On tournament inversion

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Abstract

An *inversion* of a tournament T is obtained by reversing the direction of all edges with both endpoints in some set of vertices. Let $\text{inv}_k(T)$ be the minimum length of a sequence of inversions using sets of size at most k that result in the transitive tournament. Let $\text{inv}_k(n)$ be the maximum of $\text{inv}_k(T)$ taken over n-vertex tournaments. It is well-known that $\text{inv}_2(n) = (1+o(1))n^2/4$ and it was recently proved by Alon et al. that $\text{inv}(n) := \text{inv}_n(n) = n(1+o(1))$. In these two extreme cases ($k = 2$ and $k = n$), random tournaments are extremal objects. It is proved that $\text{inv}_k(n)$ is *not* attained by random tournaments when $k \ge k_0$ and conjectured that $\text{inv}_3(n)$ is (only) attained by (quasi) random tournaments. It is further proved that $(1 + o(1))\text{inv}_3(n)/n^2 \in [\frac{1}{12}, 0.0992)$ and $(1 + o(1))\text{inv}_k(n)/n^2 \in [\frac{1}{2k(k-1)} + \delta_k, \frac{1}{2\lfloor k^2/2 \rfloor} - \epsilon_k]$ where $\epsilon_k > 0$ for all $k \ge 3$ and $\delta_k > 0$ for all $k \ge k_0$.

Keywords: tournament; inversion; triangle

1 Introduction

In this paper we mainly consider oriented graphs, which are digraphs without loops, digons, or parallel edges. In particular, we consider tournaments, which are oriented complete graphs. For an oriented graph D and a set $X \subseteq V(D)$, the *inversion* of X in D is the oriented graph obtained from D by reversing the direction of the edges with both endpoints in X; synonymously, we view an inversion as an *operation* on D and say that we *invert* X in D. Notice that inverting a sequence $X_1, \ldots, X_t$ results in the same oriented graph for any permutation of that sequence. If inverting a sequence $X_1, \ldots, X_t$ results in an acyclic digraph, we say that $\{X_1, \ldots, X_t\}$ forms a *decycling set* of D. The *inversion number of* D, denoted $\text{inv}(D)$, is the minimum size of a decycling set. If each element of a decycling set has size at most k, we say that the set is k*-decycling* and let $\text{inv}_k(D)$ denote the minimum size of a k-decycling set. We observe that $\text{inv}(D) = \text{inv}_n(D)$ where $|V(D)| = n$, and that $\text{inv}_2(D)$ is the size of a minimum feedback edge set of D. The extremal parameters of interest are $\text{inv}(n) = \text{inv}_n(n)$ and $\text{inv}_k(n)$ which, respectively, are the maximum of $\text{inv}(D)$ and $\text{inv}_k(D)$ taken over all oriented graphs with n vertices. When studying these extremal parameters, we may and will restrict to the class of n-vertex tournaments, as adding edges to D cannot decrease $\text{inv}_k(D)$.

The parameter $\text{inv}_2(n)$ is asymptotically well-understood. It is straightforward that any digraph can be made acyclic by inverting (equivalently, removing) at most half of its edges. Spencer [17] proved that $\text{inv}_2(n) \le \frac{1}{2}\binom{n}{2} - \Omega(n^{3/2})$. A probabilistic construction of Spencer [18], later simplified with an improved constant by de la Vega [9], shows that $\text{inv}_2(n) \ge \frac{1}{2}\binom{n}{2} - O(n^{3/2})$, hence

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$\text{inv}_2(n) = (1+o(1))n^2/4$ and the growth rate below the $n^2/4$ threshold is $\Theta(n^{3/2})$. In fact, it was shown by Chung and Graham [7] that an n-vertex tournament T has $\text{inv}_2(T) = (1+o(1))n^2/4$ if and only if it is quasi-random [1].

The study of $\text{inv}(D)$ was initiated by Belkhechine [5] and followed by several papers that considered its computational and extremal aspects [4, 6, 19]. It is not difficult to show that $\text{inv}(n) \leq n(1+o(1))$ [2, 4, 6] but obtaining an asymptotically matching lower bound is more involved and was only recently independently resolved by Alon et al. and by Aubian et al. [2, 3] who proved that $\text{inv}(n) \geq n - \sqrt{2n\log(n)}$ [2] for large n, thus establishing that $\text{inv}(n) = n(1+o_n(1))$. Moreover, their proof shows that a random n-vertex tournament T almost surely has $\text{inv}(T) = n(1+o(1))$.

As the extreme cases ($k=2$ and $k=n$) are solved up to low order terms, and as random tournaments are extremal objects in both of these cases, one wonders what happens for other k. Mainly, what is the asymptotic behavior of $\text{inv}_k(n)$? Is it always the case that random tournaments are extremal objects?

Fix $k \geq 3$. It is readily seen that $\text{inv}_k(n) \geq (1+o(1))n^2/2k(k-1)$. Indeed, as the minimum feedback edge set of a tournament T has size $\text{inv}_2(T)$, and since each element in a k-decycling set of T changes the direction of at most $\binom{k}{2}$ edges, we have that $\text{inv}_k(T) \geq \text{inv}_2(T)/\binom{k}{2}$ and the claim holds by recalling that $\text{inv}_2(n) = (1+o_n(1))n^2/4$. In fact, it is not difficult to prove that random tournaments attain this bound whp [3], as the following proposition shows.

Proposition 1.1. *For a random n-vertex tournament T, $\text{inv}_k(T) = (1+o(1))n^2/2k(k-1)$ whp.*

We conjecture that this bound is asymptotically tight for $k=3$ and is attained only by quasi-random (hence also by random) tournaments.

Conjecture 1.2. $\text{inv}_3(n) = (1+o(1))\frac{n^2}{12}$. *Moreover, an n-vertex tournament T has* $\text{inv}_3(T) = (1+o(1))\frac{n^2}{12}$ *if and only if it is quasi-random.*

Note that Conjecture 1.2 actually consists of two distinct assertions and a proof of each one does not necessarily imply the other. Determining the asymptotic behavior of $\text{inv}_3(n)$ seems challenging. Our first main result gives an upper bound which is not far from the conjectured value.

Theorem 1.3. $\text{inv}_3(n) \leq \frac{257}{2592}(1+o(1))n^2$.

Since $\text{inv}_3(n) \geq (1+o(1))\frac{n^2}{12}$, we have that $(1+o(1))\text{inv}_3(n)/n^2 \in [\frac{1}{12}, 0.0992)$.

As for larger fixed k, we are able to show that starting from some given k_0, the lower bound $(1+o(1))n^2/2k(k-1)$ is, perhaps surprisingly, *not* tight. This is a consequence of the proof of our second main result, which gives upper and lower bounds for $\text{inv}_k(n)$.

Theorem 1.4. *For all $k \geq 3$ there exists $\epsilon_k > 0$ such that* $\text{inv}_k(n) \leq (\frac{1}{2\lfloor k^2/2 \rfloor} - \epsilon_k)(1+o(1))n^2$. *On the other hand, there exists k_0 such that for all $k \geq k_0$ there exists $\delta_k > 0$ such that* $\text{inv}_k(n) \geq (\frac{1}{2k(k-1)} + \delta_k)(1+o(1))n^2$.

Note that whenever $k \geq k_0$ in Theorem 1.4, we have $\delta_k > 0$, so together with Proposition 1.1 this implies that for $k \geq k_0$, tournaments that attain $\text{inv}_k(n)$ are far from random. On the other hand, notice that Conjecture 1.2 asserts that we cannot have $k_0 = 3$ in Theorem 1.4.

