H-packing of k-chromatic graphs

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Abstract

For graphs H and G, let $p_H(G)$ denote the maximum number of edges covered by a set of edge-disjoint copies of H in G. We prove that if H is k-chromatic, then $p_H(G) \geq p_{K_k}(G) - o(|V(G)|^2)$. The error term cannot be improved much, as for any $\delta > 0$ there are graphs H with $\chi(H) = k$ such that for all n sufficiently large, there are graphs G with n vertices for which $p_H(G) \leq p_{K_k}(G) - n^{2-\delta}$. We present several applications of this result in extremal graph theory.
Keywords: edge-packing, chromatic number, H-packing

1 Introduction

All graphs considered here are finite, undirected, and simple. For standard graph-theoretic terminology the reader is referred to [3].

Finding sufficient conditions for the existence of copies of a given graph H as a subgraph of some larger graph lies at the heart of extremal graph theory. Many of the classical results that consider the case $H = K_k$ were subsequently generalized to hold for arbitrary H. We recall two sets of results where this has occurred.

Turán's Theorem asserts that a graph that has more edges than any complete $(k-1)$-partite graph with the same amount of vertices must contain K_k as a subgraph. Turán's Theorem was later generalized by a result of Erdős and Stone [6] to any graph H with chromatic number $\chi(H) = k \geq 3$. They proved that $ex(n,H) = (1 - 1/(k-1) + o(1))\binom{n}{2}$, where $ex(n,H)$ is the maximum number of edges in a graph with n vertices that does not contain a copy of H. The Hajnal-Szemerédi Theorem asserts that a graph with n vertices and minimum degree $n(1-1/k)$ has a K_k-*factor*, assuming $k|n$, where a K_k-factor is a set of n/k pairwise vertex-disjoint copies of K_k. This result was later generalized by Alon and Yuster [2] who proved that the same result holds asymptotically if $\chi(H) = k$. They proved that a minimum degree of $n(1 - 1/k + o(1))n$ guarantees an H-factor, assuming $|V(H)||n$. The $o(n)$ error term was later improved to a constant depending on H [11]. We see that in these two sets of results, the chromatic number plays the main role, as they show that fixed graphs H with $\chi(H) = k$ asymptotically behave "no worse" than K_k.

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In this paper we consider another natural problem of this type: the problem of edge-packing. An (edge) *H-packing* of a graph G is a set of pairwise edge-disjoint subgraphs of G that are isomorphic to H. The H-packing number of G, denoted by $\nu_H(G)$, is the maximum size of an H-packing. Equivalently, one can define $p_H(G)$ to be the number of edges covered by the elements of a maximum size H-packing, observing that $p_H(G)/\nu_H(G) = |E(H)|$. So, in analogy to the above sets of results, we would like to determine the relationship between $\nu_H(G)$ and $\nu_{K_k}(G)$ for a fixed graph H with $\chi(H) = k$. Optimally, we would expect $p_H(H)$ and $p_{K_k}(G)$ to be close. Indeed, this is our main result.

Theorem 1.1. *Let H be a graph with $\chi(H) = k$ and let $\epsilon > 0$. There exists $N = N(H, \epsilon)$ such that for all $n > N$ and for all graph G with n vertices,*

$$\nu_H(G) \geq \frac{\binom{k}{2}}{|E(H)|}\nu_{K_k}(G) - \epsilon n^2 .$$

The main ingredients in the proof of Theorem 1.1 are Szemerédi's regularity lemma [14], and an extension of a result of Haxell and Rödl that relates H-packings with their fractional relaxation [9].

The error term ϵn^2 in Theorem 1.1 cannot be significantly improved, as can be seen from the following theorem. Let $T(k, q)$ denote the complete k-partite graph with q vertices in each part. Clearly, $\chi(T(k, q)) = k$.

Theorem 1.2. *For every n, there are graphs G with n vertices such that*

$$\nu_{T(k,q)}(G) \leq \frac{\binom{k}{2}}{|E(T(k,q))|}\nu_{K_k}(G) - \Theta\left(n^{2-\frac{kq-2}{\binom{k}{2}q^2-1}}\right) .$$

In particular, Theorem 1.2 shows that for any $\delta > 0$, there are graphs H with $\chi(H) = k$ such that for all n sufficiently large, there are graphs G with n vertices for which $p_H(G) \leq p_{K_k}(G) - n^{2-\delta}$.

Theorem 1.1 has several interesting applications, as there are several known results (and conjectures) in extremal graph theory that guarantee the existence of a large K_k-packing. These results can therefore be directly extended to the more general setting of H-packing of graphs with $\chi(H) = k$.

