EDGE-DISJOINT CLIQUES IN GRAPHS WITH HIGH MINIMUM DEGREE

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Abstract. For a graph G and a fixed integer $k \geq 3$, let $\nu_k(G)$ denote the maximum number of pairwise edge-disjoint copies of K_k in G. For a constant c, let $\eta(k,c)$ be the infimum over all constants γ such that any graph G of order n and minimum degree at least cn has $\nu_k(G) \geq \gamma n^2(1-o_n(1))$. By Turán's Theorem, $\eta(k,c)=0$ if $c \leq 1-1/(k-1)$ and by Wilson's Theorem, $\eta(k,c) \to 1/(k^2-k)$ as $c \to 1$. We prove that for any $1 > c > 1-1/(k-1)$,

$$\eta(k,c) \geq \frac{c}{2} - \frac{(\binom{k}{2}-1)c^{k-1}}{2\Pi_{i=1}^{k-2}((i+1)c-i)+2(\binom{k}{2}-1)c^{k-2}}$$

while it is conjectured that $\eta(k,c)=c/(k^2-k)$ if $c \geq k/(k+1)$ and $\eta(k,c)=c/2-(k-2)/(2k-2)$ if $k/(k+1) > c > 1-1/(k-1)$.

The case $k=3$ is of particular interest. In this case the bound states that for any $1 > c > 1/2$,

$$\eta(3,c) \geq \frac{c}{2} - \frac{c^2}{4c-1} .$$

By further analyzing the case $k=3$ we obtain the improved lower bound

$$\eta(3,c) \geq \frac{\left(12\,c^2-5\,c+2-\sqrt{240\,c^4-216\,c^3+73\,c^2-20\,c+4}\right)(2\,c-1)}{32\,(1-c)\,c} .$$

This bound is always at most within a fraction of $(20-\sqrt{238})/6 > 0.762$ of the conjectured value which is $\eta(3,c)=c/6$ for $c \geq 3/4$ and $\eta(3,c)=c/2-1/4$ if $3/4 > c > 1/2$.

Our main tool is an analysis of the value of a natural fractional relaxation of the problem.

Keywords: triangles, packing, fractional

1. Introduction. All graphs considered here are finite and simple undirected graphs. For standard graph-theoretic terminology the reader is referred to [1].

The problem of computing a maximum set of pairwise edge-disjoint subgraphs of a graph G that are isomorphic to a given fixed graph H is a fundamental problem in extremal graph theory and in design theory. Perhaps the most studied case is when H is the complete graph $H=K_k$, dating back to a classical result of Kirkman [15]. See [3, 4, 8, 9, 11, 14, 12, 18, 21, 23, 24] for some representative works in this area.

For a graph G and a fixed integer $k \geq 3$, let $\nu_k(G)$ denote the maximum number of pairwise edge-disjoint copies of K_k in G. If G is sufficiently sparse, then we might have $\nu_k(G)=0$. In fact, Turán's Theorem states that $\nu_k(G)=0$ only if $e(G) \leq t(n,k-1)$. Here $t(n,r)$ denotes the number of edges of the graph $T(n,r)$ which is the complete r-partite graph with n vertices and either $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$ vertices in each partite class. As the minimum degree of $T(n,r)$ is at most $n(1-1/r)$, a graph G with minimum degree cn might have $\nu_k(G)=0$ when $c \leq 1-1/(k-1)$. On the other hand, the simple greedy algorithm shows that $\nu_k(G)=\Theta(n^2)$ for any constant $c > 1-1/(k-1)$. In fact, by Wilson's Theorem [23], K_n has a set of $\binom{n}{2}/\binom{k}{2}$ edge-disjoint copies of K_k as long as n is sufficiently large and satisfies some necessary divisibility conditions. Hence, when $c \to 1$, graphs with minimum degree at least cn have $\nu_k(G)$ close to $\binom{n}{2}/\binom{k}{2}$. It is therefore of interest to determine $\nu_k(G)$ as a function of the minimum degree of G.

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More formally, for a real constant c and an integer $k \geq 3$, let $\eta(k,c)$ be the infimum over all constants γ such that any graph G with n vertices and minimum degree at least cn has $\nu_k(G) \geq \gamma n^2(1-o_n(1))$. By the discussion in the previous paragraph, Turán's Theorem states that $\eta(k,c)=0$ if $c \leq 1-1/(k-1)$ and, more generally, $\eta(k,c) \to 0$ as $c \to 1-1/(k-1)$. On the other hand, Wilson's Theorem asserts that $\eta(k,c) \to 1/(k^2-k)$ as $c \to 1$. Observe also the trivial upper bound $\eta(k,c) \leq c/(k^2-k)$ since a cn-regular graph cannot have more than $cn^2/2$ pairwise edge-disjoint copies of K_k.

A longstanding conjecture of Nash-Williams [18] states that $\eta(3,c) = c/6$ for all $c \geq 3/4$. In fact, the conjecture of Nash-Williams is sharper in the sense that if, in addition, some divisibility conditions hold (the degree of each vertex is even and the overall number of edges is a multiple of 3), then a graph with minimum degree at least $3n/4$ has a triangle decomposition. Gustavsson, in his Ph.D. Thesis [7] proved that $\eta(3,c)=c/6$ for $c \geq 1-10^{-24}$. A special case of a recent breakthrough paper of Keevash [13] gives an alternative proof to Gustavsson's result. The results of Gustavsson and Keevash, just like the conjecture of Nash-Williams, are decomposition results. The value of c for which $\eta(3,c)=c/6$ was improved by the author in [24] to $c \geq 1-1/10^5$ and recently by Dukes [2] to $c \geq 1-1/162$. The conjecture of Nash Williams is sharp in the sense that it cannot be improved. A simple interpolation argument provides a construction showing that $\eta(3,c) \leq c/2-1/4$ for $1/2 \leq c \leq 3/4$. We state this upper bound as the conjectured value for $\eta(3,c)$.

Conjecture 1.1.

$$\eta(3,c) = \begin{cases} \frac{c}{6} & \textit{if} \quad \frac{3}{4} \leq c < 1, \\ \left(\frac{c}{2}-\frac{1}{4}\right) & \textit{if} \quad \frac{1}{2} \leq c \leq \frac{3}{4}, \\ 0 & \textit{if} \quad 0 \leq c \leq \frac{1}{2}. \end{cases}$$

As shown in Gustavsson's thesis, it is not difficult to generalize the construction of Nash Williams, and his conjecture, to larger values of k. The analogous conjecture in this case is that $\eta(k,c)=c/(k^2-k)$ for all $c \geq 1-1/(k+1)$. An interpolation argument, given in Section 7, provides a construction showing that $\eta(k,c) \leq c/2-(k-2)/(2k-2)$ for $1-1/(k-1) \leq c \leq 1-1/(k+1)$. We state this upper bound as the conjectured value for $\eta(k,c)$.

Conjecture 1.2.

$$\eta(k,c) = \begin{cases} \frac{c}{k^2-k} & \textit{if} \quad 1-\frac{1}{k+1} \leq c < 1, \\ \left(\frac{c}{2}-\frac{k-2}{2k-2}\right) & \textit{if} \quad 1-\frac{1}{k-1} \leq c \leq 1-\frac{1}{k+1}, \\ 0 & \textit{if} \quad 0 \leq c \leq 1-\frac{1}{k-1}. \end{cases}$$

Our main results are lower bounds for $\eta(k,c)$ in general and $\eta(k,3)$ in particular. As far as we know, these are the first nontrivial lower bounds that apply to the entire spectrum of c. Our first main result is a general lower bound for $\eta(k,c)$.

