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ORIGINAL PAPER

Flip colouring of graphs

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Abstract

It is proved that for integers b, r such that $3 \leq b < r \leq \binom{b+1}{2} - 1$, there exists a red/blue edge-colored graph such that the red degree of every vertex is r, the blue degree of every vertex is b, yet in the closed neighbourhood of every vertex there are more blue edges than red edges. The upper bound $r \leq \binom{b+1}{2} - 1$ is best possible for any $b \geq 3$. We further extend this theorem to more than two colours, and to larger neighbourhoods. A useful result required in some of our proofs, of independent interest, is that for integers r, t such that $0 \leq t \leq \frac{r^2}{2} - 5r^{3/2}$, there exists an r-regular graph in which each open neighbourhood induces precisely t edges. Several explicit constructions are introduced and relationships with constant linked graphs, (r, b)-regular graphs and vertex transitive graphs are revealed.

Keywords Local vs global properties · Cayley graphs · (r, c)-constant graphs

Mathematics Subject Classification 05C15 · 05C70

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1 Introduction

Local versus global phenomena are widely considered both in graph theory (combinatorics in general) and in social sciences [1–6]. Such phenomena occur in the most elementary graph theory observations as well as in highly involved theorems and conjectures.

A simple example dates back to Euler: Every degree is even (a local property), if and only if each component has an Eulerian circuit (a global property). More involved examples are Turán-type problems [7–9], broadly showing that if a graph does not contain some fixed graph (local property), then its number of edges cannot be too large (a global property). As an illustration of a somewhat counter-intuitive example, we recall a famous theorem of Erdős [10] stating that for any $h, k \geq 3$, there is a graph whose shortest cycle has length at least h (a locally verifiable property), yet its chromatic number is at least k (a global property).

We now introduce an "umbrella" for the local–global problem considered in this paper. Recall that $N^G(v)$ denotes the set of neighbours of a vertex v in G (its open neighbourhood) and $N^G[V] = N^G(v) \cup \{v\}$ denotes the closed neighbourhood. We omit the superscript when the graph is clear from context. Some further notation follows.

- For a colouring $f : E(G) \to \{1, \dots, k\}$, $k \geq 1$, let $E(j) = \{e \in E(G) : f(e) = j\}$ and $e_j(G) = |E(j)|$. For $k = 1$ we use $e(G)$.
- For a vertex v, $\deg_j(v) = |\{e : e$ incident with v and $f(e) = j\}$. For $k = 1$ we use $\deg(v)$.
- For a vertex v, $e_j[v] = |E(j) \cap E(N[v])|$. For $k = 1$ we use $e[v]$.
- For a vertex v, $e_j(v) = |E(j) \cap E(N(v))|$. For $k = 1$ we use $e(v)$.

We now state the general flip colouring problem: Given a graph G and an integer $k \geq 2$, does there exist a colouring $f : E(G) \to \{1, \dots, k\}$ such that:

- for every vertex v, $\deg_j(v) > \deg_i(v)$ for $1 \leq i < j \leq k$ (in particular, forcing global majority $e_j(G) > e_i(G)$ for $1 \leq i < j \leq k$);
- for every vertex v, $e_j[v] < e_i[v]$ for $1 \leq i < j \leq k$ (forcing an opposite local majority).

If such an edge-colouring exists, then G is said to be a *k-flip graph* and the colouring f is a *k-flip colouring*.

We shall mostly deal with a version of this problem restricted to regular edge-coloured subgraphs (hence regular graphs), as this question already captures the essence of the problem and reduces notation overload. Namely, given $k \geq 2$, a d-regular graph G and a strictly increasing positive integer sequence $(a_1, \dots, a_k)$ such that $d = \sum_{j=1}^{k} a_j$, does there exist a colouring $f : E(G) \to \{1, \dots, k\}$ such that:

- $E(j)$ spans an a_j-regular subgraph, i.e, $\deg_j(v) = a_j$ for every $v \in V(G)$;
- for every vertex $v \in V(G)$, $e_k[v] < e_{k-1}[v] < \cdots < e_1[v]$.

If such an edge-colouring exists then G is said to be an *$(a_1, \dots, a_k)$-flip graph* (in particular, a k-flip graph) and $(a_1, \dots, a_k)$ is called a *flip sequence* of G. An illustrative example is given in Fig. 1.

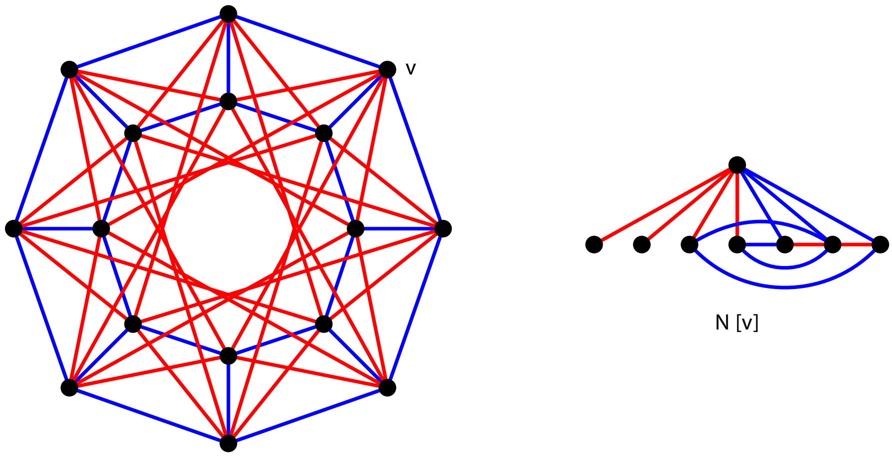


Fig. 1 Smallest known (3, 4)-flip graph having 16 vertices, with the subgraph induced by the closed neighbourhood of any vertex v illustrated on the right

1.1 Major problems concerning flip graphs

The notion of flip graph gives rise to several natural problems:

1. *Characterise flip sequences:* Given a strictly increasing positive integer sequence $(a_1, \ldots, a_k)$, is it a flip sequence of some graph?
2. *Smallest order of an* $(a_1, \ldots, a_k)$*-flip graph:* Given a flip sequence $(a_1, \ldots, a_k)$, determine the smallest order of a graph realising it.
3. Devise explicit constructions for flip graphs with two or more colours.
4. *Interval flip.* Find k-flip sequences of the form $(1 + t, 2 + t, \ldots, k + t)$.
5. Extend the notion of flip from counting colour occurrences in closed neighbourhoods to counting colour occurrences in t-closed neighbourhoods (a vertex is in the t-closed neighbourhood of v if its distance from v is at most t).
6. *Complexity of recognising a k-flip graph.* Given a (possibly regular) graph G and an integer $k \geq 2$, determine whether it is a k-flip graph. The problem is clearly in NP, but is it NP-complete?

We shall consider most of these questions in the sequel. The paper is organised as follows.

Section 2 introduces the coloured Cartesian product technique which is used several times along this paper, together with a preliminary application using the family of (r, c)-constant graphs, which are r-regular graphs such that for every vertex v, the induced graph on $N(v)$ contains exactly c edges.

Section 3 is about the flip problem with two colours. We develop techniques for constructing (b, r)-flip graphs using Cartesian products. We prove that a necessary and sufficient condition for (b, r) to be a flip sequence is $3 \leq b < r \leq \binom{b+1}{2} - 1$. This theorem completely answers Problems 1, 4 above for flip sequences of length two, supplies constructions as requested in Problem 3, and gives an upper bound for Problem 2.

Section 4 concerns the case of three or more colours. In particular, we prove that if (a_1, a_2, a_3) is a flip sequence, then $a_3 \leq 2(a_1)^2$. We provide a construction that comes close to this bound. Interestingly, it is revealed that when at least four colours are used, there are k-flip sequences where a_k can be arbitrarily large even if a_1 is fixed.

Section 5 concerns (r, c)-constant graphs and their applications to Problem 4 (interval flip). We prove the second theorem mentioned in the abstract and use it to prove that the interval $[b, \ldots, \frac{b^2}{4} - \frac{5b^{3/2}}{2}]$ is a flip sequence. We also propose a simple construction showing that for $b \geq 3$, the interval $[b, \ldots, 2b-2]$ is a flip sequence, which is useful with regards to the above result for small values of b.

Section 6 concerns Problem 5. We prove several results concerning the extension of flip graphs to larger neighbourhoods.

Finally, Sect. 7 summarises the current work and offers further open problems.

2 The coloured Cartesian product technique

2.1 Cartesian products of edge-coloured graphs

The Cartesian product of graphs will be useful in the construction of flip-graphs. Due to the additive nature of the degree and closed-neighbourhood sizes under the Cartesian product, this allows us to consider its factors independently. Before doing so, we recall the definition of Cartesian product and outline a number of its properties, including edge-colouring inheritance.

