Intersecting Designs

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Abstract

We prove the intersection conjecture for designs: For any complete graph K_r there is a finite set of positive integers $M(r)$ such that for every $n > n_0(r)$, if K_n has a K_r-decomposition (namely a 2-$(n, r, 1)$ design exists) then there are two K_r-decompositions of K_n having exactly q copies of K_r in common for every q belonging to the set $\{0, 1, \ldots, \binom{n}{2}/\binom{r}{2}\} \setminus \{\binom{n}{2}/\binom{r}{2} - m \mid m \in M(r)\}$. In fact, this result is a special case of a much more general result, which determines the existence of k distinct K_r-decompositions of K_n which have q elements in common, and all other elements of any two of the decompositions share at most one edge in common.

1 Introduction

All graphs considered here are finite, undirected, and have no loops or multiple edges. For the standard graph-theoretic and design-theoretic notations the reader is referred to [7] and [11] respectively. An *H-decomposition* of a graph G is a set L of edge-disjoint H-subgraphs of G, such that each edge of G appears in some element of L. Thus, L contains $e(G)/e(H)$ elements, where $e(X)$ denotes the number of edges of a graph X. It is straightforward to see that a necessary condition for the existence of an H-decomposition is that $e(H)$ divides $e(G)$. Another obvious requirement is that $gcd(H)$ divides $gcd(G)$ where the gcd of a graph is the greatest common divisor of the degrees of its vertices.

In general, it is NP-Complete to determine whether a given graph G has an H-decomposition for every fixed graph H containing more than two edges in some connected component. This has been proved by Dor and Tarsi [13]. However, a seminal result of Wilson [28], is that the existence of the two necessary conditions mentioned above is also sufficient to guarantee an H-decomposition of K_n for every $n > n_0(H)$, and this result holds for every fixed nonempty graph H. In terms of design-theory, Wilson's Theorem states that the necessary conditions are sufficient for the existence

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of a $2-(v,r,1)$-design, provided that v is sufficiently large (in fact, it is sufficient for the existence of a $2-(v,r,\lambda)$-design).

In order to present our result in the exact context we shall switch momentarily to the language of design-theory. Since the appearance of the seminal work of Wilson, the notion of repeated blocks in a $t-(v,r,\lambda)$ design became a central issue in design theory. We refer the reader to [27] and [11] which are major comprehensive sources for design theory and the emergence of the repeated-block issue. For research papers on this subject see [2, 3, 14, 20]. One major problem that has been developed from the study of designs with non-repeated blocks is the *intersection problem.* This problem asks for the existence of a $2-(v,r,2)$ design in which exactly $q \geq 0$ blocks are used twice. Extensions of this problem to $2-(v,r,\lambda)$ designs in which exactly $q \geq 0$ blocks are used λ times while any other block is used at most once were considered as well. In fact, this line of research has been extended to include small graphs and simple structured trees instead of just complete graphs as the blocks of the design. We refer the reader to [3, 4, 5, 6, 8, 12, 14, 17, 18, 20, 23] for various papers on the intersection problem, and to [19] as one of the first papers where the problem was raised explicitly. These works also have an obvious connection to the famous works of Lu [21, 22] and Teirlinck [24, 25, 26] on the existence of large Steiner triple systems where, clearly, $q=0$ in the above notation.

The intersection conjecture for design states that for every r, there exists a set of constantly many integers $M(r)$, such that for every integer $0 \leq q \leq \binom{n}{2}/\binom{r}{2}$ for which $\binom{n}{2}/\binom{r}{2} - q \notin M(r)$, there exist two distinct $2-(v,r,1)$ designs which have exactly q blocks in common, whenever there exists a $2-(v,r,1)$-design. Note that one cannot have $M(r)=\emptyset$, since, clearly, $1 \in M(r)$. This is because two decompositions cannot differ only in one block.