[1] More formally, when discussing quasi-randomness we need to consider infinite *sequences* of tournaments; see [7].
[2] Whenever the base of a logarithm is not specified, it is assumed to be 2.
[3] Throughout this paper, *whp* means "with probability tending to one as n tends to infinity".

Problem 1.5. *Find the smallest k_0 for which the lower bound statement in Theorem 1.4 holds.*

The rest of this paper proceeds as follows: In Section 2 we introduce some definitions and collect some known tools that are needed for the proofs. We consider the case $k = 3$ in Section 1.3 where we prove Theorem 1.3. In Section 4 we consider larger k and prove Theorem 1.4.

2 Preliminaries

This section presents several definitions and tools required for the proofs of our main results.

2.1 Hypergraph coloring

Recall that a k-uniform hypergraph is a collection of k-sets (the edges) of some n-set (the vertices). The *degree* $d(x)$ of a vertex x in a hypergraph is the number of edges containing x and the *co-degree* $d(x, y)$ of a pair of distinct vertices x, y is the number of edges containing both. A *matching* in a hypergraph is a set of pairwise disjoint edges. The *chromatic index* of a hypergraph H, denoted $\chi'(H)$, is the smallest integer q such that the set of edges of H can be partitioned into q matchings. The following result of Pippenger and Spencer [16] gives sufficient conditions on H which guarantee that $\chi'(H)$ is close to the maximum degree of H.

Lemma 2.1 ([16]). *For an integer $k \geq 3$ and a real $\gamma > 0$ there exists a real $\beta = \beta(k, \gamma)$ so that the following holds: If a k-uniform hypergraph H has the following properties for some t:*
(i) $(1 - \beta)t < d(x) < (1 + \beta)t$ holds for all vertices,
(ii) $d(x, y) < \beta t$ for all distinct x and y,
then $\chi'(H) \leq (1 + \gamma)t$. □

2.2 Digraphs, permutations, and random graphs

The edge-set of every digraph D is the disjoint union of the edge sets of two directed acyclic subgraphs. Indeed, consider some permutation π of $V(D)$ (here a permutation is a bijective function $\pi : V(D) \to [|V(D)|]$). Let $D_L(\pi)$ be the spanning subgraph of D where $(i, j) \in E(D_L(\pi))$ if and only if $(i, j) \in E(D)$ and $\pi(i) < \pi(j)$. Let $D_R(\pi)$ be the spanning subgraph of D where $(i, j) \in E(D_R(\pi))$ if and only if $(i, j) \in E(D)$ and $\pi(i) > \pi(j)$. Since $E(D_R(\pi)) \cup E(D_L(\pi)) = E(D)$, we can cover the edges of D using just two directed acyclic subgraphs of D. When referring to $D_L(\pi)$ and $D_R(\pi)$ it it convenient to view them as *undirected* simple graphs, but recalling that they correspond to edges of T going from left to right in the case of $D_L(\pi)$ or from right to left in the case of $D_R(\pi)$.

An n-vertex *random tournament* is the probability space $T(n)$ of tournaments on vertex set $[n]$, obtained by orienting the edges of K_n at random (i.e., each direction is decided with a fair coin flip) and all $\binom{n}{2}$ choices are independent. By definition, for each given permutation π of $[n]$, if $T \sim T(n)$ then each of $T_L(\pi)$ and $T_R(\pi)$ is distributed as the binomial random graph $G(n, \frac{1}{2})$.

Proof of Proposition 1.1. Fix a permutation π of $[n]$. Let $T \sim T(n)$ and notice that since $T_L(\pi) \sim G(n, \frac{1}{2})$, it has $(1 + o(1))n^2/4$ edges, whp. By the result of Frankl and Rödl [11] Theorem 1.1, applied to the hypergraph obtained from $G(n, \frac{1}{2})$ where the vertices of the hypergraph are the edges of $G(n, \frac{1}{2})$ and the edges of the hypergraph are the copies of K_k, $G(n, \frac{1}{2})$ can be almost entirely packed with pairwise edge-disjoint copies of K_k, whp. Equivalently, this means that whp we can find a collection C of $((1+o(1))n^2/4)/\binom{k}{2}$ sets of vertices, each of size either k or 2, such that (i) any pair of sets in C intersect in at most one vertex; (ii) each edge of $T_L(\pi)$ is contained in precisely one set of C; (iii) each edge of $T_R(\pi)$ is not contained in any set of C. Therefore, C forms a decycling set of T. $\square$

2.3 Fractional packing

For an undirected graph G, let $\binom{G}{k}$ denote the set of all K_k copies of G (namely, subgraphs of G that are isomorphic to K_k). A function ϕ from $\binom{G}{k}$ to $[0,1]$ is a *fractional K_k-packing* of G if for each edge of G, the sum of the values of ϕ taken over all K_k-copies that contain that edge is at most 1. The *value* of ϕ is

$$|\phi| = \sum_{H \in \binom{G}{k}} \phi(H)$$

and $\nu_k^*(G)$ is the maximum of $|\phi|$ taken over all fractional K_k-packings of G. A *K_k-packing* of G is a fractional K_k-packing whose image is included in $\{0,1\}$. Equivalently, it is a set of pairwise edge-disjoint copies of K_k. Letting $\nu_k(G)$ denote the maximum value of a K_k-packing of G, we have $\nu_k^*(G) \geq \nu_k(G)$. An important result of Haxell and Rödl [13] (see also [20]) shows that the converse inequality holds up to an additive error term which is negligible for dense graphs.

Lemma 2.2 ([13])**.** *For every $\varepsilon > 0$ and for every positive integer $k \geq 3$ there exists $N = N(k,\varepsilon)$ such that for any graph G with $n > N$ vertices, $\nu_k^*(G) - \nu_k(G) \leq \varepsilon n^2$.* $\square$

3 Triangle inversions

In this section we prove Theorem 1.3. We begin with a lemma that will be useful to "finish off" decycling a tournament which already has a relatively small feedback edge set, using triangle inversions.

Lemma 3.1. *Let G be a directed graph whose underlying undirected graph has a (not necessarily induced) 4-cycle on vertices a,b,c,d. Then, there are two sets of vertices X,Y of three vertices each, such that inverting $\{X,Y\}$ reverses the direction of the edges of this 4-cycle in G without affecting the direction of any other edge of G.*

Proof. Assume that the edges of the 4-cycle are ab, bc, cd, ad. Then take $X = \{a,b,c\}$, $Y = \{a,c,d\}$, and observe that inverting $\{X,Y\}$ reverses the direction of the edges of the 4-cycle without affecting the direction of any other edge of G. $\square$

Corollary 3.2. *Let T^* be an n-vertex tournament and suppose that $|E(T_L^*(\pi))| \leq \alpha n^2$. Then, $\mathrm{inv}_3(T^*) \leq \alpha n^2/2 + o(n^2)$.*

Proof. By Lemma 3.1 we can repeatedly invert pairs of three-sets of vertices each, until we obtain a tournament T^{**} for which $T_L^{**}(\pi)$ has no four-cycle. The number of inversions performed starting at T^* and arriving at T^{**} is therefore precisely $(|E(T_L^*(\pi))| - |E(T_L^{**}(\pi))|)/2$. By the Kovári-Sós-Turán Theorem, we have $|E(T_L^{**}(\pi))| \leq (n^{3/2}+n)/2$, hence the corollary. $\square$

A high level approach to proving Theorem 1.3 follows.