Theorem 1.3. *Let $k \geq 3$ be an integer and let c be a real satisfying* $1-1/(k-1) < c < 1$. *Then*

$$\eta(k,c) \geq \frac{c}{2} - \frac{(\binom{k}{2}-1)c^{k-1}}{2\Pi_{i=1}^{k-2}((i+1)c-i)+2(\binom{k}{2}-1)c^{k-2}}\,.$$

Observe first that if $c > 1-1/(k-1)$, then Theorem 1.3 asserts that $\eta(k,c) > 0$ and if $c \to 1$, then Theorem 1.3 asserts that $\eta(k,c) \to 1/(k^2-k)$, which is consistent with the discussion above. The following corollary is a restatement of Theorem 1.3 for the cases $k = 3$ and $k = 4$, given for reference.

COROLLARY 1.4.

$$\eta(3, c) \geq \frac{c}{2} - \frac{c^2}{4c-1} \quad \text{for } 1 > c > \frac{1}{2}\,, \tag{1.1}$$

$$\eta(4, c) \geq \frac{c}{2} - \frac{5c^3}{22c^2 - 14c + 4} \quad \text{for } 1 > c > \frac{2}{3}\,. \tag{1.2}$$

Let us compare (1.1) to some other lower bounds that can be derived from existing results. A classical result of Goodman [6] on the triangle densities of graphs, states that a graph with $c\binom{n}{2}$ edges has $c(2c-1)\binom{n}{3} - o(n^3)$ triangles. From this fact, together with the fact that K_n has $n(n-1)/6 - o(n)$ edge-disjoint triangles, one easily obtains that a graph with $c\binom{n}{2}$ edges has $c(2c-1)n^2/6 - o(n^2)$ edge-disjoint triangles. Thus, the easy lower bound $\eta(3, c) \geq c^2/3 - c/6$ follows immediately. Observe that the latter bound only relies on the fact that the edge density of the graph is c and does not use the assumption that the minimum degree is cn. If we do use the minimum degree condition, together with an observation that one can assume that any edge is contained in at most cn triangles (see more details on the latter assumption in Section 3) and together with the fact that, in dense graphs, a maximum fractional triangle packing has asymptotically the same value as a maximum integral one (see details on this fact in Section 2), then Goodman's result implies that $\eta(3, c) \geq c/3 - 1/6$. However, observe that the lower bound (1.1) is *always* better than $c/3 - 1/6$ for $1 > c > 1/2$. It is important to note that Goodman's result on the density of triangles is optimal for values of c of the form $1 - 1/t$ where $t \geq 2$ is an integer, but is not optimal for other values of c. The optimal bound was recently obtained in a breakthrough paper of Razborov [20]. If we use Razborov's bound instead of Goodman's bound (for values of c not of the form $1 - 1/t$), then the bound $c/3 - 1/6$ is slightly improved, but still falls short of our lower bound (1.1) for all $1 > c > 1/2$. As an illustrative example, consider the case $c = 5/8$. Goodman's bound would then imply $\eta(3, 5/8) \geq 1/24 = 5/120$. Razborov's bound[1] would imply $\eta(3, 5/8) \geq 5/108$, while (1.1) shows that $\eta(3, 5/8) \geq 5/96$. Finally, note that Conjecture 1.1 states that $\eta(3, 5/8) = 1/16 = 6/96$.

A generalization of Goodman's result for arbitrary $k \geq 3$ was obtained by Lovász and Simonovits [16] (see also [17]). Nikiforov [19] generalized Razborov's result to $k = 4$. Their bound can be used to prove a lower bound for $\eta(k, c)$ using a similar reasoning as for triangles. Again, the obtained bound using this approach is always smaller than the one given in Theorem 1.3. For example, if $k = 4$, then there are always $c(2c-1)(3c-2)\binom{n}{4} - o(n^4)$ copies of K_4. This can be used to give the bound $\eta(4, c) \geq c(2c-1)(3c-2)/12$. However, observe that the lower bound (1.2) is always better than $c(2c-1)(3c-2)/12$ for $1 > c > 2/3$.

Theorem 1.3 is a general lower bound for all k and c. However, for the case $k = 3$, it is possible to obtain a non-negligible improvement using a more detailed analysis. This is our second main result.

THEOREM 1.5. *Let c be a constant satisfying $1/2 < c < 1$. Then*

$$\eta(3, c) \geq \frac{\left(12\,c^2 - 5\,c + 2 - \sqrt{240\,c^4 - 216\,c^3 + 73\,c^2 - 20\,c + 4}\right)(2\,c - 1)}{32\,(1 - c)\,c}\,. \tag{1.3}$$

[1] For $c \in (1/2, 2/3)$ the density of triangles was established already in [5].

Observe that if $c > 1/2$, then Theorem 1.3 asserts that $\eta(3,c) > 0$ and if $c \to 1$, then Theorem 1.3 asserts that $\eta(3,c) \to 1/6$, which is consistent with the discussion above. As shown in Section 6, the bound (1.3) is always better than the bound (1.1) for $1 > c > 1/2$. One should therefore compare (1.3) with Conjecture 1.1 (recall that the conjectured bound is also an upper bound). As shown in Section 6, the ratio between the lower bound (1.3) and the conjectured bound is never smaller than $(20 - \sqrt{238})/6 > 0.762$ for any $1 > c > 1/2$.

To prove our main results, we first reduce the problem of computing $\eta(k,c)$ to the problem of computing its fractional relaxation. The fact that such reductions carry asymptotically no loss is an important result of Haxell and Rödl [11]. We then consider a particular fractional assignment that we call the *natural fractional packing*. We prove that the natural fractional packing always attains the values claimed in Theorem 1.3 and Theorem 1.5. Proving this requires some detailed combinatorial and analytical considerations.

The rest of this paper is organized as follows. In Section 2 we review the reduction from integral to fractional packings and define the notion of a natural fractional packing. Section 3 contains the proof of Theorem 1.3 for the case of triangles. The general case of Theorem 1.3, which contains significantly more notation but otherwise uses the same ideas as given in Section 3 is proved in Section 4. The proof of the improved bound for $\eta(3,c)$ (namely, Theorem 1.5) appears in Section 5. The quality of the lower bounds obtained in Theorems 1.3 and 1.5 with respect to the conjectured values for $k = 3, 4$ is established in Section 6. Constructions of the upper bounds that coincide with Conjecture 1.2 are provided in Section 7. The final section contains some concluding remarks.

2. Preliminaries. Let $\binom{G}{H}$ denote the set of copies of a graph H in a graph G. A function ϕ from $\binom{G}{H}$ to $[0,1]$ is a *fractional H-packing* of G if $\sum_{H' \in \binom{G}{H} : e \in H'} \phi(H') \le 1$ for each $e \in E(G)$. For a fractional H-packing ϕ, let $|\phi| = \sum_{H' \in \binom{G}{H}} \phi(H')$. The *fractional H-packing number*, denoted by $\nu^*_H(G)$, is the maximum value of $|\phi|$ ranging over all fractional H-packings ϕ. An H-packing of G is a set of pairwise edge-disjoint copies of H in G. Let $\nu_H(G)$ denote the maximum size of an H-packing of G. In the case $H = K_k$ we set $\nu_k(G) = \nu_{K_k}(G)$ and $\nu^*_k(G) = \nu^*_{K_K}(G)$.

Trivially, $\nu^*_H(G) \ge \nu_H(G)$, as an H-packing is a special case of a fractional H-packing. An important result of Haxell and Rödl [11] (see also [25]), which relies on Szemerédi's regularity lemma [22], shows that the converse is also asymptotically true, up to an additive error term which is negligible for sufficiently dense graphs.

Lemma 2.1. *For every $\epsilon > 0$ and every graph H there exists $N = N(H,\epsilon)$ such that for all $n > N$, if G is a graph with n vertices, then $\nu^*_H(G) - \nu_H(G) \le \epsilon n^2$.*

For a real constant c and an integer $k \ge 3$, let $\eta^*(k,c)$ be the infimum over all constants γ such that any graph G with n vertices and minimum degree at least cn has $\nu^*_k(G) \ge \gamma n^2(1 - o_n(1))$. The following is an immediate corollary of Lemma 2.1.