In this paper, this conjecture is solved in the asymptotic sense, namely, in the form which is analog to Wilson's Theorem:

Theorem 1.1 *Let $r \geq 3$ be an integer. There exists $N=N(r)$ and a fixed set of positive integers $M(r)$, such that for every $n > N$, there exist two distinct $2-(v,r,1)$-designs with exactly $0 \leq q \leq \binom{n}{2}/\binom{r}{2}$ blocks in common if and only if there exists a $2-(v,r,1)$-design and $\binom{n}{2}/\binom{r}{2} - q \notin M(r)$.*

Theorem 1.1 is a corollary of a much more general theorem which solves a generalized version of the intersection problem. The generalization is twofold. We require k distinct $2-(v,r,1)$-designs (instead of just two distinct $2-(v,r,1)$-designs) which share q blocks in common, and we also require that any other two distinct blocks in any two of the designs share at most one edge in common (thus, they are almost edge-disjoint). This generalized problem is solved in the following theorem. We state it in the language of graph theory, since this is the language used in the proof.

Theorem 1.2 *Let $r \geq 3$ and $k \geq 2$ be integers. There exists $N = N(r,k)$ and a fixed set of positive integers $M(r,k)$, such that for every $n > N$:*
there exist k distinct K_r-decompositions of K_n which have $0 \leq q \leq \binom{n}{2}/\binom{r}{2}$ copies of K_r in common, and any other two distinct copies of K_r in any two of the decompositions share at most one edge **if and only if** *K_n has a K_r-decomposition and $\binom{n}{2}/\binom{r}{2} - q \notin M(r,k)$.*

By taking $k = 2$ we can show that theorem 1.1 is an immediate corollary of Theorem 1.2 (see the final section for details). In fact, we are able to characterize the sets $M(r,k)$ of Theorem 1.2 and $M(r)$ of Theorem 1.1 precisely, as will be shown in the proof of Theorem 1.2. The proof of Theorem 1.2 is based on two major ingredients. The first is Gustavsson's Theorem [15] which gives necessary and sufficient conditions for the existence of H-decompositions in very dense and large (although not necessarily complete) graphs G. The second is the recent proof of the authors [10] of the existence of k distinct orthogonal K_r-decompositions of K_n (n sufficiently large), where a k-orthogonal K_r-decomposition is a set of k distinct K_r-decompositions of K_n where any two copies in any two decompositions share at most one edge. Other ingredients used in the proof are Dirichlet's theorem for primes in arithmetic progressions and the Theorem of Hajnal and Szememrédi. The proof is presented in the following section.

2 Proof of the main result

We shall first present the basic tools which are used in the proof of Theorem 1.1. The first is due to Gustavsson [15], which can be viewed as an extension of Wilson's Theorem to H-decompositions of graphs which are not necessarily complete, although they must still be very dense (this requirement is not surprising, recalling the NP-Completeness result of Dor and Tarsi).

Lemma 2.1 (Gustavsson [15]) *Let H be a fixed nonempty graph. There exists a positive integer $n_0 = n_0(H)$, and a small positive constant $\gamma = \gamma(H)$, such that if G is a graph with $n > n_0$ vertices, and $\delta(G) \geq (1-\gamma)n$, and G satisfies the necessary conditions for an H-decomposition, then G has an H-decomposition.*

We note here that the constant $\gamma(H)$ used in Gustavsson's proof is very small. In fact, even for the case where H is a triangle, Gustavsson's proof uses $\gamma = 10^{-24}$. Thus, the graph G is very dense.

A *k-orthogonal* K_r-decomposition of K_n, is a set of k distinct K_r-decompositions of K_n, such that any two copies of K_r in any two of the decompositions share at most one edge. The next tool that we use is the recent proof of the authors [10] which state that a k-orthogonal K_r-decomposition of K_n exists whenever a K_r-decomposition of K_n exists, provided that n is sufficiently large:

Lemma 2.2 (Caro and Yuster [10]) *Let $r \geq 3$ and $k \geq 2$ be integers. There exists $n_1 = n_1(r,k)$ such that for every $n > n_1$, there exists a k orthogonal K_r-decomposition of K_n if and only if $n \equiv 1, r \bmod r(r-1)$.*

We note here that the proof of Lemma 2.2 is based mostly on probabilistic arguments.