Suppose that T is a given n-vertex tournament. We prove that a randomly chosen permutation π has the following two properties whp: (i) $T_L(\pi)$ has roughly $n^2/4$ edges; (ii) $T_L(\pi)$ has many (say, roughly βn^2) pairwise edge-disjoint triangles. Once we show these properties, we can reverse the claimed set of pairwise edge-disjoint triangles to obtain a tournament T^* for which $T^*_L(\pi)$ has roughly αn^2 edges where $\alpha = \frac{1}{4} - 3\beta$. We then apply Corollary 3.2 to reverse the edges of $T^*_L(\pi)$ and obtain a decycling set of T of order roughly $\beta n^2 + \alpha n^2/2 = (\frac{1}{8} - \frac{\beta}{2})n^2$.

Proving that a random permutation satisfies (i) is a standard argument, but proving (ii) for a reasonable value of β requires careful analysis, detailed later. The main difficulty stems from the fact that we do not know much about the structure of $T_L(\pi)$.

For the rest of this section we fix a tournament T on vertex set $[n]$. We start by proving (i) above.

Lemma 3.3. *Let π be a randomly chosen permutation of $[n]$. Whp, $|E(T_L(\pi))| = (1 + o(1))n^2/4$.*

Proof. Let $X(i,j)$ denote the indicator random variable for the event that $ij \in E(T_L(\pi))$. We observe that $X(i,j) \sim \text{Bernoulli}(\frac{1}{2})$ and that $|E(T_L(\pi))|$ is the sum of $X(i,j)$ taken over all edges $(i,j) \in E(T)$, so its expected value is $\binom{n}{2}/2 = (1+o(1))n^2/4$. To show that $|E(T_L(\pi))|$ is concentrated, we upper-bound its variance. Notice that if $\{i,j\} \cap \{i',j'\} = \emptyset$, then $X(i,j)$ and $X(i',j')$ are independent and notice that there are fewer than n^3 ordered pairs $(\{i,j\},\{i',j'\})$ for which $\{i,j\} \cap \{i',j'\} \neq \emptyset$. Hence, $\text{Var}[|E(T_L(\pi))|] \leq \mathbb{E}[|E(T_L(\pi))|] + n^3 = o(\mathbb{E}[|E(T_L(\pi))|]^2)$. The lemma now follows from Chebyshev's inequality. □

We now turn to our second task, i.e., showing that $T_L(\pi)$ has many pairwise edge-disjoint triangles, whp. Let us first see a way to obtain some nontrivial bound for this quantity. In a recent paper, Gruslys and Letzter [12], improving an earlier result of Keevash and Sudakov [14], and thereby confirming a conjecture of Erdős, proved that in any two-coloring of the edges of K_n, there are at least $(1 - o(1))n^2/12$ pairwise edge-disjoint monochromatic triangles. Observing that coloring the edges of $T_L(\pi)$ red and coloring the edges of $T_R(\pi) = T_L(\pi^{\text{reverse}})$ blue corresponds to a two-coloring of the edges of K_n, we have by Lemma 3.3 and by the result in [12] that there exists a permutation π such that $|E(T_L(\pi))| = (1 + o(1))n^2/4$ and $T_L(\pi)$ has at least $(1 + o(1))n^2/24$ pairwise edge-disjoint triangles. Notice that in this argument, we are only using a weaker form of the result in [12]: that in any two-coloring of the edges of K_n, there are many pairwise edge-disjoint monochromatic triangles all of the same color. In fact, it was conjectured by Jacobson (see [10]) that there are always at least $(1 + o(1))n^2/20$ such triangles (there are examples showing that, if true, the constant $\frac{1}{20}$ is optimal), but this is still open, although it was proved in [12] that the correct constant for this question must be strictly larger than $\frac{1}{24}$. Notice, however, that even applying the latter form of the question (i.e., Jacobson's conjectured value) may be stronger than what we need. Indeed, we will show that this is provably the case; we will show that whp a random permutation gives that $T_L(\pi)$ has $(1+o(1))\beta n^2$ edge-disjoint triangles, where the obtained constant β is significantly larger than (the proven) $\frac{1}{24}$ and (the conjectured) $\frac{1}{20}$ constants of the aforementioned monochromatic triangles questions.

We require some further notation and definitions. Let $q \geq 4$ be an integer parameter to be set later (it will be small, but not too small). Let $\mathcal{T}_q$ be the set of all tournaments on q vertices. For a tournament $Q \in \mathcal{T}_q$ and a permutation σ of $V(Q)$, recall from Subsection 2.3 that $\nu_3^*(Q_L(\sigma))$ is the maximum value of a fractional triangle packing of $Q_L(\sigma)$.

Definition 3.4. *Let P be a family of permutations of $V(Q)$. Let* $\text{avg}_P(Q)$ *be the average of* $\nu_3^*(Q_L(\sigma))$ *where σ is taken over all permutations in P. Let* $\text{avg}(Q) := \text{avg}_{S_q}(Q)$ *where S_q is the family of all possible $q!$ permutations.*

It is possible that for some families, $\text{avg}_P(Q)$ is larger than the overall average $\text{avg}(Q)$ while for other families, it is smaller. It will be beneficial to assign to each tournament $Q \in \mathcal{T}_q$, a family P for which $\text{avg}_P(Q)$ is as large as possible, under some restrictions whose necessity will be apparent later (for instance, to make computations feasible, we would like $|P|$ to be small). To formalize these restrictions, we need the following definition.

Definition 3.5. *Let P be a family of permutations of a set X with $|X| = q$. We call P* orthogonal *if for any ordered pair u, v of elements of X and any two positions $1 \leq i < j \leq q$, there is exactly one $\sigma \in P$ such that $\sigma(u) = i$ and $\sigma(v) = j$.*

We note that the definition implies that the size of an orthogonal family is $q(q-1)$ (namely, much smaller than $q!$). For a prime power q, a construction of an orthogonal family of permutations can be obtained from certain constructions of $q-1$ mutually orthogonal Latin squares (aka MOLS). For example, for $q = 9$, such a family is shown in Table 1 and is obtained from the (columns of the) 8 pairwise orthogonal Latin squares of order 9 given in [8], Page 164. An obvious but useful observation is that if P is an orthogonal family of permutations of X, and σ^* is any permutation of $[q]$, the family $\{\sigma^*\sigma : \sigma \in P\}$ of permutations of X is also orthogonal. In particular, by double counting, this means that if there is an orthogonal family of permutations of $V(Q)$ for a tournament Q, then there is some such orthogonal family P for which $\text{avg}_P(Q) \geq \text{avg}(Q)$.

012345678	120453786	201534867	345678012	453786120	534867201
678012345	786120453	867201534	021687354	102768435	210876543
354021687	435102768	543210876	687354021	768435102	876543210
036471825	147582603	258360714	360714258	471825036	582603147
603147582	714258360	825036471	048723561	156804372	237615480
372156804	480237615	561048723	615480237	723561048	804372156
057138246	138246057	246057138	381462570	462570381	570381462
624705813	705813624	813624705	063852417	174630528	285741306
306285741	417063852	528174630	630528174	741306285	852417063
075264183	183075264	264183075	318507426	426318507	507426318
642831750	750642831	831750642	084516732	165327840	273408651
327840165	408651273	516732084	651273408	732084516	840165327

Table 1: An orthogonal family of permutations of $X = \{0, \ldots, 8\}$.