Corollary 2.2. *$\eta^*(k,c) = \eta(k,c)$.*

For an edge $e \in E(G)$ let $f_H(e)$ denote the number of elements of $\binom{G}{H}$ that contain e. We sometimes omit the subscript H if it can be determined from context. Let $\psi = \psi_H(G)$ be the fractional H-packing of G defined by

$$\psi(X) = \frac{1}{\max_{e \in E(X)} f_H(e)} \; .$$

In other words, for each $X \in \binom{G}{H}$, we look at all the edges of X, take the edge that appears in the largest amount of elements of $\binom{G}{H}$, and assign to X the value which is reciprocal to this amount. We call ψ the *natural fractional H-packing* of G. Observe that ψ is indeed a valid fractional H-packing of G as for each edge, the sum of the weights of the copies of H containing it is at most $f_H(e) \cdot (1/f_H(e)) = 1$.

It may be that $|\psi_H(G)| < \nu^*_H(G)$. For example, let $G = K_5^-$ and let $H = K_3$. In this case, $\binom{G}{H}$ contains 7 triangles. Each triangle contains at least one edge that appears in three triangles, so each triangle receives the weight $1/3$ in the natural fractional triangle packing of K_5^-. The value of ψ is therefore $7/3$. On the other hand, it is easy to verify that $\nu^*_3(K_5^-) = 3$.

The proofs of Theorems 1.3 and 1.5 are based on lower-bounding the value of the natural fractional K_k-packing of a graph with minimum degree cn, thereby establishing a lower bound for $\eta^*(k,c)$. By Corollary 2.2, this also establishes a lower bound for $\eta(k,c)$.

3. Proof of Theorem 1.3 for triangles. Let $1 > c > 1/2$ be fixed. Let G be a graph with n vertices and minimum degree $\delta(G) \geq cn$. By Corollary 2.2, it suffices to prove that ψ, the natural fractional triangle packing of G, satisfies

$$|\psi| \geq \left(\frac{c}{2} - \frac{c^2}{4c-1}\right) n^2 \ .$$

We may assume that for any edge $(u,v) \in E(G)$, either $d(u) = \delta(G)$ or $d(v) = \delta(G)$, as otherwise we may remove (u,v) and prove the theorem for the resulting subgraph, which still has the same minimum degree.

Let m denote the number of edges of G and observe that $m \geq cn^2/2$. Recall that for an edge $e \in E(G)$, $f(e) = f_{K_3}(e)$ denotes the number of triangles containing e. Hence, by our assumption that any edge is incident with a vertex of degree $\delta(G)$ and by the fact that the number of common neighbors of any two vertices is at least $2\delta(G) - n$, we have

$$(2c-1)n \leq f(e) \leq \delta(G) - 1 < cn \ . \tag{3.1}$$

Let $\mathcal{T}(G)$ denote the set of triangles of G and let $\alpha = |\mathcal{T}(G)|/n^3$. For a triangle $T \in \mathcal{T}(G)$, we have that

$$\psi(T) = \frac{1}{\max_{e \in E(T)} f(e)} \geq \frac{1}{cn} \ .$$

Consider first the case where

$$\alpha \geq \frac{c^2}{2} - \frac{c^3}{4c-1} \ .$$

Observe that indeed $\alpha > 0$ since $c > 1/2$. In this case we easily have

$$|\psi| = \sum_{T \in \mathcal{T}(G)} \psi(T) \geq \alpha n^3 \cdot \frac{1}{cn} \geq \left(\frac{c}{2} - \frac{c^2}{4c-1}\right) n^2 \ . \tag{3.2}$$

Consider from here onwards the case that $0 < \alpha \leq c^2/2 - c^3/(4c-1)$. Order $E(G)$ as a sequence $e_1, \ldots, e_m$ where $f(e_i) \geq f(e_{i+1})$ for $i = 1, \ldots, m-1$. Now consider the non-decreasing sequence $B = \{b_1, b_2, \ldots\}$ of $3\alpha n^3 = 3|\mathcal{T}(G)|$ rationals, consisting of m blocks, where the first $f(e_1)$ elements of B are $1/f(e_1)$ (this is the first block), the next $f(e_2)$ elements are $1/f(e_2)$ (the second block), and so on, where the final block consists of $f(e_m)$ elements which are $1/f(e_m)$. The sum of the elements of B is, therefore, m, since the sum of each block is 1. Observe that any element of the sequence B can be mapped to a triangle $T \in \mathcal{T}(G)$. Indeed, the $f(e_j)$ elements of B in block j are bijectively mapped to the $f(e_j)$ triangles containing e_j. Denote this mapping by $M : B \to \mathcal{T}(G)$. Also notice that for each $T \in \mathcal{T}(G)$ there are precisely three elements of B that are mapped to T, one for each edge of T. The first of these three elements in the sequence B is called the *leading element* with respect to T. Suppose that $E(T) = \{e_i, e_j, e_k\}$ and that $f(e_i) \geq f(e_j) \geq f(e_k)$. The leading element with respect to T has value $1/f(e_i) = \psi(T)$.

Let B' be the subsequence of B consisting of the leading elements, one for each $T \in \mathcal{T}(G)$. By definition of ψ, and since each element of B' corresponds to the weight given by ψ to a triangle, we have that the sum of the elements of B' is precisely $|\psi|$. But B' is some subsequence of $|\mathcal{T}(G)|$ elements of the nondecreasing sequence B, so in particular, its sum is at least the sum of the first $|\mathcal{T}(G)| = \alpha n^3$ elements of B. Hence,

$$|\psi| \geq \sum_{\ell=1}^{\alpha n^3} b_\ell \ .$$

Summarizing, we have that B is a nondecreasing sequence of $3\alpha n^3$ rationals, whose sum is m where $m \geq cn^2/2$. Each element in the sequence is at least $1/(cn)$ and at most $1/((2c-1)n)$, and we wish to lower bound the minimum possible sum of the first αn^3 elements of this sequence.

Let therefore $S = \{s_1, s_2, \ldots\}$ be any nondecreasing sequence of $3\alpha n^3$ rationals, whose sum is m, and whose elements are between $1/(cn)$ and $1/((2c-1)n)$ and for which the sum of the first αn^3 elements is minimized. We may assume that all the first αn^3 elements of S are equal, as otherwise we can just replace each of them with their average.

So, we may assume that the first αn^3 elements of S are all equal to some value $1/(xn)$, and that the average value of the remaining $2\alpha n^3$ elements of S is some value $1/(yn)$, and recall that we have $1/(yn) \leq 1/((2c-1)n)$. It follows that

$$m = \sum_{i=1}^{3\alpha n^3} s_i = \alpha n^3 \cdot \frac{1}{xn} + 2\alpha n^3 \cdot \frac{1}{yn} \ .$$

Using $m \geq cn^2/2$ and $y \geq 2c-1$ we obtain from the last equation that

$$\frac{c}{2} \leq \frac{\alpha}{x} + \frac{2\alpha}{2c-1} \ .$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{\alpha n^3} s_i &= \alpha n^3 \cdot \frac{1}{xn} \\
&\geq n^2 \left(\frac{c}{2} - \frac{2\alpha}{2c-1} \right) \\
&\geq n^2 \left(\frac{c}{2} - \frac{2(c^2/2 - c^3/(4c-1))}{2c-1} \right) \\
&= \left(\frac{c}{2} - \frac{c^2}{4c-1} \right) n^2 \ .
\end{aligned} \tag{3.3}$$

By the minimality of S we have that

$$|\psi| \geq \sum_{i=1}^{\alpha n^3} b_i \geq \sum_{i=1}^{\alpha n^3} s_i \geq \left(\frac{c}{2} - \frac{c^2}{4c-1}\right) n^2$$

as required.

4. Proof of Theorem 1.3. The proof of Theorem 1.3 can be generalized to other fixed complete graphs K_k by lower-bounding the natural fractional K_k-packing. We outline the differences between the proof for K_3 given in the previous section and its more general form given here.

