# Edge coloring complete uniform hypergraphs with many components

Yair Caro [*] Raphael Yuster [†]

**Abstract**

Let $H$ be a hypergraph. For a $k$-edge coloring $c : E(H) \to \{1,\ldots,k\}$ let $f(H,c)$ be the number of components in the subhypergraph induced by the color class with the least number of components. Let $f_k(H)$ be the maximum possible value of $f(H,c)$ ranging over all $k$-edge colorings of $H$. If $H$ is the complete graph $K_n$ then, trivially, $f_1(K_n) = f_2(K_n) = 1$. In this paper we prove that for $n \geq 6$, $f_3(K_n) = \lfloor n/6 \rfloor + 1$ and supply close upper and lower bounds for $f_k(K_n)$ in case $k \geq 4$. Several results concerning the value of $f_k(K_n^r)$, where $K_n^r$ is the complete $r$-uniform hypergraph on $n$ vertices, are also established.

## 1 Introduction

All graphs and hypergraphs considered here are finite, unordered and simple. For standard terminology the reader is referred to [8]. Let $H$ be a hypergraph. For a $k$-edge coloring $c : E(H) \to \{1,\ldots,k\}$ let $f(H,c)$ be the number of components in the subhypergraph induced by the color class with the least number of components. Isolated vertices are not considered as components in a subhypergraph induced by edges. To avoid trivialities we always assume $k \leq e(H)$. Let $f_k(H)$ be the maximum possible value of $f(H,c)$ ranging over all $k$-edge colorings of $H$. Trivially, if $H$ has $n$ vertices then $1 \leq f_k(H) \leq \lfloor n/r \rfloor$ where $r$ is the minimum cardinality of an edge of $H$, and $f_1(H) = c(H)$ where $c(H)$ is the number of components of $H$ that are not isolated vertices.

In case $H$ is the complete $r$-uniform hypergraph $K_n^r$ it is not difficult to prove (see last part of Theorem 1.3) that $f_k(K_n^r) = 1$ for $k \leq r$. In fact, this is a generalization of the graph theoretic case, $r = 2$, where $f_2(K_n) = 1$ is merely the well-known fact that either a graph or its complement is connected. For fixed $k \geq r+1$ it is not difficult to show that $f_k(K_n^r)$ is linear in $n$. However, determining the exact value is a nontrivial task.

[*]Department of Mathematics, University of Haifa at Oranim, Tivon 36006, Israel.
e–mail: yairc@macam98.ac.il

[†]Department of Mathematics, University of Haifa at Oranim, Tivon 36006, Israel.
e–mail: raphy@research.haifa.ac.il

This paper contains several results concerning the parameter $f_k(K_n^r)$. In the graph-theoretic case, we completely settle the case $k=3$ and the cases $k=n-1$ and $k=n$. For other fixed values of $k$ we supply close upper and lower bounds that are also valid for the hypergraph case.

Our main results are summarized in the following theorems. For simplicity we use the notation $f(n,k,r)$ instead of $f_k(K_n^r)$, and $f(n,k)$ for the graph-theoretic case $r=2$. The first theorem on $f(n,k)$ is an exact result dealing with the lower end of the scale of colors, namely $k=3$.

**Theorem 1.1** *$f(4,3)=f(5,3)=2$. Otherwise $f(n,3)=\lfloor n/6\rfloor+1$.*

The next theorem is an exact result dealing with values of $k$ in the upper end of the scale.

**Theorem 1.2**

1. *$f(n,n-1)=\lfloor n/2\rfloor$.*
2. *$f(n,n)=\lfloor (n-1)/2\rfloor$.*
3. *If $k\geq n-1$ and $k$ divides $\binom{n}{2}$ then $f(n,k)=n(n-1)/(2k)$.*
4. *Suppose $t\geq 1$ and $rt$ divides $n$, then for $k=t(n-1)!/((r-1)!(n-r)!)$, $f(n,k,r)=n/(rt)$.*
5. *If $k\geq\binom{n}{r}-\binom{n-r}{r}$ then $f(n,k,r)=\lfloor\binom{n}{r}/k\rfloor$.*

Our next theorem supplies close upper and lower bounds for all fixed values of $k$. Before we state the theorem we need a few definitions. Given an edge-coloring $c$ of $K_n^r$, let $z(n,r,c,s)$ denote the fraction of the vertices incident with at least one edge whose color is $s$. Let $z(n,r,c)$ denote the maximum value of $z(n,r,c,s)$ taken over all colors appearing in $c$. For $k\geq 1$, let $z_k(n,r)$ denote the minimum possible value of $z(n,r,c)$ taken over all colorings that use at most $k$ colors. Finally, let $z_{k,r}$ denote the infimum of $z_k(n,r)$ taken over all $n\geq r$. For $r=2$ denote $z_k=z_{k,2}$. The following theorem relates $z_{k,r}$ with the function $f(n,k,r)$.

**Theorem 1.3** *Let $k\geq r+1$. Then,*

$$n\left(\frac{1}{r}-\frac{1}{rk^{1/r}}\right)(1+o_n(1))\geq f(n,k,r)\geq n\left(\frac{1}{r}-\frac{z_{k,r}}{r}\right)(1+o_n(1)).$$

*If $k\leq r$ then $z_{k,r}=1$ and $f(n,k,r)=1$.*

We conjecture that the lower bound in the last theorem is the correct one:

