The Characterization of Zero-Sum (mod 2) Bipartite Ramsey Numbers

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Abstract

Let G be a bipartite graph, with $k \mid e(G)$. The *zero-sum bipartite Ramsey number* $B(G, Z_k)$ is the smallest integer t such that in every Z_k-coloring of the edges of $K_{t,t}$, there is a zero-sum mod k copy of G in $K_{t,t}$. In this paper we give the first proof which determines $B(G, Z_2)$ for all possible bipartite graphs G. In fact, we prove a much more general result from which $B(G, Z_2)$ can be deduced: Let G be a (not necessarily connected) bipartite graph, which can be embedded in $K_{n,n}$, and let F be a field. A function $f : E(K_{n,n}) \to F$ is called *G-stable* if every copy of G in $K_{n,n}$ has the same weight (the weight of a copy is the sum of the values of f on its edges). The set of all G-stable functions, denoted by $U(G, K_{n,n}, F)$ is a linear space which is called the $K_{n,n}$ *uniformity space* of G over F. We determine $U(G, K_{n,n}, F)$ and its dimension, for all G, n and F. Utilizing this result in the case $F = Z_2$, we can compute $B(G, Z_2)$, for all bipartite graphs G.

1 Introduction

All graphs and hypergraphs considered here are finite, undirected and have no loops or multiple edges. For the standard graph-theoretic notations the reader is referred to [5]. Let Z_k denote the cyclic additive group of order k. A *Z_k-coloring* of the edges of a graph $G = (V, E)$ is a function $f : E(G) \to Z_k$. If $\sum_{e \in E(G)} f(e) = 0$ in Z_k, we say that G is a *zero-sum* graph mod k with respect to f. The concepts of zero-sum Ramsey numbers and bipartite zero-sum Ramsey numbers were first introduced by Bialostocki and Dierker in [3] and [2], and by Caro in [7]. If $k \mid e(G)$ then the *zero-sum Ramsey number* $R(G, Z_k)$ is the smallest integer t such that in every Z_k-coloring of K_t there exists a zero-sum mod k copy of G in K_t. If $k \mid e(G)$ and G is bipartite then the *zero-sum bipartite Ramsey number* $B(G, Z_k)$ is the smallest integer t such that in every

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Z_k-coloring of $K_{t,t}$ there exists a zero-sum mod k copy of G in $K_{t,t}$. The existence of $R(G, Z_k)$ and $B(G, Z_k)$ follows from the trivial inequalities $R(G, Z_k) \leq R(G, k)$ and $B(G, Z_k) \leq B(G, k)$ where $R(G, k)$ and $B(G, k)$ are, respectively, the classical Ramsey number and Bipartite Ramsey number using k colors. Good approximations of $R(K_n, Z_k)$ have been derived in [1]. In [6], the first author has determined $R(G, Z_2)$ for all possible graphs G. The exact values of $R(G, Z_k)$ constitute an open problem for all $k \geq 3$. In this paper we determine $B(G, Z_2)$ for all possible graphs G. Upper and lower bounds for $B(G, Z_2)$ which differ by at most one were given in [8]. Hence, this paper closes the gap. The exact result is given in Theorem 1.1. For a bipartite graph G define $m(G) = \min\{|A|, V(G) = A \cup B, |A| \geq |B|\}$ where the minimum is taken over all the representations of G as a bipartite graph with classes A and B (e.g., $m(K_{1,n}) = n$, $m(K_{2,3} \cup K_{4,7}) = 9$). Clearly, $B(G, Z_k) \geq m(G)$. A graph is called a $(0, 1)$-graph if in every representation of G as a bipartite graph with classes A and B, where $|A| = m(G)$, all the vertices of A have odd degree, and all the vertices of B have even degree (e.g $K_{3,4}$ and $K_{1,4} \cup K_{2,3}$ are $(0, 1)$-graphs).

Theorem 1.1 *Let $G = (V, E)$ be a bipartite graph with an even number of edges, and with no isolated vertices. Then:*

1. *If $G = K_{n,n}$ then $B(G, Z_2) = m(G) + 1$.*
2. *If $G = K_{a,n}$ where $a < n$ then $B(G, Z_2) = m(G) + 1$ if a is odd, and $B(G, Z_2) = m(G)$ if a is even.*
3. *If $G = K_{a,b} \cup K_{n-a,n-b}$ then $B(G, Z_2) = m(G) + 1$ if n is even and at least one of a or b is odd. Otherwise, $B(G, Z_2) = m(G)$.*
4. *If G is none of the above, and all the degrees of G are even then $B(G, Z_2) = m(G)$.*
5. *If G is none of the above, and all the degrees of G are odd, then if $|V| = 2m(G)$ then $B(G, Z_2) = m(G) + 1$, and if $|V| < 2m(G)$ then $B(G, Z_2) = m(G)$.*
6. *If G is none of the above, and G is a $(0, 1)$-graph, then $B(G, Z_2) = m(G) + 1$.*
7. *If G is none of the above, then $B(G, Z_2) = m(G)$.*

Furthermore, given $f : E(K_{n,n}) \to Z_2$ where $n = B(G, Z_2)$, one can find a zero-sum copy of G in $K_{n,n}$ in $O(n^4)$ time.

The proof of Theorem 1.1 is, in fact, an application of a special case of a much more general result which we now describe.

In [9] the authors define the concept of uniformity space of graph-theoretic problems. One such problem they consider is the following: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs where G_1 is a subgraph of G_2. A function $f : E_2 \to F$ is called G_1-*stable* if all the copies of G_1 in G_2 have the same weight (the weight of a copy is the sum of the values of f on the edges of the copy). Let $U(G_1, G_2, F)$ be the set of all G_1-stable functions. Clearly, $U(G_1, G_2, F)$ is a linear vector space. We call it the *uniformity space* of G_1 in G_2, over F. Let $udim(G_1, G_2, F)$ be the dimension of $U(G_1, G_2, F)$. Clearly, $udim(G_1, G_2, F) \leq |E_2|$. In Theorem 1.1 of [9], a basis for $U(G, K_n, F)$ is determined for all graphs G with at most n vertices, and for all fields, although it is shown there that computing $udim(G_1, G_2, F)$ is, in general, NP-Complete. It is also shown that determining whether a given $f : E(K_n) \to F$ is G-stable, can be done in $O(n^4)$ time, and if it is not stable, two copies of G in K_n with different weights can be produced in $O(n^4)$ time. By considering the case $F = Z_2$, and utilizing Theorem 1.1 of [9], it is shown how to obtain an alternative proof for the Theorem in [6], which determines $R(G, Z_2)$. The advantage of the alternative proof over the original one is due to the fact that it is *algorithmic*. In analogy to the proof in [9], in this paper we determine $U(G, K_{n,n}, F)$ for all bipartite graphs G which are subgraphs of $K_{n,n}$ (i.e., with $m(G) \leq n$). Before stating this result, we need to state a few definitions and facts. If G is a subgraph of $K_{n,n}$, then by adding to G isolated vertices we can obtain a graph G' with exactly $2n$ vertices, which is a *spanning subgraph* of $K_{n,n}$. Trivially, $U(G, K_{n,n}, F) = U(G', K_{n,n}, F)$, and therefore we may always assume that G is a spanning subgraph of $K_{n,n}$. Let p denote the characteristic of F. A graph G is called a-*regular* mod p if the degree of every vertex, mod p, is a. In case $p = 0$, this means that G is a regular, in the usual sense. A bipartite graph $G = (V, E)$ is called (a, b)-*regular* mod p if in every partition of V into two n-vertex classes, the degree of each vertex in one vertex class, mod p, is a, and the degree of each vertex in the other vertex class, mod p, is b. Clearly, if G is a-regular mod p then it is also (a, a)-regular mod p. If $p = 0$, there does not exist an (a, b)-regular graph, unless $a = b$. For example, $G = K_{1,2} \cup K_{3,2}$ is $(1, 0)$-regular mod 2. On the other hand, $G = 2K_{1,2} \cup 2K_{2,3}$ is not (a, b)-regular mod 2 for any a and b. Recall that E_m denotes the empty graph on m vertices.