Definition 1 (Cartesian product) The Cartesian product $G \square H$ of the graphs G and H is the graph such that $V(G \square H) = V(G) \times V(H)$ and there is an edge $\{uv, u'v'\}$ in $G \square H$ if and only if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

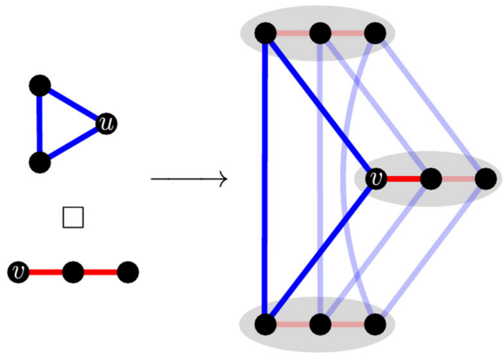
The Cartesian product of graphs is commutative and associative, so the Cartesian product of a finite set of graphs is well-defined. It also enjoys a number of additional properties. In particular, it is vertex-transitive if and only if each of its factors is vertex-transitive. More so, with an appropriate choice of generating set, the Cartesian product of Cayley graphs is also a Cayley graph.

Let $G_1, \ldots, G_r$ be edge-coloured graphs. We extend the edge-colourings of the G_i's to an edge-colouring of their Cartesian product in the natural way: the colour of $e = \{u_1u_2 \cdots u_r, u'_1u'_2 \cdots u'_r\}$ is the colour of the unique edge $u_iu'_i \in E(G_i)$ for which $u_i \neq u'_i$. This colouring inheritance is illustrated in Fig. 2, with its properties summarised in Lemma 1.

Lemma 1 *Let $G_1, \ldots, G_r$ be edge-coloured by colour set $[k] = \{1, \ldots, k\}$. Let G be their edge-coloured Cartesian product with the inherited colouring from its factors. Then for any $i \in [k]$ and $u_1 \cdots u_r \in V(G)$:*

$$\deg_i(u_1 \cdots u_r) = \sum_{j=1}^{r} \deg_i^{G_j}(u_j)\,,$$

Fig. 2 Illustration of the edge-colouring inheritance in $K_3 \square P_3$ from its factors, where K_3 is coloured blue and P_3 is coloured red



$$e_i\left[u_1 \cdots u_r\right]=\sum_{j=1}^{r} e_i^{G_j}\left[u_j\right].$$

2.2 (*r*, *c*)-constant graphs and their sub-families

We recall constant link graphs, (r, b)-regular graphs, and introduce (r, c)-constant graphs. *Constant link* graphs are those for which all subgraphs induced by open neighbourhoods are isomorphic to some fixed graph H (the link). Problems concerning which graphs H can be links is an old (mostly unsolved) problem stated first by Zikov in 1964, which received attention over the years. For some references see [11–14].

Graphs are called (r, b)-*regular* if they are r-regular and all open neighbourhoods induce a b-regular graph (hence $e[v] = r + \frac{br}{2}$). For a recent article and further references see [12].

Generalising these two families, we define (r, c)-*constant graphs* to be r-regular graphs in which for every vertex v, $e(v) = c$, hence $e[v] = r + c$.

Figure 3 illustrates the hierarchy of several families of graphs in relation to (r, c)-constant graphs.

2.3 Coloured Cartesian product lemmas

This subsection explores the useful tool of coloured Cartesian products (abbreviated as CCP) via two lemmas. The first Lemma 2 describes how to construct a k-flip graph from a family containing regular graphs satisfying prescribed conditions. The second Lemma 3 describes how to construct k-flip graphs, k-flip sequences, (r, c)-constant graphs, (r, b)-regular and constant-link graphs, from such existing graphs, respectively. We also demonstrate some initial applications of these lemmas.

Lemma 2 *(CCP Lemma I) Let $k \geq 2$ be an integer. Suppose $H_1, \ldots, H_k$ are $a_1, \ldots, a_k$ regular graphs respectively, with $a_i < a_j$ for $1 \leq i < j \leq k$. Furthermore, suppose*

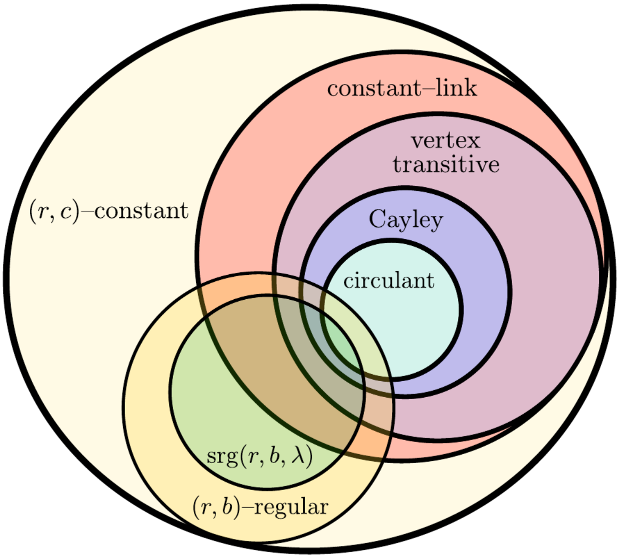


Fig. 3 Hierarchy of (r, c)-constant graphs and their sub-families of interest

that for $1 \leq i < k$,

$$\max_{u \in V(H_{i+1})} e[u] < \min_{v \in V(H_i)} e[v].$$

Then $G = \square_{j=1}^{k} H_j$ *is a* $(a_1, \ldots, a_k)$*-flip graph.*

Proof Colour the edges of H_j using colour j, for $1 \leq j \leq k$. Let $G = \square_{j=1}^{k} H_j$ be the corresponding CCP. Clearly then for every vertex w in G, given any $1 \leq i < k$, $\deg_i(w) = a_i < a_{i+1} = \deg_{i+1}(w)$ and

$$e_{i+1}[w] \leq \max_{u \in V(H_{i+1})} e[u] < \min_{v \in V(H_i)} e[v] \leq e_i[w]$$

and consequently G is an $(a_1, \ldots, a_k)$-flip graph. □

Lemma 3 *(CCP Lemma II) Let* $k \geq 2$ *be an integer.*

1. *If, for* $1 \leq j \leq q$, $(a_{j,1}, \ldots, a_{j,k})$ *are* k*-flip sequences then* $(a_1, \ldots, a_k)$ *is a* k*-flip sequence where* $a_i = \sum_{j=1}^{q} a_{j,i}$.
2. *If* H_j *for* $1 \leq j \leq q$ *are* (r_j, c_j)*-constant graphs, then there exists a graph* G *which is a* $\left(\sum_{j=1}^{q} r_j, \sum_{j=1}^{q} c_j\right)$*-constant graph.*
3. *If* H_j *for* $1 \leq j \leq q$ *are* (r_j, b)*-regular graphs, then there exists a graph* G *which is a* $\left(\sum_{j=1}^{q} r_j, b\right)$*-regular graph.*
4. *If, for* $1 \leq j \leq q$, G_j *is a constant-link graph with link* H_j, *then* $\square_{j=1}^{q} G_j$ *is a constant-link graph with link* $\cup_{j=1}^{q} H_j$.

Proof We prove (i) for the case $q = 2$; the other claims and the extension to larger q follow the same outline. Let $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ be k-flip sequences, so there exists a graph G which is a $(a_1, \ldots, a_k)$-flip graph, and a graph H which is a $(b_1, \ldots, b_k)$-flip graph.

Consider the CCP graph $F = G \,\Box\, H$. The colour degrees of a vertex v in $V(F)$ are $c_i = a_i + b_i$ for $1 \leq i \leq k$ and note that $c_i < c_j$ for $1 \leq i < j \leq k$. Also for $v = (x, y) \in V(F)$, we have that

$$e_i[v] = e_i[x] + e_i[y] > e_j[x] + e_j[y] = e_j[v]$$

for $1 \leq i < j \leq k$. Hence F is a $(c_1, \ldots, c_k)$-flip graph. □

Regular graphs with constant link are useful for flip graph construction, as if G is b-regular with constant link H where $e(H) = c$, then for any r such that $b < r < b + c$, and for any triangle free r-regular graph F, the CCP graph $G \,\Box\, F$, with G coloured blue and F coloured red, is an $(r + b)$-regular graph with r red edges and b blue edges incident with every vertex v and yet in $N[v]$ there are just r red edges but $b + c > r$ blue edges. Hence $G \,\Box\, F$ is a (b, r)-flip graph.

Similarly, (b, c)-regular graphs are useful for flip graph construction, as if G is a (b, c)-regular graph and $b < r < b + \frac{bc}{2}$ then for any triangle free r-regular graph F, the CCP graph $G \,\Box\, F$, with G coloured blue and F coloured red, is an $(r + b)$-regular graph with r red edges and b blue edges incident with every vertex v and yet in $N[v]$ there are just r red edges but $b + \frac{bc}{2} > r$ blue edges. Hence $G \,\Box\, F$ is a (b, r)-flip graph.

Clearly if G is a (b, c)-constant regular graph and $b < r < r + c$, then, as above, for any r such that $b < r < b + c$, and for any triangle free r-regular graph F, the CCP graph $G \,\Box\, F$ is a (b, r)-flip graph.

Therefore, we have an abundance of known (b, c)-regular graphs, Cayley graphs, vertex-transitive graphs and graphs with constant link as well as strongly regular graphs, which can be used to construct a panoramic collection of flip-graphs via the coloured Cartesian product.

3 Existence of (b, r)-flip graphs

This section settles the question: which pairs (b, r) form a flip sequence? We also establish an upper bound for $h(b, r)$, the smallest possible order of a (b, r)-flip graph.