The first modification is equation (3.1) which gives upper and lower bounds for $f(e)$. Now $f(e)$ is the number of K_k that contain e. Clearly, $f(e)$ is at most the number of copies of K_{k-2} in the common neighborhood of u and v where $e = (u, v)$. An upper bound for $f(e)$ follows from the fact that at least one of the endpoints has degree $\delta(G) \leq cn$. So, there could be at most $\binom{cn}{k-2}$ copies of K_{k-2} in the common neighborhood of u and v. Thus, $f(e) \leq (cn)^{k-2}/(k-2)!$.

On the other hand, u and v has at least $(2c-1)n$ common neighbors. For each such common neighbor x_1, there are at least $(3c-2)n$ common neighbors of u, v, x_1. For each such common neighbor x_2 there are at least $(4c-3)n$ common neighbors of u, v, x_1, x_2. Continuing in this way for $k-2$ times, and noticing that each K_k is counted at most $(k-2)!$ times in this way, we obtain that

$$n^{k-2}\frac{\Pi_{i=1}^{k-2}((i+1)c-i)}{(k-2)!} \leq f(e) \leq n^{k-2}\frac{c^{k-2}}{(k-2)!} \ . \tag{4.1}$$

Let $\mathcal{K}(G)$ denote the set of K_k of G and let $\alpha = |\mathcal{K}(G)|/n^k$. For a copy $K \in \mathcal{K}(G)$, we have that

$$\psi(K) = \frac{1}{\max_{e \in E(T)} f(e)} \geq \frac{(k-2)!}{c^{k-2}n^{k-2}} \ .$$

Consider first the case where

$$\alpha \geq \frac{\frac{c^{k-1}}{2(k-2)!}}{1 + \frac{(\binom{k}{2}-1)c^{k-2}}{\Pi_{i=1}^{k-2}((i+1)c-i)}} \ . \tag{4.2}$$

In this case we get the trivial bound

$$\begin{aligned}|\psi| &= \sum_{K \in \mathcal{K}(G)} \psi(K) \geq \alpha n^k \cdot \frac{(k-2)!}{c^{k-2}n^{k-2}} \\ &\geq \left(\frac{c}{2} - \frac{(\binom{k}{2}-1)c^{k-1}}{2\Pi_{i=1}^{k-2}((i+1)c-i) + 2(\binom{k}{2}-1)c^{k-2}}\right) n^2 \ .\end{aligned}$$

For α which is at most the r.h.s. of (4.2), we construct the rational sequence B as in the proof of the previous section. Now B has $\binom{k}{2}\alpha n^k = \binom{k}{2}|\mathcal{K}(G)|$ elements, whose sum is m. Each element of the sequence B is mapped to some $K \in \mathcal{K}(G)$ and for each $K \in \mathcal{K}(G)$ there are $\binom{k}{2}$ elements of B that are mapped to K, one for each edge of K. The first of these $\binom{k}{2}$ elements in the sequence B is now the leading element with respect to K.

Let B' be the subsequence of B consisting of the leading elements. By definition of ψ, the sum of the elements of B' is precisely $|\psi|$. But B' is some subsequence of $|\mathcal{K}(G)|$ elements of the nondecreasing sequence B, so in particular, its sum is at least the sum of the first $|\mathcal{K}(G)| = \alpha n^k$ elements of B. Hence,

$$|\psi| \geq \sum_{\ell=1}^{\alpha n^k} b_\ell \ .$$

Summarizing, we have that B is a nondecreasing sequence of $\binom{k}{2}\alpha n^k$ rationals, whose sum is m where $m \geq cn^2/2$. Each element in the sequence is at least $\frac{(k-2)!}{c^{k-2}n^{k-2}}$ and at most $\frac{(k-2)!}{n^{k-2}\Pi_{i=1}^{k-2}((i+1)c-i)}$ and we wish to lower bound the minimum possible sum of the first αn^k elements of this sequence.

Let $S = \{s_1, s_2, \ldots\}$ be any nondecreasing sequence of $\binom{k}{2}\alpha n^k$ rationals, whose sum is m, whose elements are between the two stated bounds, and for which the sum of the first αn^k elements is minimized. As in the previous section, we may assume that all the first αn^k elements of S are equal. So, we may assume that the first αn^k elements of S are all equal to some value $1/(xn^{k-2})$, and that the average value of the remaining $(\binom{k}{2}-1)\alpha n^k$ elements of S is some value $1/(yn^{k-2})$, and recall that we have $1/(yn^{k-2}) \leq \frac{(k-2)!}{n^{k-2}\Pi_{i=1}^{k-2}((i+1)c-i)}$. It follows that

$$m = \sum_{i=1}^{\binom{k}{2}\alpha n^k} s_i = \alpha n^k \cdot \frac{1}{xn^{k-2}} + (\binom{k}{2} - 1)\alpha n^k \cdot \frac{1}{yn^{k-2}} \ .$$

Using $m \geq cn^2/2$ and $y \geq \Pi_{i=1}^{k-2}((i+1)c-i)/((k-2)!)$ we obtain from the last equation that

$$\frac{c}{2} \leq \frac{\alpha}{x} + \frac{(\binom{k}{2}-1)\alpha}{\Pi_{i=1}^{k-2}((i+1)c-i)/((k-2)!)} \ .$$

Hence,

$$\sum_{i=1}^{\alpha n^k} s_i = \alpha n^k \cdot \frac{1}{xn^{k-2}} \geq n^2 \left(\frac{c}{2} - \frac{(\binom{k}{2}-1)\alpha}{\Pi_{i=1}^{k-2}((i+1)c-i)/((k-2)!)}\right) \ .$$

Plugging in the r.h.s. of (4.2) which is an upper bound on α, we obtain

$$\sum_{i=1}^{\alpha n^k} s_i \geq \left(\frac{c}{2} - \frac{(\binom{k}{2}-1)c^{k-1}}{2\Pi_{i=1}^{k-2}((i+1)c-i) + 2(\binom{k}{2}-1)c^{k-2}}\right) n^2 \ .$$

By the minimality of S we have that

$$|\psi| \geq \sum_{i=1}^{\alpha n^k} b_i \geq \sum_{i=1}^{\alpha n^k} s_i \geq \left(\frac{c}{2} - \frac{(\binom{k}{2}-1)c^{k-1}}{2\Pi_{i=1}^{k-2}((i+1)c-i) + 2(\binom{k}{2}-1)c^{k-2}}\right) n^2$$

as required.

5. Improved lower bound for $\eta(3,c)$. In this section we prove Theorem 1.5. Let $1 > c > 1/2$ be fixed. Let G be a graph with n vertices and minimum degree $\delta(G) \ge cn$. By Corollary 2.2, it suffices to prove that ψ, the natural fractional triangle packing of G, satisfies

$$|\psi| \ge \frac{\left(12\,c^2 - 5\,c + 2 - \sqrt{240\,c^4 - 216\,c^3 + 73\,c^2 - 20\,c + 4}\right)(2\,c-1)}{32\,(1-c)\,c} n^2 - o(n^2)\ .$$

As shown in Section 3, we may assume that for any edge $(u,v) \in E(G)$, either $d(u) = \delta(G)$ or $d(v) = \delta(G)$. We will also use (3.1), as well as the following notations from Section 3:

1. The number of edges of G is m where $m \ge cn^2/2$.
2. $\mathcal{T}(G)$ is the set of triangles of G and $\alpha = |\mathcal{T}(G)|/n^3$.
3. The ordering of $E(G)$ as a sequence $e_1, \ldots, e_m$ where $f(e_i) \ge f(e_{i+1})$ for $i = 1, \ldots, m-1$.
4. The non-decreasing sequence $B = \{b_1, b_2, \ldots\}$ of $3\alpha n^3$ rationals consisting of m blocks where block j has $f(e_j)$ elements that are all equal to $1/f(e_j)$.
5. The mapping $M : B \to \mathcal{T}(G)$ which bijectively maps the block of B corresponding to e_j to the $f(e_j)$ triangles containing e_j.
6. The notion of a *leading element* with respect to $T \in \mathcal{T}(G)$.
7. The subsequence B' consisting of the leading elements, one for each $T \in \mathcal{T}(G)$. Recall that the sum of the elements of B' is precisely $|\psi|$.