Theorem 1.2 *Let $G = (V, E)$ be a nonempty spanning subgraph of $K_{n,n}$, where $n \geq 2$, and let F be a field of characteristic p. Then:*

1. *If $n \geq b \geq a \geq 1$ and $G = K_{a,b} \cup E_{2n-a-b}$ then:*
   (a) *If $b = n$ and $a = n$ then $udim(G, K_{n,n}, F) = n^2$.*
   (b) *If $b = n$ and at least one of a or b is not* $0 \bmod p$ *(or if $p = 0$) then $udim(G, K_{n,n}, F) = n^2 - 2n + 2$.*

(c) If $b = n$ and $a = 0 \bmod p$ and $b = 0 \bmod p$ then udim$(G, K_{n,n}, F) = n^2 - 2n + 3$.

(d) If $b < n$ and $a = 0 \bmod p$ and $b = 0 \bmod p$ then udim$(G, K_{n,n}, F) = 2n - 1$.

(e) If $b < n$ and at least one of a or b is not $0 \bmod p$ (or $p = 0$) then udim$(G, K_{n,n}, F) = 1$.

2. If $p = 2$ and $G = K_{a,b} \cup K_{n-a,n-b}$ where $1 \leq a \leq b < n$ then:

(a) If n is odd then udim$(G, K_{n,n}, F) = n^2 - 2n + 2$.

(b) If n is even and $a = b \bmod 2$ then udim$(G, K_{n,n}, F) = n^2 - 2n + 3$.

(c) If n is even and $a \neq b \bmod 2$ then udim$(G, K_{n,n}, F) = n^2 - 2n + 2$.

3. Otherwise:

(a) If G is a-regular $\bmod p$ then udim$(G, K_{n,n}, F) = 2n - 1$.

(b) If G is (a, b)-regular $\bmod p$, where $a \neq b \bmod p$, then udim$(G, K_{n,n}, F) = 2n - 2$.

(c) If G is non of the above then udim$(G, K_{n,n}, F) = 1$.

In all cases, a basis of $U(G, K_{n,n}, F)$ can be computed in $O(n^4)$ time. Furthermore, given $f : E(K_{n,n}) \to F$, one can decide in $O(n^4)$ time if f is G-stable, and if it is not, two copies of G in $K_{n,n}$ having different weights can be produced.

Note that Theorem 1.2 covers all nonempty spanning subgraphs of $K_{n,n}$, where $n \geq 2$ (the case $n = 1$ is trivial). The reader should be aware that although there are some similarities between the proof characterizing $U(G, K_n, F)$ in [9] and the proof of Theorem 1.2 characterizing $U(G, K_{n,n}, F)$, there are also many differences. We therefore provide a complete, self-contained proof of Theorem 1.2 in Section 2, which is independent of the proof in [9]. In section 3 we show how to utilize Theorem 1.2, in the case where $F = Z_2$, in order to deduce Theorem 1.1.

2 The uniformity space of bipartite graphs

In this section we prove Theorem 1.2. We shall assume that $G = (V, E)$ is a nonempty spanning subgraph of $K_{n,n}$, $n \geq 2$. The graph G may contain isolated vertices. Isolated vertices are called *trivial* connected components. The degree of a vertex $v \in G$ is denoted by $d(v)$. A field is denoted by F, and p denotes the characteristic of F. It will be convenient to denote the vertices of $K_{n,n}$ by the numbers $1, \ldots, 2n$, where the first n numbers constitute the *left* vertex class, and the remaining numbers constitute the *right* vertex class. Using this convention, we may identify a copy of G in $K_{n,n}$ with a one-to-one mapping $g : V(G) \to \{1, \ldots, 2n\}$, which defines the obvious isomorphism between G and its copy in $K_{n,n}$. We denote by $g^{-1}(i)$ the vertex of G which maps by g to i. For a weight function $f : E(K_{n,n}) \to F$, and for a copy g of G in $K_{n,n}$, let $w(f,g)$ be the sum of the values of f on the edges of the copy g (the summation is performed in the field F). Thus, if f is G-stable, $w(f,g_1) = w(f,g_2)$ for any two copies g_1 and g_2.

In our proofs we shall make use of several explicit functions which we now designate:

- We denote by $\overline{f} : E(K_{n,n}) \to F$ the all-one function. Note that $\overline{f}$ is always G-stable.
- We denote by $f^* : E(K_{n,n}) \to F$ the function which assigns 1 to the edges $(i, n+i)$ for $i = 1,\dots,n$, and 0 to all other edges.
- For $i,j = 1,\dots,n$ we define the functions $f_{i,j} : E(K_{n,n}) \to F$ as follows: $f_{i,j}(e) = 1$ if e is incident with i or with $n+j$, but not both. If $e = (i, n+j)$ then $f_{i,j}(e) = 2$. (The constant 2 means $1+1$ in the field F. In particular $2 = 0$ if $p = 2$). Otherwise, $f_{i,j}(e) = 0$.
- For $i = 1,\dots,2n$ we define the functions $f_i : E(K_{n,n}) \to F$ as follows: f_i assigns 1 to all the edges of $K_{n,n}$ adjacent to i, and 0 otherwise.
- For $i = 2,\dots,n$ and $j = n+2,\dots,2n$ we define the functions $f_{1,n+1,i,j} : E(K_{n,n}) \to F$ as follows: $f_{1,n+1,i,j}$ assigns 1 to the edges of the cycle $(1, n+1, i, j)$, and 0 to all other edges.