Theorem 4 *Let $r, b \in \mathbb{N}$. If $3 \leq b < r \leq \binom{b+1}{2} - 1$ then there exists a (b, r)-flip graph, and both the upper bound and lower bound are sharp.*

We split the proof into two parts. We first prove that we must have $r < \binom{b+1}{2}$ (in particular, since $r > b$, this implies $b \geq 3$). Then we prove the rest of the theorem. Before proceeding further, we need some notation describing edge-coloured triangles rooted at some vertex.

Let $B = \mathit{blue} = 1$ and $R = \mathit{red} = 2$. In a graph G with edges coloured from $\{B, R\}$ and $X, Y, Z \in \{B, R\}$ a triangle rooted at a vertex v is said to be of *type* XYZ

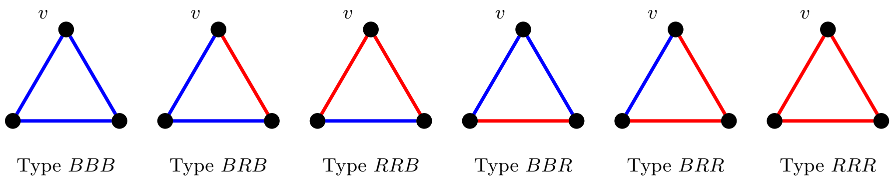


Fig. 4 All possible triangle types

at v if the two edges incident to v are coloured X and Y, and the third edge is coloured Z (the types BRR and RBR are considered identical and the types RBB and BRB are also considered identical). Hence, a triangle rooted at v can have one of six possible types, illustrated in Fig. 4.

Let $T_{XYZ}(v)$ be the number of triangles of type XYZ rooted at v. We need the following simple lemma.

Lemma 5 *In a graph G with edges coloured from* $\{B, R\}$*, we have that*

$$2\sum_{v\in V} T_{RRB}(v) = \sum_{v\in V} T_{BRR}(v) \text{ and } 2\sum_{v\in V} T_{BBR}(v) = \sum_{v\in V} T_{BRB}(v).$$

Proof Consider a triangle with two edges coloured R and a single edge coloured B. Each such triangle is of BRR-type for two vertices and is of RRB-type for a single vertex. Summing over all such triangles yields the first equality. The second equality is symmetrical by considering triangles with two edges coloured B and a single edge coloured R. □

Proposition 6 *In a* (b, r)*-sequence, we must have* $r < \binom{b+1}{2}$.

Proof Suppose that $G = (V, E)$ is a (b, r)-flip graph equipped with a suitable colouring. Note that since $e_B[v] = e_B(v) + b$ and $e_R[v] = e_R(v) + r$, we have from the definition of a flip graph that $e_B(v) - e_R(v) > r - b$. For any vertex v, the number of edges coloured B (respectively R) in the open neighbourhood is equal to the number of triangles rooted at v which have an edge coloured B (respectively R) not incident to v, so:

$$\begin{aligned} e_B(v) &= T_{BBB}(v) + T_{BRB}(v) + T_{RRB}(v)\,, \\ e_R(v) &= T_{RRR}(v) + T_{BBR}(v) + T_{BRR}(v)\,. \end{aligned}$$

Using these equalities and Lemma 5, we obtain:

$$\begin{aligned} &\sum_{v\in V} e_B(v) - e_R(v) \\ &= \sum_{v\in V} T_{BBB}(v) + T_{BRB}(v) + T_{RRB}(v) - T_{RRR}(v) - T_{BBR}(v) - T_{BRR}(v) \\ &\leq \sum_{v\in V} T_{BBB}(v) + \left(\sum_{v\in V} T_{BRB}(v) - T_{BBR}(v)\right) + \left(\sum_{v\in V} T_{RRB}(v) - T_{BRR}(v)\right) \end{aligned}$$

$$= \sum_{v \in V} T_{BBB}(v) + \sum_{v \in V} T_{BBR}(v) - \sum_{v \in V} T_{RRB}(v)$$
$$\leq \sum_{v \in V} T_{BBB}(v) + T_{BBR}(v)$$

where the next to last step follows from Lemma 5. Observe that $T_{BBB}(v) + T_{BBR}(v)$ is the number of edges incident with two blue neighbours of v hence is at most $\binom{b}{2}$, thus

$$\sum_{v \in V} e_B(v) - e_R(v) \leq |V| \binom{b}{2}.$$

It follows that there exists a vertex $v \in V$ such that $e_B(v) - e_R(v) \leq \binom{b}{2}$. Recalling that $r - b < e_B(v) - e_R(v)$ we have that $r - b < \binom{b}{2}$, so $r < \binom{b+1}{2}$. □

Proof (Proof of Theorem 4) That $3 \leq b < r \leq \binom{b+1}{2} - 1$ follows immediately from Proposition 6. We show that given such r and b, a (b, r)-flip graph exists.

Consider the CCP graph $G = K_{r,r} \square K_{b+1}$ where the edges of K_{b+1} are coloured blue and the edges of $K_{r,r}$ are coloured red. By Lemma 1, it follows that every vertex v in G has $\deg_B(v) = b$ and $\deg_R(v) = r$. Moreover, we have that $e_B[v] = \binom{b+1}{2}$ and $e_R[v] = r$. Hence G is a (b, r)-flip graph. □

3.1 Upper bounds on $h(b, r)$

As the graph G in the proof of Theorem 4 has $2r(b + 1)$ vertices, it follows that $h(b, r) \leq 2r(b + 1)$. On the other hand, we have already seen in Fig. 1 an example of a $(3, 4)$-flip graph with 16 vertices. Hence $h(3, 4) \leq 16$, so the aforementioned bound is not tight. The next result offers a more general construction, considerably improving the $2r(b + 1)$ upper bound.

Theorem 7 *Let $b, r \in \mathbb{N}$ such that $3 \leq b < r \leq \binom{b+1}{2} - 1$. Then,*

$$h(b, r) \leq \min \left\{ 2(r + x)(b + 1 - x) \colon x \in \mathbb{Z},\ 0 \leq x \leq b,\ x + \binom{b + 1 - x}{2} > r \right\}.$$

Proof Let x be an integer satisfying the theorem's condition. Consider an edge-colouring of $K_{r+x,r+x}$ such that an x-factor is coloured B and an r-factor is coloured R. Also consider K_{b+1-x} where all the edges are coloured B. Every vertex v of the CCP graph $G = K_{r+x,r+x} \square K_{b+1-x}$ has $\deg_B(v) = b - x + x = b$ and $\deg_R(v) = r$. Moreover, $e_B[v] = x + \binom{b+1-x}{2}$ and $e_R[v] = r$. By our choice of x, it follows that $e_B[v] > e_R[v]$. □

The upper bound in Theorem 7 warrants further analysis. We first require a lemma.

Lemma 8 *Let* $b, r \in \mathbb{N}$ *such that* $3 \le b < r \le \binom{b+1}{2} - 1$. *Let* $x_0 = \lceil b - (1 + \sqrt{1 + 8(r - b)})/2 \rceil - 1$. *Then,* $2(r + x_0)(b + 1 - x_0)$

$$= \min \left\{ 2(r + x)(b + 1 - x) \colon x \in \mathbb{Z},\ 0 \le x \le b,\ x + \binom{b + 1 - x}{2} > r \right\}.$$

Proof Let $g(z) = z + \binom{b+1-z}{2} - r$ be a real-valued function. As $2(r + x)(b + 1 - x)$ is strictly decreasing in $[0, \infty)$, the claimed minimum, in integer value, is attained for the largest possible integer $0 \le x_0 \le b$, such that $g(x_0) > 0$. Rearranging, $g(z)$ can be written as a quadratic in z,

$$g(z) = \binom{b + 1}{2} - r + \left(\frac{1}{2} - b\right) z + \frac{z^2}{2}$$

which has a minimum, as well as distinct roots $z_{\pm} = b - \frac{1 \mp \sqrt{1+8(r-b)}}{2}$.

Then $g(z) > 0$ whenever $z < z_-$ or $z > z_+$. Since the integer x_0 we are seeking must satisfy $x_0 \le b$, then the only admissible case when $g(x_0) > 0$ is when $x_0 < z_-$. Since we seek the largest such integer, then $x_0 = \left\lceil b - \frac{1+\sqrt{1+8(r-b)}}{2} \right\rceil - 1$. What remains is to ensure that $x_0 \ge 0$. It suffices to show that $b > \frac{1+\sqrt{1+8(r-b)}}{2}$. Rearranging, we require that $\frac{(2b-1)^2-1}{8} > r - b$. Indeed,

$$\frac{(2b - 1)^2 - 1}{8} = \frac{b^2}{2} - \frac{b}{2} = \binom{b + 1}{2} - b > r - b.$$

Hence x_0 is the largest integer for which the minimum is attained, as required. □

Substituting the value x_0 obtained in Lemma 8 and using Theorem 7 we obtain the following easily verified corollary:

Corollary 9 *Let* $b, r \in \mathbb{N}$ *such that* $3 \le b < r \le \binom{b+1}{2} - 1$. *Then*

$$h(b, r) \le 2 \left(r + b + 1 - \left\lfloor \frac{5 + \sqrt{1 + 8(r - b)}}{2} \right\rfloor \right) \left\lfloor \frac{5 + \sqrt{1 + 8(r - b)}}{2} \right\rfloor.$$

Notice that for the case $b = r - 1$, valid for all $r \ge 4$, the minimum is obtained at $x_0 = r - 4$. The flip graph $G = K_{2r-4,2r-4} \square K_4$ outlined in Theorem 7 is an $(r, r-1)$-flip graph with $16r - 32$ vertices. Even so, $(r - 1, r)$-flip graphs exist with fewer than $16r - 32$ vertices as seen from the (3, 4)-flip graph in Fig. 1.

