# Ramsey numbers for degree monotone paths

Yair Caro
Department of Mathematics
University of Haifa-Oranim
Israel

Raphael Yuster
Department of Mathematics
University of Haifa
Israel

Christina Zarb
Department of Mathematics
University of Malta
Malta

**Abstract**

A path $v_1, v_2, \ldots, v_m$ in a graph $G$ is *degree-monotone* if $deg(v_1) \le deg(v_2) \le \cdots \le deg(v_m)$ where $deg(v_i)$ is the degree of $v_i$ in $G$. Longest degree-monotone paths have been studied in several recent papers. Here we consider the Ramsey type problem for degree monotone paths. Denote by $M_k(m)$ the minimum number $M$ such that for all $n \ge M$, in any $k$-edge coloring of $K_n$ there is some $1 \le j \le k$ such that the graph formed by the edges colored $j$ has a degree-monotone path of order $m$. We prove several nontrivial upper and lower bounds for $M_k(m)$.

## 1 Introduction

A path $v_1, v_2, \ldots, v_m$ in a graph $G$ is *degree-monotone* if $deg(v_1) \le deg(v_2) \le \cdots \le deg(v_m)$ where $deg(v_i)$ is the degree of $v_i$ in $G$. The maximum order over all degree-monotone paths in $G$ is denoted by $mp(G)$. General monotone path problems were systematically treated long ago by Chvatal and Komlós [9], who related oriented graphs and oriented paths to various path monotonicity problems, motivated by the famous Erdős-Szekeres Theorem [11, 13] on monotone sub-sequences, and by the Gallai-Hasse-Roy-Vitaver Theorem (see [20]). Another famous monotone path problem is suggested by Graham and Kleitman [15] in which the edges of $K_n$ are bijectively labeled by $[1, \ldots, \binom{n}{2}]$ and the problem is to determine the minimum over all possible labellings of a maximum monotone path.

The study of degree monotone paths and $mp(G)$ was explicitly suggested and developed in connection with certain domination problems by Deering et al. [10] and further developed by Caro et al. [5, 6] who studied $mp(G)$ and related parameters in the context of extremal Turán type results.

One important observation which is immediate from the Gallai-Hasse-Roy-Vitaver Theorem is that $mp(G) \ge \chi(G)$. Indeed, if we orient an edge from a low degree vertex to a high degree vertex (breaking ties arbitrarily), then a directed path in the resulting oriented graph corresponds to a degree-monotone path in the original undirected graph, and the Gallai-Hasse-Roy-Vitaver Theorem asserts that in any orientation, the order of a longest directed path is at least as large as the chromatic number. Hence $mp(G)$ is a nontrivial upper bound for the chromatic number, which is sometimes tight.

In Ramsey theory, some interesting and active research is about $R(P_1, \ldots, P_k)$, the Ramsey number for $k$-edge-colored complete graphs that forces a monochromatic path $P_j$ in the edges colored $j$ , for some $1 \leq j \leq k$ (see for example [17, 19]). In this paper we study the corresponding Ramsey type problem for monotone paths where monotonicity is determined by the most basic parameter, the degree of a vertex. A formal definition follows.

A *k-edge coloring* is a coloring of the edges of a graph where each edge is given one of $k$ distinct colors. Denote by $M = M(m_1, m_2, \ldots, m_k)$ the minimum number $M$ such that for all $n \geq M$, in any $k$-edge coloring of $K_n$, for some $j$ where $1 \leq j \leq k$, the spanning monochromatic graph $G_j$ formed by the edges colored $j$ satisfies $mp(G_j) \geq m_j$. In the diagonal case $m = m_1 = \cdots = m_k$, we write $M_k(m)$. We refer to a monochromatic degree-monotone path in this context as an *mdm-path* for short. We will always assume that $k \geq 2$ and $m \geq 3$ and (in the non-diagonal case) $m_i \geq 3$ for all $i = 1, \ldots, k$ to avoid the trivial cases.

As we shall see, an upper bound for $M(m_1, \ldots, m_k)$ can be obtained via some classical techniques (summarized in Lemma 2.1) developed around the multicolor version of the famous Nordhaus-Gaddum Theorem [14, 18]. However, in several cases this upper bound is not sharp, and getting better upper bounds seems as a highly non-trivial task requiring new ideas, among them some characterization of certain bipartite graphs with a constrained degree sequence. Also, we may not assume monotonicity in the sense explained in the following paragraph, hence to get a lower bound construction we have to overcome this difficulty. The open problems mentioned in the end of the paper indicate the various interesting directions opened by the Ramsey degree-monotone path problem.

One should observe a subtlety in the definitions of $M_k(m)$ (as well as $M(m_1, \ldots, m_k)$). It is not clear that if $n$ is the smallest integer for which $K_n$ satisfies the stated property, then $M_k(m) = n$. This is because being true for $n$, does not a priori imply it for $n + 1$ as the parameter $mp(G)$ is not hereditary. For example, $mp(K_{2,3}) = 2$ whereas for its induced subgraph $K_{2,2}$ we have $mp(K_{2,2}) = 4$. This issue occurs in the setting of edge colorings of $K_n$ as well. Consider a 2-edge coloring of $K_5$ with color 1 inducing a $K_{2,3}$. Then there is no monotone path of order 4 in any of the colors, while the colored $K_4$ subgraph obtained by removing a vertex incident with two edges of color 1 has a monotone path of order 4 in color 1. Hence the requirement in the definition that $M$ is the smallest integer such that for *all* $n \geq M$ the stated property holds, is important. These sort of Ramsey-degree problems (with the related subtle monotonicity problem just mentioned) originated in some papers by Albertson [1, 2] and Albertson and Berman [3], and were further developed shortly afterward by Chen and Schelp [7] and Erdős et al. [12]. We mention the following interesting result that appeared in [12].

