# Fractional decompositions of dense hypergraphs


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**Abstract**

Let $H_0$ be a fixed hypergraph. A *fractional $H_0$-decomposition* of a hypergraph $H$ is an assignment of nonnegative real weights to the copies of $H_0$ in $H$ such that for each edge $e \in E(H)$, the sum of the weights of copies of $H_0$ containing $e$ is precisely one. Let $k$ and $r$ be positive integers with $k > r > 2$, and let $K_k^r$ denote the complete $r$-uniform hypergraph with $k$ vertices. We prove that there exists a positive constant $\alpha = \alpha(k,r)$ such that every $r$-uniform hypergraph with $n$ (sufficiently large) vertices in which every $(r-1)$-set is contained in at least $n(1-\alpha)$ edges has a fractional $K_k^r$-decomposition. Using our result together with a recent result of Rödl, Schacht, Siggers and Tokushige, we obtain the following corollary. For every $r$-uniform hypergraph $H_0$, there exists a positive constant $\alpha = \alpha(H_0)$ such that every $r$-uniform hypergraph $H$ in which every $(r-1)$-set is contained in at least $n(1-\alpha)$ edges has an $H_0$-packing that covers $|E(H)|(1-o_n(1))$ edges.


## 1 Introduction

A *hypergraph* $H$ is an ordered pair $H = (V, E)$ where $V$ is a finite set (the *vertex set*) and $E$ is a family of distinct subsets of $V$ (the *edge set*). A hypergraph is *$r$-uniform* if all edges have size $r$. In this paper we only consider $r$-uniform hypergraphs where $r \geq 2$ is fixed. Let $H_0$ be a fixed hypergraph. For a hypergraph $H$, the *$H_0$-packing number*, denoted $\nu_{H_0}(H)$, is the maximum number of pairwise edge-disjoint copies of $H_0$ in $H$. A function $\psi$ from the set of copies of $H_0$ in $H$ to $[0,1]$ is a *fractional $H_0$-packing* of $H$ if $\sum_{e \in H_0} \psi(H_0) \leq 1$ for each $e \in E(H)$. For a fractional $H_0$-packing $\psi$, let $|\psi| = \sum_{H_0 \in \binom{H}{H_0}} \psi(H_0)$. The *fractional $H_0$-packing number*, denoted $\nu^*_{H_0}(H)$, is defined to be the maximum value of $|\psi|$ over all fractional $H_0$-packings $\psi$. Notice that, trivially, $e(H)/e(H_0) \geq \nu^*_{H_0}(H) \geq \nu_{H_0}(H)$. In case $\nu_{H_0}(H) = e(H)/e(H_0)$ we say that $H$ has an *$H_0$-decomposition.* In case $\nu^*_{H_0}(H) = e(H)/e(H_0)$ we say that $H$ has a *fractional $H_0$-decomposition.* It is well known that computing $\nu_{H_0}(H)$ is NP-Hard already when $H_0$ is a 2-uniform hypergraph


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(namely, a graph) with more than two edges in some connected component [4]. It is well known that computing $\nu^*_{H_0}(H)$ is solvable in polynomial time for every fixed hypergraph $H_0$ as this amounts to solving a (polynomial size) linear program.

For fixed integers $k$ and $r$ with $k > r \geq 2$, let $K_k^r$ denote the complete $r$-uniform hypergraph with $k$ vertices. For $n > k$ it is trivial that $K_n^r$ has a fractional $K_k^r$-decomposition. However, it is far from trivial (and unknown for $r > 2$) whether this fractional decomposition can be replaced with an integral one, even when necessary divisibility conditions hold. In the graph-theoretic case this is known to be true (for $n$ sufficiently large), following the seminal result of Wilson [10]. Solving an old conjecture of Erdős and Hanani, Rödl proved in [8] that $K_n^r$ has a packing with $(1-o_n(1))\binom{n}{r}/\binom{k}{r}$ copies of $K_k^r$ (namely, an asymptotically optimal $K_k^r$-packing). In case we replace $K_n^r$ with a dense and large $n$-vertex $r$-uniform hypergraph $H$, it was not even known whether a fractional $K_k^r$-decomposition of $H$ exists, or whether an asymptotically optimal $K_k^r$-packing exists. In this paper we answer both questions affirmatively. We note that the easier graph theoretic case has been considered by the author in [12].

In order to state our density requirements we need a few definitions. Let $H = (V, E)$ be an $r$-uniform hypergraph. For $S \subset V$ with $1 \leq |S| \leq r-1$, let $deg(S)$ be the number of edges of $H$ that contain $S$. For $1 \leq d \leq r-1$ let $\delta_d(H) = \min_{S \subset V, |S|=d} deg(S)$ be the *minimum $d$-degree* of $H$. Usually, $\delta_1(H)$ is also called the *minimum degree* and $\delta_2(H)$ is also called the *minimum co-degree*. The analogous maximum $d$-degree is denoted by $\Delta_d(H)$. For $0 \leq \alpha \leq 1$ we say that $H$ is *$\alpha$-dense* if $\delta_d(H) \geq \alpha\binom{n-d}{r-d}$ for all $1 \leq d \leq r-1$. Notice that $K_n^r$ is 1-dense and that $H$ is $\alpha$-dense if and only if $\delta_{r-1}(H) \geq \alpha(n-r+1)$.