**Conjecture 1.4** *Let $k\geq r+1$. Then,*

$$f(n,k,r)=n\left(\frac{1}{r}-\frac{z_{k,r}}{r}\right)(1+o_n(1)).$$

In section 2 we analyze the parameters $z_{k,r}$ and $z_k$. Infinitely many values of $z_{k,r}$ are known, and the values of infinitely many others are open problems. In particular, we determine (with varying difficulty of proofs depending on $k$) the following specific values: $z_{r+1,r} = r/(r+1)$, $z_4 = 3/5$, $z_5 = 5/9$, $z_6 = 1/2$, $z_7 = 3/7$, $z_{12} = 1/3$, $z_{6,3} = 2/3$, $z_{14,3} = 1/2$. More generally, $z_{p^2+p+1} = (p+1)/(p^2+p+1)$ whenever $p$ is a prime power, and $z_{p^2+p} = 1/p$ whenever $p$ is a prime power. We note that the values of $z_k$ for $k = 8, 9, 10, 11$ can be determined precisely using the extensive case analysis appearing in the result of Mills [7]. It is also not difficult to bound $z_k$ from below. In fact, we show $z_k \geq 1/\lceil \sqrt{k+1/4} - 1/2 \rceil$. Since, by definition, $z_k$ is monotone decreasing and since for every integer $s$ there is a prime (moreover a prime power) between $s$ and $s + O(s^{2/3})$ [6] we have that $z_k = 1/\sqrt{k} + O(1/k)$. Together with Theorem 1.3 we have, in the graph-theoretic case:

**Corollary 1.5** *Let $k \geq 3$ and let $p$ be the largest prime power satisfying $p^2+p+1 \leq k$ then*

$$n\left(\frac{1}{2} - \frac{1}{2\sqrt{k}}\right)(1+o_n(1)) \geq f(n,k) \geq n\left(\frac{1}{2} - \frac{p+1}{2(p^2+p+1)}\right)(1+o_n(1)).$$

The last corollary, together with the argument concerning density of primes show that the upper and lower bounds in the last corollary are very close. Table 1 summarizes the best upper and lower bounds that we currently have for $\limsup f(n,k)/n$ and $\liminf f(n,k)/n$ respectively, for some small values of $k$. For each specific $k$ (except $k = 3$), the upper bound follows from Theorem 1.3 and in all cases, the lower bound follows from the best known upper bound for $z_k$ (all these upper bounds are consequences of constructions that appear in Section 2). We note that it may be possible to prove Conjecture 1.4 without determining the precise value of $z_k$ or $z_{k,r}$ for all $k$. On the other hand, although we know that, say, $z_4 = 3/5$, and $z_{6,3} = 2/3$, proving that $\lim f(n,4)/n = 1/5$ or $\lim f(n,6,3)/n = 1/9$ is still an open problem. Currently, Conjecture 1.4 is open, in the graph theoretic case, for all $k \geq 4$, and for $r \geq 3$ it is open for all $k \geq r+1$.

In Section 3 we prove the theorems and also consider the bipartite analog, namely, the parameter $f_k(K_{n,n})$.

# 2 Localized edge-coloring of complete hypergraphs

## 2.1 Lower bounds for $z_{k,r}$

We consider first the case $k \leq r$. We show that in this case $z_{k,r} = 1$. Namely, there must be a color that is incident with all vertices. In fact, we will show something somewhat stronger. There is always a color such that the subgraph of $K_{n,r}$ induced by the edges with this color is connected and spanning. In other words:

**Lemma 2.1** *If $k \leq r$ then $f(n,k,r) = 1$.*

| $k$ | $\limsup f(n,k)/k \leq$ | $\liminf f(n,k)/k \geq$ | $z_k \leq$ | $z_k \geq$ |
|---|---|---|---|---|
| 3 | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 4 | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{3}{5}$ |
| 5 | 0.277 | 0.222 | $\frac{5}{9}$ | $\frac{5}{9}$ |
| 6 | 0.296 | 0.25 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 7 | 0.311 | 0.285 | $\frac{3}{7}$ | $\frac{3}{7}$ |
| 8 | 0.324 | 0.285 | $\frac{3}{7}$ | $\frac{3}{8}$ |
| 9 | $\frac{1}{3}$ | $\frac{3}{10}$ | $\frac{2}{5}$ | $\frac{1}{3}$ |
| 10 | 0.342 | 0.3 | $\frac{2}{5}$ | $\frac{1}{3}$ |
| 11 | 0.35 | 0.318 | $\frac{4}{11}$ | $\frac{1}{3}$ |
| 12 | 0.356 | 0.333 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 13 | 0.362 | 0.346 | $\frac{4}{13}$ | $\frac{4}{13}$ |

Table 1: Asymptotic upper and lower bounds for small values of $k$

**Proof:** Clearly it suffices to prove $f(n,r,r) = 1$. We shall prove this by induction on $r$. For $r = 2$ we trivially have $f(n,2) = f(n,2,2) = 1$ since either a graph or its complement is connected. Assume the theorem holds for $r-1$, and we prove it for $r$. The proof for $r$ also proceeds by induction on $n$. If $n = r+1$ then exactly one color appears in two (intersecting) edges and hence this color must be incident with $r+1 = n$ vertices and we trivially have $f(r+1,r,r) = 1$. Assume it hold for $n-1$ and we prove it for $n$. Fix a vertex $v$ and consider the subhypergraph $K_{n-1}^r$ obtained by removing $v$. By the induction hypothesis there is a color $c$ such that the subhypergraph of $K_{n-1}^r$ induced by this color is spanning and connected. If $c$ also appears in some edge of $K_n^r$ that contains $v$ we are done. Otherwise, consider the hypergraph $K_{n-1}^{r-1}$ obtained by removing $v$ and coloring each $(r-1)$-set $f$ with the color of the edge $f \cup v$ in $K_n^r$. This coloring does not use the color $c$ and hence is an $r-1$ coloring of $K_{n-1}^{r-1}$. By the induction hypothesis it has a color $c'$ such that the subhypergraph induced by this color is connected and spanning. By the definition of the coloring of $K_{n-1}^{r-1}$, the subhypergraph of $K_n^r$ induced by color $c'$ must also be connected and spanning. $\square$.