The rest of this paper is organized as follows. In Section 2 we establish the tools that are required for the proof of our theorems. Section 3 contains the proof of Theorem 1.1 and Section 4 contains the proof of Theorem 1.2. Several applications of Theorem 1.1 are given in Section 5.

2 Preliminaries

Let $\binom{G}{H}$ denote the set of copies of a graph H in a graph G. A function ψ from $\binom{G}{H}$ to $[0, 1]$ is a *fractional H-packing* of G if $\sum_{H' \in \binom{G}{H} : e \in H'} \psi(H') \leq 1$ for each $e \in E(G)$. For a fractional H-packing ψ, let $w(\psi) = \sum_{H' \in \binom{G}{H}} \psi(H')$. The *fractional H-packing number*, denoted by $\nu^*_H(G)$, is the maximum value of $w(\psi)$ ranging over all fractional H-packings ψ. Trivially, $\nu^*_H(G) \geq \nu_H(G)$, as an H-packing is a special case of a fractional H-packing. An important result of Haxell and Rödl [9] asserts that the converse is also asymptotically true, up to a small error term.

Lemma 2.1. *For every $\eta > 0$ and every graph H there exists $N_1 = N_1(H, \eta)$ such that for all $n > N_1$, if G is a graph with n vertices, then $\nu^*_H(G) - \nu_H(G) \le \eta n^2$.*

We assume that the reader is familiar with the statement of Szemerédi's regularity lemma [14], and in particular with the definitions of bipartite density, ϵ-regularity, and equitable partitions, as these are now a standard concepts in Combinatorics. The survey [12] is an excellent source for an overview of the regularity lemma and its applications in graph theory.

Both the results of Haxell and Rödl [9] and of Yuster [16] are based on the following idea. Given a graph H with $|V(H)| = k$ vertices, any sufficiently large graph G has a set of k-partite subgraphs that are pairwise edge-disjoint and have the following property. Each pair of parts in these k-partite subgraphs is ϵ-regular, and if the pair corresponds to an edge of H, then it has edge density very close to some common value δ. Otherwise, its density is zero. Furthermore, if one takes a maximum H-packing in each of these subgraphs, then the sum of their sizes is very close to $\nu^*_H(G)$.

The statement in the last paragraph is a by-product of the proofs in [9, 16], and is not stated as a separate result in these papers. It will be important for us to state and quantify it accurately as a separate lemma. For simplicity, we state it for the complete graph K_k, as this case will be sufficient for the proof of Theorem 1.1.

We require the following definition. Let b and k be positive integers, and let ϵ and δ_0 be positive reals. We say that a graph P is a $(b, k, \delta_0, \epsilon)$-graph if P is a k-partite graph with b vertices in each part, and each pair of parts induces an ϵ-regular graph with density in the range $\delta \pm \epsilon$ where $\delta \ge \delta_0$.

Lemma 2.2. *For every $\zeta > 0$ and $k \ge 2$ there exist $\delta_0 = \delta_0(\zeta, k)$ and $\Gamma = \Gamma(\zeta, k)$ such that for all $0 < \gamma \le \Gamma$ there exists $N = N(\zeta, k, \gamma)$ and $M = M(\zeta, k, \gamma)$ such that any graph G with $n > N$ vertices has a set $\mathcal{P}$ of pairwise edge-disjoint (b, k, δ_0, γ)-graphs with $b \ge n/M$. Furthermore,*

$$\sum_{P \in \mathcal{P}} \nu_{K_k}(P) \ge \nu^*_{K_k}(G) - \zeta n^2 \ .$$

Outline of proof. We outline the proof from [9]. Let δ_0 and Γ be sufficiently small constants depending only on k and ζ, and let $\gamma \le \Gamma$. Let M and N be sufficiently large constants depending on ζ, k, γ. For a graph G with $n > N$, let ψ be a fractional K_k-packing of G with $w(\psi) = \nu^*_{K_k}(G)$. We apply the regularity lemma and obtain a $\gamma/2$-regular partition into m parts $V_1, \ldots, V_m$ with $1/\gamma < m < M$, so we may assume that each part has b vertices where $b \ge n/M$. The total contribution to $w(\psi)$ of copies of K_k that contain an edge with both endpoints in the same part, or between pair of parts with density smaller than δ_0, or between pairs that are not $\gamma/2$-regular, is easily shown to be at most $\zeta n^2/4$, so we can ignore such bad copies of K_k, and call the other copies *good*.