Our first main result is given in the following theorem.

**Theorem 1.1** *Let $k$ and $r$ be integers with $k > r \geq 3$. There exists a positive $\alpha = \alpha(k, r)$ and an integer $N = N(k, r)$ such that if $H$ is a $(1-\alpha)$-dense $r$-uniform hypergraph with more than $N$ vertices then $H$ has a fractional $K_k^r$-decomposition.*

We note that the constant $\alpha = \alpha(k, r)$ that we obtain is *only* exponential in $k$ and $r$. It is not difficult to show that our proof already holds for $\alpha(k, r) = 6^{-kr}$ although we make no effort to optimize the constant. We note the the proof in the graph-theoretic case given in [12] yields $\alpha(k, 2) \leq 1/9k^{10}$. However, the proof in the graph-theoretic case is quite different for the most part and cannot be easily generalized to the hypergraph setting.

Although Theorem 1.1 is stated only for $K_k^r$, it is easy to see that a similar theorem also holds for any $k$-vertex $r$-uniform hypergraph $H_0$. Indeed, if $H_0$ has $k$ vertices then, trivially, $K_k^r$ has a fractional $H_0$-decomposition. Thus, any hypergraph which has a fractional $K_k^r$-decomposition also has a fractional $H_0$-decomposition. We note that in the very special case where $H_0$ is an $r$-uniform simple hypertree then exact decomposition results are known [11].

Our second result is, in fact, a corollary obtained from Theorem 1.1 and a theorem of Rödl, Schacht, Siggers and Tokushige [9] who proved that the $H_0$-packing number and the fractional $H_0$-packing number are very close for dense $r$-uniform hypergraphs (an earlier result of Haxell, Nagle and Rödl [6] asserted this for the case $r = 3$). The exact statement of their result is the following.

**Theorem 1.2** *[Rödl, Schacht, Siggers and Tokushige [9]] For any fixed $r$-uniform hypergraph $H_0$, if $H$ is an $n$-vertex $r$-uniform hypergraph then $\nu^*_{H_0}(H) - \nu_{H_0}(H) = o(n^r)$.* ∎

From Theorem 1.1 and the comments after it, and from Theorem 1.2, we immediately obtain the following.

**Theorem 1.3** *Let $H_0$ be a fixed $r$-uniform hypergraph. There exists a positive constant $\alpha = \alpha(H_0)$ such that if $H = (V, E)$ is a $(1-\alpha)$-dense $r$-uniform hypergraph with $n$ vertices then $H$ has an $H_0$-packing that covers $|E|(1 - o_n(1))$ edges.* ∎

In the next section we prove Theorem 1.1. The final section contains some concluding remarks and open problems.

## 2 Proof of Theorem 1.1

Let $\mathcal{F}$ be a fixed family of $r$-uniform hypergraphs. An *$\mathcal{F}$-decomposition* of an $r$-uniform hypergraph $H$ is a set $L$ of subhypergraphs of $H$, each isomorphic to an element of $\mathcal{F}$, and such that each edge of $H$ appears in precisely one element of $L$. Let $H(t, r)$ denote the complete $r$-uniform hypergraph with $t$ vertices and with one missing edge. For the remainder of this section we shall use $t = k(r+1)$. Let $\mathcal{F}(k, r) = \{K^r_k\ ,\ K^r_t\ ,\ H(t, r)\}$. The proof of Theorem 1.1 is a corollary of the following stronger theorem.

**Theorem 2.1** *For all $k > r \geq 3$ there exists a positive $\alpha = \alpha(k, r)$ and an integer $N = N(k, r)$ such that every $r$-uniform hypergraph with $n > N$ vertices which is $(1-\alpha)$-dense has an $\mathcal{F}(k, r)$-decomposition.*

Clearly $K^r_t$ has a fractional $K^r_k$-decomposition, since $t > k$. Thus, in order to prove that Theorem 1.1 is a corollary of Theorem 2.1 is suffices to prove that $H(t, r)$ has a fractional $K^r_k$-decomposition. This is done in the following two lemmas.

**Lemma 2.2** *Let $A$ be an upper triangular matrix of order $r$ satisfying $A_{j,j} > 0$ and $A_{i,j} \geq 0$ for all $1 \leq i \leq j \leq r$ and $A_{i,j} \geq A_{i-1,j}$ for all $2 \leq i \leq j \leq r$. Let $J$ be the all-one column vector of length $r$. Then, in the unique solution of $Ax = J$ all coordinates of $x$ are nonnegative.*