A well-known theorem in Number Theory is Dirichlet's Theorem, which states that if $gcd(a,d) = 1$ then there are infinitely many primes of the form $a + kd$ where k ranges over the integers. We use this to prove the following lemma:

Lemma 2.3 *Let $r \geq 3$ be an integer. There exist infinitely many primes p which satisfy $p \equiv 1 \bmod r(r-1)$ and $gcd(\binom{p}{2}, \binom{p+r-1}{2}) = \binom{r}{2}$.*

Proof: Let p be a prime satisfying $p \equiv 1 \bmod r(r-1)$. According to Dirichlet's Theorem there are infinitely many values suitable for p. Thus, $p = br(r-1)+1$ for some positive integer b. Now $\binom{p}{2} = (br(r-1)+1)br(r-1)/2 = bp[r(r-1)/2]$. On the other hand, $\binom{p+r-1}{2} = (br(r-1)+r)(br(r-1)+(r-1))/2 = [b(r-1)+1][br+1][r(r-1)/2]$. Since p is prime we have that bp and $(b(r-1)+1)(br+1)$ are relatively prime. Thus, $gcd(\binom{p}{2}, \binom{p+r-1}{2}) = \binom{r}{2}$. $\square$

Finally, we shall require the Theorem of Hajnal and Szemerédi which gives sufficient conditions guaranteeing that a graph G has a K_r-factor (i.e. n/r vertex-disjoint copies of K_r):

Lemma 2.4 (Hajnal and Szemerédi [16]) *Let G be a graph with n vertices, Let r be a positive integer which divides n, and assume that $\delta(G) \geq (1-1/r)n$. Then G has a K_r-factor.*

Corollary 2.5 *If r divides n then K_n has $\lceil n/r^2 \rceil$ edge-disjoint K_r-factors.*

Proof: By deleting t edge-disjoint K_r-factors from K_n we obtain a regular graph of degree $n-1-t(r-1)$, so as long as $n-1-t(r-1) \geq n(1-1/r)$ we can delete another K_r-factor. Since $t = \lfloor n/r^2 \rfloor$ satisfies $n-1-t(r-1) \geq n(1-1/r)$ the corollary follows. $\square$

We are now ready to proceed with the proof of Theorem 1.2:

Proof of Theorem 1.2: Let $r \geq 3$ and $k \geq 2$ be fixed positive integers. Let $p > n_1(r,k)$ (n_1 is defined in Lemma 2.2) be a prime which satisfies $p \equiv 1 \bmod r(r-1)$ and $gcd(\binom{p}{2}, \binom{p+r-1}{2}) = \binom{r}{2}$. According to Lemma 2.3, p exists. Now define the graph $H_p = K_p \cup K_{p+r-1}$ (i.e. H_p is the vertex-disjoint union of two complete graphs of order p and $p+r-1$). H_p has the following properties:

1. $gcd(H_p) = gcd(p-1, p-1+r-1) = r-1$.

2. $gcd(e(K_p), e(K_{p+r-1})) = \binom{r}{2}$.

3. Both K_p and K_{p+r-1} have a k-orthogonal K_r-decomposition, and thus, in particular, H_p has a k orthogonal K_r-decomposition. This follows from Lemma 2.2.

Let $\epsilon = \min\{1/r^2, \gamma(H_p)/2, 1/(3p)\}$ where γ is defined in Lemma 2.1. The next lemma shows that if m is large enough and $m \equiv 1, r \bmod r(r-1)$ then K_m has k distinct K_r-decompositions with exactly q elements in common, where q is bounded by a small fraction (namely ϵ) of the overall number of copies of K_r.

Lemma 2.6 *Let*

$$m > \max\{n_0(H_p)\ ,\ \frac{8r}{\epsilon}\ ,\ p^3+p\ ,\ \frac{2p}{\epsilon}\}$$

satisfy $m \equiv 1, r \bmod r(r-1)$ *(n_0 is defined in Lemma 2.1). Let* $0 \le q \le \epsilon\binom{m}{2}/\binom{r}{2}$ *be an integer. Then, there exist k distinct K_r-decompositions of K_m which have exactly q elements in common, and any other two distinct copies of K_r in any two of the decompositions share at most one edge.*