We next define the "supergraph" S with vertices $\{1, \ldots, m\}$ where vertex i represents V_i. An edge ij represents a $\gamma/2$-regular pair (V_i, V_j) with density at least δ_0. By scaling down each weight of a good copy by a factor of n^2/m^2 we obtain a fractional K_k-packing ψ' of S with $w(\psi') \geq m^2 w(\psi)/n^2 - \zeta m^2/4$. Another important observation in [9] is that one can modify ψ' so that each copy H of K_K in S for which $\psi'(H) > 0$ actually has $\psi'(H) \geq \delta_0$, where after this modification we can still have $w(\psi') \geq m^2 w(\psi)/n^2 - \zeta m^2/2$.

The problem is that the densities between pairs of parts corresponding to an edge $ij \in E(S)$, although guaranteed to be at least δ_0, may still vary significantly. To overcome this problem, for each edge $e = ij \in E(S)$, the set of edges of the pair (V_i, V_j) is "sliced" into γ-regular graphs with edge sets $E_{ij}(H)$, one for each copy of K_k in S with $\psi'(H) \geq \delta_0$, such that $E_{ij}(H)$ has density $\psi'(H)$. Therefore each such copy H of K_k in S with vertex set $\{i_1, \ldots, i_k\}$ corresponds to a subgraph P_H of G with vertex set $V_{i_1} \cup \cdots \cup V_{i_k}$ and edge set $\cup_{ij \in E(H)} E_{ij}(H)$ where $E_{ij}(H)$ is γ-regular of density $\psi'(H)$. In other words, each such P_H is a (b, k, δ_0, γ)-graph. The γ-regularity of the pairs of parts of P_H is then used to show that, in fact, $\nu_{K_k}(P)$ absorbs almost all edges of P_H, implying that its total value is very close to $\psi'(H)n^2/m^2$. Summing over all H we obtain a value which is larger than $w(\psi')n^2/m^2 - \zeta n^2/2$. Recalling that $w(\psi') \geq m^2 w(\psi)/n^2 - \zeta m^2/2$, the result now follows. □

For an h-partite graph G with parts $V_1, \ldots, V_h$ and for a graph H with vertices $\{w_1, \ldots, w_h\}$, we say that a subgraph H' of G is *partite-isomorphic* to H, if H' is isomorphic to H and w_i is mapped to a vertex of H' in V_i. We also need the following lemma from [9], which relies on a result from [4], and which estimates the number of partite-isomorphic subgraphs containing a given edge of a certain h-partite graph. For a graph X, let $e(X) = |E(X)|$.

Lemma 2.3. *Let H be a graph with vertices $\{w_1, \ldots, w_h\}$ and with r edges. Let real numbers $\lambda > 0$ and $\delta_0 > 0$ be given. Then there exists $\gamma = \gamma(H, \lambda, \delta_0)$ such that the following holds. Let R be an h-partite graph with vertex classes $V_1, \ldots, V_h$ with $|V_i| = b$, satisfying*
(i) for each $w_i w_j \in E(H)$ we have that (V_i, V_j) is γ-regular with density in $\delta \pm \gamma$ where $\delta \geq \delta_0$,
(ii) for each $w_i w_j \notin E(H)$ we have that $V_i \cup V_j$ is an independent set.
Then there exists a subgraph $R' \subset R$ with vertex classes $V_1', \ldots, V_h'$, $V_i' \subset V_i$, and at least $(1-\lambda)e(R)$ edges such that for each edge e of R',

$$|c_H(e) - \delta^{r-1} b^{h-2}| < \lambda \delta^{r-1} b^{h-2},$$

where $c_H(e)$ denotes the number of subgraphs of R' containing e which are partite-isomorphic to H in $R'[V_1', \ldots, V_h']$.

In order to simplify the proof, it will be convenient to prove Theorem 1.1 for the case where H is a balanced complete partite graph. We next show why such an assumption suffices for the proof of Theorem 1.1, when combined with Lemma 2.1. Recall that $T(k, q)$ denotes the complete k-partite graph with q vertices in each part. Suppose H is any graph with $\chi(H) = k$ and $|V(H)| = q$. We may trivially embed a copy of H in $T(k, q)$ (actually, this already holds if q is the size of the largest vertex class in a k-coloring H). However, we note that a stronger property holds.