**Proof:** Clearly $Ax = J$ has a unique solution since $A$ is upper triangular and the diagonal consists of nonzero entries. Let $x^t = (x_1, \ldots, x_r)$ be the unique solution. Clearly, $x_r = 1/A_{r,r} > 0$. Assuming $x_{i+1} \geq 0$ we prove $x_i \geq 0$. Indeed,

$$x_i = \frac{1}{a_{i,i}}(1 - \sum_{j=i+1}^{r} a_{i,j}x_j) \geq \frac{1}{a_{i,i}}(1 - \sum_{j=i+1}^{r} a_{i+1,j}x_j) = 0.$$

■

**Lemma 2.3** *For all $k \geq r \geq 2$, $H(t,r)$ has a fractional $K_k^r$-decomposition.*

**Proof:** Let $A = \{u_1, \ldots, u_r\}$ be the unique set of vertices of $H(t,r)$ for which $A$ is not an edge, and let $B$ denote the set of the remaining $t-r$ vertices. For $i = 0, \ldots, r-1$, we say that an edge of $H(t,r)$ is of *type* $i$ if it intersects $i$ elements of $A$. For $j = 0, \ldots, r-1$ we say that a copy of $K_k^r$ in $H(t,r)$ is of *type* $j$ if it intersects $j$ elements of $A$. For $j \geq i$, each edge of type $i$ lies on precisely

$$f(i,j) = \binom{r-i}{j-i}\binom{t-2r+i}{k-r-j+i}$$

copies of $K_k^r$ of type $j$. We now prove that there are nonnegative real numbers $x_0, \ldots, x_{r-1}$ such that by assigning the value $x_j$ to each copy of $K_k^r$ of type $j$, we obtain a fractional $K_k^r$ decomposition, namely we must show that for each $i = 0, \ldots, r-1$,

$$\sum_{j=i}^{r-1} x_j f(i,j) = 1.$$

Indeed, consider the upper triangular matrix $A$ of order $r$ with $A_{i,j} = f(i-1, j-1)$. By Lemma 2.2 it suffices to show that $f(j,j) > 0$ and $f(i,j) \geq 0$ for all $0 \leq i \leq j \leq r-1$ and $f(i,j) \geq f(i-1,j)$ for all $1 \leq i \leq j \leq r-1$. Indeed, by definition $f(i,j) \geq 0$. Furthermore,

$$f(j,j) = \binom{t-2r+j}{k-r} > 0$$

and

$$\frac{f(i,j)}{f(i-1,j)} = \frac{(t-2r+i)(j-i+1)}{(r-i+1)(k-r-j+i)} \geq \frac{t-2r}{(r+1)(k-r)} = \frac{kr+k-2r}{kr+k-r-r^2} \geq 1.$$

■

Our goal in the remainder of this section is to prove Theorem 2.1. Our first tool is the following powerful result of Kahn [7] giving an upper bound for the minimum number of colors in a proper edge-coloring of a uniform hypergraph (his result is, in fact, more general than the one stated here).

**Lemma 2.4 (Kahn [7])** *For every $r^* \geq 2$ and every $\gamma > 0$ there exists a positive constant $\rho = \rho(r^*, \gamma)$ such that the following statement is true:*
*If $U$ is an $r^*$-uniform hypergraph with $\Delta_1(U) \leq D$ and $\Delta_2(U) \leq \rho D$ then there is a proper coloring of the edges of $U$ with at most $(1+\gamma)D$ colors.* ■

Our second Lemma quantifies the fact that in a dense $r$-uniform hypergraph every edge appears on many copies of $K_t^r$.

**Lemma 2.5** *Let $t \geq r \geq 3$ and let $\zeta > 0$. Then, for all sufficiently large $n$, if $H$ is a $(1-\zeta)$-dense $r$-uniform hypergraph with $n$ vertices then every edge of $H$ appears on at least $\frac{1}{(t-r)!}n^{t-r}(1-\zeta t 2^t)$ copies of $K_t^r$.*

Fix an edge $e = \{u_1, \ldots, u_r\}$. We prove the lemma by induction on $t$. Our base cases are $t = r, \ldots, 2r-1$ for which we prove the lemma directly. The case $t = r$ is trivial. If $r+1 \leq t \leq 2r-1$, then for any $(t-r)$-subset $S$ of $V(H) - e$, the set of $t$-vertices $S \cup e$ is *not* a $K_t^r$ if and only if there exists some $f \subset e$ with $2r - t \leq |f| \leq r-1$ and some $g \subset S$ with $|g| = r - |f|$ such that $f \cup g$ is not an edge. For any $f \subset e$ with $2r - t \leq |f| \leq r-1$, the number of non-edges containing $f$ is at most $\zeta\binom{n-|f|}{r-|f|}$. For each such non-edge $e'$, if $g = e' - f$ then $g$ appears in at most $\binom{n}{t-r-|g|} = \binom{n}{t-2r+|f|}$ possible $(t-r)$-subsets $S$ of $V(H) - e$. It follows that $e$ appears on at least

$$\binom{n-r}{t-r} - \sum_{d=2r-t}^{r-1}\binom{r}{d}\zeta\binom{n-d}{r-d}\binom{n}{t-2r+d} > \frac{n^{t-r}}{(t-r)!}(1-\zeta 2^t)$$

copies of $K_t^r$.