Lemma 2.1 *If $p \neq 0$ and G is (a,b)-regular* mod p*, then udim$(G, K_{n,n}, F) \geq 2n-2$. Furthermore, a set Q_1 of $2n-2$ linearly-independent G-stable functions can be constructed in $O(n^3)$ time.*

Proof: Consider the set $Q_1 = \{f_{1,2}, f_{2,2}, f_{2,3}, f_{3,3}, \dots, f_{n-1,n}, f_{n,n}\}$. This set contains exactly $2n-2$ distinct functions. Clearly, each member of Q_1 can be constructed in $O(|E(K_{n,n})|) = O(n^2)$ time, and Q_1 is therefore constructed in $O(n^3)$ time. Note that each member of Q_1 is G-stable since for every copy g, $w(f_{i,j}, g) = a + b \bmod p$. It remains to show that Q_1 is a linearly independent set. Indeed, assume that

$$c_{1,2}f_{1,2} + c_{2,2}f_{2,2} + \ldots + c_{n-1,n}f_{n-1,n} + c_{n,n}f_{n,n} = 0.$$

Consider the edge $(1, n+1)$. It is assigned 0 in every member of Q_1 except $f_{1,2}$. Thus, $c_{1,2} = 0$. Now consider the edge $(1, n+2)$. It is assigned 0 in every member of Q_1 except $f_{1,2}$ and $f_{2,2}$. Since $c_{1,2}$ is already 0, this means $c_{2,2} = 0$. Now by considering $(2,2)$, we obtain in the same manner, that $c_{2,3} = 0$. Continuing in the same way we obtain that all the coefficients are 0. □

In case G is a-regular mod p, we can obtain a result which is slightly sharper than Lemma 2.1.

Lemma 2.2 *If G is a-regular* mod p*, then udim$(G, K_{n,n}, F) \geq 2n-1$. Furthermore, a set Q_2 of $2n-1$ linearly-independent G-stable functions can be constructed in $O(n^3)$ time.*

Proof: Consider the set $Q_2 = \{f_i \mid i = 1, \ldots, 2n-1\}$ which contains $2n-1$ distinct functions. Each member of Q_2 can be constructed in $O(|E(K_{n,n})|) = O(n^2)$ time, and Q_2 is therefore constructed in $O(n^3)$ time. Each member of Q_2 is G-stable since for every copy g, $w(f_i, g) = a \bmod p$. It remains to show that Q_2 is a linearly independent set. Indeed, assume that

$$c_1 f_1 + \ldots c_{2n-1} f_{2n-1} = 0.$$

Consider the edge $(i, 2n)$, where $1 \leq i \leq n$. It is assigned 0 in every member of Q except f_i. Thus, $c_i = 0$ for $i = 1, \ldots, n$. Now consider the edge (n, i) where $n+1 \leq i \leq 2n-1$. It is assigned 0 in every member of Q except f_n and f_i. Thus $c_n + c_i = 0$. Since we already know that $c_n = 0$, this means $c_i = 0$ for $i = n+1 \ldots, 2n-1$. □

Lemma 2.3 *Let $f : E(K_{n,n}) \to F$. Assume that at least one of the following two conditions holds:*

1. *G has a connected component which is not complete bipartite.*
2. *$p \neq 2$ and G has at least two nontrivial connected components.*

If f is G-stable then for every four vertices a, b, c, d of $K_{n,n}$, with a, c in the left vertex class and b, d in the right vertex class,

$$f(a,b) + f(c,d) = f(b,c) + f(d,a) \tag{1}$$

holds. If f is not G-stable, and one is given four vertices a, b, c, d which violate (1), then two copies of G in $K_{n,n}$ with different weights can be produced in $O(n^2)$ time.

Proof: Consider any vertex partition of G into two vertex classes, A and B, with $|A| = |B| = n$. If G has a connected component which is not complete bipartite then G must have an *induced* path on four vertices, (y, w, x, z). Otherwise, $p \neq 2$ and G has two nontrivial connected components where (y, w) and (x, z) are two edges in different connected components. In both cases, (y, z) is not an edge of G. In the first case, (w, x) is an edge, and in the second case, it is not. Assume, w.l.o.g. that $x, y \in A$ and $z, w \in B$. Put $N(y) = \{y_1, \ldots, y_r\}$ where $w = y_1$. Put $N(x) \setminus N(y) = \{x_1, \ldots, x_s\}$ where $x_1 = z$. We may assume that $\{y_1, \ldots, y_t\}$ are also neighbors of x for some $0 \leq t \leq r$. Note that $t \geq 1$ in case $p = 2$, since in this case $y_1 = w$ is a neighbor of x. Fix any four vertices a, b, c, d of $K_{n,n}$, with $1 \leq a < c \leq n$ and $n+1 \leq b < d \leq 2n$. Consider a copy g_1 of G in $K_{n,n}$ for which $g_1(x) = a$, $g_1(y) = c$, $g_1(z) = b$, $g_1(w) = d$. g_1 maps the $n-2$ remaining vertices of A to the remaining $n-2$ numbers of the left vertex class of $K_{n,n}$ in some arbitrary way, and the remaining $n-2$ vertices of B to the remaining $n-2$ numbers of the right vertex class of $K_{n,n}$ in some arbitrary way. Now consider a copy g_2 of G which coincides with g_1 on all vertices except x and y, which are permuted with respect to g_1. Thus, $g_2(x) = c$ and $g_2(y) = a$. If f is G-stable we must have

$$0 = w(f, g_1) - w(f, g_2) = (\sum_{i=1}^{s} f(a, g_1(x_i)) + \sum_{i=t+1}^{r} f(c, g_1(y_i))) - (\sum_{i=1}^{s} f(c, g_1(x_i)) + \sum_{i=t+1}^{r} f(a, g_1(y_i))). \tag{2}$$

We now define two additional copies, g_3 and g_4, of G in $K_{n,n}$. g_3 coincides with g_1 on all vertices except w and z, which are permuted. Thus, $g_3(z) = d$ and $g_3(w) = b$. g_4 coincides with g_3 on all vertices except x and y, which are permuted. Thus $g_4(x) = c$ and $g_4(y) = a$. Once again, if f is G-stable,

$$0 = w(f, g_3) - w(f, g_4) = (\sum_{i=1}^{s} f(a, g_3(x_i)) + \sum_{i=t+1}^{r} f(c, g_3(y_i))) - (\sum_{i=1}^{s} f(c, g_3(x_i)) + \sum_{i=t+1}^{r} f(a, g_3(y_i))). \tag{3}$$