Hereafter we assume that q is such that a family of orthogonal permutations of sets of size q exists (e.g., if $q = 9$ this holds by Table 1). Let $\zeta \geq 0$ be a real value to be computed later. Suppose that for each $Q \in \mathcal{T}_q$ we can find an orthogonal family of permutations $P = P(Q)$ (note: distinct Q may be assigned distinct P) such that $\text{avg}_{P(Q)}(Q) \geq \zeta$. We will show how to lower-bound the number of pairwise edge-disjoint triangles in $T_L(\pi)$ in terms of ζ and n. To this end, we need to define a certain hypergraph, which depends on T, on π, and on the assignments $P(Q)$ for each $Q \in \mathcal{T}_q$. We next formalize the definition of this hypergraph (Definition 3.8).

Definition 3.6. *For each q-subset X of vertices of T, consider the sub-tournament $T[X] \in \mathcal{T}_q$ it induces. We say that X is successful if $\sigma = \pi|_X \in P(T[X])$. Otherwise, it is unsuccessful.*

Observation 3.7. *Since π is a random permutation, so is its restriction $\sigma = \pi|_X$, hence the probability that X is successful is precisely $P(T[X])/q! = 1/(q-2)!$.*

Definition 3.8. *Let $H(T, \pi)$ be the hypergraph whose vertex set is $E(T)$ and each edge of $H(T, \pi)$ corresponds to the edges of $T[X]$ where X is a successful q-subset (so the number of edges of $H(T, \pi)$ is the number of successful q-subsets).*

Notice that $H(T, \pi)$ is $\binom{q}{2}$-uniform. The following lemma establishes some important properties of $H(T, \pi)$ that hold whp. Its proof relies crucially on the orthogonality property of $P(T[X])$.

Lemma 3.9. *Let π be a randomly chosen permutation of $[n]$. Whp, $H(T, \pi)$ has an induced subhypergraph H' with at least $\binom{n}{2} - 3n^{1.9}$ vertices such that:*

(i) The degree of each vertex of H' is $(1 + o(1))\frac{\binom{n-2}{q-2}}{(q-2)!}$;

(ii) The co-degree of each pair of vertices of H' is at most n^{q-3}.

Proof. We start with the second assertion, which is not probabilistic. Consider two vertices of $H(T, \pi)$, i.e., two edges of T, say (u, v) and (w, z). The total number of q-subsets of vertices of T that contain both of these edges is $\binom{n-4}{q-4}$ if $\{u, v\} \cap \{w, z\} = \emptyset$ and is $\binom{n-3}{q-3}$ if $\{u, v\} \cap \{w, z\} \neq \emptyset$. In any case, we see that the number of q-sets of vertices containing both (u, v) and (w, z) is less than n^{q-3} and in particular, the co-degree of (u, v) and (w, z) in $H(T, \pi)$, which only counts successful q-sets, is less than n^{q-3}.

For the first assertion, fix a vertex of $H(T, \pi)$, i.e., an edge $e = (u, v)$ of T. Let $d(e)$ be the random variable corresponding to the degree of e in $H(T, \pi)$. Let $\mathcal{X}$ be the q-sets of vertices of T that contain both u and v and observe that $|\mathcal{X}| = \binom{n-2}{q-2}$. For $X \in \mathcal{X}$, consider the indicator random variable $d(X)$ for the event that X is successful. We have that $d(e) = \sum_{X \in \mathcal{X}} d(X)$. By Observation 3.7, we have that $d(X) \sim \text{Bernoulli}(\frac{1}{(q-2)!})$ so we obtain that

$$\mathbb{E}[d(e)] = \frac{\binom{n-2}{q-2}}{(q-2)!} \leq n^{q-2} .$$

We show that $d(e)$ is concentrated by considering its variance. To this end, fix two elements of $\mathcal{X}$, say X and Y, and consider $\text{Cov}(d(X), d(Y))$. Notice that as each of X and Y contain both u and v, we have that $|X \cap Y| \geq 2$. Now, if $|X \cap Y| \geq 3$ we shall use the trivial bound $\text{Cov}(d(X), d(Y)) \leq 1$ (recall that $d(X)$ and $d(Y)$ are indicators). So, suppose that $|X \cap Y| = 2$, i.e., they to not have common elements other than u and v. Now, suppose that we are given $d(X)$, i.e., we are told whether X is successful or not. Moreover, suppose that we are revealed all the values of π on $V(T) \setminus (Y \setminus \{u, v\})$ (notice that this data reveals all the values of π on X, so in particular, it reveals $d(X)$). So, we know $\pi(u)$ and $\pi(v)$, we know the positions in π occupied by $Y \setminus \{u, v\}$ but we do not know the internal ordering of the elements of $Y \setminus \{u, v\}$ within these positions. As the family $P(T[Y])$ is orthogonal, there is precisely *one* possible ordering of $Y \setminus \{u, v\}$ for which $\pi|_Y \in P(T[Y])$, i.e., for which Y is successful. Thus, the probability that Y is successful, given $d(X)$, is precisely $1/(q-2)!$, i.e., the same a priori probability, so $d(X)$ and $d(Y)$ are independent.

In particular, $\mathrm{Cov}(d(X), d(Y)) = 0$. We now have

$$
\begin{aligned}
\mathrm{Var}[d(e)] &= \sum_{X\in\mathcal{X}} \mathrm{Var}[d(X)] + 2\sum_{\substack{X,Y\in\mathcal{X}\\ X\neq Y}} \mathrm{Cov}(d(X), d(Y)) \\
&\le \mathbb{E}[d(e)] + 2\sum_{t=2}^{q-1} \sum_{\substack{X,Y\in\mathcal{X}\\ |X\cap Y|=t}} \mathrm{Cov}(d(X), d(Y)) \\
&= \mathbb{E}[d(e)] + 2\sum_{t=3}^{q-1} \sum_{\substack{X,Y\in\mathcal{X}\\ |X\cap Y|=t}} \mathrm{Cov}(d(X), d(Y)) \\
&\le \mathbb{E}[d(e)] + 2\sum_{t=3}^{q-1} n^{2q-2-t} \\
&\le \mathbb{E}[d(e)] + 2qn^{2q-5} \\
&\le 3qn^{2q-5}
\end{aligned}
$$

where in the last step we have used that $q \ge 3$ and in the third step we have used that the number of unordered pairs $X, Y \in \mathcal{X}$ with $|X \cap Y| = t$ is smaller than $n^t n^{q-1-t} n^{q-1-t}$. We may now apply Chebyshev's inequality and obtain that

$$
\Pr\left[\, |d(e) - \mathbb{E}[d(e)]\,| \ge n^{q-2.1} \right] \le \frac{3qn^{2q-5}}{n^{2q-4.2}} = \Theta(n^{-0.8}) \ .
$$