Lemma 2.4. *If $k = \chi(H)$ and $q = |V(H)|$, then $\nu_H^*(T(k, q)) = \binom{k}{2} q^2 / e(H)$.*

Proof. By symmetry, each edge of $T(k,q)$ lies on the same number of copies of H. Denote this number by s. Assigning to each copy of H in $T(k,q)$ the value $1/s$ defines a valid fractional H-packing, which is also a fractional H-decomposition. Hence, its value is $e(T(k,q))/e(H) = \binom{k}{2}q^2/e(H)$. $\square$

A result of Frankl and Rödl [7] on near perfect matchings of uniform hypergraphs will be useful for both the proof of Theorem 1.1 and the proof of Theorem 1.2. Recall that if x, y are two vertices of a hypergraph, then $deg(x)$ denotes the number of edges that contain x and $deg(x,y)$ denotes the number of edges that contain both x and y. We use the following extension of a theorem of Frankl and Rödl, proved by Pippenger (see [8]).

Lemma 2.5. *For an integer $r \geq 2$ and a real $\beta > 0$ there exists $\mu = \mu(r,\beta) > 0$ such that: If an r-uniform hypergraph L on t vertices has the following properties for some d:*
(i) $(1-\mu)d < deg(x) < (1+\mu)d$ holds for all vertices,
(ii) $deg(x,y) < \mu d$ holds for all distinct vertices x and y,
then L has a matching of size at least $(t/r)(1-\beta)$.

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first need to prove a slight extension of Lemma 2.3. Let G be a k-partite graph with vertex parts $V_1,\ldots,V_k$. A q-blowup of G, denoted by G^q, is obtained by cloning each V_i precisely q times, to obtain copies $V_{i,1}\ldots,V_{i,q}$ where each $v \in V_i$ has a copy v_ℓ in $V_{i,\ell}$. There is an edge of G^q between each copy of u and each copy of v if and only if $uv \in E(G)$. Two edges of G^q are *equivalent* if they correspond to the same edge of G. Likewise, two vertices of G^q are equivalent if they are clones of the same vertex of G. Each edge of G gives rise to q^2 equivalent edges of G^q and each vertex of G gives rise to q equivalent vertices of G^q. Observe that G^q is an h-partite graph where $h = kq$, having $|V(G)|q$ vertices and $e(G)q^2$ edges.

Lemma 3.1. *If the h-partite graph R of Lemma 2.3 is a q-blowup of some k-partite graph P (namely $R = P^q$), then the subgraph R' of R obtained by the lemma has the property that $e \in E(R')$ if and only if all q^2 edges of R that are equivalent to e are also in $E(R')$. Likewise, $v \in V(R')$ if and only if all q vertices of R that are equivalent to v are also in $V(R')$.*

Proof. Lemma 2.3 states that the set of edges $E(R')$ is obtained by removing a set E' of at most $\lambda e(R)$ edges of R. The proof of Lemma 2.3 constructs the set E' of removed edges in two stages. In the first stage, one actually deletes vertices from V_i to obtain the subset V_i'. The deleted vertices $V_i \setminus V_i'$ are decided upon only by examining the *neighborhood* of a vertex. Hence, all q vertices of R that are clones of the same original vertex of P are either all deleted or otherwise all are not deleted. Hence, the first set of removed edges (namely, those edges that are incident to the removed vertices $\cup_{i=1}^h V_i \setminus V_i'$) has the property that if an edge is removed then all q^2 edges equivalent to it are also removed. In the second stage, the set of removed edges are what are classified as *bad* edges. Again, the definition of a bad edge xy in the proof of Lemma 2.3 is only a function of the common neighborhoods of x and y. Since two equivalent edges have isomorphic common neighborhoods, an edge xy is bad if and only if all edges equivalent to xy are also bad. $\square$

For a q-blowup graph $R = P^q$, we say that a subgraph of R is *pure* if it contains precisely one vertex from each vertex class of R and it does not contain two equivalent vertices. Observe that a pure subgraph of R corresponds uniquely to a subgraph of P containing precisely q vertices in each vertex class of P.

Lemma 3.2. *Suppose that a k-partite graph P contains c distinct copies of $T(k,q)$. Then P^q contains $c(q!)^k$ distinct pure copies of $T(k,q)$. Furthermore, if $e \in E(P)$ belongs to $c(e)$ distinct copies of $T(k,q)$ then each of the q^2 clones of e in P^q belongs to $c(e)(q!)^k/q^2$ distinct pure copies of $T(k,q)$.*