We now subtract (3) from (2). In case $t = 0$, the subtraction gives

$$0 = (w(f, g_1) - w(f, g_2)) - (w(f, g_3) - w(f, g_4)) = f(a, b) - f(c, b) - f(a, d) + f(c, d)$$

which implies $f(a, b) + f(c, d) = f(b, c) + f(d, a)$, as required. In case $t = 0$, which can only happen if $p \neq 2$, the subtraction gives

$$0 = (w(f, g_1) - w(f, g_2)) - (w(f, g_3) - w(f, g_4)) =$$

$$f(a, b) + f(c, d) - f(c, b) - f(a, d) - f(a, d) - f(c, b) + f(c, d) + f(a, b)$$

which implies $f(a, b) + f(c, d) = f(b, c) + f(d, a)$, as required.
If f is not G-stable, and we are given four vertices a, b, c, d which violate (1), then we can create the two copy pairs (g_1, g_2) and (g_3, g_4) as before, and compute their weights, in $O(n^2)$ time. By the above equalities, we must have that either $w(f, g_1) \neq w(f, g_2)$ or $w(f, g_3) \neq w(f, g_4)$. $\square$

Let S denote the set of edges of $K_{n,n}$ which are adjacent to vertex 1 or vertex $n + 1$. That is,

$$S = \{(1, n + 1), \ldots, (1, 2n), (2, n + 1), \ldots, (n, n + 1)\}$$

Note that S contains $2n - 1$ edges, but has no cycle. Also note that if $e \in E(K_{n,n})$ and $e \notin S$, then $S \cup \{e\}$ contains a four-cycle. The following is an immediate corollary of Lemma 2.3.

Corollary 2.4 *Let $f : E(K_n) \to F$ be such that for any four vertices a, b, c, d of $K_{n,n}$ with a, c in the left vertex class, and b, d in the right vertex class, $f(a, b) + f(c, d) = f(b, c) + f(d, a)$ holds. Then f is determined by its values on S. In particular, if $1 \leq i \leq n$ and $n + 1 \leq j \leq 2n$ then $f(i, j) = f(1, j) + f(i, n + 1) - f(1, n + 1)$.* $\square$

Now, corollary 2.4, Lemma 2.3, and the fact that S has $2n-1$ elements imply the following corollary:

Corollary 2.5 *If at least one of the two conditions in Lemma 2.3 holds, then* $udim(G, K_{n,n}, F) \leq 2n-1$.

It is interesting to note the connection between C_4-saturated graphs and the set S. A graph G is called H-saturated if it does not contain a copy of H, and if we add any edge to G, we obtain a copy of H. In [4], Bollobás has addressed the question of minimal-saturated bipartite graphs. Let $sat(m,n,s,t)$ denote the minimal number of edges in a bipartitte graph $G \subset K_{m,n}$ which is $K_{s,t}$-saturated. Note that due to the fact that S has $2n-1$ edges, Corollary 2.5 and Lemma 2.2 we have that $sat(n,n,2,2) = 2n-1$. This coincides with the general value for $sat(n,m,s,t) = m(t-1)+n(s-1)-(s-1)(t-1)$ proved in [4]. See also [11] for further results on saturated graphs.

Lemma 2.6 *Assume that at least one of the two conditions in Lemma 2.3 holds, and that G is* not *(a,b)-regular* mod *p, for any a and b, and that if $p=0$ then G is not regular. Let $f : E(K_{n,n}) \to F$. Then, f is G-stable iff f is constant. If f is not constant, one can find two copies of G in $K_{n,n}$, with different weights, in $O(n^4)$ time.*

Proof: Clearly, a constant function is always G-stable. Assume, therefore, that f is G-stable. According to Lemma 2.3 and corollary 2.4 we know that f is determined by its values on the set S. Furthermore, according to corollary 2.4 it suffices to show that f is constant on S. Since G is not (a,b)-regular mod p, there exists a vertex partition of G into two vertex classes A and B with $|A| = |B| = n$, and two vertices $x \in A$ and $y \in A$ with $d(x) \neq d(y) \bmod p$. Put $N(x) \setminus N(y) = \{x_1, \ldots, x_s\}$, and $N(y) \setminus N(x) = \{y_1, \ldots, y_r\}$. Hence, $r \neq s \bmod p$ (in case $p=0$ this simply means that $r \neq s$). Consider two copies of G in $K_{n,n}$, that differ only in their values on x and y. One of the copies, say g_1, has $g_1(x) = 1$ and $g_1(y) = i$ for some $2 \leq i \leq n$, while the other copy, g_2, has $g_2(x) = i$ and $g_2(y) = 1$. For any other vertex z, we have $g_1(z) = g_2(z)$. Since f is stable it follows that

$$0 = w(f,g_1) - w(f,g_2) = \left(\sum_{j=1}^{s} f(1, g_1(x_j)) + \sum_{j=1}^{r} f(i, g_1(y_j))\right) - \left(\sum_{j=1}^{s} f(i, g_1(x_j)) + \sum_{j=1}^{r} f(1, g_1(y_j))\right). \quad (4)$$

According to Corollary 2.4, we know that $f(i, g_1(x_j)) = f(1, g_1(x_j)) + f(i, n+1) - f(1, n+1)$, and $f(i, g_1(y_j)) = f(1, g_1(y_j)) + f(i, n+1) - f(1, n+1)$. Placing these two equalities in (4) we get:

$$(s-r)(f(1,n+1) - f(i,n+1)) = 0.$$

This implies that $f(i,n+1) = f(1,n+1)$ for $i = 2,\dots,n$. In a similar way we can define two copies g_3 and g_4 where $g_3(x) = n+1$ and $g_3(y) = i$ for some $n+2 \le i \le 2n$, $g_4(x) = i$ and $g_4(y) = n+1$, and $g_3(z) = g_4(z)$ for any other vertex z. A similar equality shows that $(s-r)(f(1,n+1) - f(1,i)) = 0$, which implies $f(1,i) = f(1,n+1)$ for $i = n+2\dots,2n$. Thus, f is constant on S.
Now, if f is not constant, then f is not G-stable. If there are four vertices a,b,c,d in $K_{n,n}$ which violate (1) (the existence of four such vertices can be determined in $O(n^4)$ time by considering all possible subsets of four vertices), then one can generate two copies with different weights according to Lemma 2.3. Otherwise, we know by Corollary 2.4 that f cannot be constant on S. We may assume w.l.o.g. that $f(1,n+1) \neq f(i,n+1)$ (otherwise we may rename the vertices of $K_{n,n}$ such that this holds). Hence, according to the first part of the proof of our lemma, we must have that the copies g_1 and g_2 have different weights. These copies are easily created in $O(n^2)$ time. $\Box$

Lemma 2.7 *If G has only one non-trivial connected component, and this component is $K_{a,n}$, where $1 \le a \le n$, then the following holds:*

1. *If $a = n$ then $udim(G, K_{n,n}, F) = n^2$.*
2. *If $a \neq 0 \bmod p$ or $n \neq 0 \bmod p$ then $udim(G, K_{n,n}, F) = n^2 - 2n + 2$.*
3. *If $a = 0 \bmod p$ and $n = 0 \bmod p$ then $udim(G, K_{n,n}, F) = n^2 - 2n + 3$.*

In all cases, a basis Q_3 for $U(G, K_{n,n}, F)$ can be computed in $O(n^4)$ time. Furthermore, given $f : E(K_{n,n}) \to F$, one can decide in $O(n^2)$ time if f is G-stable, and if not, produce two copies with different weights.