Assume the lemma holds for all $t' < t$ and that $t \geq 2r$. Let $H^*$ be the subhypergraph of $H$ induced on $V(H) - e$. $H^*$ has $n - r$ vertices. Since $n$ is chosen large enough, the deletion of a constant (namely $r$) vertices from a $(1-\zeta)$-dense $n$-vertex hypergraph has a negligible affect on the density. In particular, the density of $H^*$ is larger than $(1-2\zeta)$. By the induction hypothesis, each edge of $H^*$ appears in at least

$$\frac{(n-r)^{t-2r}}{(t-2r)!}(1-2\zeta(t-r)2^{t-r})$$

copies of $K_{t-r}^r$ in $H^*$. Since $H^*$ is $(1-2\zeta)$-dense it has at least $\binom{n-r}{r}(1-2\zeta)$ edges. As each copy of $K_{t-r}^r$ has $\binom{t-r}{r}$ edges, we have that $H^*$ contains at least

$$\frac{(n-r)^{t-2r}}{(t-2r)!}(1-2\zeta(t-r)2^{t-r})\binom{n-r}{r}(1-2\zeta)\frac{1}{\binom{t-r}{r}} > \frac{n^{t-r}}{(t-r)!}(1-2\zeta(t-r)2^{t-r})(1-3\zeta)$$

copies of $K_{t-r}^r$. If $S$ is the set of vertices of some $K_{t-r}^r$ in $H^*$ we say that $S$ is *good* if $S \cup e$ is the set of vertices of a $K_t^r$, otherwise $S$ is *bad*. We can estimate the number of bad $S$ in a similar fashion to the estimation in the base cases of the induction. Indeed, $S$ is bad if and only if there exists some $f \subset e$ with $1 \leq |f| \leq r-1$ and some $g \subset S$ with $|g| = r - |f|$ such that $f \cup g$ is not an edge. It follows that the number of bad $S$ is at most

$$\sum_{d=1}^{r-1} \binom{r}{d} \zeta \binom{n-d}{r-d} \binom{n}{t-2r+d} < \frac{n^{t-r}}{(t-r)!} \zeta 2^t.$$

It follows that the number of good $S$, and hence the number of $K_t^r$ of $H$ containing $e$, is at least

$$\frac{n^{t-r}}{(t-r)!}((1-2\zeta(t-r)2^{t-r})(1-3\zeta) - \zeta 2^t) > \frac{n^{t-r}}{(t-r)!}(1-\zeta t 2^t)$$

as required. ■

**Proof of Theorem 2.1** Let $k > r \geq 3$ be fixed integers. We must prove that there exists $\alpha = \alpha(k,r)$ and $N = N(k,r)$ such that if $H$ is an $r$-uniform hypergraph with $n > N$ vertices and $\delta_d(H) \geq \binom{n-d}{r-d}(1-\alpha)$ for all $1 \leq d \leq r-1$ then $H$ has an $\mathcal{F}(k,r)$-decomposition.

Let $\epsilon = \epsilon(k,r)$ be a constant to be chosen later (in fact, it suffices to take $\epsilon = (2kr)^{-2r}$ but we make no attempt to optimize $\epsilon$). Let $\eta = (2^{-H(\epsilon)}0.9)^{1/\epsilon}$ where $H(x) = -x \log x - (1-x)\log(1-x)$ is the entropy function. Let $\alpha = \min\{(\eta/2)^2 \ , \ \epsilon^2/(t^2 4^{t+1})\}$. Let $\gamma > 0$ be chosen such that $(1-\alpha t 2^t)(1-\gamma)/(1+\gamma)^2 > 1 - 2\alpha t 2^t$. Let $r^* = \binom{t}{r}$. Let $\rho = \rho(r^*, \gamma)$ be the constant from Lemma 2.4. In the proof we shall assume, whenever necessary, that $N$ is sufficiently large as a function of these constants.

Let $H = (V,E)$ be an $r$-uniform hypergraph with $n > N$ vertices and $\delta_d(H) \geq \binom{n-d}{r-d}(1-\alpha)$ for all $1 \leq d \leq r-1$.

Our first step is to color the edges of $H$ such that the spanning subhypergraph on each color class has some "nice" properties. We shall use $q$ colors where $q = n^{1/(4\binom{t}{r}-4)}$ (for convenience we ignore floors and ceilings as they do not affect the asymptotic nature of our result). Each $e \in E$ selects a color from $[q]$ uniformly at random. The choices are independent. Let $H_i = (V, E_i)$ denote the subhypergraph whose edges received the color $i$. Let $S \subset V$ with $1 \leq |S| \leq r-1$. Clearly, the degree of $S$ in $H_i$, denoted $deg_i(S)$, has binomial distribution $B(deg(S), 1/q)$. Thus, $E[deg_i(S)] = deg(S)/q$. By a large deviation inequality of Chernoff (cf. [2], Appendix A) it follows that the probability that $deg_i(S)$ deviates from its mean by more than a constant fraction of the mean is exponentially small in $n$. In particular, for $n$ sufficiently large,

$$\Pr\left[\left|deg_i(S) - \frac{deg(S)}{q}\right| > \gamma \frac{deg(S)}{q}\right] < \frac{1}{4qrn^r}. \quad (1)$$