Proof. Let $V_1, \dots, V_k$ denote the partite classes of P. Suppose that H is a copy of $T(k,q)$ in P. Observe that H is uniquely defined by k subsets $W_1, \dots, W_k$ with $|W_i| = q$ and $W_i \subset V_i$. Consider the partite sets of P^q denoted by $V_{i,\ell}$ for $i = 1, \dots, k$ and $\ell = 1, \dots, q$. A pure copy of $T(k,q)$ in P^q that corresponds to H selects from each of $V_{i,1}, \dots, V_{i,q}$ a unique clone of a vertex from W_i. As there are precisely $q!$ ways to select these clones, and as this holds for all $i = 1, \dots, k$, there are precisely $(q!)^k$ copies that correspond to H. Hence there are $c(q!)^k$ distinct pure copies of $T(k,q)$. For the second part, suppose that $e \in E(P)$ and let H be a copy of $T(k,q)$ in P that contains e. Each clone of e (recall that there are q^2 such clones) corresponds to a $1/q^2$ fraction of the pure copies of $T(k,q)$ in P^k that correspond to H, since each such copy contains a unique clone of e. $\square$

We restate Theorem 1.1 for the special case where $H = T(k,q)$ and then obtain Theorem 1.1 as a corollary.

Theorem 3.3. *For integers $q \ge 1$, $k \ge 2$ and for $\epsilon > 0$ there exists $N = N(k,q,\epsilon)$ such that for all $n > N$, if G is a graph with n vertices, then*

$$\nu_{T(k,q)}(G) \ge \frac{1}{q^2}\nu_{K_k}(G) - \epsilon n^2 .$$

Proof of Theorem 1.1. Let H be a graph with $\chi(H) = k$ and let $\epsilon > 0$. Let $\eta = \epsilon/2$ and $N_1 = N_1(H,\eta)$ be the constants from Lemma 2.1. Let $q = |V(H)|$ and let $N = N(k,q,\epsilon/2)$ be the constant from Theorem 3.3. Observe also the trivial fact that $\nu^*_H(G) \ge \nu^*_H(Q)\nu_Q(G)$ for any graph Q. We therefore have by Lemmas 2.1, 2.4, and Theorem 3.3 that for all $n > \max\{N, N_1\}$,

$$\begin{aligned}
\nu_H(G) &\ge \nu^*_H(G) - \frac{\epsilon}{2}n^2 \\
&\ge \nu^*_H(T(k,q))\nu_{T(k,q)}(G) - \frac{\epsilon}{2}n^2 \\
&= \frac{\binom{k}{2}q^2}{e(H)}\nu_{T(k,q)}(G) - \frac{\epsilon}{2}n^2 \\
&\ge \frac{\binom{k}{2}q^2}{e(H)} \cdot \frac{1}{q^2}\nu_{K_k}(G) - \frac{\epsilon}{2}n^2 - \frac{\epsilon}{2}n^2 \\
&= \frac{\binom{k}{2}}{e(H)}\nu_{K_k}(G) - \epsilon n^2 .
\end{aligned}$$

□

Proof of Theorem 3.3. We start be setting some constants that are functions of the given constants k, q, ϵ.

1. Let $h = kq$ and $r = \binom{k}{2}q^2$ denote the number of vertices and the number of edges of $T(k,q)$.
2. Let $\zeta = \epsilon/2$ and $\beta = \epsilon/8$.
3. Let $\mu = \mu(r, \beta)$ be the constant from Lemma 2.5.
4. Let $\lambda = \min\{\mu/2, \epsilon/8\}$.
5. Let $\delta_0 = \delta_0(\zeta, k)$ and $\Gamma = \Gamma(\zeta, k)$ be the constants from Lemma 2.2.
6. Let $\gamma = \min\{\Gamma\ ,\ \delta_0/2\ ,\ \gamma(T(k,q), \lambda, \delta_0)\}$ where $\gamma(T(k,q), \lambda, \delta_0)$ is the constant from Lemma 2.3.
7. Let $M = M(\zeta, k, \gamma)$ and $N = \max\{N(\zeta, k, \gamma)\ ,\ \frac{kq^2M}{\lambda\delta_0^{r-1}}\ ,\ \frac{(q!)^kM}{q^2\delta_0^{r-1}\mu}\}$ where $N(\zeta, k, \gamma)$ and $M = M(\zeta, k, \gamma)$ are the constants from Lemma 2.2.

Suppose that G is a graph with $n > N$ vertices. We apply Lemma 2.2 and obtain a set $\mathcal{P}$ of pairwise edge-disjoint (b, k, δ_0, γ)-graphs of G with $b \geq n/M$, and which satisfy the statement of Lemma 2.2.