Proof: Let $m' \le m$ be the largest integer which is a multiple of r. Note that $m' > m - r$. We first show that $\lceil qr/m' \rceil \le q\lceil m'/r^2 \rceil$. Clearly, it suffices to show that $q \le m'^2/r^3$, and this follows from the fact that $q \le \epsilon\binom{m}{2}/\binom{r}{2}$, $m' > m-r$, $r \ge 3$ and $m \ge 8r/\epsilon$. Thus, according to Corollary 2.5, any set of m' vertices of K_m contains $\lceil qr/m' \rceil$ edge-disjoint K_r-factors, and thus the overall number of copies of K_r in all these factors is $m'/r\lceil qr/m' \rceil \ge q$. It follows that we can delete from K_m a set Q of q edge-disjoint copies of K_r such that the resulting graph G has

$$\delta(G) \ge m-1-(r-1)\lceil\frac{qr}{m'}\rceil \ge m-r-\frac{qr(r-1)}{m'} \ge m-r-\frac{m(m-1)\epsilon}{m-r} \ge m(1-1.5\epsilon)$$

where the last inequality follows from $m \ge 8r/\epsilon$. Obviously, $e(G) = \binom{m}{2} - q\binom{r}{2}$. Let $e(G) \equiv a \bmod e(H_p)$ where $0 \le a < e(H_p)-1$. Since $e(G)$, a and $e(H_p)$ are all multiples of $\binom{r}{2}$ and since $gcd(\binom{p}{2}, e(H_p)) = gcd(\binom{p}{2}, \binom{p+r-1}{2}) = \binom{r}{2}$ there exists $0 \le t < e(H_p)/\binom{r}{2}$ such that $e(G) - t\binom{p}{2} \equiv 0 \bmod e(H_p)$. We claim that we can delete from G a set of t vertex-disjoint copies of K_p. Let $m'' \le m$ be the largest integer which is a multiple of p. Note that $m'' > m-p \ge p^3$. Since $t < e(H_p)/\binom{r}{2} < p^2$ it suffices to show that any set of m'' vertices of G has a K_p-factor. Since any set of m'' vertices of G induces a subgraph whose minimum degree is at least $\delta(G)-p$, we only need to show, by lemma 2.4, that $\delta(G)-p \ge m''(1-1/p)$. Since $m'' \le m$ and since $\delta(G) \le m(1-1.5\epsilon)$ it suffices to show that $1.5m\epsilon \le m/p - p$, and this follows from the fact that $\epsilon \le 1/(3p)$ and $m \ge 2p^2$. Let G' be the graph obtained from G after deleting a set T of t vertex-disjoint copies of K_p. Clearly, since $m \ge 2p/\epsilon$, we have

$$\delta(G') \ge \delta(G) - p + 1 > m(1-1.5\epsilon) - p \ge m(1-2\epsilon) \ge m(1-\gamma(H_p)).$$

Also note that $e(G') \equiv 0 \bmod e(H_p)$ and that $gcd(G')$ is a multiple of $r-1$, since we have only deleted from K_m copies of K_r and copies of K_p, each having degree which is a multiple of $r-1$. Thus, according to Lemma 2.1, G' has an H_p-decomposition. Denote this decomposition by L. Each member of L is a copy of H_p.

We now use the sets Q, T and L to define the k desired K_r-decompositions. Each member $C \in T \cup L$ is either a K_p or an H_p. In any case, C has a k-orthogonal K_r-decomposition. Denote by $C_1, \ldots, C_k$ a set of k-orthogonal K_r decompositions of C. Now define k distinct K_r decompositions of K_m, denoted $S_1, \ldots S_k$, as follows:

$$S_i = Q \cup \{C_i \mid C \in T \cup L\}.$$

Note that S_i is indeed a K_r decomposition, since Q contains edge-disjoint members of K_r and for each $C \in T \cup L$ we have that C_i is, by definition, a K_r-decomposition of C, and obviously, C is edge-disjoint with any member of Q, and any two distinct members C and C' of $T \cup L$ are edge-disjoint. Now note that Q is a set of q copies of K_r which are common to all the S_i's. Any K_r copy X of S_i not belonging to Q belongs to some C_i. X has at most one common edge with any member of S_j belonging to C_j since C_i and C_j are orthogonal. X is completely edge-disjoint from any member of S_j belonging to C'_j if $C' \neq C$, since C and C' are two distinct members in the H_p decomposition of G'. Thus, $\{S_1, \ldots, S_k\}$ satisfy the statement of the lemma. $\Box$