**Theorem 1.1.** *In any 2-coloring of the edges of $K_n$, where $n \geq R(m, m)$, there is a monochromatic copy of $K_m$ with vertices $v_1, \ldots, v_m$ such that in the host monochromatic graph $G$,*

$$\max\{deg(v_i) : i = 1, \ldots, m)\} - \min\{deg(v_i) : i = 1, \ldots, m\} \leq R(m, m) - 2 ,$$

*and this is sharp for $n \geq 4(r - 1)(r - 2)$ where $r = R(m, m)$.*

Having all these facts in mind we are now ready to state our first main result, which provides general upper and lower bounds for $M_k(m)$.

**Theorem 1.2.** *Let $k \geq 2$ and $m \geq 3$ be integers. Then:*

$$\frac{(m-1)^k}{2} + \frac{m-1}{2} + 1 \leq M_k(m) \leq (m-1)^k + 1\ .$$

*In fact, more generally, if $m_i \geq 3$ for all $i = 1, \ldots, k$, then $M(m_1, \ldots, m_k) \leq \prod_{i=1}^{k}(m_i - 1) + 1$.*

Notice that the upper and lower bounds for $M_k(m)$ differ by a factor smaller than 2.

As usual in most Ramsey type problems, proving tighter bounds, or even computing exact small values, turns out to be a difficult task already in the first, and perhaps most interesting, case of paths of order 3, namely $M_k(3)$. This case can also be interpreted as requiring that the degree of every vertex of a graph with no isolated edges is a local extremum (either strictly smaller than the degree of all its neighbors or strictly larger than the degree of all its neighbors). Observe that Theorem 1.2 gives $2^{k-1} + 2 \leq M_k(3) \leq 2^k + 1$. Our next theorem improves both upper and lower bounds.

**Theorem 1.3.** *$M_2(3) = 4$, $M_3(3) = 8$ and $\frac{3}{4}2^k + 2 \leq M_k(3) \leq 2^k - 1$ for $k \geq 4$.*

We note that while the upper bound is only a mild improvement over the one provided by Theorem 1.2, its proof turns out to be somewhat involved.

The first off-diagonal nontrivial case is $M(3, m)$ for which we prove:

**Theorem 1.4.** *For all $m \geq 3$, $M(3, m) = 2(m - 1)$.*

In the rest of this paper we prove the general bounds in Section 2, the more involved tighter bounds for paths of order 3 are proved in Section 3, and the proof of Theorem 1.4 appears in Section 4. The final section contains some specific open problems. Our notation follows that of [20], and will otherwise be introduced when it first appears.

## 2 General upper and lower bounds

In this section we prove Theorem 1.2. The upper bound in Theorem 1.2 is a consequence of the following result proved independently by Gyárfás and Lehel [16], Bermond [4], and Chvatal [8]. They used an observation of Zykov [21] that states that in any edge coloring of a complete graph with more than $\prod_{i=1}^{k}(m_i - 1)$ vertices with $k$ colors, there is a color $i$ that induces a graph whose chromatic number is at least $m_i$, together with the Gallai-Hasse-Roy-Vitaver Theorem to deduce:

**Lemma 2.1.** *In any $k$-coloring of the edges of a tournament on more than $\prod_{i=1}^{k}(m_i - 1)$ vertices, there is a directed path of order $m_i$, all of whose edges are colored $i$. The bound $\prod_{i=1}^{k}(m_i - 1)$ is tight. Furthermore, in any extremal example, the chromatic number of the graph whose edges are colored with color $i$ is $m_i - 1$ and any proper $(m_i - 1)$-vertex coloring of it is equitable (all vertex classes have equal size).*

The upper bound in Theorem 1.2 is a consequence of Lemma 2.1, as shown in the following proposition.

**Proposition 2.2.** *If $n \geq t > \prod_{i=1}^{k}(m_i - 1)$, then for any $k$-edge coloring of $K_n$, and for any subset $T$ of $t$ vertices of $K_n$, there is an mdm-path of order $m_i$ in color $i$ for some $i$, all of whose vertices are in $T$. In particular, $M(m_1, \dots, m_k) \leq \prod_{i=1}^{k}(m_i - 1) + 1$. Furthermore, if $M(m_1, \dots, m_k) = 1 + \prod_{i=1}^{k}(m_i - 1)$, then any extremal example must have that $mp(G_i) = m_i - 1$ and that any $(m_i - 1)$-vertex coloring of $G_i$ is equitable.*

