isomorphic to a given graph*, Congressus Numerantium XV (1975), 647–659.

[8] R. Yuster, *Asymptotically optimal K_k-packings of dense graphs via fractional K_k-decompositions*, Journal of Combinatorial Theory, Series B 95 (2005), 1–11.

[9] R. Yuster, *Integer and fractional packing of families of graphs*, Random Structures and Algorithms 26 (2005), 110–118.