Lemma 2.6 is still far from what we want since the lemma only gives us that if n is sufficiently large and $n = 1, r \bmod r(r-1)$ and q is in the range $0, \ldots, \epsilon\binom{n}{2}/\binom{r}{2}$, then Theorem 1.2 holds. However, we want the theorem to hold for any q in the complete range from 0 to $\binom{n}{2}/\binom{r}{2}$ (except for finitely many values). Our next goal is, therefore, to extend the range for q significantly.

Lemma 2.7 *There exist positive integers $N' = N'(r,k)$ and $Q' = Q'(r,k)$ such that for every $n > N'(r,k)$ which satisfies $n = 1, r \bmod r(r-1)$ and for every $0 \le q \le \binom{n}{2}/\binom{r}{2} - Q'(r,k)$, there exist k distinct K_r-decompositions of K_n which have exactly q elements in common, and any other two distinct copies of K_r in any two of the decompositions share at most one edge.*

Proof: Let $m \equiv 1 \bmod r(r-1)$ be a prime which satisfies $gcd(\binom{m}{2}, \binom{m+r-1}{2}) = \binom{r}{2}$, and also satisfies the conditions of Lemma 2.6. According to Lemma 2.3, m exists. Define the graph $H_m = K_m \cup K_{m+r-1}$. Note that H_m has similar properties to those of the graph H_p defined previously. Namely:

1. $gcd(H_m) = gcd(m-1, m-1+r-1) = r-1$.

2. $gcd(e(K_m), e(K_{m+r-1})) = \binom{r}{2}$.

3. Since $m > p > n_1(r,k)$, both K_m and K_{m+r-1} have a k-orthogonal K_r-decomposition, and thus, in particular, H_m has a k orthogonal K_r-decomposition. This follows from Lemma 2.2.

4. K_m satisfies the statement of Lemma 2.6.

We now define the values N' and Q' :

$$N' = \max\{m\lceil\frac{3}{\epsilon}\rceil + m^3\ ,\ \lceil\frac{m}{\gamma(H_m)}\rceil\}.$$

$$Q' = \frac{\binom{m}{2}}{\binom{r}{2}}(m^2 + \lceil\frac{3}{\epsilon}\rceil).$$

Recall that p is a function of r and k, ϵ is a function of r and p, and m is a function of r, p and ϵ. Thus, indeed, $N' = N'(r,k)$ and $Q' = Q'(r,k)$. Let $n > N'$ satisfy $n = 1, r \bmod r(r-1)$. Let $0 \le b \le e(H_m) - 1$ satisfy

$$b \equiv \binom{n}{2} - \binom{m}{2}\lceil\frac{3}{\epsilon}\rceil \bmod e(H_m).$$

Since $e(H_m)$, $\binom{n}{2}$ and $\binom{m}{2}$ are all multiples of $\binom{r}{2}$, so is b. Furthermore, $gcd(\binom{m}{2}, e(H_m)) = gcd(\binom{m}{2}, \binom{m+r-1}{2}) = \binom{r}{2}$. Thus, there exists $0 \le t < e(H_m)/\binom{r}{2}$ such that

$$\binom{n}{2} - \binom{m}{2}(t + \lceil\frac{3}{\epsilon}\rceil) \equiv 0 \bmod e(H_m).$$