Let β and r be parameters satisfying

$$\frac{c}{2} > \beta > \frac{1}{4}\ , \qquad r = \frac{\beta(4\beta-1)}{3} \le \alpha\ . \tag{5.1}$$

Let $F \subset E$ be $F = \{e_m, e_{m-1}, \ldots, e_{m-\beta n^2+1}\}$. Thus, F consists of the βn^2 edges having the lowest values of f. The following is an immediate consequence of the result of Goodman [6] mentioned in the introduction.

Lemma 5.1. *Let $G[F]$ be the spanning subgraph of G consisting of the edges of F. Then $G[F]$ contains at least $rn^3 - o(n^3)$ triangles where $r = \beta(4\beta-1)/3$.*

Proof. By Goodman's theorem, a graph with n vertices and $\rho\binom{n}{2}$ edges has $\rho(2\rho-1)\binom{n}{3} - o(n^3)$ triangles. Our graph $G[F]$ has βn^2 edges so we can use $\rho = 2\beta - o(1)$ and the lemma follows. ∎

Let B_F be the subsequence of B consisting of the last βn^2 blocks (namely, the blocks corresponding to the elements of F). By Lemma 5.1, we know that at least $rn^3 - o(n^3)$ of the leading elements fall in B_F. In other words, the subsequence B' of the leading elements has $|B' \cap B_F| \ge rn^3 - o(n^3)$ while $|B' \cap (B \setminus B_F)| \le (\alpha - r)n^3 + o(n^3)$. Recall that our goal is to lower bound the sum of the elements of B'. As B is non-decreasing, if we replace B' with the subsequence B'' (of the same cardinality αn^3) consisting of the first $(\alpha - r)n^3$ elements of B and the first rn^3 elements of B_F then the sum of the elements of B'' is at most the sum of the elements of B' (up to the negligible error term $o(n^2)$).

Let, therefore, $\mathcal{S}(n,c,\alpha,\beta)$ denote the set of all non-decreasing sequences with the following properties for each $S \in \mathcal{S}$:

1. S contains $3\alpha n^3$ rational elements, partitioned into m blocks, where $\binom{n}{2} \ge m \ge cn^2/2$.
2. The overall sum of the sequence S is m.
3. Each element is between $1/(cn)$ and $1/((2c-1)n)$.
4. The sum of the elements in the last βn^2 blocks is βn^2.

5. Each S is occupied with a sub-sequence S'' consisting of the first $(\alpha - r)n^3$ elements and also with the first rn^3 elements of the subsequence of S consisting of the final βn^2 blocks.
6. The value of S, denoted $Val(S)$, is the sum of the elements of S''.

Observe that $B \in \mathcal{S}(n, c, \alpha, \beta)$ and that $Val(B)$ is just the sum of the elements of B'' which is what we want to lower bound. Let, therefore,

$$Val(n, c, \alpha, \beta) = \min_{S \in \mathcal{S}(n,c,\alpha,\beta)} Val(S) .$$

Observe that if $S \in \mathcal{S}$ satisfies $Val(S) = Val(n, c, \alpha, \beta)$, then as shown in Section 3, we may assume that the first $(\alpha - r)n^3$ elements of S (which are also the first elements of S'') are all equal to some value x/n where $x \geq 1/c$. Likewise, the remaining rn^3 elements of S'' are all equal to some value z/n where $1/(2c-1) \geq z \geq x$. The elements of S starting with the element at location $(\alpha - r)n^3$ and ending at the last element of block $m - \beta n^2$ have values which are at least x/n and at most z/n. Denote the number of such elements by tn^3. The remaining $(2\alpha - t)n^3$ elements (which are also the last elements of S) have values which are at least z/n and at most $1/((2c-1)n)$. With this notation we have that $Val(n, c, \alpha, \beta)$ is of the form

$$Val(n, c, \alpha, \beta) = Val(S) = n^2((\alpha - r)x + rz)$$

but we must still compute x and z which minimize the r.h.s. of the last equation, subject to their constraints.

LEMMA 5.2. *The solution to program $P(c, \alpha, \beta)$ of Figure 1 is a lower bound for $Val(n, c, \alpha, \beta)/n^2$.*

$P(c, \alpha, \beta)$:	
minimize	$(\alpha - r)x + rz$
s.t.	
(1)	$(\alpha - r)x \geq \frac{c}{2} - \beta - 2\alpha z + (2c-1)\beta z - (2c-1)rz^2$,
(2)	$\frac{1}{c} \leq x \leq z \leq \frac{1}{2c-1}$.

FIG. 1. *The program $P(c, \alpha, \beta)$ whose solution is a lower bound for $Val(c, \alpha, \beta)$.*

Proof. Let S be as in the previous paragraph, namely $Val(n, c, \alpha, \beta) = Val(S) = n^2((\alpha - r)x + rz)$. The target value of $P(c, \alpha, \beta)$ then follows from the fact that $Val(n, c, \alpha, \beta)/n^2$ is of the form $(\alpha - r)x + rz$. Constraint (2) is also obvious from the constraints for x and z stated in the paragraph preceding the lemma. To see the Constraint (1) we observe the following. Notice first that we must have

$$(\alpha - r)x + tz \geq \frac{c}{2} - \beta . \tag{5.2}$$

Indeed, the sum of the elements of S in the first $m - \beta n^2$ blocks is precisely $m - \beta n^2$. On the other hand by the discussion in the paragraph before the lemma, it is at least $n^3(\alpha - r)(x/n) + (tn^3)(z/n)$. Dividing by n^2 and using $m \geq cn^2/2$, (5.2) follows. Notice next that we must have

$$rz + \frac{1}{2c-1}(2\alpha - t) \geq \beta . \tag{5.3}$$

Indeed, the sum of the elements of S in the last βn^2 blocks is precisely βn^2. On the other hand by the discussion in the paragraph before the lemma, it is at least

$n^3 r(z/n) + (2\alpha - t)n^3(1/((2c-1)n))$. Dividing by n^2, (5.3) follows. Now, constraint (1) is obtained by combining inequalities (5.2) and (5.3). □

Let $p(c,\alpha,\beta)$ denote the solution for $P(c,\alpha,\beta)$. Let

$$p(c,\alpha) = \max_{\beta \text{ satisfying } (5.1)} p(c,\alpha,\beta) .$$

Corollary 5.3.

$$\eta(3,c) \geq \min_{0 \leq \alpha \leq \frac{1}{6}} \max \left\{ p(c,\alpha) \ , \ \frac{\alpha}{c} \ , \ \frac{c}{2} - \frac{2\alpha}{2c-1} \right\} .$$

Proof. Let G be any n-vertex graph with minimum degree cn and with αn^3 triangles. The fact that $0 \leq \alpha \leq 1/6$ is trivial. Lemma 5.2 proves that for any β satisfying (5.1), $p(c,\alpha,\beta)n^2 - o(n^2)$ is a lower bound for the natural fractional triangle packing of G. Thus, $p(c,\alpha)n^2 - o(n^2)$ is a lower bound for the natural fractional triangle packing of G. The other two bounds in the statement of the corollary have already been proved in Section 3. The bound $(\alpha/c)n^2 - o(n^2)$ is the trivial lower bound proved in (3.2), and is useful when α is relatively large. The bound $(\frac{c}{2} - \frac{2\alpha}{2c-1})n^2 - o(n^2)$ is the other lower bound for the fractional natural triangle packing of G proved in (3.3) and is useful when α is relatively small. In fact, observe that the bound for $\eta(3,c)$ proved in Section 3 is just the minimum over all α of the maximum of the two bounds $\frac{\alpha}{c}$ and $\frac{c}{2} - \frac{2\alpha}{2c-1}$. □