Proof: If $a = n$ then $G = K_{n,n}$, and every function is G-stable. Thus, $udim(G, K_{n,n}, F) = n^2$, and the standard basis is a basis for $U(G, K_{n,n}, F)$ in this case. We may now assume that $a < n$. We first show that if i and j belong to the same vertex class of $K_{n,n}$, and $f : E(K_{n,n}) \to F$ is G-stable, then the sum of weights of the edges adjacent to i is equal to the sum of weights of the edges adjacent to j. Indeed, assume w.l.o.g. that $1 \le i < j \le n$. Let g_1 be a copy of G which assigns the a vertices of G with degree n, to a set $A \subset \{1,\dots,n\}$ of vertices of $K_{n,n}$ where $i \in A$ but $j \notin A$. Let g_2 be identical to g_1 except that $g_2(g_1^{-1}(i)) = j$, and $g_2(g_1^{-1}(j)) = i$. Now,

$$0 = w(f,g_1) - w(f,g_2) = \sum_{k=n+1}^{2n} f(i,k) - \sum_{k=n+1}^{2n} f(j,k).$$

We have thus shown that if f is G-stable, then there exist $w_1 = w_1(f), w_2 = w_2(f)$ such that for each $i = 1,\dots,n$, $s_i = \sum_{k=n+1}^{2n} f(i,k) = w_1$, and for each $i = n+1,\dots,2n$, $s_i = \sum_{k=1}^{n} f(k,i) = w_2$.

Consider first the G-stable functions for which $w_1 = w_2 = 0$. These functions form a subspace U' of $U(G, K_{n,n}, F)$, and satisfy the $2n$ equations $s_i = 0$ for $i = 1, \ldots, 2n$. Note, however, that the set of $2n-1$ equations $s_i = 0$ for $i = 1, \ldots, 2n-1$ are *linearly independent* (the proof of independence is identical to the one in Lemma 2.2), and that s_{2n} depends on $\{s_1, \ldots, s_{2n-1}\}$ since $s_1 + \ldots + s_n = s_{n+1} + \ldots s_{2n}$. Thus, $dim(U') = n^2 - (2n-1) = n^2 - 2n + 1$. A basis Q' of U' can be computed by solving a set of $2n$ linear equations in n^2 variables, which can be done in $O(n^4)$ time by Gaussian elimination. Now consider the function f^* which assigns 1 to the edges $(i, n+i)$ for $i = 1, \ldots, n$. Note that f^* is G-stable as $w(f^*, g) = a \bmod p$ for every copy g of G in $K_{n,n}$. However, f^* clearly does not belong to U', since for f^* we have $w_1(f^*) = w_2(f^*) = 1$. Thus, $udim(G, K_{n,n}, F) \geq n^2 - 2n + 2$ and $Q'' = Q' \cup \{f^*\}$ is linearly independent in $U(G, K_{n,n}, F)$.
If f is any G-stable function, then by summing the values of f on all edges in two ways (once from the A side and once from the side of B), we obtain $nw_1(f) = nw_2(f)$. Also note that if g is a copy of G which assigns the a vertices of G with degree n to vertices of the left vertex class of $K_{n,n}$ then $w(f, g) = aw_1$. Similarly, if g' assigns the a vertices of G with degree n to vertices of the right vertex class of $K_{n,n}$ then $w(f, g') = aw_2$. Thus, $aw_1 = aw_2$. It follows that if $a \neq 0 \bmod p$ or $n \neq 0 \bmod p$ then $w_1 = w_2$. Furthermore, by putting $f' = f - w_1 f^*$ we have that $f' \in U'$, and therefore f is a linear combination of Q''. Thus, $udim(G, K_{n,n}, F) = n^2 - 2n + 2$, and putting $Q_3 = Q''$, we have that Q_3 is a basis of $U(G, K_{n,n}, F)$. We remain with the case $a = 0 \bmod p$ and $n = 0 \bmod p$. Consider the function f_1 (recall that this functions assigns 1 to the edges adjacent to vertex 1 in $K_{n,n}$). Clearly, f_1 is G-stable as $w(f_1, g) = 0 \bmod p$, for every copy g. Also, $w_1(f_1) = 0$ while $w_2(f_1) = 1$. Thus, f_1 is independent of Q''. Put $Q_3 = Q'' \cup \{f_1\}$. Let f be any G-stable function. We show that f depends on Q_3. Put $f' = f - w_1(f)f^* + (w_1(f) - w_2(f))f_1$. f' is G-stable with $w_1(f') = w_1(f) - w_1(f)w_1(f^*) + (w_1(f) - w_2(f))w_1(f_1) = 0$, and $w_2(f') = w_2(f) - w_1(f)w_2(f^*) + (w_1(f) - w_2(f))w_2(f_1) = 0$. Thus, $f' \in U'$, and therefore f is a linear combination of Q_3. It follows that Q_3 is a basis for $U(G, K_{n,n}, F)$ and $udim(G, K_{n,n}, F) = n^2 - 2n + 3$.
Given a function $f : E(K_{n,n}) \to F$ one can compute, for all $i \in K_{n,n}$, the sum of weights of the edges adjacent to i in $O(n^2)$ time. If there are two vertices, i and j, in the right vertex class, with different sums of adjacent weights, we construct the copies g_1 and g_2 defined in the first part of the proof, in $O(n^2)$ time, and these copies must have different weights. Similar considerations apply to the right vertex class. If all the weights in the left vertex class are w_1, and all the weights in the right vertex class are w_2, and $w_1 = w_2$, then f is G-stable. Otherwise, $w_1 \neq w_2$, and we must have $n = 0 \bmod p$. If $a = 0 \bmod p$ then f is also G-stable. If $a \neq 0 \bmod p$ then two copies g and g' with different weights $w(f, g) = aw_1$ and $w(f, g') = aw_2$, can be produced in $O(n^2)$ time. □

The final lemma of this section determines $U(G, K_{n,n}, F)$ and $udim(G, K_{n,n}, F)$ in case $p = 2$

and G is a union of exactly two non-trivial complete bipartite graphs.

Lemma 2.8 *If $G = K_{a,b} \cup K_{n-a,n-b}$ where $1 \le a \le b < n$, and $p = 2$ then:*

1. *If n is odd then $udim(G, K_{n,n}, F) = n^2 - 2n + 2$.*
2. *If n is even and $a = b \bmod 2$ then $udim(G, K_{n,n}, F) = n^2 - 2n + 3$.*
3. *If n is even and $a \neq b \bmod 2$ then $udim(G, K_{n,n}, F) = n^2 - 2n + 2$.*

In all cases, a basis Q_4 for $U(G, K_{n,n}, F)$ can be computed in $O(n^4)$ time. Furthermore, given $f : E(K_{n,n}) \to F$, one can decide in $O(n^2)$ time if f is G-stable, and if not, produce two copies with different weights.