As T has fewer than n^2 edges, we obtain from the last inequality and from Markov's inequality that whp, all but $O(n^{1.2}) < n^{1.5}$ vertices e of $H(T, \pi)$ have $|d(e) - \mathbb{E}[d(e)]| \le n^{q-2.1}$. Consider then the set F of at most $n^{1.5}$ vertices e of $H(T, \pi)$ which violate the last inequality. For a vertex $v \in V(T)$ we say that v is *dangerous* if v is an endpoint of at least $n^{0.6}$ elements of F. So, there are fewer than $2n^{0.9}$ dangerous vertices. Remove from $H(T, \pi)$ all elements of F and also remove all vertices e of $H(T, \pi)$ such that e is an endpoint of a dangerous vertex of T. Let H' be the induced subhypergraph of $H(T, \pi)$ obtained after the removal. As we remove at most $|F| + 2n^{1.9}$ vertices from $H(T, \pi)$, we have that H' contains at least $\binom{n}{2} - 3n^{1.9}$ vertices. Clearly, the co-degree of any two vertices in H' is not larger than it is in H. By how much might a degree of a vertex $e = (u, v)$ in H' decrease? It might belong to a successful q-subset which contains a dangerous vertex of T, but there are only at most $2n^{0.9}n^{q-3} = 2n^{q-2.1}$ such q-subsets. As u and v are non-dangerous, there may additionally be at most $2n^{0.6}$ vertices (x, y) of $H(T, \pi)$ where $(x, y) \in F$ and $|\{x, y\} \cap \{u, v\}| = 1$. But then these may cause a further reduction of at most $2n^{0.6}n^{q-3} < n^{q-2.1}$ (where the inequality holds for all $n \ge 11$) in the degree of e. Additionally, it may be that (u, v) belongs to q-subsets which contain an element (x, y) of F such that $\{x, y\} \cap \{u, v\} = \emptyset$. But then these may cause a further reduction of at most $n^{1.5}n^{q-4} < n^{q-2.1}$ in the degree of e. It now follows that all the vertices of H' have degree $\mathbb{E}[d(e)] \pm 5n^{q-2.1}$, which is $\mathbb{E}[d(e)](1 + o(1))$. $\square$

We use Lemma 3.9 to lower-bound the number of pairwise edge-disjoint triangles in $T_L(\pi)$ in terms of ζ and n.

Lemma 3.10. *Let $q \geq 4$ and suppose that $\zeta \geq 0$ is such that for each $Q \in \mathcal{T}_q$ it holds that $\text{avg}_{P(Q)}(Q) \geq \zeta$. Then whp it holds that $T_L(\pi)$ has at least $(1-o(1))\frac{\zeta n^2}{q(q-1)}$ pairwise edge-disjoint triangles.*

Proof. Let π be a randomly chosen permutation of $[n]$. Define the following random variable for each q-subset X of vertices of T:

$$X^* = \begin{cases} \text{avg}_{P(T[X])}(T[X]) & \text{if } X \text{ is successful,} \\ 0 & \text{otherwise}. \end{cases}$$

Let R denote the sum of X^* taken over all q-subsets of vertices T. Notice that by its definition, $\text{avg}_{P(T[X])}(T[X])$ is the expected value of $\nu_3^*(T[X]_L(\pi|_X))$ taken over all permutations of the orthogonal family $P(T[X])$, so R is the expected sum of $\nu_3^*(T[X]_L(\pi|_X))$ taken over all successful q-subsets of vertices. By Observation 3.7 and by our assumption that $\text{avg}_{P(T[X])}(T[X]) \geq \zeta$ we have that

$$\mathbb{E}[R] = \sum_X \mathbb{E}[X^*] = \sum_X \frac{\text{avg}_{P(T[X])}(T[X])}{(q-2)!} \geq \binom{n}{q}\frac{\zeta}{(q-2)!}\,.$$

We next show that R is concentrated by considering its variance. To this end, notice that R is a sum of $\binom{n}{q} = \Theta(n^q)$ nonnegative random variables, each bounded from above by the constant $q(q-1)/6$. Indeed, a fractional triangle packing of a graph on q vertices cannot be more than $\frac{1}{3}$ of the number of its edges, so $\text{avg}_{P(T[X])}(T[X]) \leq q(q-1)/6$. To show that $\text{Var}[R] = o(\mathbb{E}[R]^2)$ it is therefore enough to show that the number of pairs X, Y such that $\text{Cov}(X^*, Y^*) \neq 0$ is $o(n^{2q})$. Indeed, observe that the permutations $\pi|_X$ and $\pi|_Y$ are independent whenever X and Y are disjoint sets of vertices. As each X is not disjoint with at most $q\binom{n-1}{q-1}$ possible Y, we have that the number of pairs X, Y such that $\text{Cov}(X^*, Y^*) \neq 0$ is only $O(n^{2q-1})$, as required. As we have shown that $\text{Var}[R] = o(\mathbb{E}[R]^2)$, it follows from Chebyshev's inequality that $R - \mathbb{E}[R]$ is $o(\mathbb{E}[R])$ whp and in particular, $R \geq (1-o(1))\binom{n}{q}\frac{\zeta}{(q-2)!}$ whp.

By Lemma 3.9 and from the previous paragraph we have that whp, π is such that: (a) $R \geq (1-o(1))\binom{n}{q}\frac{\zeta}{(q-2)!}$; (b) $H(T,\pi)$ has an induced subhypergraph H' on at least $\binom{n}{2} - 3n^{1.9}$ vertices satisfying both items of Lemma 3.9. For the remainder of the proof we assume that π is such that (a) and (b) hold.

Notice that H' satisfies the conditions of Lemma 2.1 with $k = \binom{q}{2}$, $t = \binom{n-2}{q-2}/(q-2)!$ and $\gamma = o(1)$. Indeed, H' has $\binom{n}{2}(1-o(1))$ vertices and by Lemma 3.9, the degree of every vertex of H' is $(1+o(1))\binom{n-2}{q-2}/(q-2)!$ while the co-degree of every pair of vertices of H' is much smaller, only at most n^{q-3}. So by Lemma 2.1, $\chi'(H') \leq (1+o(1))\binom{n-2}{q-2}/(q-2)!$. By the definition of $H(T,\pi)$ and H', this means that there is a subset $W \subseteq E(T)$ of at least $\binom{n}{2} - 3n^{1.9}$ edges, such that the family of all successful q-sets whose edges are entirely contained in W can be partitioned into $\chi'(H')$ parts, say $M_1, \ldots, M_{\chi'(H')}$ where for each M_i, all q-subsets contained in it are pairwise edge-disjoint.

For $1 \leq i \leq \chi'(H')$ let R_i denote the contribution to R of the q-subsets in M_i and let R_0 denote the contribution to R of the q-subsets that contain an edge in $E(T) \setminus W$. Hence, $R = \sum_{i=0}^{\chi'(H')} R_i$. We first observe that R_0 is negligible. As there are only at most $3n^{1.9}$ edges not in W, there are at most $3n^{1.9}n^{q-2}$ q-subsets that contribute to R_0, and recall that they each contribute at most

$q(q-1)/6$, so in total $R_0 = o(n^q)$. Since $R \ge (1-o(1))\binom{n}{q}\frac{\zeta}{(q-2)!}$ we may assume w.l.o.g. that

$$R_1 \ge \frac{1}{\chi'(H')}(1-o(1))\binom{n}{q}\frac{\zeta}{(q-2)!} \ge (1-o(1))\binom{n}{q}\frac{\zeta}{\binom{n-2}{q-2}} = (1-o(1))\frac{\zeta n^2}{q(q-1)} \ .$$

Recalling that the q-sets in M_1 are pairwise edge disjoint, we have that $T_L(\pi)$ has fractional triangle packing of size $\nu_3^*(T_L(\pi)) \ge (1-o(1))\frac{\zeta n^2}{q(q-1)}$. By Lemma 2.2 we therefore obtain that $\nu_3(T_L(\pi)) \ge (1-o(1))\frac{\zeta n^2}{q(q-1)}$ as well. □

Corollary 3.11. *Let $q \ge 4$ and suppose that for each $Q \in \mathcal{T}_q$ we can find an orthogonal family of permutations $P = P(Q)$ such that* $\text{avg}_{P(Q)}(Q) \ge \zeta$. *Then,* $\text{inv}_3(n) \le (1+o(1)n^2\left(\frac{1}{8} - \frac{\zeta}{2q(q-1)}\right)$.