In order to lower bound $p(c,\alpha)$ it would be simpler to compute $p(c,\alpha,\beta)$ at the particular point $\beta = 2\alpha/(2c-1)$. The reason for choosing this particular value of β will be made apparent later. However, we must make sure that choosing $\beta = 2\alpha/(2c-1)$ does not violate (5.1). For this to hold, α must therefore satisfy

$$\frac{c}{2} > \frac{2\alpha}{2c-1} > \frac{1}{4} \ , \qquad r = \frac{\frac{2\alpha}{2c-1}(\frac{8\alpha}{2c-1} - 1)}{3} \leq \alpha .$$

This is equivalent to requiring

$$\frac{2c-1}{8} < \alpha < \frac{c(2c-1)}{4} . \tag{5.4}$$

Thus, by the definition of $p(c,\alpha)$ as a maximum over all plausible β of $p(c,\alpha,\beta)$, we obtain that $p(c,\alpha) \geq p(c,\alpha,2\alpha/(2c-1))$ whenever α satisfies (5.4). For notational convenience, define

$$q(c,\alpha) = \begin{cases} p(c,\alpha,2\alpha/(2c-1)) & \text{if } \ \frac{2c-1}{8} < \alpha < \frac{c(2c-1)}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can restate Corollary 5.3 as

$$\eta(3,c) \geq \min_{0 \leq \alpha \leq \frac{1}{6}} \max \left\{ q(c,\alpha) \ , \ \frac{\alpha}{c} \ , \ \frac{c}{2} - \frac{2\alpha}{2c-1} \right\} . \tag{5.5}$$

The reason for using $q(c,\alpha)$ instead of $p(c,\alpha)$ becomes apparent by the fact that the program $P(c,\alpha,\beta)$ is significantly simplified by plugging in $\beta = 2\alpha/(2c-1)$. The following is an immediate corollary of Lemma 5.2.

$Q(c,\alpha)$:
minimize $\quad (\alpha - r)x + rz$
s.t.
(1) $\quad (\alpha - r)x \geq \frac{c}{2} - \frac{2\alpha}{2c-1} - (2c-1)rz^2$,
(2) $\quad \frac{1}{c} \leq x \leq z \leq \frac{1}{2c-1}$.

FIG. 2. *The program $Q(c,\alpha)$ whose solution is $q(c,\alpha)$.*

COROLLARY 5.4. *Let $\frac{2c-1}{8} < \alpha < \frac{c(2c-1)}{4}$ and $r = (\frac{2\alpha}{2c-1})(\frac{8\alpha}{2c-1} - 1)/3$. Then, $q(c,\alpha)$ is the solution to program $Q(c,\alpha)$ of Figure 2.*

Analyzing $Q(c,\alpha)$. Observe that whenever $Q(c,\alpha)$ has a feasible solution, $q(c,\alpha)$ is always at least as large as α/c. Indeed, the target function is $(\alpha - r)x + rz = \alpha x + r(z - x)$ but in any feasible solution, $z \geq x$ and $x \geq 1/c$. Similarly, $q(c,\alpha)$ is always at least as large as $\frac{c}{2} - \frac{2\alpha}{2c-1}$. To see this, notice that by Constraint (1), the target function satisfies

$$(\alpha - r)x + rz \geq \frac{c}{2} - \frac{2\alpha}{2c-1} - (2c-1)rz^2 + rz \geq \frac{c}{2} - \frac{2\alpha}{2c-1}$$

since $z \leq \frac{1}{2c-1}$. So, we now specify a valid range of α where $Q(c,\alpha)$ has a feasible solution, and where we can analytically *compute* this solution, and we are guaranteed that in this range, $q(c,\alpha)$ dominates the three terms in the max expression of (5.5). We define two points $g(c)$ and $h(c)$ (where $g(c) < h(c)$) as follows. The point $h(c)$ is the smallest point for which we can take $x = z = 1/c$ as a feasible (and hence optimal) solution. For this to hold, constraint (1) must satisfy

$$\frac{(\alpha - r)}{c} \geq \frac{c}{2} - \frac{2\alpha}{2c-1} - \frac{(2c-1)r}{c^2} .$$

Solving the last inequality for α (and recall that $r = (\frac{2\alpha}{2c-1})(\frac{8\alpha}{2c-1} - 1)/3$) we have that

$$h(c) = \frac{1}{32} \cdot \frac{\left(12c^2 - 5c + 2 - \sqrt{240c^4 - 216c^3 + 73c^2 - 20c + 4}\right)(2c-1)}{1-c} . \tag{5.6}$$

In particular, at the point $\alpha = h(c)$ (and above it) we have that $q(c,\alpha)$ coincides with α/c. The point $g(c)$ is the largest point for which we can take $x = 1/c$ and still have a feasible solution. For this to hold, constraint (1) must satisfy

$$\frac{(\alpha - r)}{c} \geq \frac{c}{2} - \frac{2\alpha}{2c-1} - \frac{(2c-1)r}{(2c-1)^2} .$$

Solving the last inequality for α we get that

$$g(c) = \frac{1}{32} \frac{\left(-24\,c^2 + 16\,c - 1 + \sqrt{384\,c^4 - 480\,c^3 + 208\,c^2 - 32\,c + 1}\right)(2\,c - 1)}{1-c} . \tag{5.7}$$

Since we have shown that $q(c,a)$ dominates the two other terms in the max expression of (5.5) in the range $[g(c), h(c)]$ we have, in particular, that

$$\eta(3,c) \geq \min_{0 \leq \alpha \leq \frac{1}{6}} \begin{cases} \frac{c}{2} - \frac{2\alpha}{2c-1} & \text{if} \quad \alpha \leq g(c), \\ q(c,\alpha) & \text{if} \quad g(c) \leq \alpha \leq h(c), \\ \frac{\alpha}{c} & \text{if} \quad \alpha \geq h(c). \end{cases} \tag{5.8}$$

Also notice from the above discussion that at point $\alpha = h(c)$ we have $q(c,\alpha) = \alpha/c$ and at point $\alpha = g(c)$ we have $q(c,\alpha) = \frac{c}{2} - \frac{2\alpha}{2c-1}$.

Solving $Q(c,\alpha)$ **in** $[g(c),h(c)]$. We can analytically compute $q(c,\alpha)$ at the range $\alpha \in [g(c),h(c)]$. Let $f(z) = \frac{c}{2} - \frac{2\alpha}{2c-1} - (2c-1)rz^2 + rz$. By Constraint (1) in Figure 2 we have that $(\alpha - r)x + rz \geq f(z)$. Now, $f(z)$ is concave and, in addition, gets its minimum when z is as large as possible subject to constraint (2) in Figure 2. But constraint (1) shows that in order to make z as large as possible, x should be as small as possible. Hence, the solution to $Q(c,\alpha)$ is when $x = \frac{1}{c}$ and z satisfies the quadratic equation $(\alpha - r)\frac{1}{c} = \frac{c}{2} - \frac{2\alpha}{2c-1} - (2c-1)rz^2$. Thus,

$$z = \frac{1}{\sqrt{r}} \cdot \sqrt{\left(\frac{c}{4c-2} - \frac{2a}{(2c-1)^2} - \frac{a-r}{c\,(2c-1)}\right)}$$

and

$$q(c,\alpha) = \frac{\alpha - r}{c} + \sqrt{r\left(\frac{c}{4c-2} - \frac{2a}{(2c-1)^2} - \frac{a-r}{c\,(2c-1)}\right)}.$$

By taking second derivative (recall that r is also a function of α) we have that $q(c,\alpha)$, as a function of α, is concave in $[g(c),h(c)]$. Thus, its minimum is obtained either at $q(c,g(c))$ or $q(c,h(c))$. Comparing the two values we obtain that the minimum of $q(c,a)$ in $[g(c),h(c)]$ is obtained at $\alpha = h(c)$. Thus, by (5.8) we have that

$$\eta(3,c) \geq \frac{h(c)}{c} = \frac{\left(12\,c^2 - 5\,c + 2 - \sqrt{240\,c^4 - 216\,c^3 + 73\,c^2 - 20\,c + 4}\right)(2\,c - 1)}{32\,(1-c)\,c}.$$

■

We end this section by graphically illustrating the three functions on the r.h.s. of (5.8) for the case $c = 3/4$. In this case we have $h(c) = h(3/4) = 5/16 - \sqrt{238}/64 \approx 0.0714$ and $g(c) = g(3/4) = \sqrt{13}/16 - 5/32 \approx 0.0691$. Figure 3 shows that in the range $[g(3/4),h(3/4)]$ the function $q(3/4,\alpha)$ is concave and dominates both $\frac{\alpha}{c} = \frac{4\alpha}{3}$ and $\frac{c}{2} - \frac{2\alpha}{2c-1} = \frac{3}{8} - 4\alpha$. The minimum in (5.8) is attained for $\alpha = h(3/4)$.