3.2 Weak flip graphs

While $(2, r)$-flip graphs do not exist, weakening the flip constraint from $e_1[v] > e_2[v]$ to $e_1[v] \ge e_2[v]$ (yet still requiring $r > b$), an admissible colouring can be found such that $E(1)$ spans a b-regular subgraph and $E(2)$ spans an r-regular subgraph. We term such graphs (b, r)-*weak*-flip graphs.

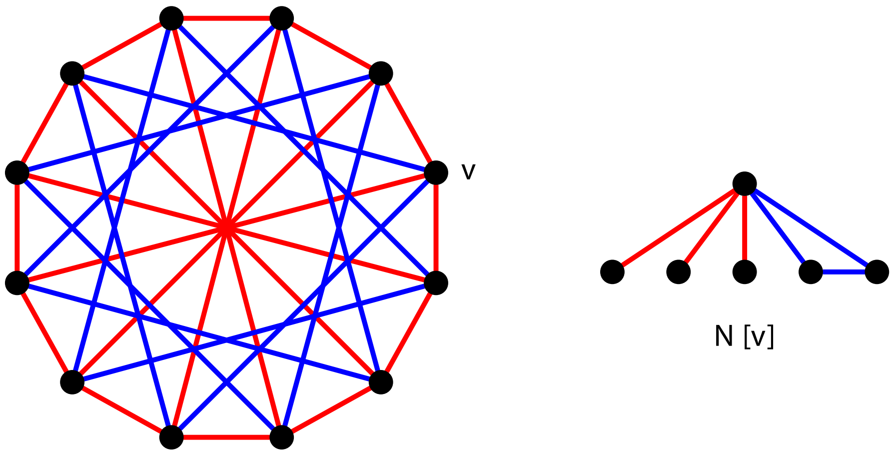


Fig. 5 Smallest known (2, 3)-weak-flip graph having 12 vertices, with the subgraph induced by the closed neighbourhood of any vertex v on the right

The CCP graph $K_{3,3} \square K_3$, with $K_{3,3}$ coloured red and K_3 coloured blue, is a (2, 3)-weak-flip graph on 18 vertices. Figure 5 illustrates the existence of a smaller (2, 3)-weak-flip graph, having just 12 vertices.

Likewise, $(1, r)$-flip graphs do not exist. However, not even $(1, r)$-weak-flip graphs exist.

Proposition 10 *Let $r \in \mathbb{N}$ such that $r > 1$. There is no $(1, r)$-weak-flip graph.*

Proof Consider first the case $r > 2$. Suppose that a $(1, r)$-weak-flip graph G exists. Let v be a vertex in such a graph. Then $e_1[v] \leq 1 + \lfloor \frac{r}{2} \rfloor < r \leq e_2[v]$ since $r > 2$. In the case when $r = 2$, suppose that a (1, 2)-weak-flip graph G exists. Then by the above argument, it follows that $e_1[v] = e_2[v] = 2$. Consider a vertex v with neighbours u, w and x such that $\{v, w\}$ and $\{v, u\}$ are coloured 2 and $\{v, x\}$ is coloured 1. Clearly $\{u, w\}$ must be coloured 1 since $e_1[v] = e_2[v] = 2$. But then u has some neighbour y different from v and w, such that $\{u, y\}$ is coloured 2. Hence $e_2[u] \geq 3$, a contradiction. □

4 Flipping with three or more colours

Unlike the case of two colours where (b, r)-flip sequences are completely characterised in Theorem 4, for $k \geq 3$ colours we do not have a full characterisation of all k-flip sequences. Our first result, nonetheless, establishes a necessary condition for 3-flip sequences.

Theorem 11 *If (a_1, a_2, a_3) is a 3-flip sequence, then $a_3 < 2(a_1)^2$.*

Proof Suppose on the contrary that (a_1, a_2, a_3) is a flip sequence realised by some graph G, but that $2a_1^2 \leq a_3$. We shall prove that for some vertex v of G, $e_1[v] \leq e_2[v]$ or $e_1[v] \leq e_3[v]$.

For $i \in \{1, 2, 3\}$, let $N_i(v) = \{u \in V : \{u, v\} \in E(i)\}$. Clearly, there is a set of (at least) $e_1[v] - (a_1)^2$ edges of $N[v]$ coloured 1, having both of their endpoints in $N_2(v) \cup N_3(v)$. We may assume $e_1[v] - (a_1)^2 \geq 0$, otherwise we are done.

Now each of these $e_1[v] - (a_1)^2$ edges forms a unique triangle with v, where only one edge of the triangle is coloured 1. The other two edges of such a triangle contribute two edges coloured using either 2 or 3, to some open neighbourhood. This means that the total number of edges coloured using 2 or 3 in all open neighbourhoods is at least $\sum_{v \in V} 2\left(e_1[v] - (a_1)^2\right)$. Hence there is some vertex v with at least $2\left(e_1[v] - (a_1)^2\right)$ edges coloured using 2 or 3 in its open neighbourhood. Therefore,

$$e_2[v] + e_3[v] \geq a_3 + a_2 + 2e_1[v] - 2(a_1)^2.$$

But $a_3 + a_2 + 2e_1[v] - 2(a_1)^2 \geq 2e_1[v]$ since $a_3 + a_2 - 2(a_1)^2 \geq a_3 - 2(a_1)^2 \geq 0$.

Hence $e_3[v] + e_2[v] \geq 2e_1[v]$. But this means that $e_1[v] \leq e_2[v]$ or $e_1[v] \leq e_3[v]$, which is a contradiction since G is a flip graph. □

In view of Theorem 11, it is of interest to find construction of 3-flip sequences in which a_3 is quadratic in a_1. To this end, we use the following proposition.

Proposition 12 *Let $a_1, a_2, a_3 \in \mathbb{N}$ such that $a_1 < a_2 < a_3$.*

1. *If H is an (a_2, a_3)-flip graph for which $e_2[v] < \binom{a_1+1}{2}$ for each vertex v, then (a_1, a_2, a_3) is a 3-flip sequence.*
2. *Suppose that H_1, H_2 and H_3 are, respectively, a_1, a_2 and a_3 regular. If for $i \in \{1, 2\}$, we have that*

$$\max_{u \in V(H_{i+1})} e[u] < \min_{v \in V(H_i)} e[v]$$

then (a_1, a_2, a_3) is a 3-flip sequence.

Proof To prove (i), consider the CCP graph $G = H \,\square\, K_{a_1+1}$ where H is coloured using 2 and 3, and K_{a_1+1} is coloured using 1. Then G is a graph with colour degrees a_i for colour $i \in \{1, 2, 3\}$. Moreover, $e_3[v] < e_2[v] < \binom{a_1+1}{2} = a_1[v]$. Hence G is an (a_1, a_2, a_3)-flip graph and we are done.

For (ii), consider the CCP graph $G = H_1 \,\square\, H_2 \,\square\, H_3$, where H_i is coloured using i for $i \in \{1, 2, 3\}$. The result immediately follows by the conditions necessitated in (ii). □

Constructing 3-flip sequences (a_1, a_2, a_3) using Proposition 12 is relatively easy when a_2 and a_3 are not too far apart. For example, through Theorem 7, we have seen the existence of $(r - 1, r)$-flip graphs with $e_1[v] = r + 2$ and $e_2[v] = r$. Hence with $a_2 = r - 1$, $a_3 = r$ and $a_1 < a_2$ such that $r + 2 < \binom{a_1+1}{2}$, it follows that (a_1, a_2, a_3) is a 3-flip sequence. Note that we can take $a_1 = \lceil \sqrt{2r} \rceil$ for $r \geq 9$. This shows hat we can have $a_3 \approx 0.5(a_1)^2$, so Theorem 11 is tight up to a constant factor.

4.1 Unbounded gap *k*-flip sequences

We have seen that for two or three colours, the largest element in a flip sequence must be bounded above quadratically in the smallest element. A natural question is whether this extends to four or more colours. We shall see that the answer is negative—indeed, we shall see constructions where the smallest element is constant and yet the largest element may be arbitrarily large.

We must first recall another handy graph product, the *strong product*, along with a way to inherit an edge colouring from its factors.

Definition 2 (Strong product)The strong product $H \boxtimes K$ of two graphs H and K is the graph such that $V(H \boxtimes K) = V(H) \times V(K)$ and there is an edge $\{(u, v), (u', v')\}$ in $H \boxtimes K$ if and only if either $u = u'$ and $vv' \in E(K)$, or $v = v'$ and $uu' \in E(H)$, or $uu' \in E(H)$ and $vv' \in E(K)$.

Let H and K be two graphs with an edge-colouring from a set of colours C. We extend the edge-colourings of H and K to an edge-colouring of $H \boxtimes K$ as follows. Consider edge $e = \{(u, v), (u', v')\}$ in $H \boxtimes K$; if $u = u'$ then e inherits the colouring of the edge vv' in K, otherwise if $u \neq u'$ the colouring of the edge vv' in H is inherited. Note that the inherited colourings of $H \boxtimes K$ and $K \boxtimes H$ are different, even though the two uncoloured graphs are isomorphic.