The following lemma supplies a lower bound for $z_{k,r}$ in case $k \geq r+1$.

**Lemma 2.2** *Let $k \geq r+1$, and let $d \geq 2$ be a positive integer. Then,*

$$z_{k,r} \geq \max\left\{ \min\left\{ \left(\frac{d}{k}\right)^{\frac{1}{r-1}} \;,\; \frac{1}{d-1} \right\} \;,\; \min\left\{ \frac{d}{k} \;,\; z_{d-1,r-1} \right\} \right\}.$$

**Proof:** Let $d \geq 2$ be a positive integer. Consider a coloring of $K_n^r$ with at most $k$ colors. For $i = 1, \ldots, k$, let $s_i$ denote the number of $(r-1)$-sets that are contained in an edge whose color is $i$. For an $(r-1)$-set $U$, let $b_U$ denote the number of distinct colors that appear in edges that contain $U$. Clearly, $\sum_{U \in \binom{[n]}{r-1}} b_U = s_1 + \cdots + s_k$. If $b_U \geq d$ for all $U \in \binom{[n]}{r-1}$ then $s_1 + \cdots + s_k \geq \binom{n}{r-1} d$

and hence at least some $s_i$ satisfies $s_i \geq \binom{n}{r-1}d/k$. Thus, if $x$ is the number of vertices appearing in an edge colored $i$, we must have $\binom{x}{r-1} \geq s_i \geq \binom{n}{r-1}d/k$. This implies that

$$\left(\frac{x}{n}\right)^{r-1} \geq \frac{x(x-1)\cdots(x-r+2)}{n(n-1)\cdots(n-r+2)} \geq \frac{d}{k}$$

and hence $x \geq n(d/k)^{1/(r-1)}$. Thus, the fraction of vertices incident with color $i$ is at least $(d/k)^{1/(r-1)}$. Otherwise, there exists an $(r-1)$-subset $U$ such that $b_U \leq d-1$. Assume, without loss of generality, that only the colors $1,\ldots,d-1$ appear in edges that contain $U$. Then each vertex of $K_n^r$ appears in an edge whose color is one of the colors $1,\ldots,d-1$. Thus, for some color $i$ the fraction of vertices appearing in an edge colored $i$ is at least $1/(d-1)$. We have shown that $z_{k,r} \geq \min\{(d/k)^{1/(r-1)}\ ,\ 1/(d-1)\}$.
Next, we show that $z_{k,r} \geq \min\{d/k, z_{d-1,r-1}\}$. For completeness, define $z_{t,1} = 1/t$ (in a $t$-coloring of $n$ singletons at least $\lceil n/t \rceil$ elements obtain the same color, and this bound is realized). If there is a vertex $v$ of $K_n^r$ that is incident only with $t \leq d-1$ colors then consider a coloring of $K_{n-1}^{r-1}$ obtained by removing vertex $v$ and coloring each $(r-1)$-set $U$ with the color of the edge $U \cup \{v\}$ in $K_n^r$. This defines a $t$-coloring of $K_{n-1}^{r-1}$ and hence, by definition of $z_{t,r-1}$, some color is incident with at least $(n-1)z_{t,r-1}$ vertices. Since this color is also incident with $v$ and since $z_{d-1,r-1} \leq z_{t,r-1}$ we have found a color incident with a fraction of at least $z_{d-1,r-1}$ vertices of $K_n^r$. Otherwise, each vertex of $K_n^r$ is incident with at least $d$ colors, and hence there is a color that is incident with at least $nd/k$ vertices. In any case we have shown $z_{k,r} \geq \min\{d/k, z_{d-1,r-1}\}$. $\Box$

In the case $r = 2$, by choosing $d = \lceil \sqrt{k+1/4} + 1/2 \rceil$ and using the fact that for this choice of $d$, $d/k \geq 1/(d-1)$, we get

**Corollary 2.3** *For all $k \geq 3$, $z_k \geq 1/\lceil \sqrt{k+1/4} - 1/2 \rceil$.*

Notice that for $r \geq 3$ it is best to take $d \approx k^{1/r}$ in Lemma 2.2, except for small values of $k$ where the lower bound of $\min\{d/k, z_{d-1,r-1}\}$ does better.

## 2.2 Upper bounds and precise values of $z_{k,r}$

Upper bounds for $z_k$ and $z_{k,r}$ are demonstrated by construction. Consider a trivial coloring of $K_n^r$ with $\binom{n}{r}$ colors, each edge colored with a unique color. The fraction of vertices incident with each color is trivially $r/n$. This shows, in particular, that

**Corollary 2.4** $z_{\binom{n}{r},r} \leq r/n$ , $z_{n(n-1)/2} \leq 2/n$ , $z_{r+1,r} \leq r/(r+1)$. $\Box$

Corollary 2.4, together with Lemma 2.2 yield the following:

**Corollary 2.5** $z_3 = 2/3$ , $z_{r+1,r} = r/(r+1)$ , $z_6 = 1/2$.

**Proof:** Upper bounds follow from Corollary 2.4. $z_{r+1,r} \geq r/(r+1)$ follows from Lemma 2.2 by taking $k = r+1$ and $d = r$ (and recalling that $z_{r-1,r-1} = 1$ by Lemma 2.1). $z_3$ is a special case of $z_{r+1,r}$. $z_6 \geq 1/2$ by taking $r = 2$, $k = 6$ and $d = 3$ in Lemma 2.2. $\square$