6. Quality of lower bounds. We start by analyzing the quality of the worst case ratio between the lower bounds given in Corollary 5.3 (namely, Theorem 1.3 in the cases $k = 3,4$) and the conjectured values given in Conjectures 1.1 for $k = 3$ and 1.2 for $k = 4$ (recall that the conjectured values are also upper bounds).

Consider first the case $k = 3$. As the conjectured value is $c/6$ for $c \geq 3/4$ and $c/2 - 1/4$ when $1/2 \leq c \leq 3/4$, the worst case ratio is

$$\min\left\{\inf_{\frac{1}{2}<c\leq\frac{3}{4}} \frac{\frac{c}{2} - \frac{c^2}{4c-1}}{\frac{c}{2} - \frac{1}{4}},\ \min_{\frac{3}{4}\leq c\leq 1} \frac{\frac{c}{2} - \frac{c^2}{4c-1}}{\frac{c}{6}}\right\}.$$

The left expression is a monotone decreasing function in $(1/2,3/4]$ and the right expression is a monotone increasing function in $[3/4,1]$. Hence, $c = 3/4$ minimizes both expression and the obtained ratio is 0.75.

Consider next the case $k = 4$. As the conjectured value is $c/12$ for $c \geq 4/5$ and $c/2 - 1/3$ when $2/3 \leq c \leq 4/5$, the worst case ratio is

$$\min\left\{\inf_{\frac{2}{3}<c\leq\frac{4}{5}} \frac{\frac{c}{2} - \frac{5c^3}{22c^2-14c+4}}{\frac{c}{2} - \frac{1}{3}},\ \min_{\frac{4}{5}\leq c\leq 1} \frac{\frac{c}{2} - \frac{5c^3}{22c^2-14c+4}}{\frac{c}{12}}\right\}.$$

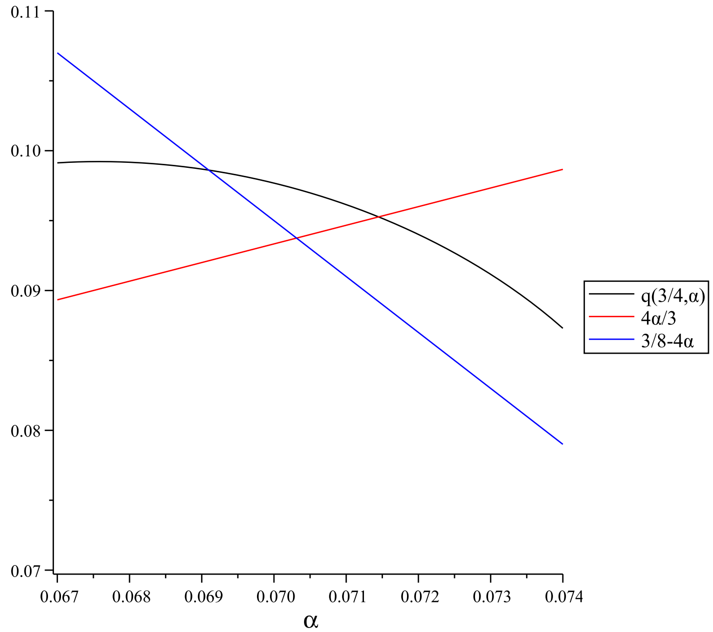

FIG. 3. *The three functions in the r.h.s. of (5.8) in the case* $c = 3/4$.

By taking derivatives, it is not difficult to see that the left expression is a monotone increasing function in $(2/3, 4/5]$ and the right expression is also monotone increasing in $[4/5, 1]$. Both expressions are equal at $c = 4/5$. Hence, the minimum is when $c = 2/3$ (which yields the infimum of the left expression) and the obtained ratio is 0.3.

We now analyze the quality of the worst case ratio between the lower bounds given in Theorem 1.5 and the conjectured value given in Conjecture 1.1. Here we need to compute

$$\min\left\{ \inf_{\frac{1}{2}<c\leq\frac{3}{4}} x(c) \ , \quad \inf_{\frac{3}{4}\leq c<1} y(c) \right\}$$

where

$$x(c) = \frac{\frac{\left(12\,c^2-5\,c+2-\sqrt{240\,c^4-216\,c^3+73\,c^2-20\,c+4}\right)(2\,c-1)}{32(1-c)c}}{\frac{c}{2}-\frac{1}{4}}$$

$$y(c) = \frac{\frac{\left(12\,c^2-5\,c+2-\sqrt{240\,c^4-216\,c^3+73\,c^2-20\,c+4}\right)(2\,c-1)}{32(1-c)c}}{\frac{c}{6}} \ .$$

Using derivatives, it is not difficult (though tedious) to see that $x(c)$ is a monotone decreasing function in $(1/2, 3/4]$ (in fact, in $(1/2, 1)$) and that $y(c)$ is a monotone increasing function in $[3/4, 1)$ (in fact, in $(1/2, 1)$); see Figure 4. They are (trivially) both equal at $c = 3/4$ which is, therefore, the minimum point of their upper envelope. The value at this point, and hence the obtained ratio, is $x(3/4) = y(3/4) = (20 - \sqrt{238})/6 > 0.762$.

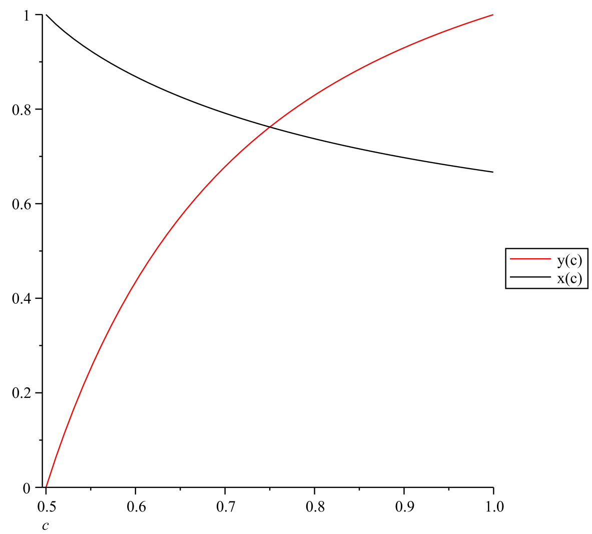

FIG. 4. *The decreasing function $x(c)$ and the increasing function $y(c)$ shown in the range* $(0.5, 1)$.

7. Upper bounds. We start by giving a simple proof showing that the values stated in Conjecture 1.2 are, in fact, upper bounds for $\eta(k, c)$.