*Proof.* Consider an edge coloring of $K_n$ with $k$ colors, and in each colored graph $G_i$, orient an edge $uv$ colored with $i$ from $u$ to $v$ if $deg_i(u) > deg_i(v)$ where $deg_i(x)$ is the degree of $x$ in $G_i$ (break ties arbitrarily). We then obtain a coloring of a tournament with $k$ colors. Now, if $T$ is a subset of $t$ vertices and $n \geq t > \prod_{i=1}^{k}(m_i - 1)$, Lemma 2.1 asserts that there is a monochromatic directed path of order $m_i$ all of whose edge are colored $i$, in the induced $K_t$ on the vertices of $T$. This path is, by construction, an mdm-path in $G_i$. This proves the first part of the claim and that $M(m_1, \dots, m_k) \leq 1 + \prod_{i=1}^{k}(m_i - 1)$ and in particular, $M_k(m) \leq 1 + (m-1)^k$. Observe that a construction showing tightness in Lemma 2.1 is not necessarily relevant in our setting as it may not imply tightness for the degree-monotone problem. Nevertheless, as the lemma states, it does imply that *if* the bound $1 + \prod_{i=1}^{k}(m_i - 1)$ is tight, then any extremal example on $\prod_{i=1}^{k}(m_i - 1)$ vertices must have that $mp(G_i) = m_i - 1$ and that any $(m_i - 1)$-vertex coloring of $G_i$ is equitable. □

The next lemma proves the lower bound in Theorem 1.2.

**Lemma 2.3.** *Let $k \geq 2$ and $m \geq 3$ be integers. Then: $M_k(m) \geq \frac{(m-1)^k}{2} + \frac{m-1}{2} + 1$.*

*Proof.* We will prove the stronger claim that for each integer $n$ of the form $\frac{(m-1)^k}{2} + \frac{m-1}{2} - t$ for $t = 0, \dots, m-1$, there is an edge coloring of $K_n$ with $k$ colors and with no mdm-path of order $m$.

We proceed by induction on $k$, starting with $k = 2$. Let $X_1, \dots, X_{m-1}$ be sets such that $|X_j| = j$ for $j = 1, \dots, m-1$. Form a complete graph on $V = \cup_{j=1}^{m-1} X_j$ by coloring an edge with both endpoints in the same set with color 1 and an edge with endpoints in distinct sets with color 2. As the color 1 induces a graph $G_1$ whose components are cliques of order at most $m-1$, there is no path on $m$ vertices in $G_1$. For the color 2, observe that any path on $m$ vertices in the graph $G_2$ must contain two non-consecutive vertices from the same set $X_j$ for some $j$. But any two vertices in $X_j$ have the same degree in $G_2$ and this degree is distinct from the degree in $G_2$ of any vertex not in $X_j$. Hence there is no mdm-path of order $m$ in $G_2$. As the number of vertices is $|V| = \sum_{j=1}^{m-1} j = m(m-1)/2$, the claim holds for $k = 2$ with $t = 0$. However, notice that the same argument holds if we take a smaller union $V \setminus X_t$ for $t = 1, \dots, m-1$ (just take the same coloring and omit $X_t$). Hence, the claim holds for $k = 2$ and $t = 0, \dots, m-1$.

Now assume we have proved that using $k$ given colors, there are complete graphs on $\frac{(m-1)^k}{2} + \frac{m-1}{2} - t$ vertices for $t = 0, \dots, m-1$, and a $k$-edge coloring of each of them with no mdm-path of order $m$. We proceed with the induction step proving that the same statement holds for $k+1$. Denote such colored complete graphs by $X_0, \dots, X_{m-1}$ where $X_t$ has $\frac{(m-1)^k}{2} + \frac{m-1}{2} - t$ vertices. Let $Y_t$ be the complete graph obtained by taking the disjoint union of $X_0, X_1, \dots, X_{t-1}, X_{t+1}, \dots, X_{m-1}$ (using the existing $k$-coloring in each component) and color any two vertices with endpoints in distinct $X_j$ with color $k+1$. By induction, there is no mdm-path of order $m$ on colors $1, \dots, k$ and there is also no mdm-path of order $m$ on color $k+1$ since any path on $m$ vertices in the graph $G_{k+1}$ (the subgraph of $Y_t$ on the edges colored $k+1$) must contain two non-consecutive vertices from the same subgraph $X_j$ for some $j$. But any two vertices in the same $X_j$ have the same degree in $G_{k+1}$ and this degree is distinct from the degree in $G_{k+1}$ of any vertex not in $X_j$. Hence $Y_t$ has no mdm-path of order $m$. Now notice that

$$\begin{aligned} |V(Y_{m-1-t})| &= \left(\sum_{s=0}^{m-1}(\frac{(m-1)^k}{2}+\frac{m-1}{2}-s)\right)-\left(\frac{(m-1)^k}{2}+\frac{m-1}{2}-(m-1-t)\right) \\ &= \frac{(m-1)^{k+1}}{2}+\frac{m-1}{2}-t \end{aligned}$$

proving the induction step for $k+1$ and $t=0,\dots,m-1$. □

**Remark 1**: we observe that once we have $m$ consecutive integers $t, t-1, t-2, \dots, t-m+1$ for which it is possible to $k$-color the edges of $K_{t-j}$, $j=0,\dots,m-1$ without an mdm-path of order $m$, then we can $(k+1)$-color the graph $K_{q-j}$ for $j=0,\dots,m-1$ without an mdm-path of order $m$, where $q=(m-1)t-\frac{(m-1)(m-2)}{2}$ and the process can be continued. So whenever we have an improvement of the basic lower bound, we can carry over this new better bound. We shall use this to prove the lower bound for $M_k(3)$ obtained in Theorem 1.3.