We are now in a position to prove the main result of this section.

Theorem 13 *Let $k \in \mathbb{N}, k > 3$. Then there is some constant $m = m(k) \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, there exists a k-flip sequence $(a_1, a_2, \ldots, a_k)$ such that $a_1 = m$ and $a_k > N$.*

Proof Let K be the complete graph K_{2n} where $n > \dfrac{k(k^2 - 2k + 1)}{4(k - 3)}$. Since K is a complete graph on an even number of vertices, K has a 1-factorisation. For $1 < i < k$, let $k - i$ 1-factors be coloured using colour i and let the remaining edges be coloured 1. It follows then that every vertex v in K_{2n} has

$$\deg_1^K(v) = 2n - 1 - \binom{k-1}{2}$$

incident edges coloured 1 and $\deg_i^K(v) = k - i$ incident edges coloured i for $1 < i < k$. For convenience, define $\deg_k^K(v) = 0$. Observe that the sequence $\deg_1^K(v), \ldots, \deg_k^K(b)$ is strictly decreasing, noting that by the choice of n,

$$\deg_1^K(v) - \deg_2^K(v) > \frac{k(k-1)}{k-3} > 0.$$

Since K is a complete graph and each vertex v has the same number of incident edges coloured i, then $e_i^K[v] = n \deg_i^K(v)$ for $1 \leq i \leq k$.

We now show that for every vertex v in K, $(k-1)(e_1^K[v] - e_2^K[v]) > 4n^2$. Rearranging and substituting for $e_1^K[v]$ and $e_2^K[v]$ in terms of n and k, we must show that $n > \dfrac{k(k^2-2k+1)}{4(k-3)}$. This follows by our choice of n.

Consider $t \in \mathbb{N}$ such that

$$t \geq \frac{4n^2}{(k-1)\min\limits_{1<i<k}\{e_i^K[v] - e_{i+1}^K[v]\}} = \frac{4n}{k-1}.$$

Let $\rho = \dfrac{(k-1)(2t+k-2)}{2}$ and let H be a ρ-regular bipartite graph. For $0 \leq i \leq k-2$, let $t+i$ matchings of H be coloured using colour $2+i$.

Let G be the graph $H \boxtimes K$, inheriting the edge-colourings of H and K respectively. For a vertex u in H, let G_u be the subgraph of G induced by the vertices $\{(u, w)\colon w \in V(K)\}$ in G; note that G_u is isomorphic to K.

If an edge ux in H is coloured i, then all the edges in G between $V(G_u)$ and $V(G_x)$ are coloured i, by the inherited edge-colouring. Hence, if u in H and v in K have $\deg_i^H(u)$ and $\deg_i^K(v)$ incident edges coloured i, respectively, then the vertex (u, v) in G has $\deg_i^K(u) + |V(K)| \deg_i^H(v)$ incident edges coloured i. Consequently, by construction each vertex (u, v) in G has:

$$\deg_i^G\big((u,v)\big) = \begin{cases} 2n-1-\binom{k-1}{2} & i=1 \\ (k-i)+2n(t+i-2) & 2 \leq i \leq k \end{cases}$$

which is strictly increasing.

Now, consider the vertex (u, v) in G. Let $u_1, \ldots, u_\rho$ be the neighbours of u in H. In G we have that every vertex in G_u has a neighbour in G_{u_i} for $1 \leq i \leq \rho$, but for $1 \leq i < j \leq \rho$, we have that there are no edges between G_{u_i} and G_{u_j} in G, since H is bipartite and u_i and u_j belong to the same partite set. Therefore, the edges coloured i in the closed neighbourhood of (u, v) are:

1. those edges coloured i in the closed neighbourhood of v in each of the $\rho+1$ copies of K (namely $G_u, G_{u_1}, \ldots, G_{u_\rho}$), totalling $(\rho+1)e_i^K[v]$ edges,
2. and a further $e_i^H[u]|V(K)|^2$ edges from the matchings coloured i in H.

This is clearly exemplified in Fig. 6. By construction of H and K, it follows that for any vertex (u, v) of G,

$$e_i^G[(u,v)] = \begin{cases} (\rho+1)e_1^K[v] & i=1 \\ (\rho+1)e_i^K[v] + 4n^2(t+i-2) & 2 \leq i \leq k \end{cases}$$

which we now show to be strictly decreasing.

Firstly note that since $\rho + 1 = (k-1)t + \binom{k-1}{2} + 1$, then there exists $\kappa \in \mathbb{R}$ such that $\kappa > 1$ and $\rho + 1 = (k-1)t\kappa$. Now, recall that K has the property that $(k-1)(e_1^K[v] - e_2^K[v]) > 4n^2$.

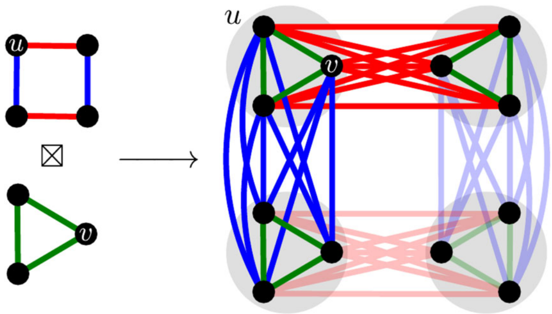


Fig. 6 Illustration of the edge-colouring inheritance in $K_{2,2} \boxtimes K_3$ from its factors, where $K_{2,2}$ has red and blue coloured 1-factor, and K_3 is coloured green. The edges in the closed neighbourhood of (u, v) are highlighted

Since $\kappa > 1$, it follows that

$$(\rho + 1)\left(e_1^K[v] - e_2^K[v]\right) = (k-1)\left(e_1^K[v] - e_2^K[v]\right)(t\kappa) > 4n^2t$$

and therefore since $e_1^K[v] > e_2^K[v]$, we have that $(\rho+1)e_1^K[v] > (\rho+1)e_2^K[v] + 4n^2t$. Consequently, $e_1^G[(u, v)] > e_2^G[(u, v)]$ as required.

Now consider $2 \leq i \leq k-1$. By the choice of t and $\kappa > 1$, we have that

$$\begin{aligned}(\rho + 1)\left(e_i^K[v] - e_{i+1}^K[v]\right) &= (k-1)\left(e_i^K[v] - e_{i+1}^K[v]\right)(t\kappa)\\ &> 4n^2\\ &= 4n^2(t+i-1) - 4n^2(t+i-2)\end{aligned}$$

and therefore $e_i^G[(u, v)] > e_{i+1}^G[(u, v)]$.

It follows that G is a flip graph on $k > 3$ colours, such that for any vertex (u, v), $\deg_1\left((u, v)\right)$ is only dependent on k and $\deg_k\left((u, v)\right)$ increases with t. Since t is not bounded from above in the construction, then given any $N \in \mathbb{N}$, a sufficiently large t can be found such that $\deg_k\left((u, v)\right) > N$. □

5 (r, c)-Constant graphs, long flipping intervals, and sufficient conditions for k-flip sequences

This section explores the notion of (r, c)-constant graphs, first introduced in Sect. 2, and their use in constructing long flipping intervals. This will then allow us to deduce a sufficient condition for k-flip sequences.

5.1 Existence of (r, c)-constant graphs

We recall that an (r, c)-constant graph H is an r-regular graph such that for every vertex $v \in V(H)$, $e(v) = c$. We have already seen, as summarised in Fig. 3, that many familiar classes of graphs are subclasses of (r, c)-constant graphs.

An inevitable problem regarding (r, c)-constant graphs is: given a positive integer r, for which integers c, $0 \le c \le \binom{r}{2}$, does there exist an (r, c)-constant graph?

The *spectrum* of r, denoted by $\mathsf{spec}(r)$, is the set of all such integers c such that an (r, c)-constant graph exists. The following theorem shows that $\mathsf{spec}(r)$ contains nearly all c in the interval $\left[0, \binom{r}{2}\right]$.

Theorem 14 *(Existence of (r, c)-constant graphs) Let $r \in \mathbb{N}$.*

1. *For every integer c such that $0 \le c \le \dfrac{r^2}{2} - 5r^{\frac{3}{2}}$, $c \in \mathsf{spec}(r)$.*
2. *Suppose $k \in \mathbb{N}$ and $r \ge 3k$. Then $\binom{r}{2} - k \notin \mathsf{spec}(r)$.*

Proof First recall that for every $r, c \in \mathbb{N}$ such that $0 \le c \le r^2/2 - 5r^{3/2}$, there are positive integers $a_1, \ldots, a_s$ summing to r such that $\sum_{j=1}^{s} \binom{a_j}{2} = c$; see [15] and [16], Lemma 3.4. Consider the Cartesian product $G = \square_{j=1}^{s} K_{a_j+1}$. Then G is r-regular with $r = \sum_{j=1}^{s} a_j$, and $e(v) = \sum_{j=1}^{s} \binom{a_j}{2} = c$, so (i) follows.

We proceed to prove (ii). Suppose the contrary, and let H be an r-regular graph where each vertex v has $e(v) = \binom{r}{2} - k$. Then every vertex in $N(v)$ has at least $r - k$ neighbours in $N[v]$.