PROPOSITION 7.1.

$$\eta(k,c) \leq \begin{cases} \frac{c}{k^2-k} & \text{if} \quad 1-\frac{1}{k+1} \leq c < 1, \\ \left(\frac{c}{2} - \frac{k-2}{2k-2}\right) & \text{if} \quad 1-\frac{1}{k-1} \leq c \leq 1-\frac{1}{k+1}, \\ 0 & \text{if} \quad 0 \leq c \leq 1-\frac{1}{k-1}. \end{cases}$$

Proof. If $c \leq 1 - \frac{1}{k-1}$, then $\eta(k,c) = 0$ by Turán's Theorem. If $c \geq 1 - \frac{1}{k+1}$ then the upper bound $\frac{c}{k^2-k}$ is trivial as any cn-regular graph cannot have more than $(cn^2/2)/\binom{k}{2}$ edge-disjoint copies of K_k. It remains to consider the case $1 - \frac{1}{k-1} \leq c \leq 1 - \frac{1}{k+1}$. Start with the Turán graph $T(n, k-1)$ which has no copy of K_k. We may assume than $k-1$ divides n and that cn is an integer, by the asymptotic nature of the definition of $\eta(k,c)$. Let H be any graph that has $n/(k-1)$ vertices and is regular of degree $(c + \frac{1}{k-1} - 1)n$. Embed a copy of H in each vertex class of $T(n, k-1)$. The resulting graph G is regular of degree $(c + \frac{1}{k-1} - 1)n + (1 - \frac{1}{k-1})n = cn$. Now, each copy of K_k in G must contain an edge from one of the $k-1$ embedded copies of H. Thus

$$\nu_k(G) \leq (k-1)e(H) = (k-1) \cdot \frac{n^2}{2(k-1)} \left(c + \frac{1}{k-1} - 1\right) = n^2 \left(\frac{c}{2} - \frac{k-2}{2k-2}\right) .$$

□

Observe that the proof of Proposition 7.1 actually also gives the same upper bound for $\eta^*(k, c)$.

Interestingly, the following proposition shows that in order to prove Conjecture 1.2, it suffices to prove it for $\eta(k, 1 - \frac{1}{k+1})$.

PROPOSITION 7.2. *If $\eta(k, 1 - \frac{1}{k+1}) = \frac{1}{k^2-1}$, then Conjecture 1.2 holds.*

Proof. Assume that indeed $\eta(k, 1-\frac{1}{k+1}) = \frac{1}{k^2-1}$. Consider first the case $c > 1-\frac{1}{k+1}$. Let G be a graph with minimum degree cn. We may assume that n is divisible by k, by the asymptotic nature of the definition of $\eta(k,c)$. By a theorem of Hajnal and Szemerédi [10], any graph with minimum degree at least $n(1-1/k)$ contains a K_k-factor (assuming n is divisible by k), namely n/k vertex-disjoint copies of K_k. We may therefore greedily remove from G a set of r pairwise edge-disjoint K_k-factors, as long as $cn-(k-1)r \geq n(1-1/k)$. In fact, we will only use

$$r = n\frac{c+\frac{1}{k+1}-1}{k-1} \ .$$

After deleting r edge-disjoint K_k-factors, we obtain a graph G' with minimum degree $(1-\frac{1}{k+1})n$. Assuming $\eta(k, 1-\frac{1}{k+1}) = \frac{1}{k^2-1}$, we have that G' has $n^2/(k^2-1)-o(n^2)$ edge-disjoint copies of K_k. Thus, G has at least

$$r\frac{n}{k} + \frac{n^2}{k^2-1} - o(n^2) = \frac{c}{k^2-k}n^2 - o(n^2)$$

edge-disjoint copies of K_k. Hence, $\eta(k,c) \geq c/(k^2-k)$.

We remain with the case $1-\frac{1}{k-1} < c < 1-\frac{1}{k+1}$. Let G be a graph with minimum degree cn. We first need to establish the following claim: in order to increase the minimum degree of the graph by 1 (and remain with a simple graph), it suffices to add at most $n/2+o(n)$ edges. We prove this claim in the next paragraph.

Let $A \subset V(G)$ denote the set of vertices with degree cn and let $B = V(G) \setminus A$ be the vertices with higher degree. The number of edges of the cut (B,A) is therefore larger than $(cn-|B|)|B|$. On the other hand, the sum of the degrees of the vertices of A is $cn|A|$. It follows that $e(A) > (cn|A| - (cn-|B|)|B|)/2$. Denoting $|A| = \alpha n$ and denoting by $G^c[A]$ the complement of the subgraph of G induced by A we have:

$$e(G^c[A]) \geq \binom{|A|}{2} - e(A) \geq n^2\left(\frac{\alpha^2}{2} - \frac{c\alpha-(c-1+\alpha)(1-\alpha)}{2}\right) - o(n^2) \ .$$

Simplifying the last expression we obtain

$$e(G^c[A]) \geq \left(\alpha - \frac{1}{2}\right)(1-c)n^2 - o(n^2) \ .$$

The maximum degree of $e(G^c[A])$ is not larger than the maximum degree of G^c which is at most $(1-c)n$. Summarizing, $G^c[A]$ is a graph with αn vertices, maximum degree at most $(1-c)n$, and its number of edges is at least $(\alpha-1/2)(1-c)n^2 - o(n^2)$. A simple application of Tutte's Theorem (see, e.g., [1]) gives that $G^c[A]$ has a matching M of size at least $(\alpha-1/2)n - o(n)$ edges. The number of unmatched edges of $G^c[A]$ is therefore at most $\alpha n - 2|M| = (1-\alpha)n + o(n)$. For each unmatched vertex $v \in A$, pick an arbitrary edge of G^c incident with v. Denote the set of selected edges by P. Hence, $P \cup M$ is a set of edges whose addition to G increases the minimum degree by 1. But

$$|P \cup M| = |P| + |M| \leq (1-\alpha)n + (\alpha-1/2)n + o(n) = \frac{n}{2} + o(n) \ .$$

Having proved our claim, we see that given a graph G with minimum degree cn, we can add to it at most $(1-1/(k+1)-c)n \cdot (n/2+o(n))$ edges and obtain a graph G' with minimum degree at least $(1-1/(k+1))n$. By our assumption that $\eta(k, 1-\frac{1}{k+1}) = \frac{1}{k^2-1}$ we have that G' contains at least $n^2/(k^2-1)-o(n^2)$ edge-disjoint copies of K_k. At most $e(G')-e(G)$ of these copies of K_k contain an edge that does not appear in G. It follows that G has at least

$$\frac{n^2}{k^2-1} - o(n^2) - \left(1 - \frac{1}{k+1} - c\right) n \cdot \left(\frac{n}{2} + o(n)\right) = \left(\frac{c}{2} - \frac{k-2}{2k-2}\right) n^2 - o(n^2)$$

edge-disjoint copies of K_k. Hence, $\eta(k,c) \geq \frac{c}{2} - \frac{k-2}{2k-2}$, as required. □

8. Concluding remarks. The proofs of Theorem 1.3 and Theorem 1.5 are algorithmic. Moreover, given a graph with minimum degree at least cn, one can find a K_k-packing of size at least $\eta(k,c)n^2(1-o_n(1))$ in polynomial time. This follows directly from the fact that Lemma 2.1 (the theorem of Haxell and Rödl [11]) is algorithmic, as they show that the optimal fractional packing (found via linear programming) can be converted to an integral one with only a negligible additive loss, in polynomial time.

Although our main results consider packings with complete graphs, they have a natural extension to arbitrary fixed graphs H. Let $\eta(H,c)$ be the infimum over all constants γ such that any graph G of order n and minimum degree at least cn has $\nu_H(G) \geq \gamma n^2(1-o_n(1))$. As proved in [26], if $\chi(H) = k$ and $|E(H)| = r$, then $r \cdot \nu_H(G) \geq \binom{k}{2}\nu_k(G) - o(n^2)$. Thus, one can use the lower bound in Theorem 1.3, multiplied by $\binom{k}{2}/r$, and obtain a lower bound for $\eta(H,c)$. Likewise, the lower bound in Theorem 1.5, multiplied by $3/r$, serves as a lower bound for $\eta(H,c)$ where H is a 3-chromatic graph.

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