Let $e \in E_i$. Let $C(e)$ denote the set of $K_t^r$ copies of $H$ that contain $e$ and let $c(e) = |C(e)|$. Trivially, $c(e) \leq \binom{n-r}{t-r}$. Thus, by Lemma 2.5 with $\zeta = \alpha$ we have

$$\frac{1}{(t-r)!} n^{t-r} \geq c(e) \geq \frac{1}{(t-r)!} n^{t-r}(1-\alpha t 2^t).$$

Let $C_i(e)$ denote the set of $K_t^r$ copies of $H_i$ containing $e$, and put $c_i(e) = |C_i(e)|$. Clearly, $E[c_i(e)] = c(e)q^{-\binom{t}{r}+1} = c(e)n^{-1/4}$. Therefore,

$$\frac{1}{(t-r)!}n^{t-r-1/4} \geq E[c_i(e)] \geq \frac{1}{(t-r)!}n^{t-r-1/4}(1-\alpha t 2^t).$$

However, this time we cannot simply use Chernoff's inequality to show that $E[c_i(e)]$ is concentrated around its mean, since, given that $e \in E_i$, two elements of $C(e)$ are *dependent* if they contain another common edge in addition to $e$. However, we can overcome this obstacle using the fact that the dependence is *limited*. This is done as follows. Consider a graph $G$ whose vertex set is $C(e)$ and whose edges connect two elements of $C(e)$ that share at least one edge (in addition to $e$). For $X \in C(e)$ the degree of $X$ in $G$ is clearly at most $(\binom{t}{r}-1)\binom{n-r-1}{t-r-1}$ since given $f \in E(X)$ with $f \neq e$ we have $|f \cup e| \geq r+1$ and thus there are at most $\binom{n-|f\cup e|}{t-|f \cup e|}$ copies of $K_t^r$ containing both $f$ and $e$. In particular, $\Delta(G) = O(n^{t-r-1})$. On the other hand $|V(G)| = c(e) = \Theta(n^{t-r})$. Notice also that the chromatic number of $G$ is $\chi = \chi(G) = O(n^{t-r-1})$. Consider a coloring of $G$ with $\chi(G)$ colors. If $X$ and $X'$ are in the same color class then, given that $e \in E_i$, the event that $X \in C_i(e)$ is *independent* of the event that $X' \in C_i(e)$. For $z = 1, \ldots, \chi(G)$, let $C^z(e)$ denote the elements of $C(e)$ colored with $z$ and put $c^z(e) = |C^z(e)|$. Put $C_i^z(e) = C^z(e) \cap C_i(e)$ and let $c_i^z(e) = |C_i^z(e)|$. Clearly, $c_i(e) = \sum_{z=1}^{\chi} c_i^z(e)$ and $E[c_i^z(e)] = c^z(e)n^{-1/4}$. Whenever $|c^z(e)| > n^{1/2}$ we can use Chernoff's inequality to show that $c_i^z(e)$ is highly concentrated around its mean (that, is, the probability that it deviates from its mean by any given constant fraction of the mean is exponentially small in $n$). Whenever $|c^z(e)| \leq n^{1/2}$ we simply notice that the overall number of elements of $C(e)$ belonging to these small color classes is at most $\chi n^{1/2} = O(n^{t-r-1/2}) << n^{t-r-1/4}$. We therefore have that for $n$ sufficiently large,

$$\Pr[c_i(e)] < (1-\gamma)\frac{1}{(t-r)!}n^{t-r-1/4}(1-\alpha t 2^t) < \frac{1}{4\binom{n}{r}}, \tag{2}$$

$$\Pr[c_i(e)] > (1+\gamma)\frac{1}{(t-r)!}n^{t-r-1/4} < \frac{1}{4\binom{n}{r}}. \tag{3}$$

Since the overall number of subsets $S$ with $1 \leq |S| \leq r-1$ is less than $rn^r$, and since $|E| \leq \binom{n}{r}$ we have, by (1), (2) and (3) that with probability at least $1 - qrn^r/(4qrn^r) - 2\binom{n}{r}/(4\binom{n}{r}) \geq 1/4$, a random $q$-coloring of the edges of $H$ satisfies the following:

*A*. For all $S \subset V$ with $1 \leq |S| \leq r-1$, and for all $i = 1, \ldots, q$, $|deg_i(S) - \frac{deg(S)}{q}| \leq \gamma \frac{deg(S)}{q}$.

*B*. For each $e \in E$, if $e \in E_i$ then $c_i(e) \geq (1-\gamma)\frac{1}{(t-r)!}n^{t-r-1/4}(1-\alpha t 2^t)$ and $c_i(e) \leq (1+\gamma)\frac{1}{(t-r)!}n^{t-r-1/4}$.

We therefore fix an edge coloring and the resulting spanning subhypergraphs $H_1, \ldots, H_q$ satisfying properties $A$ and $B$.