# 3 Paths of order 3

## 3.1 A structural property

We consider certain conditions imposed on the degrees of bipartite graphs, and then use the structural properties of these bipartite graphs when such graphs exists, and the non-existence of such graphs otherwise, to prove the upper bound in Theorem 1.3.

A bipartite graph with bipartition $V=A\cup B$ is said to be *illusive* if:

- $|A|>|B|$, $A$ has no isolated vertices, and for every vertex $v\in A$, $deg(v)\geq deg(u)$ for all vertices $u\in N(v)$, or
- $|A|=|B|$, $A$ has no isolated vertices, and for every vertex $v\in A$, $deg(v)\geq deg(u)$ for all vertices $u\in N(v)$. Furthermore, there exists $v\in A$ such that $deg(v)>deg(u)$, for some $u\in N(v)$.

**Lemma 3.1.** *Illusive graphs do not exist.*

*Proof.* We consider first the case $|A|>|B|$ and assume by contradiction that $G$ is a minimum counter-example, namely $G$ is an illusive graph with a minimum number of vertices, such that $|A|>|B|$. Let $|A|=p$ and $|B|=q$, $p>q$. Let us order the vertices $v_1, v_2, \cdots, v_p \in A$ such that $deg(v_1)\geq\cdots\geq deg(v_p)$.

Consider the set of vertices $S=\{v_1,\dots,v_q\}\subset A$. Now if $S$ has a matching to $B$ in $G$, then we can arrange the vertices $u_1,\dots,u_q\in B$ such that $v_i$ is adjacent to $u_i$ for $i=1,\dots,q$, and clearly $deg(v_i)\geq deg(u_i)$ as they are neighbors in $G$.

Now as $deg(v_p)\geq 1$, we clearly have

$$|E(G)|=\sum_{i=1}^{i=p}deg(v_i)>\sum_{i=1}^{i=q}deg(u_i)=|E(G)|,$$

a contradiction. Hence, there is no matching between $S$ and $B$. Now by Hall's Theorem [20], there exists $Q \subset S$ such that $|N(Q)| < |Q|$. Consider the subgraph $H$ of $G$ induced by $Q \cup N(Q)$. Clearly $|V(H)| < |V(G)|$. $H$ is an illusive graph because any vertex in $Q$ has the same degree in $H$ as in $G$. Also, since no vertex in $A$ is isolated, it follows that no vertex in $H$ is isolated as all the neighbors of the vertices in $Q$ are in $N(Q)$. Hence, $H$ is an illusive graph with $|V(H)| < |V(G)|$, a contradiction to the minimality of $G$.

Now consider the case $|A| = |B|$ and assume by contradiction that $G$ is a minimum counter-example, namely $G$ is an illusive graph with a minimum number of vertices, such that $|A| = |B| = p$ and furthermore from all such graphs with $|A| = |B| = p$, let $G$ have the minimum number of edges. Let us order the vertices $\{v_1, \ldots, v_p\}$ in $A$ such that $deg(v_1) \geq \cdots \geq deg(v_p)$.

If there is a matching in $G$ from $A$ to $B$, then we can rearrange the vertices $\{u_1, \ldots, u_p\}$ of $B$ such that every vertex $v_i$ is adjacent to $u_i$ for $i = 1, \ldots, p$, and clearly $deg(v_i) \geq deg(u_i)$ as they are neighbors in $G$.

Let us delete the edges $(v_i, u_i)$ $i = 1, \ldots, p$ to obtain $G^*$ such that $V(G^*) = V(G)$ but $|E(G^*)| < |E(G)|$. Every vertex in $G^*$ has degree one less than that in $G$, so we still have $deg(v_i) \geq deg(u)$ for every $v_i \in A$ and every $u \in N(v_i)$.

Now if some vertex in $A$ in $G^*$ has degree 0, then its neighbor in the matching in $B$ must also have degree 0. We consider the following two cases:

1. In $G^*$ there are more vertices of degree 0 in $B$ than in $A$. Let us delete all the vertices of degree 0 from $A$ and $B$ to get $A^*$ and $B^*$, and $H = A^* \cup B^*$, with $|V(A^*)| > |V(B^*)|$. But then $H$ is illusive of the type which we proved to be impossible in the first part of the proof.

2. There are exactly the same number of vertices of degree 0 in $A$ and in $B$ (possibly no isolated vertices at all). Let us delete all the vertices of degree 0 from $A$ and $B$ to get $A^*$ and $B^*$, and $H = A^* \cup B^*$ a subgraph of $G^*$, with $|V(A^*)| = |V(B^*)|$. Recall that there exists a vertex $v \in A$ such that there is a vertex $u \in N(v)$ with $deg(v) > deg(u)$. Now if in the matching in $G$, $v$ is matched with $u$, then in $G$

$$|E(G)| = \sum_{i=1}^{i=p} deg(v_i) > \sum_{i=1}^{i=p} deg(u_i) = |E(G)|,$$

   which is not possible. Hence $v$ is not matched to $u$. But then $deg(v) \geq 2$ and $v$ is still connected to $u$ in $H$, and hence in $H$, $deg(v) > deg(u)$ and $u$ and $v$ are adjacent, which implies that $H$ is a smaller illusive graph, a contradiction.