Proof. By Lemma 3.3 and Lemma 3.10, for every n-vertex tournament T, a random permutation π of $V(T)$ satisfies whp: (i) $|E(T_L(\pi))| = (1+o(1))n^2/4$. (ii) $\nu_3(T_L(\pi)) \ge (1-o(1))\frac{\zeta n^2}{q(q-1)}$. Fixing such a π which satisfies both requirements, defining $\beta := \zeta/q(q-1)$ and using

$$\text{inv}_3(T) \le (1+o(1))n^2(\frac{1}{8} - \frac{\beta}{2})$$

as shown in the paragraph following Corollary 3.2, the present corollary follows. □

Following Corollary 3.11, our remaining task is to find $q \ge 4$ and ζ such that $\zeta/q(q-1)$ is as large as feasibly computable (note: not all q are possible; if q is not a prime power, we do not have a construction of an orthogonal family of permutations). Let us be more formal about our computational task: Suppose that $q \ge 4$ is such that an orthogonal family of permutations of sets of order q exists. For $Q \in \mathcal{T}_q$, let $\zeta(Q)$ be the maximum of $\text{avg}_{P(Q)}(Q)$ where P ranges over all orthogonal families of permutations of $V(Q)$. Let ζ_q be the minimum of $\zeta(Q)$ taken over all $Q \in \mathcal{T}_q$. Hence, we would like to compute ζ_q or at least obtain a close lower bound for it, as any such lower bound ζ can be applied in Corollary 3.11.

So, suppose we are given a database $\mathcal{D}_q$ of all tournaments on q vertices, each labeled on $[q]$, i.e., for each $Q \in \mathcal{T}_q$, there is precisely one element of $\mathcal{D}_q$ that is isomorphic to it. For example, such a database for all $q \le 10$ is given in `https://users.cecs.anu.edu.au/~bdm/data/digraphs.html`. Furthermore, suppose that $\mathcal{P}_q$ is the set of all orthogonal families of permutations of $[q]$. Then, for each $Q \in \mathcal{D}_q$, for each $P \in \mathcal{P}_q$ and for each $\sigma \in P$, we can easily compute the graph $Q_L(\sigma)$ and then construct a linear program to determine $\nu^*(Q_L(\sigma))$, thus determining $\text{avg}_{P(Q)}(Q)$, consequently determining $\zeta(Q)$, consequently determining ζ_q. The number of operations of this approach is at least $|\mathcal{D}_q||\mathcal{P}_q|q(q-1)L_q$ where L_q is the time to run a single linear program; the latter is non-negligible as the program may have size $\Theta(q^3)$ (the number of possible triangles in $Q_L(\sigma)$). The values of $|\mathcal{D}_q|$ follow the sequence OEIS A000568 [1], so we have, e.g., $|\mathcal{D}_{11}| = 903753248$ and $|\mathcal{D}_9| = 191536$. We see that already for $q = 11$ we need to make at least $|\mathcal{P}_{11}|99412857280$ calls to a linear program of nontrivial size (as most calls involve linear programs with over 100 variables) which is overwhelmingly huge (even if the search space were to be trimmed by employing some heuristics, e.g., as we may settle for a lower bound for ζ_q, we don't need to examine all of the utterly huge $|\mathcal{P}_{11}|$, rather just a few of its members, but even this is not feasible already for a single member). For $q = 9$ (recall, for $q = 10$ we do not know of an orthogonal family), the number of calls to a linear program

Algorithm 1 Computing a lower bound for ζ_q

Require: tournament database $\mathcal{D}_q$; orthogonal family $P \in \mathcal{P}_q$

```
ζ ← ∞
for all Q ∈ D_q do
    best ← −1
    loop ← 0
    while loop < 1000 and best < ζ do
        loop ← loop + 1
        π ← random permutation of [q]
        P* ← {πσ : σ ∈ P}
        sum ← 0
        for all σ ∈ P* do
            G ← Q_L(σ) ▷ Construct the graph Q_L(σ)
            sum ← sum + LPSolve(G) ▷ Compute ν*(Q_L(σ)) via an LP package
        avg ← sum/(q(q − 1)) ▷ Compute avg_{P*}(Q)
        if avg > best then
            best ← avg
    if best < ζ then
        ζ ← best
return ζ
```

(again, not a very small one) is $|\mathcal{P}_9|13790592$ which becomes feasible if instead of going over all $|\mathcal{P}_9|$, we only scan a few members of it. Similarly, we can do the same for $q = 4, 5, 7, 8$. Indeed, this is what our program does; its pseudocode is given in Algorithm 1 and its code can be obtained from `https://github.com/raphaelyuster/decycling/blob/main/decycling_latin.cpp`.

As can be seen from the pseudocode, as well as the code, we start with some fixed orthogonal family P. For example, if $q = 9$ we use the one in Table 1. For each $Q \in \mathcal{D}_q$ we generate a constant, say 1000, orthogonal families of the form $P^* = \{\pi\sigma : \sigma \in P\}$ where π is a random permutation of $[q]$ and for each such family we compute $\text{avg}_{P^*}(Q)$, taking the best (i.e., highest over all 1000 trials) result that we find as a lower bound for $\zeta(Q)$, and setting ζ to be the minimum of these lower bounds, taken over all $Q \in \mathcal{D}_q$. We summarize the result of the program runs in Table 2. As can be seen, for $q = 9$ we obtain a value of $\zeta = \frac{67}{18}$ which implies the constant $\frac{257}{2592}$ for $\frac{1}{8} - \frac{\zeta}{2q(q-1)}$, completing the proof of Theorem 1.3. $\square$

4 Larger k

In this section we prove Theorem 1.4. Starting with the lower bound, our aim is to construct an n-vertex tournament T for which $\text{inv}_k(T)$ is large.

Lemma 4.1. *For all sufficiently large q, there exists a tournament Q on q vertices such that:*
(i) $\text{inv}_2(Q) \geq \frac{q^2}{4} - 2q^{3/2}$ *;*
(ii) For any set R of r vertices for which $1.04^r \geq q$, it holds that $\text{inv}_2(Q[R]) \geq \frac{1}{4}\binom{r}{2}$ *.*

Proof. Let Q be a random tournament on q vertices. It suffices to prove that each of the two

q	ζ	$\frac{1}{8} - \frac{\zeta}{2q(q-1)}$
4	$\frac{1}{3}$	$\frac{1}{9} \approx 0.1111$
5	$\frac{7}{10}$	$\frac{43}{400} = 0.1075$
7	$\frac{27}{14}$	$\frac{5}{49} \approx 0.1021$
8	$\frac{153}{56}$	$\frac{631}{6272} \approx 0.1006$
9	$\frac{67}{18}$	$\frac{257}{2592} \approx 0.0992$

Table 2: The values of $\zeta \leq \zeta_q$ for various choices of q obtained from the program whose pseudocode is given in Algorithm 1, and the resulting constant from Corollary 3.11.

items in the lemma's statement holds with probability at least, say, $\frac{2}{3}$, as then they both hold with positive probability, implying Q's existence. As for the first item, the result of de la Vega [9] states that with probability $1 - o_q(1)$ it holds that $\text{inv}_2(Q) \geq \frac{1}{2}\binom{q}{2} - 1.73q^{3/2}$, implying in particular that for all q sufficiently large, with probability at least $\frac{2}{3}$ it holds that $\text{inv}_2(Q) \geq \frac{q^2}{4} - 2q^{3/2}$.