Our first task is to designate in K_n a set of $t + \lceil\frac{3}{\epsilon}\rceil$ vertex-disjoint copies of K_m. Such a set clearly exists since $t < e(H_m) < m^2$ and since $n > m(m^2 + \lceil\frac{3}{\epsilon}\rceil)$. Given such a set, let $\mathcal{A}$ be a set of t copies of K_m and let $\mathcal{B}$ be a set of $\lceil\frac{3}{\epsilon}\rceil$ copies of K_m, where any two distinct elements of $\mathcal{A} \cup \mathcal{B}$ are vertex-disjoint. Consider the graph G obtained by deleting from K_n the elements of $\mathcal{A} \cup \mathcal{B}$. G has $e(G) = \binom{n}{2} - \binom{m}{2}(t + \lceil\frac{3}{\epsilon}\rceil)$ edges. $gcd(G) = gcd(n-1, n-m)$ is divisible by $r-1$. Also, $\delta(G) = n - m > n(1 - \gamma(H_m))$ since $n > m/\gamma(H_m)$. It follows from Lemma 2.1 that G has an H_m-decomposition. Denote such a decomposition by $\mathcal{L}$. Now let q be an integer satisfying $0 \le q \le \binom{n}{2}/\binom{r}{2} - Q'$. We shall use the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}$ to define the k desired K_r-decompositions of K_n. Let $q \equiv x \bmod e(H_m)/\binom{r}{2}$, where $0 \le x \le e(H_m)/\binom{r}{2} - 1$ is an integer. Thus $q = z \cdot (e(H_m)/\binom{r}{2}) + x$, where z is an integer. Now let $x \equiv w \bmod \lfloor\epsilon\binom{m}{2}/\binom{r}{2}\rfloor$, where $0 \le w \le \lfloor\epsilon\binom{m}{2}/\binom{r}{2}\rfloor - 1$ is an integer. Thus, $x = y \cdot \lfloor\epsilon\binom{m}{2}/\binom{r}{2}\rfloor + w$ where y is an integer. Since

$$\lceil\frac{3}{\epsilon}\rceil \ge \frac{e(H_m)}{\epsilon\binom{m}{2}} + 1 \ge \frac{x}{\epsilon\binom{m}{2}/\binom{r}{2}} + 1 \ge y + 1$$

we may define $\mathcal{B}' \subset \mathcal{B}$ to be a set of y K_m-elements of $\mathcal{B}$, and define $F \in \mathcal{B} \setminus \mathcal{B}'$ to be another fixed K_m element of $\mathcal{B}$. Since

$$\frac{e(G)}{e(H_m)} = \frac{\binom{n}{2} - \binom{m}{2}(t + \lceil\frac{3}{\epsilon}\rceil)}{e(H_m)} \ge \frac{\binom{n}{2} - \binom{m}{2}(m^2 + \lceil\frac{3}{\epsilon}\rceil)}{e(H_m)} = \frac{\binom{n}{2}/\binom{r}{2} - Q'}{e(H_m)/\binom{r}{2}} \ge \frac{q}{e(H_m)/\binom{r}{2}} \ge z$$

we may define $\mathcal{L}' \subset \mathcal{L}$ to be a set of z H_m-elements of $\mathcal{L}$. Having defined the sets $\mathcal{A}$, $\mathcal{B}$ $\mathcal{B}'$, $\mathcal{L}$ $\mathcal{L}'$ and the K_m-graph F we perform the following process for each element in these sets (and for F):

1. Let $X \in \mathcal{A} \cup (\mathcal{L} \setminus \mathcal{L}') \cup \mathcal{B} \setminus (\mathcal{B}' \cup \{F\})$. Since X is either a K_m or an H_m, it has a k orthogonal K_r-decomposition. Let, therefore, $\{X_1, \ldots, X_k\}$ denote a k orthogonal K_r-decomposition of X (namely, X_i is a K_r-decomposition of X, and for $i \neq j$, any copy in X_i shares at most one edge with any copy in X_j).

2. Let $X \in \mathcal{B}'$. Since X is a K_m, and since K_m satisfies the statement of Lemma 2.6, there exists a set $\{X_1, \ldots, X_k\}$ of K_r-decompositions of X, which have exactly $\lfloor \epsilon \binom{m}{2} / \binom{r}{2} \rfloor$ copies in common, and any other two distinct copies of K_r in any two of the decompositions share at most one edge.

3. As $w \leq \lfloor \epsilon \binom{m}{2} / \binom{r}{2} \rfloor - 1$, the graph F, being a K_m, has a set $\{F_1, \ldots, F_k\}$ of K_r-decompositions, which have exactly w copies in common, and any other two distinct copies of K_r in any two of the decompositions share at most one edge.