In many cases we can find non-trivial constructions that match the lower bound that follows from Lemma 2.2. For example, the smallest nontrivial Steiner Triple System shows that $K_7$ can be decomposed into 7 triangles. In other words, there is a coloring of $K_7$ with 7 colors such that each color induces a triangle. We therefore get $z_7 \leq 3/7$. On the other hand, applying Lemma 2.2 with $r = 2$, $k = 7$ and $d = 3$ gives $z_7 \geq 3/7$. Thus, $z_7 = 3/7$. More generally we can prove the following:

**Proposition 2.6** *Let $p$ be a prime power. Then, $z_{p^2+p+1} = (p+1)/(p^2+p+1)$ and $z_{p^2+p} = 1/p$.*

**Proof:** Whenever $p$ is a prime power there exists a projective plane $PG(2,p)$. This projective plane corresponds to the existence of a $2-(p^2+p+1, p+1, 1)$ design (see, e.g., [2]), which, in turn, corresponds to the fact that $K_{p^2+p+1}$ decomposes into $p^2+p+1$ copies of $K_{p+1}$. Hence, we have that $z_{p^2+p+1} \leq (p+1)/(p^2+p+1)$ and using Lemma 2.2 with $r = 2$ and $d = p+1$ we get $z_{p^2+p+1} = (p+1)/(p^2+p+1)$. Similarly, when $p$ is a prime power there exists an affine plane $AG(2,p)$. This affine plane corresponds to the existence of a $2-(p^2, p, 1)$ design ([2]), which, in turn, corresponds to the fact that $K_{p^2}$ decomposes into $p^2+p$ copies of $K_p$ and using Lemma 2.2 with $r = 2$, $d = p+1$ we get $z_{p^2+p} = 1/p$. $\square$

Notice that Proposition 2.6 gives, in particular, $z_{13} = 4/13$ and $z_{12} = 1/3$.

Hanani has shown the existence of $3-(n,4,1)$ designs for every even $n$ not divisible by 6 (cf. [2]). In other words, $K_4^3$ decomposes $K_n^3$ for $n = 2, 4 \bmod 6$. Hence if $k = n(n-1)(n-2)/24$ and $n = 2, 4 \bmod 6$ then $z(k,3) \leq 4/n$. In particular $z(14,3) \leq 1/2$. This upper bound has a matching lower bound that follows from Lemma 2.2 by taking $r = 3$, $k = 14$ and $d = 7$ and recalling that $z_{6,2} = z_6 = 1/2$. Thus, we obtain the sporadic value $z_{14,3} = 1/2$.

In all previous constructions, all color classes induced the same clique (or hyperclique) size. Constructions using non-isomorphic color classes (and even color classes that are not cliques) are also very useful. In fact, sometimes using non-isomorphic color classes is provably an optimal strategy. Consider the case $k = 4$. Color $K_5$ with four colors as follows: For $i = 1, 2, 3$, color $i$ appears in the edges $(i,4)$ and $(i,5)$ and color 4 appears in the edges $(1,2), (1,3), (2,3)$. The edge $(4,5)$ is colored arbitrarily by one of the colors $1, 2$ or 3. Notice that each color is incident with precisely three vertices. Thus, $z_4 \leq 3/5$. Notice that we cannot match this upper bound with Lemma 2.2 so in order to prove that this is an optimal strategy we need to explicitly prove:

**Proposition 2.7** $z_4 \geq 3/5$.

**Proof:** We need to show that in any coloring of a complete graph $K_n$ with at most four colors, at least one color is incident with at least $3n/5$ vertices. Consider a coloring of $K_n$ with the colors $1, 2, 3, 4$. We use the same notations as in the proof of Lemma 2.2 (recalling that in case $r = 2$ the sets $U$ are singletons). If $b_v \geq 3$ for all $v = 1, \ldots, n$ then at least one color has $s_i \geq 3n/4 > 3n/5$. Assume, therefore, that there exist vertices with $b_v = 2$. (If there exists a vertex with $b_v = 1$ then the unique color $i$ incident with $v$ has $s_i = n$.) Let $X \subset \{1, \ldots, n\}$ be the subset of vertices with $b_v = 2$. Each $v \in X$ is associated with the unique pair of colors incident with $v$. Notice that if $v$ is associated with $(i, j)$ then no $v' \in X$ is associated with $(k, l)$ where $\{k, l\} \cap \{i, j\} = \emptyset$. Thus, there are only at most three types of associations. It follows that there are two (not necessarily mutually exclusive) cases. Either there exists a color $i$ such that each $v \in X$ is incident with $i$, or there exists a color $j$ such that no $v \in X$ is incident with $j$. Consider the first case. If $|X| \geq 3n/5$ then $s_i \geq 3n/5$ and we are done. Otherwise, $s_1 + s_2 + s_3 + s_4 \geq 3(n - |X|) + 2|X| = 3n - |X| \geq 12n/5$. Thus, some $s_i$ has $s_i \geq 3n/5$. Consider the second case. Then $s_1 + s_2 + s_3 + s_4 - s_j \geq 2n$. Hence, some $s_i$ (i $\neq j$) has $s_i \geq 2n/3$. $\square$

The coloring of $K_5$ with four colors described above yielding $z_4 \leq 3/5$ is somewhat "reducible". Indeed, if we allow weights on the vertices of $K_n$ then the following coloring of $K_4$ with four colors is more efficient: Color the edge $(i, 4)$ with color $i$ for $i = 1, 2, 3$. Color the triangle $(1, 2, 3)$ with color 4. Assign the weight $2/5$ to vertex 4, and the weight $1/5$ to each of the vertices $1, 2, 3$. Thus, the sum of weights of the vertices incident with color $i$ is $3/5$ for all $i = 1, 2, 3, 4$, as expected. Indeed, any $c$-coloring of a weighted $K_n$ (with rational weights) can be transformed to a non-weighted $c$-coloring of a larger $K_{n'}$ where each vertex of $K_n$ is "blown-up" proportionally to its weigh (an edge of $K_{n'}$ connecting two vertices corresponding to the same blown-up vertex of $K_n$ can be colored with any arbitrary color that is incident with that vertex of $K_n$).