For the second item, fix a subset $R \subseteq V(Q)$ with $r = |R|$ where $1.04^r \geq q$. As Q is a random tournament, so is its sub-tournament $Q[R]$. Let π be a permutation of R and consider $Q[R]_L(\pi)$, which, in turn is the undirected random graph $G(r, \frac{1}{2})$. The expected number of edges of $Q[R]_L(\pi)$ is therefore $\frac{1}{2}\binom{r}{2}$ and the probability of this number being smaller than $\frac{1}{4}\binom{r}{2}$ (which is precisely half the expectation) is, by Chernoff's inequality, at most $\exp(-\frac{1}{8}\binom{r}{2})$. As there are only $r!$ possible π to consider, we have that the probability that $\text{inv}_2(Q[R]) < \frac{1}{4}\binom{r}{2}$ is at most $r!\exp(-\frac{1}{8}\binom{r}{2})$. Now, for any r, the number of possible subsets R is $\binom{q}{r}$, so the probability that (ii) fails is at most

$$\Pr[(ii) \text{ fails}] \leq \sum_{r=\lceil \log_{1.04} q \rceil}^{q} \binom{q}{r} r! e^{-\frac{1}{8}\binom{r}{2}} \leq q(qr)^r e^{-\frac{1}{8}\binom{r}{2}} \leq 1.05^{r^2} e^{-\frac{1}{8}\binom{r}{2}} \ll \frac{1}{3}$$

where we have used that $1.04^r \geq q$, that $1.05e^{-1/16} < 1$ and that q is sufficiently large. ☐

Let R be an r-vertex tournament and suppose that n is a multiple of r. An n-vertex *balanced blowup* of R is obtained by replacing each vertex $i \in V(R)$ with a set V_i of size n/r (all the V_i's are pairwise disjoint), and constructing a tournament with vertex set $\cup_{i \in V(R)} V_i$ as follows. For each edge $(i, j) \in E(R)$, all edges are oriented from V_i to V_j (we call such edges *outer*), and for each V_i, all edges with both endpoints in V_i are oriented arbitrarily (we call such edges *inner*).

Lemma 4.2. *Suppose R is an r-vertex tournament having $\text{inv}_2(R) = m$ and let Z be an n-vertex balanced blowup of a R. Then for any permutation σ of $V(Z)$, the graph $Z_L(\sigma)$ contains at least mn^2/r^2 outer edges.*

Proof. Consider some permutation σ of $V(Z)$ and the corresponding $Z_L(\sigma)$. Recall that $V(Z)$ is the disjoint union of r sets V_i for $i \in V(R)$ and let W be a transversal of the V_i's. Then, by construction, $Z[W]$ is isomorphic to R. So, the subgraph of $Z_L(\sigma)$ induced by W contains at least $\text{inv}_2(R) = m$ outer edges. By double-counting over all possible transversals, we have that $Z_L(\sigma)$ contains at least $m(n/r)^2/(n/r)^{r-2} = mn^2/r^2$ outer edges. ☐

Fix a tournament Q on vertex set $[q]$ where $q = k^2$, satisfying both items of Lemma 4.1. Let T be an n-vertex balanced blowup of Q (we may assume than n is a multiple of q as removing a constant number of vertices from a tournament T on n vertices only decreases $\text{inv}_k(T)$ by $o(n^2)$ so does not affect the asymptotic claim of Theorem 1.4). We will show that $\text{inv}_k(T)$ is at least as large as the claimed lower bound in Theorem 1.4.

Proof of Theorem 1.4, lower-bound. Suppose $\{X_1, \dots, X_t\}$ is a k-decycling set of T where $t = \text{inv}_k(T)$. Listing the elements of T in the order they appear in the resulting transitive tournament corresponds to some permutation π of $V(T)$ for which all edges of $T_L(\pi)$ have been reversed. We will show that we need t to be rather large in order to reverse all edges of $T_L(\pi)$ for *any* possible π. So, fix some such π and partition the edges of $T_L(\pi)$ into two parts E_{in} and E_{out} where the former are the edges of $T_L(\pi)$ that are inner edges of T and the latter are the edges of $T_L(\pi)$ that are outer edges of T.

We next lower-bound $|E_{\text{out}}|$. By Lemma 4.2, using $R = Q$, $r = q$, $m \geq q^2/4 - 2q^{3/2}$ (which follows from Lemma 4.1 Part (i)) $Z = T$ and $\sigma = \pi$, we have

$$|E_{\text{out}}| \geq \frac{n^2}{q^2}\left(\frac{q^2}{4} - 2q^{3/2}\right) = \frac{n^2}{4} - \frac{2n^2}{k}. \tag{1}$$

Now, consider some X from the aforementioned k-decycling set. We will show that X reverses at most $\binom{k}{2} - \Omega(k^2/\log^4 k)$ edges of E_{out}. Let $|X| = x$ and suppose first that $x \leq k/2$. In this case we can use the trivial fact that X reverses at most $\binom{x}{2} \leq k^2/4$ edges. Assume therefore that $k/2 \leq x \leq k$. Recall that $V(T)$ consists of q parts $V_1, \dots, V_q$. Let $W_i = V_i \cap X$ for $1 \leq i \leq q$. Suppose next that some W_i has size at least $k/\log^2 k$. In this case, we see that X induces at least $\binom{|W_i|}{2}$ inner edges, implying that X reverses at most $\binom{k}{2} - \Omega(k^2/\log^4 k)$ edges of E_{out}. Thus, we may assume that $0 \leq |W_i| \leq k/\log^2 k$ for all $1 \leq i \leq q$.

Partition the W_i into *bunches* according to their size. For $1 \leq j \leq \lfloor \log k \rfloor$ we say that W_i is in bunch j if $2^{j-1} \leq |W_i| < 2^j$. Letting B_j be the union of all the W_i in bunch j, we have $\sum_{j=1}^{\lfloor \log k \rfloor} |B_j| = x$. Let $j^* = \lfloor \log k - 3 \log\log k \rfloor$. Consider first the case that $\sum_{j=1}^{j^*} |B_j| \leq x/2$. In this case, the last $\lceil 3 \log\log k \rceil$ bunches contain together at least $x/2$ vertices of X. But since in each such bunch it holds that $|W_i| \geq 2^{j^*-1} = \Omega(k/\log^3 k)$, it induces at least $\Omega(k^2/\log^6 k)$ inner edges. Furthermore, as each $|W_i| \leq k/\log^2 k$ and since $x/2 \geq k/4$, there are $\Omega(\log^2 k)$ such W_i in the last $\lceil 3 \log\log k \rceil$ bunches, so together they induce at least $\Omega(\log^2 k \cdot k^2/\log^6 k) = \Omega(k^2/\log^4 k)$ inner edges, implying that X reverses at most $\binom{k}{2} - \Omega(k^2/\log^4 k)$ edges of E_{out}.