Let x be a vertex in $N(v)$ with at most $r - 1$ neighbours in $N[v]$. Observe that the k non-edges in the subgraph induced by $N[v]$ span at most $2k$ vertices. Hence there are $r + 1 - 2k \ge k + 1$ vertices in $N[v]$, including v, whose r neighbours are all in $N[v]$ and therefore they are all neighbours of x.

Now consider $N[x]$, which contains some vertex w not in $N[v]$, since x has at most $r - 1$ neighbours in $N[v]$. Then this vertex w is not adjacent to at least $k + 1$ vertices in $N[x] \cap N[v]$ (those having degree r in $N[v]$). Therefore w has at most $r - k - 1$ neighbours in $N[x]$. Consequently, $e(x) \le \binom{r}{2} - k - 1$, a contradiction. □

5.2 Flipping intervals and *k*-flip sequences

The usefulness of (r, c)-constant graphs stems from the fact that together with CCP, they serve as the building blocks for long interval flips. In particular, we obtain a sufficient condition for k-flip sequences.

Theorem 15 *Let $b \in \mathbb{N}$.*

1. *If $b \ge 101$ then $\left[b, b + \left\lfloor \frac{1}{4}\left(b^2 - 10b^{\frac{3}{2}}\right)\right\rfloor\right]$ is a flipping interval.*
2. *If $b \ge 3$ then $[b, 2b - 2]$ is a flipping interval.*

Proof For (i), consider the interval $[b, b + k]$ where $k = \lfloor (b^2 - 10b^{\frac{3}{2}})/4 \rfloor$. Since $b \ge 101$, we have $k \ge 12$. Set $M_1 = \lfloor b^2/2 - 5b^{3/2} \rfloor$. For $1 \le j \le k$, set H_j to be a $(b + j - 1, M_1 - 2(j - 1))$-constant graph which exists by Theorem 14 and observe $M_1 \ge 2k \ge 2(j - 1)$ for $1 \le j \le k$. Consider the CCP graph $G = \square_{j=1}^{k} H_j$ where H_j is coloured using colour j. Then by Lemma 2, G is a $(b, b + 1, \ldots, b + k)$-flip graph and therefore (i) follows.

We now prove (ii). Consider the triple $(b + j, b + 1 - j, 2j)$ where $0 \le j \le b - 2$. Consider a regular bipartite graph H of degree $\sum_{j=1}^{b-2} 2j = (b - 1)(b - 2)$, where a $2j$-factor of H is coloured j for $1 \le j \le b - 2$.

We may take H to be $K_{n,n}$ where $n = (b-1)(b-2)$. Now consider the CCP graph $G = H \,\square\, \left(\square_{j=1}^{b-2} K_{b+1-j}\right)$ where K_{b+1-j} is coloured j for $1 \le j \le b-2$.

In every vertex v of G, for $1 \le j \le b-2$, $\deg_j(v) = b+j$ which is increasing and $e_j[v] = \binom{b+1-j}{2} + 2j$, which is decreasing. Hence G is a $(b, b+1, \ldots, 2b-2)$-flip graph. □

We note that for $b \le 107$ the interval $[b, 2b-2]$ contains $[b, b+\lfloor(b^2-10b^{\frac{3}{2}})/4\rfloor]$.

Corollary 16 *(Sufficient condition for k-flip sequences) Suppose that* $k \ge 2$. *Let* $3 \le a_1 < a_2 < \cdots < a_k$ *be a sequence of integers such that either* $a_k \le 2a_1 - 2$ *or* $a_k \le a_1 + \left\lfloor \frac{1}{4}\left((a_1)^2 - 10(a_1)^{\frac{3}{2}}\right)\right\rfloor$, *then* $(a_1, \ldots, a_k)$ *is a k-flip sequence.*

Proof In both cases, the sequence $(a_1, \ldots, a_k)$ is a subsequence of a flipping interval in Theorem 15 and hence a k-flip sequence, as we can consider from the construction of (i) or (ii) in Theorem 15 the edge-induced subgraph by the colours corresponding to the subsequence. □

Observe that the existence of (r, c)-constant graphs can be used to construct a wider class of graphs than just flip graphs. Namely k-edge coloured graphs, with prescribed colour degrees and edge-coloured neighbourhood sizes, in which for all vertices v, $\deg_j(v) = r_j$ and $e_j[v] = m_j$ for some positive integers r_j, m_j satisfying $m_j \ge r_j$, $1 \le j \le k$. To achieve this, we only need to assure the existence of (r_j, c_j)-constant graphs H_j such that $m_j = r_j + c_j$ for $1 \le j \le k$, and then consider the CCP graph $\square_{j=1}^{k} H_j$.

6 Existence of *t*-neighbourhood flip graphs

We have so far considered flip colourings with regards to the immediate neighbourhood of a vertex. A natural extension of this problem, as outlined in Problem 5 in the introduction, is to consider the neighbourhood consisting of all vertices at a distance (at most) t from v.

Before proceeding further, we require an adaptation of our existing notation. Let $k \in \mathbb{N}$ and let G be a graph with an edge-coloring from $\{1, \ldots, k\}$. As before, $E(j)$ is the set of edges coloured $j \in \{1, \ldots, k\}$ in G.

1. For two vertices u and v, $d(u, v)$ is the *distance* between u and v, i.e., the length of a shortest path between the two vertices.
2. For a vertex v and $t \in \mathbb{N}$, $N_t(v) = \{u \in V(G)\colon 1 \le d(u, v) \le t\}$ is its *open* t-neighbourhood.
3. For a vertex v and $t \in \mathbb{N}$, $N_t[v] = N_t(v) \cup \{v\}$ is its *closed* t-neighbourhood.
4. For a vertex v and $t \in \mathbb{N}$, $e_{j,t}(v) = |E(j) \cap E(N_t(v))|$.
5. For a vertex v and $t \in \mathbb{N}$, $e_{j,t}[v] = |E(j) \cap E(N_t[v])|$.

By $\Delta(G)$ and $g(H)$ we shall denote, as usual, the maximum degree of G and the girth (shortest cycle length) of G, respectively. We now extend our general flip colouring problem to t-neighbourhoods: Given a graph $G = (V, E)$, and $k, t \in \mathbb{N}$

such that $k \geq 2$, does there exist an edge-colouring $f: E(G) \to \{1, \ldots, k\}$ such that for every vertex v

- $\deg_j(v) > \deg_i(v)$ for $1 \leq i < j \leq k$ (forcing global majority $e_j(G) > e_i(G)$),
- for $1 \leq s \leq t$, $e_{j,s}[v] < e_{i,s}[v]$ for $1 \leq i < j \leq k$ (forcing opposite majority order up to a distance t from v, with respect to the global $e_j(G)$ and the local $\deg_j(v)$).

If such a colouring exists, then G is said to be a $([t], k)$-flip graph (with respect to f). When we do not concern ourselves with the number of colours used, we simply say that G is a $[t]$-flip graph.

As before, we shall mostly consider a more restricted version of this problem, where for every $j \in \{1, \ldots, k\}$, the edge set $E(j)$ spans a regular subgraph of degree a_j, where $a_1 < a_2 < \cdots < a_k$ and resulting in a $([t], (a_1, \ldots, a_k))$-flip graph.

We shall restrict to the case of two colours, demonstrating the existence of $([t], 2)$-flip graphs from two different perspectives, namely through Cayley graphs and packing arguments.

6.1 Constructing $([t], 2)$-flip graphs through Cayley graphs

This subsection is devoted to constructing of $([t], 2)$-flip graphs through Cayley graphs.

Let Γ be an abelian group and let S be a subset of Γ such that S is inverse-closed and does not contain the identity element. Recall that the *Cayley graph* $\mathsf{Cay}(\Gamma; S)$ is a graph with vertex set Γ and edge set $\{\{g, gs\}: g \in \Gamma, s \in S\}$. The set S is often termed as the *connecting set* of the Cayley graph.

By $\mathbf{0}$ we shall denote the all-zeros vector in $\mathbb{Z}_2^n$. Given any $i \in \{1, \ldots, n\}$, let $\mathbf{e}_i$ be the vector in $\mathbb{Z}_2^n$ which is 1 at the i^{th} position, and 0 everywhere else. Given any $1 \leq s \leq n$ and $0 \leq j \leq n - s$, we denote by $W_{s,j}$ the set of all binary vectors with the first s entries all zero and exactly j nonzero subsequent entries, namely

$$W_{s,j} = \left\{ \mathbf{w} \in \mathbb{Z}_2^n : \mathbf{w} \cdot \left(\sum_{i=1}^{s} \mathbf{e}_i \right) = 0 \ \wedge \ \mathbf{w} \cdot \left(\sum_{i=s+1}^{n} \mathbf{e}_i \right) = j \right\}.$$

Having established our working notation, we proceed to prove our main result for this subsection.

Theorem 17 *Let $s, t \in \mathbb{N}$, $t < s$. There exists a $([t], (2^s - 1, 2^s))$-flip graph.*

Proof Let $n = 2^s + s$. Consider $B = \operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_s\} \backslash \{\mathbf{0}\}$ and notice that $|B| = 2^s - 1$. Let $R = \{\mathbf{e}_{s+1}, \ldots, \mathbf{e}_n\}$; by our choice of n, we have that $|R| = 2^s$. Consider $\mathsf{Cay}\left(\mathbb{Z}_2^n; R \cup B\right)$ with the edge-colouring $f: E \to \{1, 2\}$ such that given $v \in \mathbb{Z}_2^n$ and $\alpha \in R \cup B$:

$$f(\{v, \alpha v\}) = \begin{cases} 1, & \alpha \in B \\ 2, & \alpha \in R \end{cases}$$

Note that $R \cup B$ spans $\mathbb{Z}_2^n$, $R \cup B$ is inverse-closed, and $R \cap B = \emptyset$. Hence, f is well-defined.