The proof for the case $k = 5$ is more complicated. For the upper bound, color $K_9$ as follows: Two colors induce each a copy of $K_5$. The two copies of $K_5$ share one vertex. The remaining 16 yet uncolored edges form a $K_{4,4}$. $K_{4,4}$ can be decomposed into two $K_{2,3}$ and one $K_{1,4}$. Thus, we have a 5-coloring of $K_9$ where each color is incident with 5 vertices. This shows that $z_5 \leq 5/9$. Again, we cannot match this upper bound with Lemma 2.2. We therefore show:

**Proposition 2.8** $z_5 \geq 5/9$.

**Proof:** We need to show that in any coloring of a complete graph $K_n$ with at most five colors, at least one color is incident with at least $5n/9$ vertices. Consider a coloring of $K_n$ with the colors $1, 2, 3, 4, 5$. We use the same notations as in the proof of Lemma 2.2. If some vertex has $b_v = 1$ we are done. Assume, therefore all vertices have $b_v \geq 2$. If at least $7n/9$ vertices have $b_v > 2$ then $s_1 + s_2 + s_3 + s_4 + s_5 \geq 3n - 2n/9$ which implies that some $i$ has $s_i \geq 5n/9$ as required. Thus, let $X \subset \{1, \ldots, n\}$ be the subset of vertices with $b_v = 2$, and we may assume $|X| \geq 2n/9$. As in Proposition 2.7, each $v \in X$ is associated with the unique pair of colors incident with $v$, and if $v$ is associated with $(i, j)$ then no $v' \in X$ is associated with $(k, l)$ where $\{k, l\} \cap \{i, j\} = \emptyset$. Thus, either there exists a color $i$ such that each $v \in X$ is incident with $i$, or there exist three colors $i, j, k$ such that only the associations $(i, j), (i, k), (j, k)$ are valid associations for the vertices of $A$. Consider the second case. In this case, each $u \notin X$ is incident with at least two of the colors $\{i, j, k\}$. To see this, notice that if $u$ is not incident with, say, both $i$ and $j$ then the edge connecting $u$ to $v \in A$ where $v$ is associated with $(i, j)$, cannot be colored. We therefore have $s_i + s_j + s_k \geq 2n$ so at least one of the colors is incident with at least $2n/3 > 5n/9$ vertices. Now consider the first case. Since color $i$ is incident with all vertices of $X$ we may assume $|X| \leq 5n/9$, otherwise we are done. Let $s$ be the number of additional colors, other than $i$, incident with vertices of $X$. Without loss of generality, assume $i = 1$ and assume all other colors incident with vertices of $X$ are $2, \ldots, s+1$. Clearly $1 \leq s \leq 4$. Let $Y$ be the subset of vertices not in $X$ and which are incident with color 1. Thus, color 1 is incident with precisely $|Y|+|X|$ vertices, and colors $2, \ldots, s+1$ are all incident with at least all vertices of $v(K_n) \setminus (X \cup Y)$, namely to at least $n-|X|-|Y|$ vertices. Assume first that $s \geq 2$. Each vertex of $Y$ is incident with at least three colors, one of which is color 1, so it is also incident with at least $s-2$ of the colors $2, \ldots, s+1$. It follows that $s_2 + \ldots + s_{s+1} \geq s(n-|X|-|Y|)+(s-2)|Y|+|X|$. Thus, some $s_i$ is incident with at least $n - |X| - 2|Y|/s + |X|/s$ vertices. It suffices to show that $\max\{|Y| + |X|\ ,\ n - |X| - 2|Y|/s + |X|/s\} \geq 5n/9$. Since the maximum is minimized when both terms are equal, and this happens when $|Y| = ns/(s+2) - |X|(2s-1)/(s+2)$ it suffices to show that $ns/(s+2) - |X|(s-3)/(s+2) \geq 5n/9$. For $s = 2$ this holds since $|X| \geq 2n/9$ and thus $n/2 + |X|/4 \geq 5n/9$. For $s = 3$ this holds since $3n/5 > 5n/9$. For $s = 4$ this holds since $|X| \leq 5n/9$ and thus $2n/3 - |X|/6 > 5n/9$. Finally, assume $s = 1$. In this case colors 1 and 2 are incident with all vertices of $K_n$ and therefore $s_1 + s_2 \geq n + |X|$. Thus, one of them is incident with at least $n/2 + |X|/2 > 5n/9$ since $|X| \geq 2n/9$. $\square$

The reader may notice the significant added complexity to the proof of Proposition 2.8 as opposed to Proposition 2.7. It is plausible that with an increasing amount of effort one may determine $z_k$ or $z_{k,r}$ for every specific $k$ and $r$ with an appropriate "ad-hoc" proof.

A non-symmetric construction in the hypergraph ($r = 3$) case that yields an exact result is the following:

**Proposition 2.9** $z_{6,3} = 2/3$.

**Proof:** The lower bound follows from Lemma 2.2 by taking $r = 3$, $k = 6$ and $d = 4$ and recalling that $z_{3,2} = z_3 = 2/3$. For the upper bound, color $K_6^3$ with six colors as follows: Color 1 appears in $(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)$. Color 2 appears in $(1, 2, 5), (1, 2, 6), (1, 5, 6), (2, 5, 6)$. Color 3 appears in $(1, 3, 5), (1, 3, 6), (3, 5, 6)$. Color 4 appears in $(2, 3, 5), (2, 4, 5), (3, 4, 5)$. Color 5 appears in $(1, 4, 5), (1, 4, 6), (4, 5, 6)$. Color 6 appears in $(2, 3, 6), (2, 4, 6), (3, 4, 6)$. Every color is incident with four vertices hence $z_{6,3} \leq 4/6 = 2/3$. $\square$