We remain with the case that $\sum_{j=1}^{j^*} |B_j| \geq x/2$, so there is some bunch j with $1 \leq j \leq j^*$ for which $|B_j| \geq x/2j^* \geq x/2\log k$, and we shall focus on that bunch. Notice that since each W_i in bunch j has size at most 2^j, this means that the number of W_i in bunch j is at least

$$\frac{x}{2^{j+1}\log k} \geq \frac{k}{2^{j+2}\log k} \geq \frac{k}{2^{j^*+2}\log k} \geq \frac{\log^2 k}{4}\ .$$

Let $R = \{i : W_i \text{ is in bunch } j\}$. Notice that $R \subseteq [q] = V(Q)$ and that $r = |R| \geq \frac{k}{2^{j+2}\log k} \geq \frac{\log^2 k}{4}$ by the last inequality. Observe that if $i \in R$ then $|W_i| \geq 2^{j-1}$, so let $W_i^* \subseteq W_i$ be chosen such that $|W_i^*| = 2^{j-1}$. By Lemma 4.1, item (ii), we have that $\text{inv}_2(Q[R]) \geq \frac{1}{4}\binom{r}{2}$. Notice also that the union of the W_i^* for $i \in R$ is an $r2^{j-1}$ balanced blowup of $Q[R]$. Hence, by Lemma 4.2, with $R = R[Q]$, $m \geq \frac{1}{4}\binom{r}{2}$, Z being the subgraph of $T[X]$ induced by the union of the W_i^*, and $\sigma = \pi^{\text{reverse}}$ we have that $Z_L(\sigma)$ contains at least $\frac{1}{4}\binom{r}{2}2^{2j-2}$ outer edges. Notice that all of these outer edges do not appear in $T_L(\pi)$. It follows that X reverses at most

$$\binom{k}{2}-\frac{1}{4}\binom{r}{2}2^{2j-2} \le \binom{k}{2}-\frac{1}{10}\left(\frac{k}{2^{j+2}\log k}\right)^2 2^{2j-2}=\binom{k}{2}-\Omega(k^2/\log^2 k)$$

edges of $E_{\rm out}$.

It now follows from (1) and form the fact that each X element in the decycling set reverses at most $\binom{k}{2}-\Omega(k^2/\log^4 k)$ edges of $E_{\rm out}$, that the number of elements of the decycling set must be

$$\text{inv}_\text{k}(n) \ge \text{inv}_\text{k}(T) \ge \frac{\frac{n^2}{4}-\frac{2n^2}{k}}{\binom{k}{2}-\Omega(k^2/\log^4 k)}\;.$$

The existence of $\delta_k > 0$ for all $k \ge k_0$ is now guaranteed since for all sufficiently large k,

$$\frac{\frac{1}{4}-\frac{2}{k}}{\binom{k}{2}-\Omega(k^2/\log^4 k)} > \frac{1}{2k(k-1)}\;.$$

□

We now turn to proving the upper bound in Theorem 1.4. We start with the following lemma which is analogous to Lemma 3.1.

Lemma 4.3. *Let G be a directed graph and suppose that $k \ge 4$ is even and that the underlying undirected graph of G has a (not necessarily induced) copy of $K_{k,k}$ or that $k \ge 5$ is odd and that the underlying undirected graph of G has a (not necessarily induced) copy of $K_{k+1,k-1}$. Let H denote the corresponding copy. Then, there are four sets of vertices X, Y, Z, W of k vertices each, such that inverting $\{X, Y, Z, W\}$ reverses the direction of the edges of H in G without affecting the direction of any other edge of G.*

Proof. Suppose first that k is even and that H is a copy of $K_{k,k}$. Consider the bipartition of H, denoting the vertices of one part by $v_1, \ldots, v_k$ and the other part by $u_1, \ldots, u_k$. Let $A = \{v_1, \ldots, v_{k/2}\}$, $B = \{v_{k/2+1}, \ldots, v_k\}$, $C = \{u_1, \ldots, u_{k/2}\}$, $D = \{u_{k/2+1}, \ldots, u_k\}$. Inverting $\{A \cup C, A \cup D, B \cup C, B \cup D\}$ reverses the direction of the k^2 edges of H without affecting the direction of any other edge of G. Suppose next that k is odd and that H is a copy of $K_{k+1,k-1}$. Consider the bipartition of H, denoting the vertices of one part by $v_1, \ldots, v_{k+1}$ and the other part by $u_1, \ldots, u_{k-1}$. Let $A = \{v_1, \ldots, v_{(k+1)/2}\}$, $B = \{v_{(k+1)/2+1}, \ldots, v_{k+1}\}$, $C = \{u_1, \ldots, u_{(k-1)/2}\}$, $D = \{u_{(k-1)/2+1}, \ldots, u_k\}$. Inverting $\{A \cup C, A \cup D, B \cup C, B \cup D\}$ reverses the direction of the $k^2 - 1$ edges of H without affecting the direction of any other edge of G. □

The next lemma is a simple consequence of Turán's Theorem and Ramsey's Theorem.

Lemma 4.4. *For every fixed $k \ge 3$ there exists $\gamma_k > 0$ such that in any 2-coloring of the edges of K_n, there are at least $(1 - o(1))\gamma_k n^2$ pairwise edge-disjoint monochromatic copies of K_k, all of the same color.*

Proof. Let $q = R(k) < 4^k$ be the diagonal Ramsey number of k. Consider some 2-coloring of the edges of K_n, and remove monochromatic edge-disjoint copies of K_k until none are left. We must have removed at least n^2/q^2 edges as otherwise by Turán's Theorem, there is a K_q on the non-removed edges, so a monochromatic K_k. The result follows for $\gamma_k = 1/q^2k^2$ as at least half of the monochromatic removed K_k are of the same color. □

Proof of Theorem 1.4, upper-bound. Let T be an n-vertex tournament. Considering some random permutation π of $V(T)$, we have that with high probability, $|E(T_L(\pi))| = (1+o(1))n^2/4$ and $|E(T_R(\pi))| = (1+o(1))n^2/4$. By Lemma 4.4 one of $T_L(\pi)$ or $T_R(\pi)$ has at least $(1-o(1))\gamma_k n^2$ pairwise edge-disjoint copies of K_k. Without loss of generality, assume this is $T_L(\pi)$. Removing a set of $(1-o(1))\gamma_k n^2$ edge-disjoint copies of K_k from $T_L(\pi)$ amounts to inverting this amount of k-sets of vertices, such that after applying these inversions we obtain a tournament T^* such that $T^*_L(\pi)$ has at most $(1+o(1))n^2/4 - (1-o(1))\binom{k}{2}\gamma_k n^2$ edges. By Lemma 4.3 we can repeatedly invert quartets of k-sets of vertices each, until we obtain a tournament T^{**} for which $T^{**}_L(\pi)$ has no $K_{k,k}$ (when k is even) or no $K_{k+1,k-1}$ (when k is odd). The number of inversions performed starting at T^* and arriving at T^{**} is therefore precisely $4(|E(T^*_L(\pi))| - |E(T^{**}_L(\pi))|)/k^2$ if k is even and precisely $4(|E(T^*_L(\pi))| - |E(T^{**}_L(\pi))|)/(k^2-1)$ if k is odd. By the Kovári-Sós-Turán Theorem [15], we have $|E(T^{**}_L(\pi))| = O(n^{2-1/k})$. We therefore obtain that

$$\mathrm{inv}_k(T) \le (1+o(1))n^2\left[\gamma_k + \frac{4}{k^2}\left(\frac{1}{4} - \binom{k}{2}\gamma_k\right)\right] = (1+o(1))n^2\left[\frac{1}{k^2} - \epsilon_k\right]$$

when k is even and

$$\mathrm{inv}_k(T) \le (1+o(1))n^2\left[\gamma_k + \frac{4}{k^2-1}\left(\frac{1}{4} - \binom{k}{2}\gamma_k\right)\right] = (1+o(1))n^2\left[\frac{1}{k^2-1} - \epsilon_k\right]$$

for a suitable $\epsilon_k > 0$. As T is an arbitrary n-vertex tournament, we obtain (unifying the even and odd cases of k) that

$$\mathrm{inv}_k(n) \le (1+o(1))n^2\left[\frac{1}{2\lfloor k^2/2\rfloor} - \epsilon_k\right] .$$

□

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