Proof: Partition G into two vertex classes A and B, where $|A| = |B| = n$. Assume w.l.o.g. that the a vertices of degree b in the $K_{a,b}$ component of G belong to A. This forces the $n-a$ vertices of degree $n-b$ in the $K_{n-a,n-b}$ component of G to belong to A. We first show, as in Lemma 2.7, that if i and j belong to the same vertex class of $K_{n,n}$, and $f : E(K_{n,n}) \to F$ is G-stable, then the sum of weights of the edges adjacent to i is equal to the sum of weights of the edges adjacent to j. Indeed, assume w.l.o.g. that $1 \le i < j \le n$. Let g_1 be a copy of G which assigns the n vertices of A to the vertices $\{1, \ldots, n\}$ of $K_{n,n}$ in some arbitrary way, and assign B to the vertices $\{n+1, \ldots, 2n\}$ of $K_{n,n}$ is some way. Now consider a copy g_2 which is identical to g_1 except that $g_2(g_1^{-1}(i)) = j$, and $g_2(g_1^{-1}(j)) = i$. Since $w(f, g_1) - w(f, g_2) = 0$ and the since negation and addition are the same when $p = 2$, we obtain, as in Lemma 2.7 that $\sum_{k=n+1}^{2n} f(i,k) = \sum_{k=n+1}^{2n} f(j,k)$. Symmetric arguments apply when $n+1 \le i < j \le 2n$. We have thus shown, as in Lemma 2.7, that if f is G-stable, then there exist $w_1 = w_1(f), w_2 = w_2(f)$ such that for each $i = 1, \ldots, n$, $s_i = \sum_{k=n+1}^{2n} f(i,k) = w_1$, and for each $i = n+1, \ldots, 2n$, $s_i = \sum_{k=1}^{n} f(k,i) = w_2$. Let U' be the subspace of the G-stable functions for which $f(w_1) = f(w_2) = 0$. As in the proof in Lemma 2.7, we know that $dim(U') \le n^2 - 2n + 1$ (unlike the previous lemma, we cannot deduce equality since there is no guarantee that functions that satisfy $s_i = 0$ for $i = 1, \ldots, 2n$ are G-stable). In order to show that, indeed, $dim(U') = n^2 - 2n + 1$, we construct a basis of U'. Consider the functions $f_{1,n+1,i,j}$ defined at the beginning of this section, and consider $Q' = \{f_{1,n+1,i,j} \mid 2 \le i \le n, n+2 \le j \le 2n\}$. Note that Q' contains $(n-1)^2 = n^2 - 2n + 1$ members, which belong to U', and which are linearly independent since the edge (i,j) is assigned 1 only in $f_{1,n+1,i,j}$. Now consider the function f^*. Clearly, f^* is G-stable and $w_1(f^*) = w_2(f^*) = 1$. Thus, $udim(G, K_{n,n}, F) \ge n^2 - 2n + 2$ and $Q'' = Q' \cup \{f^*\}$ is linearly independent in $U(G, K_{n,n}, F)$. Assume that n is odd. By counting the sum of weights on all the edges of $K_{n,n}$ in two ways we have $w_1 = w_2$. As in Lemma 2.7, any G-stable function is linearly dependent on Q'' and thus

$Q_4 = Q''$ is a basis for $U(G, K_{n,n}, F)$ and $udim(G, K_{n,n}, F) = n^2 - 2n + 2$. Now consider the case where n is even. If $a = b \bmod 2$ then, clearly, $n-a, n-b, a, b$ all have the same parity and therefore the function f_1 is G-stable and has $w_1(f_1) = 0$ and $w_2(f_1) = 1$. Thus, f_1 is independent of Q'', and putting $Q_4 = Q'' \cup \{f_1\}$ we obtain, as in Lemma 2.7, a basis for $U(G, K_{n,n}, F)$, and $udim(G, K_{n,n}, F) = n^2 - 2n + 3$. The only remaining case is when n is even and $a \neq b \bmod 2$. We show that Q'' is a basis in this case by showing that we must have $w_1 = w_2$. Indeed, assume that g_1 is any copy, and g_2 is obtained from g_1 by putting $g_2(g_1^{-1}(i)) = n + i$ and $g_2(g_1^{-1}(n+i)) = i$ for $i = 1, \ldots, n$. Let x be the sum of weights on the edges that do not appear in neither g_1 nor g_2. Note that these edges induce a complete bipartite graph with $b - a$ vertices in each vertex class. Therefore, $0 = w(f, g_1) - w(f, g_2) = ((b-a)w_1 - x) - ((b-a)w_2 - x)$. Thus, $0 = (b-a)(w_1 - w_2)$, and therefore $w_1 = w_2$. The algorithmic part of the lemma is similar to that of Lemma 2.7, and is left for the reader. □

We are now ready to prove our main result.

Proof of Theorem 1.2: Given a partition of V into vertex classes A and B with $|A| = |B| = n$, we denote by $\overline{G}(V, \overline{E})$ the *bipartite complement* of G. That is,

$$\overline{E} = \{(a, b) \mid a \in A, b \in B, (a, b) \notin E\}.$$

Note that if G is not connected, there may be more than one way to partition V into A and B, and so there may be more than one bipartite complement. However, it is clear that if $\overline{G}$ is connected, then

$$U(G, K_{n,n}, F) \subset U(\overline{G}, K_{n,n}, F) \tag{5}$$

as any G-stable function is also $\overline{G}$-stable. Furthermore, if G has a *unique* bipartite complement, then equality holds in (5). We now step through the cases in Theorem 1.2.

Cases (1a), (1b) and (1c) are covered in Lemma 2.7. Now consider cases (1d) and (1e). In these cases, G has a unique bipartite complement $\overline{G}$, which is connected, and which is not complete bipartite, thus equality holds in (5). Furthermore, $\overline{G}$ has vertices with degree n in both vertex classes. It also has b vertices with degree $n - a$ in one vertex class and a vertices with degree $n - b$ in the other vertex class. If at least one of a or b is not $0 \bmod p$, (if $p = 0$ then this is trivially true), then the conditions in Lemma 2.6 are satisfied for $\overline{G}$ and hence $udim(G, K_{n,n}, F) = 1$, proving (1e). If both a and b are $0 \bmod p$, then $\overline{G}$ is regular mod p, and so we have from Lemma 2.2 and Corollary 2.5 that $udim(G, K_{n,n}, F) = 2n - 1$, proving (1d). Cases (2a), (2b) and (2c) are determined in Lemma 2.8. Consider case 3, and assume first that at least one of the two conditions in Lemma 2.3 is satisfied. Under this assumption, (3a) follows from Lemma 2.2 and Corollary 2.5. If G is (a, b)-regular mod p, and $a \neq b \bmod p$, we have from Lemma 2.1 and Corollary 2.5