For each $H_i = (V, E_i)$ we create another hypergraph, denoted $U_i$, as follows. The vertex set of $U_i$ is $E_i$. The edges of $U_i$ are the sets of edges of copies of $K_t^r$ in $H_i$. Notice that $U_i$ is a $\binom{t}{r}$-uniform hypergraph. Let $D = (1+\gamma)((t-r)!)^{-1}n^{t-r-1/4}$. By Property $B$, $\Delta_1(U_i) \le D$. Also, we trivially have that for all $n$ sufficiently large, $\Delta_2(U_i) \le n^{t-r-1} < \rho D$. It follows from Lemma 2.4 that the set of $K_t^r$ copies of $H_i$ can be partitioned into at most $(1+\gamma)D$ packings. Denote these packings by $L_i^1, \ldots, L_i^{z_i}$ where $z_i \le (1+\gamma)D$.

We now choose a $K_t^r$-packing of $H$ as follows. For each $i = 1, \ldots, q$ we select, uniformly at random, one of the packings $\{L_i^1, \ldots, L_i^{z_i}\}$. Denote by $L_i$ the randomly selected packing. All $q$ selections are performed independently. Notice that $L = L_1 \cup \cdots \cup L_q$ is a $K_t^r$-packing of $H$. Let $M$ denote the set of edges of $H$ that do not belong to any element of $L$, and let $H[M]$ be the spanning subhypergraph of $H$ consisting of the edges of $M$. Let $p = \binom{k}{r} - 1$. We say that a $p$-subset $S = \{S_1, \ldots, S_p\}$ of $L$ is *good for* $e \in M$ if we can select edges $f_i \in E(S_i)$ such that $\{f_1, \ldots, f_p, e\}$ is the set of edges of a $K_k^r$ in $H$. We say that $L$ is *good* if for each $e \in M$ there exists a $p$-subset $S(e)$ of $L$ such that $S(e)$ is good for $e$ and such that if $e \neq e'$ then $S(e) \cap S(e') = \emptyset$.

**Lemma 2.6** *If $L$ is good then $H$ has an $\mathcal{F}(k,r)$-decomposition.*

**Proof:** For each $e \in M$, pick a copy of $K_k^r$ in $H$ containing $e$ and precisely one edge from each element of $S(e)$. As each element of $S(e)$ is a $K_t^r$, deleting one edge from such an element results in an $H(t,r)$. We therefore have $|M|$ copies of $K_k^r$ and $|M|(\binom{k}{r} - 1)$ copies of $H(t,r)$, all being edge disjoint. The remaining element of $L$ not belonging to any of the $S(e)$ are each a $K_t^r$, and they are edge-disjoint from each other and from the previously selected $K_k^r$ and $H(t,r)$. ■

Our goal in the remainder of this section is to show that there exists a good $L$. We will show that with positive probability, the random selection of the $q$ packings $L_1, \ldots, L_q$ yields a good $L$. We begin by showing that with high probability, $H[M]$ has a relatively small maximum $d$-degree, for all $1 \le d \le r-1$.

**Lemma 2.7** *With positive probability, for all $d = 1, \ldots, r-1$, $\Delta_d(H[M]) \le 2\epsilon\binom{n-d}{r-d}$.*

**Proof:** Let $S \subset V$ with $1 \le |S| \le r-1$. Let $F_i(S) \subset E_i$ denote the edges of $H_i$ containing $S$ and let $L_i(S) \subset F_i(S)$ denote those edges of $F_i(S)$ that are covered by $L_i$. For $e \in F_i(S)$, the probability that $e$ is covered by $L_i$ is $c_i(e)/z_i$. By Property $B$ and since $z_i \le (1+\gamma)D$ we have

$$\frac{c_i(e)}{z_i} \ge \frac{(1-\gamma)(1-\alpha t 2^t)}{(1+\gamma)^2} \ge 1 - 2\alpha t 2^t.$$

It follows that $E[|L_i(S)|] \ge (1 - 2\alpha t 2^t)|F_i(S)| = (1 - 2\alpha t 2^t)deg_i(S)$ and that

$$\Pr[|L_i(S)| \le (1 - \alpha^{1/2} t 2^t)deg_i(S)] \le 2\alpha^{1/2} \le \eta.$$

Since $|L_1(S)|, \ldots, |L_q(S)|$ are independent random variables it follows that the probability that at least $\epsilon q$ of them have cardinality at most $(1-\alpha^{1/2}t2^t)deg_i(S)$ is at most

$$\binom{q}{\epsilon q}\eta^{\epsilon q} < 0.9^q << \frac{1}{qrn^r}$$

where in the last inequality we used the fact that $\eta = (2^{-H(\epsilon)}0.9)^{1/\epsilon}$. It follows that there exists a choice of $L_1, \ldots, L_q$ such that for all $S$, at most $\epsilon q$ of the packings have $|L_i(S)| \leq (1-\alpha^{1/2}t2^t)deg_i(S)$. Let $deg^M(S)$ denote the degree of $S$ in $H[M]$. By Property $A$, $deg_i(S) \leq (1+\gamma)deg(S)/q$. Thus, since $\sum_{i=1}^{q} deg_i(S) = deg(S)$ we have