Finally let us assume that there is no matching between $A$ and $B$ in $G$. Again, by Hall's Theorem, there exists $Q \subset A$ such that $|N(Q)| < |Q|$. Consider the subgraph $H$ of $G$ induced by $Q \cup N(Q)$. Clearly $|V(H)| < |V(G)|$. $H$ is an illusive graph because any vertex in $Q$ has the same degree in $H$ as in $G$. Also, since no vertex in $A$ is isolated, it follows that no vertex in $H$ is isolated as all the neighbors of the vertices in $Q$ are in $N(Q)$. Hence, $H$ is an illusive graph of the type proved impossible in the first part of this proof. Hence illusive graphs do not exist. □

An immediate consequence of Lemma 3.1 is:

**Corollary 3.2.** *Let $G$ be a connected bipartite graph with bipartition $V = A \cup B$ such that $|A| \geq |B|$ and for every vertex $v \in A$, $deg(v) \geq deg(u)$ for every $u \in N(v)$. Then $|A| = |B|$ and $G$ is regular.*

**Lemma 3.3.** *Suppose $G$ is a bipartite graph with $V = A \cup B$ such that $|A| = k$ and $|B| = k+1$, and such that for every vertex $v \in A$, $deg(v) \geq 1$ and $deg(v) > deg(u)$ for every $u \in N(v)$. Then $G = K_{k,k+1}$.*

*Proof.* Let us order the vertices $u_1, \ldots, u_{k+1}$ of $B$ in non-increasing order so that $deg(u_1) \geq \cdots \geq deg(u_{k+1})$. Let $B^* = B \backslash \{u_{k+1}\}$. Suppose first that $A$ has a perfect matching to $B^*$. Then the vertices $v_1, \ldots, v_k$ of $A$ can be ordered such that $v_i$ is adjacent to $u_i$ and $deg(v_i) \geq deg(u_i) + 1$. Counting edges in $G$ we get

$$|E(G)| = \sum_{i=1}^{k} deg(v_i) = \sum_{i=1}^{k+1} deg(u_i) \leq \left( \sum_{i=1}^{k} deg(v_i) - 1 \right) + deg(u_{k+1})$$

$$= \left( \sum_{i=1}^{k} deg(v_i) \right) - k + deg(u_{k+1}) = |E(G)| - k + deg(u_{k+1}).$$

Hence $deg(u_{k+1})$ must be equal $k$ (since $|A| = k$), and since $u_{k+1}$ has minimum degree in $B$ it forces all other vertices in $B$ to have degree $k$, and hence $G = K_{k,k+1}$.

Hence suppose $A$ has no perfect matching to $B^*$. Then by Hall's Theorem there is a subset $Q$ in $A$ such that $|N(Q)| < |Q| \leq |A| = k$ . Consider the bipartite graph $H$ induced by the parts $Q$ and $N(Q) \cup \{u_{k+1}\}$. Since the only possible neighbor of the vertices of $Q$ not in $B^*$ is $u_{k+1}$, it follows that $Q$ and $N(Q) \cup \{u_{k+1}\}$ induce a bipartite graph $H$ where the degrees of all vertices in $Q$ are strictly larger than the degrees of their neighbors in $N(Q) \cup \{u_{k+1}\}$. Since $|N(Q) \cup \{u_{k+1}\}| \leq |Q|$, $H$ is an illusive graph which doesn't exists. □

## 3.2 Proof of Theorem 1.3

We start with the following proposition that yields the upper bound $M_3(k) \leq 2^k$. It will be useful to establish the small values $M_2(3)$ and $M_3(3)$.

**Proposition 3.4.** *For all $k \geq 2$, $M_k(3) \leq 2^k$.*

*Proof.* We already know that $M_k(3) \leq 2^k + 1$ from Theorem 1.2, so to establish the proposition it suffices to consider $k$-edge colorings of $K_{2^k}$. Suppose, for contradiction, that we can color the edges of $K_{2^k}$ using $k$ colors such that there is no mdm-path of order 3. Let $G_j$ be the spanning graph whose edges are colored $j$ for $j = 1, \ldots, k$. So by Proposition 2.2, our coloring is an extremal example and thus for all $j = 1, \ldots, k$ we have $\chi(G_j) = 2$ and in every bipartition of $G_j$ both parts have the same order $2^{k-1}$.

Hence, each component of $G_j$ is a bipartite graph with bipartition $A, B$ where $|A| = |B|$. Consider any such component which is not a $K_2$. Hence $|A| = |B| \geq 2$. The degrees of the vertices in any path connecting a vertex from $A$ with another vertex from $A$ form a sequence of integers with odd length and with no monotone subsequence of order 3. As any two vertices of $A$ can be connected via a path, we have that either all vertices of $A$ have degree larger than all the degrees of their neighbors is $B$ or vice versa. Assume the former. Then this component is illusive, and by Lemma 3.1, this is impossible.