The upper and lower bounds for $z_k$ in case $k = 8, 9, 10, 11$ appearing in Table 1 are obtained as follows: For $k = 8$, $z_8 \leq z_7 = 3/7$, and $z_8 \geq 3/8$ by selecting $d = 3$ in Lemma 2.2. For $k = 9$, $z_9 \leq 2/5$ since $K_4^-$ (the graph obtained from $K_4$ be deleting an edge) decomposes $K_{10}$ into 9 copies (see [2]). $z_9 \geq 1/3$ by selecting $d = 4$ in Lemma 2.2. $z_{10} \leq z_9 \leq 2/5$. $z_{10} \geq 1/3$ by selecting $d = 4$

in Lemma 2.2. $z_{11} \le 4/11$ since $K_4^-$ decomposes $K_{11}$ into 11 copies (see [2]). $z_{11} \ge 1/3$ by selecting $d = 4$ in Lemma 2.2.

Finally, infinitely many nontrivial upper bounds in the case $r = 3$ are obtained by using Möbius designs. For $q$ a prime power there exist the Möbius designs $3-(q^s, q+1, 1)$ (cf. [2]). Hence, $K^3_{q+1}$ decomposes $K^3_{q^s+1}$ and hence, for $s = 2$ we have that for $k = (q^2+1)q$, $z(k,3) \le (q+1)/(q^2+1)$. In particular, $z(10,3) \le 3/5$. This can be compared with the lower bound of $z(10,3) \ge 5/9$ that follows from Lemma 2.2 for $r = 3$, $k = 10$ and $d = 6$, and using Proposition 2.8 which states that $z_{5,2} = z_5 = 5/9$.

# 3 Proof of the main results

We start with the lower bound in theorem 1.3 as its proof will also be used to prove one direction in Theorem 1.1.

## 3.1 A lower bound for $f(n,k,r)$

We shall require the following lemma, whose proof is an immediate consequence of a theorem of Baranyai [1].

**Lemma 3.1** *Let $r \ge 2$ and $t \ge 1$ be positive integers. There exists $n_0 = n_0(r,t)$ such that for all $n \ge n_0$, $K_n^r$ contains $t$ disjoint maximum matchings (a maximum matching is a set of $\lfloor n/r \rfloor$ independent edges).* □

It is well-known that $n_0(2,t) = t+1$, since the complete graph on $t+1$ vertices decomposes into $t$ perfect matchings in case $t$ is odd, and $t+1 > t$ maximum matchings in case $t$ is even. For $r \ge 3$ the theorem of Baranyai states that if $n$ is a multiple of $r$ then $K_n^r$ decomposes into perfect matchings, and hence precisely $t = (n-1)!/((r-1)!(n-r)!)$ disjoint perfect matchings. Thus, $n_0(r,t) \approx (t(r-1)!)^{1/(r-1)}$.

We need to prove that $f(n,k,r) \ge n\left(\frac{1}{r} - \frac{z_{k,r}}{r}\right)(1+o_n(1))$. Let $\epsilon > 0$. Let $t = t(k,r,\epsilon)$ be the smallest integer such that $K_t^r$ has a coloring $C$ using at most $k$ colors, such that each color is incident with at most $t(z_{k,r}+\epsilon)$ vertices. Notice that in some cases it is possible to have $\epsilon = 0$. In Section 2 we have shown, e.g., that for $r = 2$ and $k = 3,4,5,6,7,12,13$ we can take $\epsilon = 0$ (in other words, $z_k$ is *realized* in these cases). As an example, for $r = 2$ and $k = 3$ we can pick $\epsilon = 0$ and $t = 3$.

**Lemma 3.2** *Suppose $n$ satisfies*

$$\left\lfloor \frac{n}{t} \right\rfloor \ge n_0\left(r\ ,\ k - \left\lceil \frac{1}{z_{k,r}+\epsilon} \right\rceil\right),$$

*then $f(n,k,r) \ge \lfloor \frac{n}{rt} \rfloor \lceil t(1 - z_{k,r} - \epsilon) \rceil + 1$.*

**Proof:** Assume the vertices of $K_t^r$ are $\{1,\ldots,t\}$ and assume the colors of $C$ are $\{1,\ldots,k\}$. Let $C(e)$ denote the color of edge $e$ (notice that it is possible that $C$ is not onto $\{1,\ldots,k\}$ as $c$ may use less than $k$ colors).

Partition the vertices of $K_n^r$ into $t$ equitable parts $A_1,\ldots,A_t$. The cardinality of each part is either $\lceil n/t \rceil$ or $\lfloor n/t \rfloor$ and by the assumption in the statement of the lemma, the parts are nonempty.

We now show how to color each edge of $K_n^r$. Let $f$ be an arbitrary edge, and let $U = \{i \,:\, A_i \cap f \neq \emptyset\}$. Clearly $1 \le |U| \le r$. Consider first the case $|U| > 1$. In this case, let $e \in K_t^r$ be an edge such that $U \subset e$. In this case, we color $f$ with the color $C(e)$. Consider next the case $|U| = 1$. In this case $f$ is completely within some part $A_i$. Hence, it remains to show how to color edges that are completely within some $A_i$. Let $C_i \subset \{1,\ldots,k\}$ be the subset of colors not appearing in any edge of $K_t^r$ that contains the vertex $i$. We claim that $|C_i| \le k - \lceil \frac{1}{z_{k,r}+\epsilon} \rceil$. To see this, notice that if there were less than $\frac{1}{z_{k,r}+\epsilon}$ colors incident with $i$ then at least one color would have been incident with more than $t(z_{k,r}+\epsilon)$ vertices, contradicting the assumption. We claim that we can find in $A_i$ a set of $|C_i|$ disjoint maximum matchings. Indeed this follows from Lemma 3.1, and by the assumption in the current lemma that states that $|A_i| \ge \lfloor n/t \rfloor \ge n_0(r, k - \lceil \frac{1}{z_{k,r}+\epsilon} \rceil)$. Fixing $|C_i|$ disjoint maximum matchings in $A_i$ we now color each maximum matching with a distinct color of $C_i$. It remains to show how to color the edges completely within $A_i$ that do not belong to any of the selected maximum matchings. In this case, we can color them with any color of $C \setminus C_i$ (does not matter which one).