For $P \in \mathcal{P}$ consider $R = P^q$, the q-blowup of P. We claim that R satisfies the conditions of Lemma 2.3 for the case $H = T(k,q)$ in that lemma. For convenience, denote the vertex classes of P by $V_1, \ldots, V_k$. Denote the vertices of $T(k,q)$ belonging to the i'th vertex class by $w_{i,1}, \ldots, w_{i,q}$. Denote the vertex classes of R by $V_{i,\ell}$ for $i = 1, \ldots, k$ and $\ell = 1, \ldots, q$. Now, there are no edges between $V_{i,\ell}$ and $V_{i,\ell'}$ but this is fine since $w_{i,\ell}$ and $w_{i,\ell'}$ are not adjacent in $T(k,q)$. On the other hand, the bipartite subgraph induced by $V_{i,\ell}$ and $V_{j,\ell'}$ for $i \neq j$ is isomorphic to the bipartite subgraph induced by V_i and V_j which, by the fact that P is a (b, k, δ_0, γ)-graph, means that it is γ-regular with density in $\delta \pm \gamma$ for some $\delta \geq \delta_0$. This is also fine since $w_{i,\ell}$ and $w_{j,\ell'}$ are adjacent in $T(k,q)$.

By Lemma 2.3 there exists a subgraph $R' \subset R$ with vertex classes $V'_{i,\ell} \subset V_{i,\ell}$, and at least $(1-\lambda)e(R)$ edges such that for each edge e of R',

$$|c'(e) - \delta^{r-1}b^{h-2}| < \lambda\delta^{r-1}b^{h-2}, \tag{1}$$

where $c'(e)$ denotes the number of subgraphs of R' containing e which are partite-isomorphic to $T(k,q)$ in R'. As R is a q-blowup, we have that by Lemma 3.1, R' is also a q-blowup and $R' = (P')^q$ where P' is a subgraph of P, and

$$e(P') = \frac{e(R')}{q^2} \geq (1-\lambda)\frac{e(R)}{q^2} = (1-\lambda)e(P)\ . \tag{2}$$

What we are really interested in is the number of copies of $T(k,q)$ in P' that contain a given edge. The problem is that (1) estimates the total number of partite isomorphic copies of $T(k,q)$

that contain e, including the non-pure copies. So, let us estimate the number of non-pure partite isomorphic copies of $T(k,q)$ containing a given edge of R. Assume, without loss of generality, that $e \in E(R)$ and $e = xy$ with $x \in V_{1,1}$ and $y \in V_{2,1}$. A non-pure copy selects (at least) two equivalent vertices. Namely, for some, i, and for some ℓ and ℓ', the copy contain two equivalent vertices, one from $V_{i,\ell}$ and one from $V_{i,\ell'}$. There are k choices for i and $\binom{q}{2}$ choices for ℓ and ℓ'. Observe that once we have chosen the vertex from $V_{i,\ell}$, its equivalent vertex from $V_{i,\ell'}$ is determined. Hence, the number of non-pure copies of $T(k,q)$ containing a given edge is trivially at most $k\binom{q}{2}b^{h-3}$. Thus, using (1), if $c(e)$ denotes the number of pure copies of $T(k,q)$ in R' that contain e, we have that

$$|c(e) - \delta^{r-1}b^{h-2}| < \lambda\delta^{r-1}b^{h-2} + k\binom{q}{2}b^{h-3} \leq 2\lambda\delta^{r-1}b^{h-2},$$

where in the last inequality we have used the fact than $b \geq n/M \geq kq^2/(\lambda\delta_0^{r-1})$. By Lemma 3.2 we have that for $e \in P'$, if $c(e)$ is the number of copies of $T(k,q)$ in P' that contain e, then

$$|c(e) - \frac{q^2}{(q!)^k}\delta^{r-1}b^{h-2}| < 2\lambda\frac{q^2}{(q!)^k}\delta^{r-1}b^{h-2} \; . \qquad (3)$$

We construct an r-uniform hypergraph L as follows. The vertices of L are the edges of P'. Hence L has $t = e(P')$ vertices. The edges of L correspond to the edge sets of copies of $T(k,q)$ in P'. Observe that L is indeed r-uniform and the degree of a vertex $e \in V(L)$ is $c(e)$. Observe that by setting $d = \frac{q^2}{(q!)^k}\delta^{r-1}b^{h-2}$ and since $\mu \geq 2\lambda$, equation (3) shows that L satisfies the first condition of Lemma 2.5. We need to show that the second condition also holds. For two edges of P', the number of copies of $T(k,q)$ containing both of them is trivially at most b^{h-3} which is smaller than μd, since $b > n/M \geq (q!)^k/(q^2\delta_0^{r-1}\mu)$. If follows from Lemma 2.5 that P' contains at least $(t/r)(1-\beta)$ edge-disjoint copies of $T(k,q)$. By (2) we therefore have that

$$\nu_{T(k,q)}(P) \geq (1-\beta)\frac{e(P')}{r} \geq (1-\beta)(1-\lambda)\frac{e(P)}{r} \; .$$