Hence all components of $G_j$ are $K_2$ and therefore all $G_j$ for $j = 1, \ldots, k$ are perfect matchings. So we cover $K_{2^k}$ by $k$ matchings each having precisely $2^{k-1}$ edges. Hence

$$k2^{k-1} = \frac{2^k(2^k - 1)}{2}$$

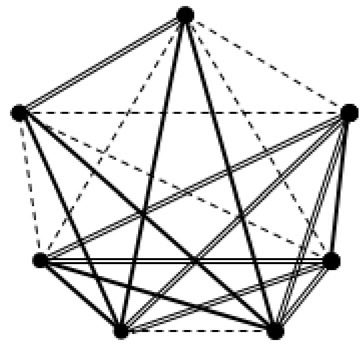

Figure 1: $K_7$ decomposed into three copies of $K_{2,3} \cup K_2$

and thus $k = 2^k - 1$ which is impossible for $k \geq 2$, a contradiction. □

**Corollary 3.5.** *$M_2(3) = 4$ and $M_3(3) = 8$.*

*Proof.* By Proposition 3.4 we have $M_2(3) \leq 4$. Trivially, if we color $K_3$ using two colors we do not have an mdm-path of order 3. Hence $M_2(3) = 4$. Similarly, we know that $M_3(3) \leq 2^3 = 8$. We can color the edges of $K_7$ in such a way that $G_i = K_{2,3} \cup K_2$ for $i = 1, 2, 3$, as shown in Figure 1. It is easy to see that $mp(K_{2,3} \cup K_2) = 2$, and hence $M_3(3) = 8$. □

As the sequence starts with $M_2(3) = 2^2$ and $M_3(3) = 2^3$ and since $M_k(3) \leq 2^k$ one may wonder whether $M_k(3) = 2^k$. The following lemma shows that this is not the case already for $k \geq 4$. Somewhat surprisingly, the proof requires some effort.

**Lemma 3.6.** *For $k \geq 4$, $M_k(3) \leq 2^k - 1$.*

*Proof.* We already know that $M_k(3) \leq 2^k$ so to establish the lemma it suffices to consider $k$-edge colorings of $K_{2^k-1}$. Let $G_j$ be the spanning graph whose edges are colored $j$ for $j = 1, \ldots, k$. If for some $j$, $G_j$ is not bipartite, then there is an mdm-path of order 3, so we may assume that each $G_j$ is bipartite.

We claim that in any bipartition of $G_j$, one side has $2^{k-1}$ vertices (and thus the other side has $2^{k-1} - 1$ vertices). Indeed, otherwise, one side would contain more than $2^{k-1}$ vertices, and induces an edge coloring with $k - 1$ colors, but since a complete graph on more than $2^{k-1}$ vertices cannot be edge-decomposed into $k - 1$ bipartite graphs, this contradicts the assumption in the previous paragraph.

So, each component of $G_j$ is a bipartite graph where the two sides have equal size, except precisely one component where the two sides differ in size by 1. (If there were more that one such component we could arrange a bipartition of $G_j$ into two sides whose sizes differ by more than 1, and we have shown that this is impossible). Now, by Lemmas 3.1 and 3.3, if there is no mdm-path of order 3, then each balanced component must be a single edge, and the non-balanced component must be $K_{b,b-1}$ for some integer $b$.

Hence $G_j$ is of the form $K_{b,b-1} \cup (2^{k-1} - b)K_2$. For each $G_j$, we call the $K_{b,b-1}$ component the *essential component* and the remaining matching on $2^{k-1} - b$ edges is the *non-essential part.*

Now, the number of edges of $K_{2^k-1}$ is $(2^k-1)(2^k-2)/2$, so the average number of edges colored with a given color is $(2^k-1)(2^k-2)/2k = (2^k-1)(2^{k-1}-1)/k$. So, if we consider the graph $G_j$ with the largest number of edges, it is of the form $K_{a,a-1} \cup (2^{k-1}-a)K_2$ where we must have $a(a-1)+2^{k-1}-a \geq (2^k-1)(2^{k-1}-1)/k$. Solving for $a$ we obtain that we must have

$$a-1 \geq \left\lceil \sqrt{(2^{k-1}-1)(2^k-k-1)/k} \right\rceil .$$

For example, if $k=5$ we must have $a \geq 10$.

Without loss of generality, let $G_k$ is the graph with the largest number of edges. So, let us consider the essential component of $G_k$. It is a complete bipartite graph with sides $A, B$ with $|A| = a$ and $|B| = a-1$. Now, each color $i$ for $i = 1, \ldots, k-1$ has the property that its essential component cannot intersect both $A$ and $B$. So either $A$ or $B$ have the property that they intersect $t \leq \lfloor (k-1)/2 \rfloor$ essential parts of the colors $1, \ldots, k-1$. We will consider the case that $B$ intersects $t \leq \lfloor (k-1)/2 \rfloor$ essential parts (the proof for $A$ is similar and in fact easier since $A$ is larger than $B$). Without loss of generality, the essential parts of $G_1, \ldots, G_t$ intersect $B$ and the essential parts of $G_{t+1}, \ldots, G_{k-1}$ do not intersect $B$.