Now consider any color $c \in C$. If $c$ is not used at all in the coloring of $K_t^r$ then the subhypergraph of $K_n^r$ induced by the edges colored $c$ is composed of isolated edges, exactly $\lfloor |A_i|/r \rfloor$ isolated edges from each $A_i$. Therefore, this subhypergraph has at least $t\lfloor n/(rt) \rfloor$ components. If $c$ is used as a color of at least one edge of $K_t^r$ then the subhypergraph induced by the edges colored $c$ in $K_n^r$ has some large components consisting of all the $s$ sets $A_i$ such that the vertex $i$ of $K_t^r$ is incident with an edge colored $c$, and isolated edges, exactly $\lfloor |A_i|/r \rfloor$ isolated edges from each of the $t-s$ sets $A_i$ where vertex $i$ of $K_t^r$ is not incident with any edge colored $c$. Thus, this subhypergraph has at least $(t-s)\lfloor n/(rt) \rfloor + 1$ components. Since $s \le (z_{k,r}+\epsilon)t$ our construction shows that $f(n,k,r) \ge \lfloor \frac{n}{rt} \rfloor \lceil t(1 - z_{k,r} - \epsilon) \rceil + 1$. $\square$

### 3.2 Proof of Theorem 1.1

Trivially $f(3,3) = 1$. $f(4,3) = 2$ as seen by coloring $K_4$ with three colors each inducing a perfect matching. $f(5,3) = 2$ as seen by coloring $K_5$ with a red triangle and a red edge vertex-disjoint with the triangle, and coloring the remaining 6 edges with three blue edges forming a $K_{1,2}$ and a $K_{1,1}$ vertex disjoint with each other, and three green edges that are now also forced to induce a $K_{1,2}$ and a $K_{1,1}$ vertex disjoint with each other. Thus, we assume $n \ge 6$.

For the lower bound we use lemma 3.2. Recall that $z_3 = 2/3$ and $K_3$ realizes $z_3$ with $\epsilon = 0$ by giving each edge of $K_3$ a distinct color. The condition in Lemma 3.2 is satisfied for all $n \ge 6$ since

$n_0(2,1) = 2$, and hence we get $f(n,3) \geq \lfloor n/6 \rfloor + 1$.

It remains to show the upper bound. Consider a coloring of $K_n$ with the three colors $1, 2, 3$. Let $A_i$ be the subset of vertices in a maximum cardinality component induced by color $i$. Put $|A_i| = a_i$ and assume $a_1 \geq a_2 \geq a_3$. Put $D = A_1 \cap A_2$ and $d = |D|$. Put $M = V \setminus (A_1 \cup A_2)$ where $V$ is the set of vertices of $K_n$, and $m = |M|$. If $a_1 \geq 2n/3$ then we are done since the subgraph induced by color 1 has at most $\lfloor 1 + (n - 2n/3)/2 \rfloor = 1 + \lfloor n/6 \rfloor$ components. Hence we assume $a_i < 2n/3$ for $i = 1, 2, 3$. Consider the following cases

1. $d = 0$. In this case every edge between $A_1$ and $A_2$ is colored 3. Thus, $a_3 \geq a_1 + a_2$, a contradiction since the $a_i$ cannot be zero.

2. $d > 0$, $m = 0$, and $d = a_1$ or $d = a_2$ (or both). In this case one of the colors 1 or 2 (or both) induces a connected spanning subgraph of $K_n$.

3. $d > 0$, $m = 0$, $d < a_1$, $d < a_2$. In this case all edges between $A_1 \setminus D$ and $A_2 \setminus D$ are colored 3. Thus, $2n/3 > a_3 \geq (a_1 - d) + (a_2 - d)$. On the other hand $(a_1 - d) + (a_2 - d) + d = n$. Thus, $d > n/3$. It follows that $a_1 + a_2 = n + d > 4n/3$. Hence, $a_1 > 2n/3$, a contradiction.

4. $d > 0$, $m > 0$, $a_1 > d$ and $a_2 > d$. As in the previous case all edges between $A_1 \setminus D$ and $A_2 \setminus D$ are colored 3. Also all edges between $M$ and $D$ are colored 3. Thus, the subgraph induced by 3 has at most two components and $2 \leq \lfloor n/6 \rfloor + 1$.

5. $d > 0$, $m > 0$, $a_2 = d$. In this case all edges between $M$ and $D$ are colored 3. Thus, $a_3 \geq d + m > d = a_2$, a contradiction to the assumption $a_3 \leq a_2$. $\square$

### 3.3 Proof of Theorem 1.2

1. If $n$ is even then $K_n$ decomposes into $n - 1$ perfect matchings. Thus, $f(n, n-1) \geq n/2$. If $n$ is odd then $K_n$ decomposes into $(n-1)/2$ Hamiltonian cycles. Each cycle further decomposes into a matching of size $(n-1)/2$ and the remaining $(n+1)/2$ edges form a subgraph with $(n-1)/2$ components (one component is a path with three vertices and the others are independent edges). Thus, $f(n, n-1) \geq (n-1)/2 = \lfloor n/2 \rfloor$. In both the even and odd cases we always have $f(n, n-1) \leq \lfloor \binom{n}{2}/(n-1) \rfloor = \lfloor n/2 \rfloor$. Thus, $f(n, n-1) = \lfloor n/2 \rfloor$.