that $2n-1 \geq udim(G, K_{n,n}, F) \geq 2n-2$. In order to show that $udim(G, K_{n,n}, F) = 2n-2$ it suffices to show that there exists a weight function on the set S (defined prior to Corollary 2.4) which cannot be extended to a G-stable function (recall that the extension is uniquely defined in Corollary 2.5). Indeed, assume that f assigns 0 to all the members of S except for the edge $(2, n+1)$ which is assigned 1. The extension of f to $E(K_{n,n})$ is exactly the function f_2 (the function which assigns 1 to the edges adjacent to vertex 2, and 0 otherwise). Clearly, f_2 is not G-stable, since if $v \in G$ has degree $a \bmod p$ and g_1 is a copy with $g_1(v) = 2$, then $w(f_2, g_1) = a \bmod p$, and if $u \in G$ has degree $b \bmod p$ and g_2 is a copy with $g_2(u) = 2$, then $w(f_2, g_2) = b \bmod p$. We have thus proved case (3b). Case (3c) follows directly from Lemma 2.6. We now need to show that cases (3a), (3b) and (3c) also apply when the two conditions in Lemma 2.3 do not hold, and cases (1a)-(1e) and (2a)-(2c) do not hold. If $p \neq 2$ this is impossible, since if G has only one nontrivial connected component, this component is not complete-bipartite (since cases (1a)-(1e) deal with this case), and so the first condition in Lemma 2.3 holds. Otherwise, G has at least two nontrivial connected components and the second condition in Lemma 2.3 holds. Thus, we must have $p = 2$, and every non-trivial connected component of G is complete bipartite, and G has at least three connected components, and at least two of them are non-trivial. This implies that every bipartite complement of G is connected, and is not complete bipartite. Fix a bipartite complement $\overline{G}$ of G. If G is not (a, b)-regular mod 2 for no a and b, then $\overline{G}$ is not (a, b)-regular mod 2 for no a and b, and so $1 \leq udim(G, K_{n,n}, F) \leq udim(\overline{G}, K_{n,n}, F) \leq 1$, proving (3c). If G is a-regular mod 2 then, by Lemma 2.2, $udim(G, K_{n,n}, F) \geq 2n-1$, and since $\overline{G}$ is $(n-a)$-regular mod 2, we have $2n-1 = udim(\overline{G}, K_{n,n}, F) \geq udim(G, K_{n,n}, F) \geq 2n-1$, proving (3a). Finally, if G is (a, b)-regular mod 2, where $a \neq b \bmod 2$ then by Lemma 2.1 $udim(G, K_{n,n}, F) \geq 2n-2$, and since $\overline{G}$ is $(n-a, n-b)$-regular mod 2 and $n-a \neq n-b \bmod 2$, we have $2n-2 = udim(\overline{G}, K_{n,n}, F) \geq udim(G, K_{n,n}, F) \geq 2n-2$, proving (3b).

The fact that in all cases a basis for $U(G, K_{n,n}, F)$ can be computed in $O(n^4)$ time follows from the constructions of the bases Q_1, Q_2, Q_3 and Q_4 in Lemmas 2.1, 2.2, 2.7 and 2.8 respectively (in case $udim(G, K_{n,n}, F) = 1$ one can construct the all-one function $\overline{f}$ as a basis, in $O(n^2)$ time). Given $f : E(K_{n,n}) \to F$, deciding whether f is G-stable, and producing two copies with different weights if it is not can be done in $O(n^4)$ time in the various cases, as shown in Lemmas 2.3, 2.6, 2.7 and 2.8.

Finally, given a graph G as an input, we need to determine which of the various cases in the theorem applies to G. This can clearly be done in $O(n^2)$ by examining the degrees of the vertices of G, unless G is not connected, and every connected component is (a, b)-regular mod p, where $a \neq b \bmod p$. In this case we can decide whether G is (a, b)-regular mod p, or not, using the standard dynamic programming algorithm for subset-sum (cf. [10]), in $O(n^3)$ time. $\Box$

3 Zero-sum mod 2 bipartite Ramsey numbers

In this section we show how to use Theorem 1.2 in order to compute the zero-sum mod 2 bipartite Ramsey numbers $B(G, Z_2)$. Namely, we prove Theorem 1.1.

Proof of Theorem 1.1: We use the same notation used in Section 2. Recall first that, trivially, $B(G, Z_2) \geq m(G)$. Note also that if $r < B(G, Z_2)$, then there exists $f : E(K_{r,r}) \to Z_2$ such that for every copy g of G in $K_{r,r}$, $w(f, g) \neq 0$. But in Z_2 this implies that $w(f, g) = 1$, and thus f is G-stable. Since f is not the constant function (as the constant function has weight 0 on every copy of G), it follows that if $udim(G, K_{n,n}, Z_2) = 1$ then $B(G, Z_2) \leq n$. More generally, if Q is a basis for $udim(G, K_{n,n}, Z_2)$ and every member of Q assign 0 to every copy of G, then $B(G, Z_2) \leq n$. We shall make extensive use of these facts.

We now analyze the different cases in Theorem 1.1. We demonstrate the algorithmic part only in the first case. The reader may verify the algorithmic part in the other cases in an analogous way.

1. $G = K_{n,n}$. Note that n is even. Consider $G' = G \cup E_2$. Clearly, $udim(G, K_{n+1,n+1}, Z_2) = udim(G', K_{n+1,n+1}, Z_2)$. Note that case (1d) of Theorem 1.2 applies to G', and so we have that $udim(G', K_{n+1,n+1}, Z_2) = 2(n+1) - 1$, and the proof of this case shows that the set $Q_2 = \{f_i \mid i = 1, \ldots, 2(n+1)-1\}$ of Lemma 2.2 is a basis. Since every member of Q_2 assigns 0 to every copy of G in $K_{n+1,n+1}$ it follows that $B(G, Z_2) \leq n+1$.

   We now show that $B(G, Z_2) > n$, thus proving $B(G, Z_2) = n+1 = m(G)+1$. This follows by considering a function $f : E(K_{n,n}) \to Z_2$ which assigns 1 to a single edge, and 0 to all other edges. Clearly, every copy of G is not zero-sum.

   We now prove the algorithmic part. According to Theorem 1.2, given an assignment $f : E(K_{n+1,n+1}) \to Z_2$ we can find in $O(n^4)$ time whether f is G-stable or not, and if it is not, we can produce two copies with different weights in $O(n^4)$ time. One of these copies has weight 0 (since the other has weight 1). If f is stable, then it is a linear combination of the elements of the basis, each element having weight 0 on each copy. Thus, f also has weight 0 on each copy, so we choose an arbitrary copy.