So, the complete graph induced on $B$ (namely, $K_{a-1}$) has the property that it is composed of $t$ spanning bipartite graphs $H_1, \ldots, H_t$, where $H_i$ is the subgraph of $G_i$ induced on $B$, and $k-t$ matchings (these matchings are from the non-essential parts of the colors $t+1, \ldots, k-1$ whose essential parts do not intersect $B$). Since $H_i$ is bipartite it has a bi-partition $L_i \cup R_i$. We associate with each vertex $v$ of $B$ a binary vector of length $t$ where the $i$'th coordinate is 1 if $v \in R_i$ and 0 if $v \in L_i$. Altogether there are $2^t$ possible vectors, distributed over the $a-1$ vertices of $B$.

So, there is a subset $B' \subset B$ of size at least $(a-1)/2^t$ such that any two vertices of $B$ are associated with the same vector. Now, consider $u, v \in B'$. The edge connecting them cannot be colored with any of the colors $1, \ldots, t$, since they received the same vector. Hence, the edge connecting them must be from one of the non-essential parts of the colors $t+1, \ldots, k-1$. But since the non-essential parts of these $k-1-t$ colors are a union of $k-1-t$ matchings, in order to get a contradiction it suffices to prove that $|B'|-1 > k-1-t$ or, if $|B'|$ is odd, it suffices to prove that $|B'|-1 \geq k-1-t$.

So we are left with the issue of verifying that $\lceil (a-1)/2^t \rceil > k-t$ or, if $\lceil (a-1)/2^t \rceil$ is odd, it suffices to show that $\lceil (a-1)/2^t \rceil \geq k-t$. Using the fact that $a-1 \geq \lceil \sqrt{(2^{k-1}-1)(2^k-k-1)/k} \rceil$ and that $t \leq \lfloor (k-1)/2 \rfloor$ this amounts to verifying the following inequality:

$$\left\lceil \frac{\lceil \sqrt{(2^{k-1}-1)(2^k-k-1)/k} \rceil}{2^{\lfloor (k-1)/2 \rfloor}} \right\rceil > k - \lfloor \frac{k-1}{2} \rfloor$$

or, if the left hand side is odd, it suffices to prove a weak inequality.

Notice that for $k \geq 10$ the strong inequality is true even if we remove the ceilings in the l.h.s. and remove the floor in the denominator of the l.h.s. For $k = 4, \ldots, 9$ we verify explicitly:

For $k = 4, 5$ we have the inequality $3 \geq 3$ which is true (here we use the fact the the l.h.s. is odd so the weak inequality suffices). For $k = 6$ we have the inequality $\lceil 18/4 \rceil > 6-2$. For $k = 7$ we have the inequality $\lceil 33/8 \rceil > 7-3$. For $k = 8$ we have the inequality $\lceil 63/8 \rceil > 8-3$. For $k = 9$ we have the inequality $\lceil 120/16 \rceil > 9-4$. Hence the lemma holds for all $k \geq 4$. □

We now turn to prove the lower bound in Theorem 1.3.

**Lemma 3.7.** $M_k(3) \geq \frac{3}{4}2^k + 2$ *for* $k \geq 3$.

*Proof.* We have already shown in Figure 1 that there is an edge coloring of $K_7$ with 3 colors without an mdm-path of order 3. Figure 2 gives constructions of edge colorings of $K_6$ and $K_5$ with 3 colors without an mdm-path of order 3 (recall: we cannot just use the coloring for $K_7$ to deduce this for $K_6$ and $K_5$ as the degree-monotone property may not necessarily be hereditary as demonstrated in the introduction). Hence by Remark 1, using $k = 3$, $m = 3$ and $t = 7$, we have that we can 4-color $K_{13}, K_{12}, K_{11}$ with no mdm-path of order 3, we can 5-color $K_{25}, K_{24}, K_{23}$ with no mdm-path of order 3, and the process continues so that we can $k$-color $K_{(3/4)2^k+1}, K_{(3/4)2^k}, K_{(3/4)2^k-1}$ with no mdm-path of order 3, so in particular $M_k(3) \geq (3/4)2^k + 2$ for $k \geq 3$. □

Theorem 1.3 now follows from Corollary 3.5, Lemma 3.6 and Lemma 3.7. □

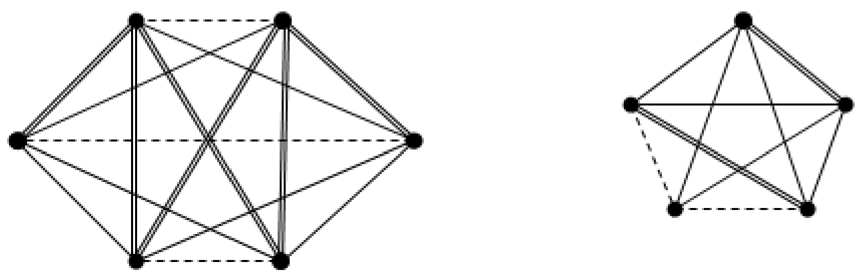

Figure 2: 3-edge-colorings of $K_6$ and $K_5$ with no mdm-path of order 3.