4. Let $X \in \mathcal{L}'$. We simply let X_1 be any K_r-decomposition of X, and put $X_i = X_1$ for $i = 2, \ldots, k$.

We can now define k distinct K_r-decompositions of K_n, denoted $\{S_1, \ldots, S_k\}$ as follows:

$$S_i = \{X_i \mid X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{L}\}$$

The fact that each S_i is a K_r-decomposition follows from the fact that the elements of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}$ decompose K_n, and each of these elements is, in turn, decomposed into K_r. Also, the S_i's have exactly q elements in common. This is because $X_i = X_j$ for $X \in \mathcal{L}'$, because X_i and X_j have exactly $\lfloor \epsilon \binom{m}{2} / \binom{r}{2} \rfloor$ elements in common for $X \in \mathcal{B}'$, and because any two distinct K_r-decompositions F_i and F_j of F have exactly w elements in common. Thus, the total number of common elements is

$$|\mathcal{L}'| \cdot \frac{e(H_m)}{\binom{r}{2}} + |\mathcal{B}'| \lfloor \epsilon \frac{\binom{m}{2}}{\binom{r}{2}} \rfloor + w = z \frac{e(H_m)}{\binom{r}{2}} + y \lfloor \epsilon \frac{\binom{m}{2}}{\binom{r}{2}} \rfloor + w = z \frac{e(H_m)}{\binom{r}{2}} + x = q.$$

Any two distinct copies of K_r in S_i and S_j are either edge-disjoint disjoint or share one edge, by our construction. This completes the proof of the lemma. $\Box$

Lemma 2.7 shows that the statement of Theorem 1.2 holds for any value of q in the range $0, \ldots, \binom{n}{2} / \binom{r}{2} - Q'(r, k)$. However, we still need to determine for which values of t in the range $0, \ldots, Q'(r, k) - 1$ it is possible to satisfy Theorem 1.2 with $q = \binom{n}{2} / \binom{r}{2} - t$. Note that although the range for t is bounded we still need to show that the set of values of t which satisfy Theorem 1.2

is *independent* of n, since it is claimed in the theorem that this set of values is $M(r,k)$ (namely, it is only a function of r and k). In order to define $M(r,k)$ we need the following definition:
A positive integer s is called (r,k)-*irreducible* if for every graph G with $s\binom{r}{2}$ edges, there is no k orthogonal K_r-decomposition of G. For example, the number 1 is (r,k)-irreducible for every $r \geq 3$ and $k \geq 2$. Trivially, if s is (r,k)-irreducible, then it is also $(r,k+1)$-irreducible. It is also not difficult to establish that if $s \leq 3$ then s is $(r,2)$-irreducible. The following lemma is a corollary of Lemma 2.7

Lemma 2.8 *If $s \geq Q'(r,k)$ then s is not (r,k)-irreducible.*

Proof: Assume $s \geq Q'(r,k)$. Let $q = \binom{n}{2}/\binom{r}{2} - s$. Thus, q satisfies the conditions in Lemma 2.7. Using the same notations of Lemma 2.7, we know that there exists n sufficiently large such that K_n has k distinct K_r-decompositions sharing exactly q elements, and any other two distinct elements in any two of the decompositions share at most one edge. Thus, if G is the graph obtained by deleting from K_n the q shared copies of K_r, we have that G has $s\binom{r}{2}$ edges, and a k orthogonal K_r-decomposition. Consequently, s is not (r,k)-irreducible. $\Box$