$$deg^M(S) = deg(S) - \sum_{i=1}^{q}|L_i(S)| \leq deg(S) - (1-\alpha^{1/2}t2^t)deg(S) + \epsilon q(1+\gamma)\frac{deg(S)}{q}$$

$$\leq deg(S)(\alpha^{1/2}t2^t + \epsilon(1+\gamma)) \leq 2\epsilon deg(S) \leq 2\epsilon\binom{n-|S|}{r-|S|}$$

where in the last inequality we used the fact that $\alpha \leq \epsilon^2/(t^2 4^{t+1})$. It follows that there is a choice of $L_1, \ldots, L_q$ such that for all $d = 1, \ldots, r-1$, $\Delta_d(H[M]) \leq 2\epsilon\binom{n-d}{r-d}$. ■

By Lemma 2.7, we may *fix* $L$ and $M$ such that $\Delta_d(H[M]) \leq 2\epsilon\binom{n-d}{r-d}$ for $d = 1, \ldots, r-1$. Let $M = \{e_1, \ldots, e_m\}$. Notice that, in particular, $m \leq 2\epsilon\binom{n}{r}$. Let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be the family of $p$-uniform hypergraphs defined as follows. The vertex set of each $U_i$ is $L$. The edges of $U_i$ are the $p$-subsets of $L$ that are good for $e_i$. A *system of disjoint representatives* (SDR) for $\mathcal{U}$ is a set of $m$ edges $S(e_i) \in E(U_i)$ for $i = 1, \ldots, m$ such that $S(e_i) \cap S(e_j) = \emptyset$ whenever $i \neq j$. Thus, $L$ is good if and only if $\mathcal{U}$ has an SDR. Generalizing Hall's Theorem, Aharoni and Haxell [1] gave a sufficient condition for the existence of an SDR.

**Lemma 2.8** *[Aharoni and Haxell [1]] Let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be a family of $p$-uniform hypergraphs. If for every $\mathcal{W} \subset \mathcal{U}$ there is a matching in $\cup_{U \in \mathcal{W}} U$ of size greater than $p(|\mathcal{W}|-1)$ then $\mathcal{U}$ has an SDR.* ■

We use Lemma 2.8 to prove:

**Lemma 2.9** *If $\Delta_d(H[M]) \leq 2\epsilon\binom{n-d}{r-d}$ for $d = 1, \ldots, r-1$ then $\mathcal{U}$ has an SDR.*

**Proof:** Let $R_i$ denote the set of $K_k^r$ copies of $H$ that contain $e_i$ and whose remaining $p$ edges are each from a distinct element of $L$. We establishing a lower bound for $|R_i|$. Let $a_i$ denote the number of copies of $K_k^r$ containing $e_i$, let $b_i$ denote the number of copies of $K_k^r$ containing $e_i$ and at least two edges from the same element of $L$. Let $c_i$ denote the number of copies of $K_k^r$ containing $e_i$ and at least another edge of $M$. Clearly, $|R_i| = a_i - b_i - c_i$.

A similar proof to that of Lemma 2.5 where we use $k$ instead of $t$ and $\zeta = \alpha$ immediately gives

$$a_i \geq \frac{1}{(k-r)!} n^{k-r}(1 - \alpha k 2^k). \tag{4}$$

Consider a pair of edges $f_1, f_2$ that belong to the same element of $L$. Suppose $|(f_1 \cup f_2) \cap e_i| = d$ then we must have $0 \leq d \leq r-1$. The overall number of choices for $f_1, f_2$ for which $|(f_1 \cup f_2) \cap e_m| = d$ is $O(n^{r-d})$ (there are $O(n^{r-d})$ choices for $f_1$, and given $f_1$ there are only $\binom{t}{r} - 1$ choices for $f_2$ in the same element of $L$). Given $f_1, f_2$, the number of $K_k^r$ containing $f_1, f_2, e_i$ is at most $O(n^{k-(2r+1-d)})$, since $|f_1 \cup f_2 \cup e_m| \geq 2r + 1 - d$. Thus, in total, we get,

$$b_i = \sum_{d=0}^{r-1} O(n^{r-d} n^{k-(2r+1-d)}) = O(n^{k-r-1}). \tag{5}$$

Consider an edge $f \in M$ with $f \neq e_i$. If $f$ and $e_i$ are independent then there are at most $\binom{n-2r}{k-2r}$ copies of $K_k^r$ containing both of them. Overall, there are less than $m\binom{n-2r}{k-2r}$ such copies. If $f$ and $e_i$ intersect in $d$ vertices then there are at most $\binom{n-2r+d}{k-2r+d}$ copies of $K_k^r$ containing both of them. However, the maximum $d$-degree of $M$ is at most $2\epsilon\binom{n-d}{r-d}$ and hence there are at most $\binom{r}{d}2\epsilon\binom{n-d}{r-d}$ choices for $f$. We therefore have that

$$c_i \leq m\binom{n-2r}{k-2r} + \sum_{d=1}^{r-1} \binom{r}{d} 2\epsilon \binom{n-d}{r-d} \binom{n-2r+d}{k-2r+d} \tag{6}$$

$$\leq \sum_{d=0}^{r-1} 2\epsilon \binom{r}{d} \binom{n-d}{r-d} \binom{n-2r+d}{k-2r+d}.$$