Altogether we obtain

$$\begin{aligned}
\nu^*_{K_k}(G) &\leq \sum_{P\in\mathcal{P}} \nu_{K_k}(P) + \zeta n^2 \\
&\leq \sum_{P\in\mathcal{P}} \frac{e(P)}{\binom{k}{2}} + \zeta n^2 \\
&\leq \frac{r}{(1-\beta)(1-\lambda)\binom{k}{2}} \sum_{P\in\mathcal{P}} \nu_{T(k,q)}(P) + \zeta n^2 \\
&= \frac{q^2}{(1-\beta)(1-\lambda)} \sum_{P\in\mathcal{P}} \nu_{T(k,q)}(P) + \zeta n^2 \\
&\leq q^2 \sum_{P\in\mathcal{P}} \nu_{T(k,q)}(P) + (2\beta+2\lambda)q^2 \sum_{P\in\mathcal{P}} \nu_{T(k,q)}(P) + \zeta n^2 \\
&\leq q^2 \sum_{P\in\mathcal{P}} \nu_{T(k,q)}(P) + (2\beta+2\lambda)n^2 + \zeta n^2 \\
&\leq q^2 \sum_{P\in\mathcal{P}} \nu_{T(k,q)}(P) + q^2\epsilon n^2 \; .
\end{aligned}$$

It follows from the last inequality that

$$\nu_{T(k,q)}(G) \geq \sum_{P \in \mathcal{P}} \nu_{T(k,q)}(P) \geq \frac{1}{q^2}\nu^*_{K_k}(G) - \epsilon n^2 \geq \frac{1}{q^2}\nu_{K_k}(G) - \epsilon n^2 .$$

□

4 Lower bound

Proof of Theorem 1.2. Let c be some small absolute constant. We show that the random graph $G = G(n,p)$ with

$$p = cn^{-\frac{kq-2}{\binom{k}{2}q^2-1}}$$

satisfies the stated claim with high probability. Let

$$z = n^{2-\frac{kq-2}{\binom{k}{2}q^2-1}} .$$

As $T(k,q)$ has kq vertices and $\binom{k}{2}q^2$ edges, the expected number of copies of $T(k,q)$ in G is less than

$$n^{kq}p^{\binom{k}{2}q^2} = c^{\binom{k}{2}q^2} z .$$

On the other hand, the expected number of edges of G is

$$p\binom{n}{2} \approx \frac{c}{2}z .$$

Since the number of edges is a binomial random variable, highly concentrated around its expectation, we obtain that for a sufficiently small c, with probability at least, say, $2/3$, any packing of G with edge-disjoint copies of $T(k,q)$ leaves at least $(c/6)z$ edges uncovered.

On the other hand, we will show that with probability at least, say, $2/3$, a maximum packing of G with edge-disjoint copies of K_k covers all but at most $(c/10)z$ edges, thereby proving the existence of an n-vertex graph G for which $p_{T(k,q)}(G) \leq p_{K_k}(G) - \Theta(z)$, as required.

For $\beta = 0.1$ and $r = \binom{k}{2}$, and let $\mu = \mu(r,\beta)$ be the constant from Lemma 2.5. Consider the r-uniform hypergraph whose vertices are the edges of G and whose edges are the copies of K_k in G. For an edge $e \in E(G)$, $deg(e)$ denotes the number of copies of K_k that contain e, and for a pair of edges $e, f \in E(G)$, $deg(e,f)$ denote the number of copies of K_k that contain both of e and f. Given that e is an edge of G, the expectation of $deg(e)$ is

$$d = \binom{n-2}{k-2}p^{\binom{k}{2}-1} .$$

A standard application of the second moment method (see, [1], Chapter 4) shows that $deg(e)$ is concentrated around its expectation d, so that for any $\mu > 0$, if n is sufficiently large, $|deg(e) - d| \leq \mu d$ holds with high probability for all edges of G. On the other hand, we trivially have $deg(e,f) \leq \mu d$

for all n sufficiently large. Hence, by Lemma 2.5, with high probability, a maximum K_k-packing of G covers all but a β-fraction of the edges. As with high probability we also have that the number of edges of G is very close to $(c/2)z$, we are guaranteed that with high probability, the number of uncovered edges in an maximum K_k-packing is at most $(c/10)z$, as claimed. □

5 Applications

For a graph H and for positive integers r and n, let $f(H,r,n)$ denote the least integer t such than in any r-coloring of the edges of K_n, there are t pairwise edge-disjoint monochromatic copies of H.