## 4 Proof of Theorem 1.4

Let $m \geq 3$. We prove that $M(3, m) = 2(m-1)$. First observe that $M(3, m) \geq 2(m-1)$ as seen by coloring the edges of $G = K_{m-1,m-2}$ with color 1 and coloring the edges of its complement with color 2. It remains to show that for $n \geq 2(m-1)$, in any 2-coloring of the edges of $K_n$ there is either a degree-monotone path of order 3 in color 1 or a degree-monotone path of ordered $m$ in color 2.

So suppose $n \geq 2(m-1)$ and $K_n$ is edge-colored using two colors 1 and 2. If $n > 2(m-1)$ the result follows from the upper bound in Theorem 1.2. So we assume $n = 2(m-1)$. Let $G_j$ be the spanning graph of $K_{2(m-1)}$ induced by the edges with color $j$, $j = 1, 2$. We will show that either $G_1$ has a has degree monotone path of order 3 or $G_2$ has a degree monotone path of order $m$.

Assume otherwise. We thus know by Proposition 2.2 that $\chi(G_1) = 2$ and $\chi(G_2) = m-1$, and further know from Proposition 2.2 that any bipartition of $G_1$ is equitable and hence $G_2$ can be split into two cliques $A$ and $B$ such that $|A| = |B| = m-1$, with possible edges between $A$ and $B$. Now if we order the vertices in both $A$ and $B$ in non-decreasing order of their degrees in $G_2$, say $A = \{v_1, v_2, \ldots, v_{m-1}\}$ and $B = \{u_1, u_2, \ldots, u_{m-1}\}$, then in $G_2$, $v_i$ is not adjacent to $u_i$ for otherwise there will be a degree monotone path of order $m$, either $v_1, \ldots, v_i, u_i, \ldots, u_{m-1}$ or $u_1, \ldots, u_i, v_i, \ldots, v_{m-1}$.

Hence in $G_1$, $v_i$ and $u_i$ are adjacent and $deg(v_i) \geq deg(v_{i+1})$, as well as $deg(u_i) \geq deg(u_{i+1})$ in $G_1$. Assume without loss of generality, that $deg(v_1)$ is maximal in $G_1$. If $deg(v_1) = 1$, then $G_1$ is a matching — but then clearly there is a monotone path of order $m$ in $G_2$. Thus, $deg(v_1) \geq 2$. We may assume $deg(v_1) > deg(u_1)$ for otherwise $deg(v_1) = deg(u_1) \geq 2$ will force a monotone path of order 3.

Now all the neighbors of $v_1$ are in $B$ and hence have degrees less than $deg(v_1)$. Consider $N(N(v_1))$ which are all in $A$ and each vertex in $N(N(v_1))$ must have a degree strictly greater than the degrees of its neighbor in $N(N(N(v_1)))$ in $B$ (otherwise there is monotone degree path of order 3). Continuing this way, then either $G_1$ is connected and illusive or contains an illusive component with balanced sides $A^*$ and $B^*$, $|A^*| = |B^*|$, guaranteed by the matching $(v_i u_i)$ for $i = 1, \dots, (m-1)$. But illusive graphs are impossible, hence $G_1$ contains a degree-monotone path of order 3. □

## 5 Some open problems

As mentioned and demonstrated in the introduction, having a degree-monotone path of a certain order is not a hereditary property. Hence the following problem seems of interest. Let $\mathcal{N}_k(m)$ be the set of all positive integers such that $n \in \mathcal{N}_k(m)$ if and only if in every $k$-coloring of the edges of $K_n$ there is a monochromatic mdm-path of order $m$.

**Problem 5.1.** *Is it true that for all $k$ and $m$, $\mathcal{N}_k(m)$ has no gaps?*

Recall that the proof in Section 3 gives that $\mathcal{N}_2(3), \mathcal{N}_3(3), \mathcal{N}_4(3)$ have no gaps. For $\mathcal{N}_4(3)$ this follows since Theorem 1.3 gives $14 \leq M_4(3) \leq 15$ and since the construction in Lemma 3.7 together with some small case analysis can be used to prove that for all $n \leq 13$, there are 4-edge colorings of $K_n$ with no mdm-path of order 3. We can also show (see below) that $\mathcal{N}_2(4)$ has no gaps.

Theorem 1.3 asserts that

$$\frac{3}{4} \leq \liminf_{k\to\infty} \frac{M_k(3)}{2^k} \leq \limsup_{k\to\infty} \frac{M_k(3)}{2^k} \leq 1 \, .$$

**Problem 5.2.** *Determine if* $\lim_{k\to\infty} \frac{M_k(3)}{2^k}$ *exists and determine it.*

The diagonal case with two colors, namely $M_2(m)$, may be the most accessible. By Theorem 1.2 we know that $M_2(m) \leq (m-1)^2 + 1$.

**Conjecture 5.3.** *For every constant $C$, if $m$ is sufficiently large, then $M_2(m) \leq (m-1)^2 - C$.*

Recall that we know that $M_2(3) = 4$ and we have also verified (using a computer) that $M_2(4) = 7$. For the latter we needed to verify that all 2-edge colorings of $K_7, K_8, K_9$ have an mdm-path of order 4 (recall that Theorem 1.2 guarantees that $7 \leq M_2(4) \leq 10$).

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