Since a Cayley graph is vertex-transitive, it suffices to consider a single vertex. Consider $\mathbf{0} \in \mathbb{Z}_2^n$; by the edge-colouring f, we have that

$$\deg_1(\mathbf{0}) = 2^s - 1 < 2^s = \deg_2(\mathbf{0}).$$

Now, $N_1(\mathbf{0}) = R \cup B$; we will add each of R and B to $N_1(\mathbf{0})$ so that we find the vertices in $N_2(\mathbf{0})$. Since $B \cup \{\mathbf{0}\}$ is a vector space, then in particular $B + B = B$. On the other hand, adding B to R results in the set $B + W_{s,1}$. Hence, $(R \cup B) + B = B + W_{s,1}$. Since neither R nor B includes $\mathbf{0}$, it follows that $(R \cup B) \cap (B + W_{s,1}) = \emptyset$ and therefore $B + W_{s,1} \subseteq N_2(\mathbf{0})$. Likewise, adding R to B gives $B + W_{s,1}$ once more and $R + R = W_{s,2}$. Therefore,

$$N_2(\mathbf{0}) = \left(B + W_{s,1}\right) \,\dot{\cup}\, W_{s,2}$$

and repeating the above argument for $1 \leq j < t$, we get that:

$$N_{j+1}(\mathbf{0}) = \left(B + W_{s,j}\right) \,\dot{\cup}\, W_{s,j+1}.$$

Note that for $\mathbf{w} \in W_{s,j}$, $B + \mathbf{w}$ is a clique isomorphic to K_{2^s-1} since $\mathbf{w}$ is not in the span of B. More so, for $\mathbf{w}_1, \mathbf{w}_2 \in W_{s,j}$ such that $\mathbf{w}_1 \neq \mathbf{w}_2$, we have that $(B + \mathbf{w}_1) \cap (B + \mathbf{w}_2) = \emptyset$. Therefore, $(2^s - 1)\binom{n-s}{j} = (2^s - 1)\binom{2^s}{j}$ edges coloured 1 arise between $W_{s,j}$ and $B + W_{s,j}$. Observe also that $(R + N_j(\mathbf{0})) \cap N_j(\mathbf{0}) = \emptyset$, and therefore the subgraph induced by $N_j(\mathbf{0})$ contains no edges coloured 2.

The edges coloured 1 between $N_j(\mathbf{0})$ and $N_{j+1}(\mathbf{0})$ arise by adding B to $W_{s,j}$ and therefore by our previous remark there are $(2^s - 1)\binom{2^s}{j}$ such edges. Meanwhile, the edges coloured 2 between $N_j(\mathbf{0})$ and $N_{j+1}(\mathbf{0})$ arise by adding R to $N_j(\mathbf{0})$. Adding R to $W_{s,j}$ results in $(2^s - j)|W_{s,j}| = (2^s - j)\binom{2^s}{j}$ edges coloured 2 between $N_j(\mathbf{0})$ and $N_{j+1}(\mathbf{0})$. On the other hand, adding R to $B + W_{s,j-1}$ maps each clique in $B + W_{s,j-1}$ to a total of $2^s - j + 1$ cliques in $B + W_{s,j}$, with a perfect matching between every such pair of cliques. Therefore, there are an additional $(2^s - 1)(2^s - j + 1)\binom{2^s}{j-1}$ edges coloured 2 between $N_j(\mathbf{0})$ and $N_{j+1}(\mathbf{0})$.

By our previous remark, the subgraph induced by $N_{j+1}(\mathbf{0})$ contains no edges coloured 2 and hence it follows that:

$$e_{2,j+1}[\mathbf{0}] = e_{2,j}[\mathbf{0}] + (2^s - 1)(2^s - j + 1)\binom{2^s}{j-1} + (2^s - j)\binom{2^s}{j} \tag{1}$$

while between the vertices in $N_{j+1}(\mathbf{0})$ there are $\binom{n-s}{j} = \binom{2^s}{j}$ cliques coloured 1 which are isomorphic to K_{2^s-1} and therefore:

$$e_{1,j+1}[\mathbf{0}] = e_{1,j}[\mathbf{0}] + \binom{2^s - 1}{2}\binom{2^s}{j} + (2^s - 1)\binom{2^s}{j} = e_{1,j}[\mathbf{0}] + \binom{2^s}{2}\binom{2^s}{j}. \tag{2}$$

Now,

$$\begin{aligned}(2^s-1)(2^s-j+1)\binom{2^s}{j-1}+(2^s-j)\binom{2^s}{j} &= (2^s-1)j\binom{2^s}{j}+(2^s-j)\binom{2^s}{j}\\ &\le (2^s-1)(j+1)\binom{2^s}{j}\\ &< \binom{2^s}{2}\binom{2^s}{j}\end{aligned}$$

where the last inequality follows from $j+1 \le t \le s-1 < \frac{2^s}{2}$. Consequently, from (1) and (2), we have for $1 \le j < t$ that if $e_{2,j}[\mathbf{0}] < e_{1,j}[\mathbf{0}]$, then

$$e_{2,j+1}[\mathbf{0}] < e_{1,j+1}[\mathbf{0}]\ . \tag{3}$$

Hence, it only remains to show that $e_{1,1}[\mathbf{0}] > e_{2,1}[\mathbf{0}]$. By choice of R and B, we have that the vertices of B in $\mathsf{Cay}\left(\mathbb{Z}_2^n; R \cup B\right)$ induce the complete graph K_{2^s-1} and therefore

$$e_{1,1}[\mathbf{0}] = (2^s-1) + \binom{2^s-1}{2} = \binom{2^s}{2}$$

while the the vertices in R are all linearly independent and hence $e_{2,1}[\mathbf{0}] = 2^s$. Hence, indeed, $e_{1,1}[\mathbf{0}] > e_{2,1}[\mathbf{0}]$. □

6.2 Constructing ([t], 2)-flip graphs through packings

In this subsection we construct $([t], 2)$-flip graphs using two classical graph theoretic results, concerned with the existence of r-regular graphs with large girth, and with graph packings.

Theorem 18 *(Erdős-Sachs [17–19]) Given integers $r \ge 2$ and $k \ge 3$, there are infinitely many connected r-regular graphs with girth at least k.*

Theorem 19 *(Sauer-Spencer-Catlin [20, 21]) Let G and H be two graphs on n vertices, such that $2\Delta(G)\Delta(H) < n$. Then there exists a packing of G and H into an n vertex set, with no overlapping edges.*

A rooted tree of is (j, b)-*perfect* if every internal vertex has b children and all leaves are at distance j from the root.

Theorem 20 *Let $t \in \mathbb{N}$. There exist $([t], 2)$-flip graphs.*

Proof Let $b, r \in \mathbb{N}$ such that for some $q \in \mathbb{N}$, $q \ge 2$, we have that $(q+1)b \ge r \ge 2b+1$ and $b \ge 2(q+3)^t$. We will construct a $\big([t], (2b, r)\big)$-flip graph.

Suppose that G^* and H^* are connected graphs such that G^* is r-regular with girth $g(G^*) > 2((q+3)b)^t$, and H^* is $b+1$ regular with sufficiently large girth. Note that $L(H^*)$, the line graph of H^*, is $2b$-regular. We shall assume subsequently that G^* and

H^* are as large as necessary. The existence of such graphs G^* and H^* is guaranteed by Theorem 18.

Let $p, p' \in \mathbb{N}$ such that $2\Delta(G^*)\Delta(L(H^*)) < p\,|V(G^*)| = p'\,|V(L(H^*))|$. Then let $G = pG^*$ and $H = p'L(H^*)$. Since G and H are the union of disjoint copies of G^* and $L(H^*)$ respectively, we have that $\Delta(G) = \Delta(G^*)$ and $\Delta(H) = \Delta(L(H^*))$. Furthermore, $g(G) = g(G^*)$. We will colour the edges of H using 1 and the edges of G using 2.

Consider the vertex $v = \{x, y\}$ in H (so xy is an edge of some copy of H^*). Since the girth of H^* is sufficiently large, and in particular much larger than t, it follows that in H^*, x and y are roots of two disjoint copies of a $(j+1, b)$-perfect tree T, for $1 \le j \le t$. Joining these two trees by the edge $\{x, y\}$, the line graph of the resulting graph is two copies of some block graph with $(b+1)$-cliques, coalesced at the vertex $\{x, y\}$. The number of $(b+1)$-cliques is, by virtue of T being a $(j+1, b)$-perfect tree,

$$2\left(\sum_{i=0}^{j-1} b^i\right) = 2\left(\frac{b^j - 1}{b - 1}\right)$$

and consequently, noting that all the edges in H are coloured 1, we have that

$$e_{j,1}^{H}[v] = 2\left(\frac{b^j - 1}{b - 1}\right)\binom{b+1}{2} = \frac{(b^{j+1} - b)(b+1)}{b - 1} > b^{j+1} \tag{4}$$

for $1 \le j \le t$.