Corollary 5.1. *If H be a fixed graph with $\chi(H)=k$ then*

$$f(H,r,n) \geq f(K_k,r,n)\frac{\binom{k}{2}}{e(H)}(1-o_n(1)) \, .$$

Furthermore, if $\chi(H)=3$, then $f(H,2,n) \geq \frac{3n^2}{13e(H)}$ for all n sufficiently large.

Proof. First observe that $f(K_k,r,n)=\Theta(n^2)$ is an immediate consequence of Ramsey's Theorem, so it suffices to prove that $f(H,r,n) \geq f(K_k,r,n)\binom{k}{2}/e(H) - o(n^2)$. Indeed, consider a maximum packing of monochromatic copies of K_k in a given r-coloring of the edges of K_n. Suppose the packing contains t_i copies of K_k with color i for $i=1,\ldots,r$. Then, by definition, $\sum_{i=1}^r t_i \geq f(K_k,r,n)$. We can also view the edges with color i as a graph G_i with n vertices, and clearly $t_i = \nu_{K_k}(G_i)$. Applying Theorem 1.1 to each G_i separately yields a packing with edge disjoint monochromatic copies of H of total size at least

$$\sum_{i=1}^{r}\left(t_i\frac{\binom{k}{2}}{e(H)} - o(n^2)\right) \geq f(K_k,r,n)\frac{\binom{k}{2}}{e(H)} - o(n^2) \, .$$

The second part of the corollary follows from a result of Keevash and Sudakov [10] who proved that $f(K_3,2,n) \geq n^2/12.9$ for all n sufficiently large. □

We note that a conjecture of Erdős (see [5]) states that $f(K_3,2,n) \geq n^2/12 - o(n^2)$. This conjecture may now be appropriately generalized to any fixed graph H with $\chi(H)=3$.

It seems reasonable that if a graph has large minimum degree, then it has a large H-packing, covering almost all the edges. Indeed, this has been proved by the author for the case of $H=K_k$ and can now be proved for all k-chromatic graphs.

Corollary 5.2. *If H be a fixed graph with $\chi(H)=k$, then any graph with n vertices and minimum degree at least $n(1-1/9k^{10})$ has an H-packing which covers all but $o(n^2)$ edges.*

Proof. The proof follows directly from Theorem 1.1 and from the result from [15] for the case $H=K_k$. □

A longstanding conjecture of Nash-Williams [13] asserts that if a graph has minimum degree $3n/4$ then it has a packing of edge-disjoint triangles covering all but $o(n^2)$ edges. In fact, the conjecture is sharper as it states that if, in addition, all the degrees are even and the number of edges is divisible by 3, then all edges can be covered (namely, a triangle decomposition exists). The 3/4-fraction for the minimum degree requirement is essential, as there is a construction showing that it cannot be replaced by a smaller constant. Clearly, Theorem 1.1 can be used to generalize the conjecture of Nash-Williams to any fixed 3-chromatic graph. The following conjecture is yet a further generalization as it also considers the case where the minimum degree is smaller than $3n/4$.

Conjecture 5.3. *For any $1 \geq \alpha \geq 0$, and any fixed graph H with $\chi(H) = 3$, if a graph G with n vertices has minimum degree at least $n/2 + \alpha n/4$, then it contains an H-packing which covers all but at most $(1-\alpha)n^2/4 + o(n^2)$ edges.*

Theorem 1.1 shows that it suffices to prove Conjecture 1.1 for the case $H = K_3$. We next show that, if true, Conjecture 5.3 is optimal, in the sense that the constant $(1-\alpha)/4$ cannot be replaced with a smaller one. Let R be any graph with $n/4$ vertices which is $\alpha n/4$-regular. Let R_1, R_2, R_3, R_4 denote four disjoint copies of R. Add all $n^2/16$ edges between R_i and R_{i+1} for $i = 1, 2, 3$ as well as between R_1 and R_4. The constructed graph G has n vertices and is $(n/2 + \alpha n/4)$-regular. Any triangle of G must contain at least one edge with both endpoints in R_i for some i. Hence, a maximum triangle packing contains at most $4e(R)$ triangles, so $\nu_{K_3}(G) \leq \alpha n^2/8$. As $e(G) = \alpha n^2/8 + n^2/4$ it follows that a maximum triangle packing fails to cover at least $\alpha n^2/8 + n^2/4 - 3\alpha n^2/8 = (1-\alpha)n^2/4$ edges.

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