2. $G = K_{a,n}$, $a < n$. Assume first that a is odd, and thus n is even. Consider $G' = G \cup E_{n-a+2}$. Case (1e) applies to G', so $udim(G, K_{n+1,n+1}, Z_2) = udim(G', K_{n+1,n+1}, Z_2) = 1$. Therefore, $B(G, Z_2) \leq n+1$. The function $f^* : E(K_{n,n}) \to Z_2$ assigns $a = 1 \bmod 2$ to every copy of G in $K_{n,n}$, thus showing $B(G, Z_2) > n$. Consequently, $B(G, Z_2) = n+1 = m(G)+1$. Now assume

that a is even. If n is odd then case (1b) applies to G, so $udim(G, K_{n,n}, Z_2) = n^2 - 2n + 2$. The functions $f_{1,n+1,i,j}$ for $i, j = 2, \ldots, n$ together with the function f^* form a basis of the space. Since each of these functions assigns 0 to every copy of G in $K_{n,n}$, we have $B(G, Z_2) \leq n$. Thus, $B(G, Z_2) = n = m(G)$. If n is even then case (1c) applies for G, so $udim(G, K_{n,n}, F) = n^2 - 2n + 3$. Using the same basis as in the case where n is odd, together with the function f_1, we obtain a basis for the space. Since every member of the basis assigns 0 to every copy of G we have, once again, $B(G, Z_2) = n = m(G)$.

3. $G = K_{a,b} \cup K_{n-a,n-b}$. Clearly, $m(G) = n$. Assume first that n is even and at least one of a or b is odd. Consider $G' = G \cup E_2$. Any partition of G' into two $n + 1$ vertex classes has the property that at least one vertex class has both an odd and an even degree vertex. Thus, case (3c) applies to G' so $udim(G', K_{n+1,n+1}, Z_2) = 1$, showing that $B(G, Z_2) \leq n + 1$. If both a and b are odd, the function f_1 gives an odd weight to every copy of G in $K_{n,n}$. If exactly one of a or b is odd, the function f^* gives an odd weight to every copy of G in $K_{n,n}$. In any case we have $B(G, Z_2) = n + 1$. Assume next that n is even and both a and b are even. Case (2b) applies to G giving $udim(G, K_{n,n}, Z_2) = n^2 - 2n + 3$. The basis of this space is explicitly determined in Lemma 2.8, and every member of the basis gives total weight 0 to every copy of G. Hence, $B(G, Z_2) = n$. Now assume that n is odd. In this case $a + b$ must be odd. Since case (2a) applies to G we have $udim(G, K_{n,n}, Z_2) = n^2 - 2n + 2$. Once again, the basis is determined in Lemma 2.8, and every member gives total weight 0 to every copy of G. Therefore, $B(G, Z_2) = n$.

4. G is none of the graphs above, and all the degrees are even. Adding $2m(G) - |V|$ isolated vertices to G we obtain a 0-regular graph G'. By case (3a), $udim(G, K_{m(G),m(G)}, Z_2) = udim(G', K_{m(G),m(G)}, Z_2) = 2m(G) - 1$. The functions $\{f_i \mid i = 1, \ldots, 2m(G) - 1\}$ form the basis, and each gives total weight 0 to every copy of G. Thus, $B(G, Z_2) = m(G)$.

5. G is none of the graphs above and all the degrees are odd. If $|V| < 2m(G)$ then by adding $2m(G) - |V|$ isolated vertices, we obtain a graph G' to which case (3c) applies, so $udim(G', K_{m(G),m(G)}, Z_2) = 1$. This shows $B(G, Z_2) = m(G)$. If $|V| = 2m(G)$ consider $G' = G \cup E_2$. Once again (3c) applies to G', so $udim(G, K_{m(G)+1,m(G)+1}, Z_2) = 1$. Hence, $B(G, Z_2) \leq m(G) + 1$. The function $f_1 : E(K_{m(G),m(G)}) \to Z_2$ gives total odd weight to every copy of G, showing $B(G, Z_2) > m(G)$. Thus, $B(G, Z_2) = m(G) + 1$.

6. G is none of the graphs above, and G is a $(0, 1)$-graph. Consider $G' = G \cup E_{2m(G)+2-|V|}$. Note that G' has $m(G) + 2$ vertices with even degree, and $m(G)$ vertices with odd degree. Thus, G' is not (a, b)-regular mod 2, so case (3c) applies to G'. This shows that $B(G, Z_2) \leq m(G) + 1$.

The functions $f_{i,j} : E(K_{m(G),m(G)}) \to Z_2$, give total odd weight to every copy of the $(0,1)$-graph G. This shows $B(G, Z_2) = m(G) + 1$.

7. G is none of the graphs above. Considering $G' = G \cup E_{2m(G)-|V|}$, we have that case (3c) applies to G', so $udim(G, K_{m(G),m(G)}, Z_2) = udim(G', K_{m(G),m(G)}, Z_2) = 1$. This shows $B(G, Z_2) = 1$. □

As a final comment, we note that given a bipartite graph $G = (V, E)$ as input, we can compute $m(G)$ in $O(|V|^3)$ time using standard dynamic programming as for the Subset-Sum problem, shown in [10]. After computing $m(G)$, deciding whether G is a $(0,1)$-graph can be done, by similar dynamic programming, in $O(|V|^3)$ time.

References

[1] N. Alon and Y. Caro, *On three zero-sum Ramsey-type problems*, J. Graph Theory 17 (1993), 177-192.

[2] A. Bialostocki, *Zero-Sum trees: a survey of results and open problems* in: "Finite and infinite combinatorics in sets and logic". N.W. Sauer, R.E. Woodrow and B. Sands Eds. Nato ASI series, vol. 411 (1993), 19-29.

[3] A. Bialostocki and P. Dierker, *Zero-sum Ramsey theorems*, Congressus Numerantium 70 (1990), 119-130.

[4] B. Bollobás, *On a conjecture of Erdös, Hajnal and Moon*, Amer. Math. Monthly 74 (1967) 178-179.

[5] J.A. Bondy and U.S. R. Murty, *Graph Theory with Applications*, Macmillan Press, 1976.

[6] Y. Caro, *A complete characterization of the zero-sum (*mod 2*) Ramsey numbers*, Journal of Combinatorial Theory, Series A 68 (1994), 205-211.

[7] Y. Caro, *A Survey on Zero-Sum Problems*, Discrete Mathematics 152 (1996), 93-113.

[8] Y. Caro, *Zero-sum bipartite Ramsey numbers*, Czechoslovak Mathematical Journal 43 (118), (1993), 107-114.

[9] Y. Caro and R. Yuster, *The uniformity space of hypergraphs and its applications*, submitted.

[10] M.R. Garey and D.S. Johnson, *Computers and Intractability, A Guide to the Theory of NP-Completeness*, W. H. Freeman and Company. New York, 1979.

[11] L. Kászonyi and Zs. Tuza, *Saturated graphs with minimal number of edges*, Journal of Graph Theory 10 (1986), 203-210.