Now, these two graphs G and H can be packed by Theorem 19 into a graph Q with no overlapping edges, while preserving their edge colourings. By this packing, Q is $r + 2b$ regular, where every vertex has $2b$ incident edges coloured 1 and r incident edges coloured 2. We will show that for any vertex v, $e_{j,1}[v] > e_{j,2}[v]$ in Q for $1 \le j \le t$, and hence Q is a $([t], (2b, r))$-flip graph.

We first compute an upper bound for $|N_j[v]|$ for any $v \in V(Q)$, observing that as Q is $r + 2b$ regular, and G^* and H^* are connected and can be arbitrarily large, it follows that $N_{j-1}[v]$ is strictly contained in $N_j[v]$ for $2 \le j \le t$. Firstly observe that for an s-regular graph we have that

$$|N_j[v]| \le 1 + s\sum_{i=0}^{j-1}(s-1)^i < 2s^j.$$

Since Q is $r + 2b$ regular and $r \le (q+1)b$, by the inequality it holds for $1 \le j \le t$ that

$$|N_j[v]| < 2(r + 2b)^j \le 2((q+3)b)^j.$$

Due to the girth condition on G, we have that for $1 \leq j \leq t$, $g(G) > |N_j[v]|$. Hence, the subgraph in Q induced by the edges coloured 2 in $N_j[v]$ is acyclic, and therefore

$$e^{Q}_{j,2}[v] < |N_j[v]| < 2((q+3)b)^j,$$

while by (4)

$$e^{Q}_{j,1}[v] \geq e^{H}_{j,1}[v] \geq b^{j+1}.$$

Therefore, for $1 \leq j \leq t$, to flip the majority in the j^{th} neighbourhood of v we require that $2((q+3)b)^j \leq b^{j+1}$, which simplifies to $2(q+3)^j \leq b$, which is the case since $2(q+3)^t \leq b$ and $j \leq t$. Hence Q is a $([t], (2b, r))$-flip graph. □

7 Concluding remarks and open problems

We have provided an in-depth treatment of most of the problems mentioned in the introduction, yet, nonetheless, several open problems remain. In particular, the complexity aspect of flip sequences and flip graphs (Problems 6). We next summarise a few additional open problems.

We have seen a comprehensive treatment of the two-colour case in Sect. 3. However, with it still remains open the determination of $h(b, r)$, the smallest order of an (b, r)-flip graph.

Problem *Determine $h(b, r)$ or at least obtain a nontrivial lower bound.*

For three or more colours, Problem 2 remains entirely open.

Regarding three colors, in Theorem 11 we have seen a necessary condition for a sequence (a_1, a_2, a_3) to be a 3-flip sequence. In view of this theorem, it is of interest to find constructions of 3-flip sequences (a_1, a_2, a_3) with as large as possible a constant c such that $a_3 = c(a_1)^2$, where we have seen that $\frac{1}{2} \leq c \leq 2$.

Problem *Determine the supremum over all constants c such that there exist infinitely many 3-flip sequences (a_1, a_2, a_3) satisfying $a_3 \geq c(a_1)^2$.*

Whilst for two and three colours we have a necessary condition for flip sequences, namely that the largest colour-degree is at most quadratic in the smallest colour-degree, we have proved in Theorem 13 that there is no such condition for $k \geq 4$ colours. In fact, we have shown that there exists some $m(k) \in \mathbb{N}$ such that given any $N \in \mathbb{N}$, there is a k-flip sequence $(a_1, \ldots, a_k)$ where $a_1 = m(k)$ and $a_k > N$. Explicitly from our construction we establish a linear upper bound on $m(k)$.

Problem *Let $k \in \mathbb{N}$, $k \geq 4$. What is the minimum value of $m(k) \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, there is a k-flip sequence $(m(k), a_2, \ldots, a_k)$ where $a_k > N$?*

We have explored the relationship between the smallest and largest colour-degrees, however for four colours and higher, in light of Theorem 13, it is worth exploring the relationship between the other intermediate colour-degrees and the largest colour-degree. We pose the following problem.

Problem *For $k \geq 4$, what is the largest integer $q(k)$, $q(k) < k$, such that there exists some $h(k) \in \mathbb{N}$ and for all $N \in \mathbb{N}$, there is a k-flip sequence $(a_1, \ldots, a_k)$ where $a_{q(k)} = h(k)$ and $a_k > N$?*

Our work in this paper establishes that $q(k) \geq 1$ for all $k \geq 4$.

Given the demonstrated usefulness of (r, c)-constant graphs, it is of interest to advance our knowledge concerning $\mathsf{spec}(r)$.

Problem *Determine $\mathsf{spec}(r)$, or at least improve upon the bounds given in Theorem 14 for membership and non-membership in $\mathsf{spec}(r)$.*

We note that the graphs constructed in the proof of Theorem 15 are substantially large, as are the graphs constructed to demonstrate the existence of (r, c)-constant graphs. It is therefore of interest to find smaller (r, c)-constant graphs, especially in light of Problem 2.

Problem *Find lower and upper bounds to*

$$g(r, c) = \min \{|V(G)| \colon G \text{ is an } (r, c)\text{−constant graph}\}.$$

Lastly, Sect. 6 dealt with the extension of the flip problem to the t^{th} neighbourhood. In particular we illustrated two distinct constructions for the case of two colours. Problems 1–5 naturally extend to this generalisation and remain to be studied. In particular, define $b(t) = \min \{b \colon ([t], (b, r))\text{−flip graph exists}\}$. Observe that for $t \geq 2$, the proof of Theorem 17 gives $b(t) \leq 2^{t+1} - 1$.

Problem *For $t \in \mathbb{N}$, $t \geq 2$ determine $b(t)$.*

Another interesting problem is to consider weakening the condition for the t-neighbourhood flipping to only require the flip to occur at select neighbourhood levels, rather than for all levels up to some distance t (as was required in Sect. 6). In particular, once can consider requiring the flip to occur:

- strictly at a select level,
- at alternating levels up to some distance t.

This allows one to consider triangle-free graphs, which cannot be considered for the flip-colouring problems considered in this paper.

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References

1. Abdullah, M.A., Draief, M.: Global majority consensus by local majority polling on graphs of a given degree sequence. Discrete Appl. Math. **180**, 1–10 (2015)
2. Caro, Y., Yuster, R.: The effect of local majority on global majority in connected graphs. Graphs and Combinatorics **34**(6), 1469–1487 (2018)
3. Chebotarev, P.l., Peleg, D.: The power of small coalitions under two-tier majority on regular graphs. Discrete Appl. Math. **340**, 239–258 (2023)
4. Fishburn, P.C., Hwang, F.K., Lee, H.: Do local majorities force a global majority? Discrete. Math. **61**(2), 165–179 (1986)
5. Lerman, K., Yan, X., Wu, X.: The "majority illusion" in social networks. PLoS ONE **11**(2), 0147617 (2016)
6. Lisonêk, P.: Local and global majorities revisited. Discret. Math. **146**(1), 153–158 (1995)
7. Füredi, Z., Simonovits, M.: The History of Degenerate (Bipartite) Extremal Graph Problems. In: Erdős Centennial, pp. 169–264. Springer, Berlin (2013)
8. Turán, P.: On an external problem in graph theory. Mat. Fiz. Lapok **48**, 436–452 (1941)
9. Turán, P.: On the theory of graphs. In Colloquium Mathematicum **1**(3), 19–30 (1954)
10. Erdős, P.: Graph theory and probability. Can. J. Math. **11**, 34–38 (1959)
11. Blass, A., Harary, F., Miller, Z.: Which trees are link graphs? J. Combin. Theory Ser. B **29**(3), 277–292 (1980)
12. Conder, M., Schillewaert, J., Verret, G.: Parameters for certain locally-regular graphs. arXiv preprint arXiv:2112.00276 (2021)
13. Hall, J.I.: Graphs with constant link and small degree or order. J. Graph Theory **9**(3), 419–444 (1985). https://doi.org/10.1002/jgt.3190090313
14. Larrión, F., Pizaña, M.A., Villarroel-Flores, R.: Small locally nk_2 graphs. ARS Comb. **102**, 385–391 (2011)
15. Reznick, B.: The sum of the squares of the parts of a partition, and some related questions. J. Number Theory **33**(2), 199–208 (1989)
16. Caro, Y., Lauri, J., Zarb, C.: The feasibility problem for line graphs. Discret. Appl. Math. **324**, 167–180 (2023). https://doi.org/10.1016/j.dam.2022.09.019
17. Dahan, X.: Regular graphs of large girth and arbitrary degree. Combinatorica **34**(4), 407–426 (2014). https://doi.org/10.1007/s00493-014-2897-6
18. Exoo, G., Jajcay, R.: Dynamic cage survey. Electron. J. Combin. DS16, 48 (2008).
19. Lazebnik, F., Ustimenko, V.A.: Explicit construction of graphs with an arbitrary large girth and of large size. Discret. Appl. Math. **60**(1), 275–284 (1995). https://doi.org/10.1016/0166-218X(94)00058-L
20. Catlin, P.: Embedding subgraphs under extremal degree conditions. PhD thesis (1976)
21. Sauer, N., Spencer, J.: Edge disjoint placement of graphs. J. Combin. Theory Ser. B **25**(3), 295–302 (1978). https://doi.org/10.1016/0095-8956(78)90005-9