We now get, using (4), (5) and (6), that for $\epsilon = \epsilon(k, r)$ sufficiently small and for $n$ sufficiently large,

$$|R_i| \geq \frac{1}{(k-r)!} n^{k-r}(1 - \alpha k 2^k) - O(n^{k-r-1}) - \sum_{d=0}^{r-1} 2\epsilon \binom{r}{d} \binom{n-d}{r-d} \binom{n-2r+d}{k-2r+d}$$

$$\geq \frac{1}{2(k-r)!} n^{k-r}.$$

Let $\mathcal{W} \subset \mathcal{U}$ with $w = |\mathcal{W}|$. Without loss of generality assume $\mathcal{W} = \{U_1, \ldots, U_w\}$. Put $M(\mathcal{W}) = \{e_1, \ldots, e_w\}$. We must show that the condition in Lemma 2.8 holds. Assume that this is not the case. Consider a maximum matching $T$ in $U_1 \cup \cdots \cup U_w$. Thus, $|T| \leq p(w-1)$. In particular, $|T|$ contains at most $p^2(w-1)$ vertices (recall that the vertices are element of $L$). Let $L' \subset L$ denote the vertices contained in $T$. Thus, $|L'| \leq p^2(w-1)$. The overall number of copies of $K_k^r$ that contain precisely one edge from $M(\mathcal{W})$ and whose other edges are in $p$ distinct elements of $L$ is

$$|R_1| + \cdots + |R_w| \geq w \frac{1}{2(k-r)!} n^{k-r}.$$

Let $F$ be the set of edges in the elements of $L'$. Hence, $|F| = |L'|\binom{t}{r}$. Let $f \in F$. Let $c(f)$ denote the number of copies of $K_k^r$ containing $f$ and precisely one edge from $M(\mathcal{W})$. For $Y \subsetneq f$, let $M_f(Y) = \{e_i \mid e_i \cap f = Y \ , \ i = 1, \ldots, w\}$. This partitions $M(\mathcal{W})$ into $2^r - 1$ classes according to the choice of $Y$. Let $c(f, Y)$ denote the number of copies of $K_k^r$ containing $f$ and precisely one edge from $M_f(Y)$. Given $e \in M_f(Y)$, the number of $K_k^r$ containing both $f$ and $e$ is at most $\binom{n-2r+|Y|}{k-2r+|Y|}$. On the other hand, since $deg^M(Y) \le 2\epsilon\binom{n-|Y|}{r-|Y|}$ we have $|M_f(Y)| \le 2\epsilon\binom{n-|Y|}{r-|Y|}$. Thus,

$$c(f, Y) \le 2\epsilon \binom{n-|Y|}{r-|Y|}\binom{n-2r+|Y|}{k-2r+|Y|} < 2\epsilon n^{k-r}.$$

It follows that

$$c(f) < 2^{r+1}\epsilon n^{k-r}.$$

Now, for $\epsilon = \epsilon(k, r)$ sufficiently small

$$\sum_{f \in F} c(f) < |L'|\binom{t}{r}2^{r+1}\epsilon n^{k-r} \le p^2(w-1)\binom{t}{r}2^{r+1}\epsilon n^{k-r}$$

$$\le w\frac{1}{2(k-r)!}n^{k-r} \le |R_1| + \cdots + |R_w|.$$

It follows that there exists a $K_k^r$ containing precisely one edge from $M(\mathcal{W})$, say, $e_i$, and whose other edges are in $p$ distinct elements of $L - L'$. The $p$ distinct elements form an edge $\{S_1, \ldots, S_p\}$ of $U_i$ and hence $\{S_1, \ldots, S_p\}$ is an edge of $\cup_{U \in \mathcal{W}} U$. Since $\{S_1, \ldots, S_p\}$ is independent of all the edges of $T$ we have that $T$ is *not* a maximal matching of $\cup_{U \in \mathcal{W}} U$, a contradiction. ■

We have now completed the proof of Theorem 2.1. ■

# 3 Concluding remarks and open problems

- A simpler version of Theorem 1.1 holds in case we assume that every edge of $K_n^r$ lies on approximately the same number of copies of $K_k^r$ (such is the case in, say, the random $r$-uniform hypergraph). In this case the statement of Theorem 1.1 follows quite easily from the result given in [3] and the result of [5]. However, our Theorem 1.1 does *not* assume these regularity conditions. It only assumes a minimum density threshold.

- Theorem 2.1 gives a nontrivial minimum density requirement which guarantees the existence of an $\mathcal{F}$-decomposition for the family $\mathcal{F} = \{K_k^r, K_t^r, H(t, r)\}$. It is interesting to find other more general families $\mathcal{F}$ for which nontrivial density conditions guarantee an $\mathcal{F}$-decomposition.

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