2. If $n$ is odd then the chromatic index of $K_n$ is $n$. Thus, $K_n$ decomposes into $n$ matchings. This forces each matching to be of size $(n-1)/2$. Thus, $f(n,n) \geq (n-1)/2$. If $n$ is even then $n+1$ is odd and we have $f(n+1, n) = n/2$ by the previous case. Deleting the additional vertex causes each induced subgraph of a color to loose at most one component. Thus, $f(n,n) \geq n/2 - 1$. In both cases we always have $f(n,n) \geq \lfloor (n-1)/2 \rfloor$. Since, trivially, $f(n,n) \leq \lfloor \binom{n}{2}/n \rfloor = \lfloor (n-1)/2 \rfloor$ we have shown $f(n,n) = \lfloor (n-1)/2 \rfloor$.

3. Assume $k \geq n-1$ and assume $t = n(n-1)/(2k)$ is an integer. Thus, $2t \leq n$ and it is well known that in this case $tK_2$ decomposes $K_n$ [5, 3]. Thus, $f(n,k) \geq t$. The other direction follows from the trivial fact that $f(n,k) \leq \binom{n}{2}/k = t$.

4. Suppose $t \geq 1$ and $rt$ divides $n$. In particular, $r$ divides $n$, and by the result of Baranyai [1], $K_n^r$ decomposes into $(n-1)!/((r-1)!(n-r)!)$ perfect matchings. Each perfect matching has cardinality $n/r$ and is further decomposed into $t$ matchings, each consisting of $n/(rt)$ edges. Hence, for $k = t(n-1)!/((r-1)!(n-r)!)$ we have $f(n,k,r) \geq n/(rt)$. Trivially, $f(n,k,r) \leq \binom{n}{r}/k = n/(rt)$.

5. Consider the line graph $L$ of $K_n^r$. $L$ has $N = \binom{n}{r}$ vertices and is regular of degree $s = \binom{n}{r} - \binom{n-r}{r} - 1$. By the theorem of Hajnal and Szemerédi [4], for all $k' > s$, $L$ has a $k'$ equipartite coloring, namely, a vertex-coloring with $k'$ colors where each color class has either $\lceil N/k' \rceil$ or $\lfloor N/k' \rfloor$ elements. Using $k = k'$ and translating this back to the original hypergraph $K_n^r$, we have a $k$-edge coloring of $K_n^r$ such that each color induces a matching of cardinality at least $\lfloor \binom{n}{r}/k \rfloor$. Thus, $f(n,k,r) \geq \lfloor \binom{n}{r}/k \rfloor$. The other direction is trivial. □

### 3.4 Proof of Theorem 1.3

In case $k \geq r+1$, the lower bound of Theorem 1.3 is shown in Lemma 3.2. For the upper bound, notice that in a $k$-edge coloring of $K_n^r$ at least one color appears in at least $\binom{n}{r}/k$ edges. Fix such a color, and let $s_1, \ldots, s_t$ denote the cardinalities of the components in the subhypergraph induced by this color. Clearly, $s_i \geq r$ for $i = 1, \ldots, t$ and

$$\binom{n - r(t-1)}{r} + (t-1) \geq \binom{s_1 + \cdots + s_t - r(t-1)}{r} + (t-1) \geq \sum_{i=1}^{t} \binom{s_i}{r} \geq \binom{n}{r} \cdot \frac{1}{k}. \qquad (1)$$

It follows that $n - r(t-1) \geq (1 + o_n(1))n/k^{1/r}$ and hence $t \leq (n/r - n/(rk^{1/r}))(1 + o_n(1))$. In case $k \leq r$, $f(n,k,r) = 1$ is shown in Lemma 2.1. □

It is not difficult to show that in case $r = 2$, the negation of the error term in the upper bound of Theorem 1.3 is at least $\Theta(1/k^{3/2})$, by considering the maximal cardinality components induced by each color, and showing that either there is a color with a "huge" component, or otherwise there must be (at least) two such maximal components (belonging to two distinct colors) that intersect in $\Theta(n)$ vertices, and thus at least one of these maximal components is far from being a clique (misses $\Theta(n^2)$ edges from being a clique), and hence the convexity argument in inequality (1) cannot be exploited to its extreme. However, this improvement is negligible. Even for $k = 4$ we could only improve the constant from 0.25 to (a little less than) 0.243. That is, $\limsup f(n.4)/4 \leq 0.243$.

### 3.5 The bipartite analog

Let $C$ be a $k$-edge coloring of $K_r$ where color $i$ is incident with $s_i$ vertices. We show how to construct a coloring $C'$ of $K_{r,r}$ where color $i$ is incident with $2s_i$ vertices. Assume the vertices of one partite class are labeled $\{a_1, \ldots, a_r\}$ and the vertices of the other partite class are labeled $\{b_1, \ldots, b_r\}$. For $i \neq j$ color $(a_i, b_j)$ with the same color as the edge $(i, j)$ of $K_r$. Color $(a_i, b_i)$ with any color incident with vertex $i$ in $K_r$. The obtained coloring has the desired property. Now, this construction together with the same "blow up" argument as in Lemma 3.2 yields the following:

$$f_k(K_{n,n}) \geq n(1 - z_k)(1 + o_n(1)).$$

An upper bound density argument similar to the one in Theorem 1.3 gives $f_k(K_{n,n}) \leq n(1 - 1/\sqrt{k})(1 + o_n(1))$. Recalling that $z_k = 1/\sqrt{k} + O(1/k)$ we get that the upper and lower bounds for $f_k(K_{n,n})$ are very close.

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