We can now complete theorem 1.2. Define

$$N(r,k) = \max\{N'(r,k), \frac{Q'(r,k)r^2}{\gamma(K_r)}\}.$$

and define $M(r,k)$ as the set of all (r,k)-irreducible numbers. Let $n > N(r,k)$ satisfy $n \equiv 1, r \bmod r(r-1)$ (if n does not satisfy this last requirement, then K_n does not have a K_r-decomposition and there is nothing to prove). Let $0 \leq q \leq \binom{n}{2}/\binom{r}{2}$, and put $s = \binom{n}{2}/\binom{r}{2} - q$. Assume first that K_n has k distinct K_r-decompositions sharing q elements, and any other two distinct copies in the decompositions sharing at most one edge. We need to show that s is not (r,k)-irreducible. Indeed, as in the proof of Lemma 2.8, let G be the graph obtained from K_n by deleting the q copies of K_r shared by all the decompositions. G has $s\binom{r}{2}$ edges and the k decompositions of K_n induce a k orthogonal K_r-decomposition of G. Thus, s is not (r,k)-irreducible. Now consider the converse. Assume that s is not (r,k)-irreducible. If $s \geq Q'(r,k)$ then we are done by Lemma 2.7, since $n \geq N'(r,k)$. If $s < Q'(r,k)$ then let G be a graph with $s\binom{r}{2}$ edges with a k orthogonal K_r-decomposition. Since we can assume G has no isolated vertices, we clearly have that G has less than $sr^2 < Q'(r,k)r^2 < N(r,k) < n$ vertices. Thus, G is a subgraph of K_n. Let G^* be obtained from K_n by deleting G. G^* has $q\binom{r}{2}$ edges, Furthermore,

$$\delta(G^*) \geq n - 1 - \delta(G) \geq n - s\binom{r}{2} \geq n - Q'(r,k)r^2 \geq n - N(r,k)\gamma(K_r) > n(1 - \gamma(K_r)).$$

Also, $r-1$ divides $gcd(G^*)$, since the degrees in both K_n and G are multiples of $r-1$. It follows from Lemma 2.1 that G^* has a K_r-decomposition, with q elements. Thus, extending each of the k decompositions of G with the decomposition of G^* we obtain a set of k K_r-decompositions of K_n sharing q copies, where any other two distinct copies in any two of the decompositions share at most one edge. □

3 Concluding remarks and open problems

1. By modifying the definition of (r,k)-irreducibility, saying that a number s is (r,k)-irreducible if any graph with $s\binom{r}{2}$ edges does not have k distinct K_r-decompositions with no repeated blocks (instead of demanding that the k decompositions be orthogonal, as in the original definition) we immediately obtain a weaker version of Theorem 1.2. Namely, we can drop the requirement that any two distinct copies share at most one edge (Thus, there are q copies shared by all the k decompositions, and the other copies in all the decompositions are distinct). Note that the proof remains completely intact. Naturally, $M(r,k)$ will be changed to reflect the set of (r,k)-irreducible numbers according to the revised definition. Note that the set $M(r) = M(r,2)$ referenced in Theorem 1.1 corresponds to this revised definition.

2. The set $M(r,k)$ appearing in the statement of Theorem 1.1 is, in fact, the set of (r,k)-irreducible numbers. Since the largest element in $M(r,k)$ is constantly bounded as a function of r and k, we obtain that Theorem 1.2, stated as an existence problem, is solvable in polynomial time. Namely, given n and q, determining whether K_n has k decompositions sharing q copies of K_r and any two distinct copies of K_r in any two of the decompositions sharing at most one edge, can be done in polynomial (in n) time. (Note that if $n \le N(r,k)$ we can use brute force to answer the question, since everything is bounded).

3. In view of Theorem 1.1 and Theorem 1.2 it is interesting to determine exactly the sets $M(r,k)$ (in both the orthogonal or non-orthogonal versions). It is known that $M(3,2) = \{1,2,3,5\}$ (note that for $r=3$, the orthogonal and non-orthogonal versions of $M(r,k)$ coincide. This is no longer true for $r=4$ since two K_4's may be distinct but still share a triangle, and thus, more than one edge). It is thus an intriguing open problem to determine $M(r,k)$ for all r and k.

4. Extensions of Theorem 1.2 are possible in two ways. The decomposing graph does not have to be complete. Namely, we may use a fixed graph H instead of K_r (However, the decomposed graph still needs to be K_n). Another generalization is the packing version of Theorem 1.2.

Namely, if n is not of the form $1, r \bmod r(r-1)$ we still have an optimal K_r-packing [9] (provided that n is sufficiently large), and thus we may extend the theorem to require k optimal packings sharing q copies instead of k decompositions sharing q copies. This extension is due to the fact that Lemma 2.2 is also valid in a packing version [10].

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