Packing Directed Cycles Efficiently *

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Abstract

Let G be a simple digraph. The *dicycle packing number* of G, denoted $\nu_c(G)$, is the maximum size of a set of arc-disjoint directed cycles in G. Let G be a digraph with a nonnegative arc-weight function w. A function ψ from the set $\mathcal{C}$ of directed cycles in G to R_+ is a *fractional dicycle packing* of G if $\sum_{e \in C \in \mathcal{C}} \psi(C) \le w(e)$ for each $e \in E(G)$. The *fractional dicycle packing number*, denoted $\nu_c^*(G,w)$, is the maximum value of $\sum_{C \in \mathcal{C}} \psi(C)$ taken over all fractional dicycle packings ψ. In case $w \equiv 1$ we denote the latter parameter by $\nu_c^*(G)$.

Our main result is that $\nu_c^*(G) - \nu_c(G) = o(n^2)$ where $n = |V(G)|$. Our proof is algorithmic and generates a set of arc-disjoint directed cycles whose size is at least $\nu_c(G) - o(n^2)$ in randomized polynomial time. Since computing $\nu_c(G)$ is an NP-Hard problem, and since almost all digraphs have $\nu_c(G) = \Theta(n^2)$ our result is a FPTAS for computing $\nu_c(G)$ for almost all digraphs.

The latter result uses as its main lemma a much more general result. Let $\mathcal{F}$ be any fixed family of oriented graphs. For an oriented graph G, let $\nu_{\mathcal{F}}(G)$ denote the maximum number of arc-disjoint copies of elements of $\mathcal{F}$ that can be found in G, and let $\nu_{\mathcal{F}}^*(G)$ denote the fractional relaxation. Then, $\nu_{\mathcal{F}}^*(G) - \nu_{\mathcal{F}}(G) = o(n^2)$. This lemma uses the recently discovered *directed regularity lemma* as its main tool.

It is well known that $\nu_c^*(G,w)$ can be computed in polynomial time by considering the dual problem. We present a polynomial algorithm that finds an optimal fractional dicycle packing ψ *yielding* $\nu_c^*(G,w)$. Our algorithm consists of a solution to a simple linear program and some minor modifications, and avoids using the ellipsoid method. In fact, the algorithm shows that a maximum fractional dicycle packing yielding $\nu_c^*(G,w)$ with at most $O(n^2)$ dicycles receiving nonzero weight can be found in polynomial time.

1 Introduction

All graphs and digraphs considered here are finite and have no loops, parallel arcs or isolated vertices. For the standard terminology used the reader is referred to [5]. We use the terms *digraph* and *dicycle* to refer to a directed graph and a directed cycle, respectively.

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We consider the following fundamental problem in algorithmic graph-theory. Given a digraph G, how many arc-disjoint cycles can be packed into G? Define the *dicycle packing number* of G, denoted $\nu_c(G)$, to be the maximum size of a set of arc-disjoint dicycles in G. We also consider the fractional relaxation of this problem. Let R_+ denote the set of nonnegative reals. A *fractional dicycle packing* of G is a function ψ from the set $\mathcal{C}$ of dicycles in G to R_+, satisfying $\sum_{e \in C \in \mathcal{C}} \psi(C) \le 1$ for each $e \in E(G)$. Letting $|\psi| = \sum_{C \in \mathcal{C}} \psi(C)$, the *fractional dicycle packing number*, denoted $\nu_c^*(G)$, is defined to be the maximum of $|\psi|$ taken over all fractional dicycle packings ψ. Since a dicycle packing is also a fractional dicycle packing, we always have $\nu_c^*(G) \ge \nu_c(G)$. The notion of a fractional dicycle packing can be extended to digraphs with nonnegative arc weights. In this case we require that $\sum_{e \in C \in \mathcal{C}} \psi(C) \le w(e)$ for each $e \in E(G)$ where $w(e)$ is the weight of e. We denote by $\nu_c^*(G, w)$ the corresponding *fractional dicycle packing number* where $w : E \to R_+$ is the weight function.

Problems concerning packing arc-disjoint or vertex-disjoint dicycles in digraphs have been studied extensively (see, e.g., [4, 14]). It is well known that computing $\nu_c(G)$ (and hence finding a maximum dicycle packing) is an NP-Hard problem. Even the very special case of deciding whether a digraph has a triangle decomposition is known to be NP-Complete (see, e.g. [6] for a more general theorem on the NP-Completeness of such decomposition problems). Currently, the best approximation algorithm for this problem [12] has an approximation ratio of $O(n^{1/2})$ which is also an upper bound for the integrality gap. Thus, it is interesting to find out when $\nu_c(G)$ and $\nu_c^*(G)$ are "close" as this immediately yields an efficient approximation algorithm for this NP-Hard problem. Our main result shows that the two parameters differ by at most $o(n^2)$, thus giving an approximation algorithm with an $o(n^2)$ additive error term.

Theorem 1.1 *If G is an n-vertex digraph then $\nu_c^*(G) - \nu_c(G) = o(n^2)$ and a set of at least $\nu_c(G) - o(n^2)$ arc-disjoint dicycles can be generated in randomized polynomial time. There are n-vertex graphs G for which $\nu_c^*(G) - \nu_c(G) = \Omega(n^{3/2})$.*

The $o(n^2)$ additive error term is only interesting if the graph G is dense and $\nu_c(G) = \Theta(n^2)$. This, however, is the case for almost all digraphs, as it is known (and easy) that the directed random graph $G(n,p)$ has $\nu_c(G) = \Theta(n^2)$ for any constant p, $0 < p < 1$ (in this model each of the $n(n-1)$ arcs has probability p of being selected). There are also many other explicit constructions of digraphs with $\nu_c(G) = \Theta(n^2)$ which do not resemble a typical element of $G(n,p)$. The second part of Theorem 1.1 shows that the $o(n^2)$ error term in Theorem 1.1 **cannot** be replaced with $o(n^{1.5})$.

The first part of Theorem 1.1 uses as its main lemma a much more general result concerning packings of oriented graphs. Recall that an oriented graph is a directed graph without 2-cycles. Let $\mathcal{F}$ be any given (finite or infinite) family of oriented graphs. For an oriented graph G, let $\nu_{\mathcal{F}}(G)$ denote the maximum number of arc-disjoint copies of elements of $\mathcal{F}$ that can be found in G, and let $\nu_{\mathcal{F}}^*(G)$ denote the respective fractional relaxation. We prove the following.

Theorem 1.2 *For any given family of oriented graphs, if G is an n-vertex oriented graph then $\nu^*_{\mathcal{F}}(G) - \nu_{\mathcal{F}}(G) = o(n^2)$. Furthermore, a set of at least $\nu_{\mathcal{F}}(G) - o(n^2)$ arc-disjoint elements of $\mathcal{F}$ can be generated in randomized polynomial time.*

The first part of Theorem 1.1 is a consequence of Theorem 1.2 by considering the family $\mathcal{F}$ of all directed cycles of length at least 3. An initial preprocessing step allows us to get rid of the 2-cycles of G.

We note that an *undirected* version of Theorem 1.2 has been recently proved by the second author [17] extending an earlier result of Haxell and Rödl [10] dealing with single element families. The proof of Theorem 1.2 makes use of the recently discovered *directed regularity lemma* which has been proved by Alon and Shapira in [2], and which enables us to overcome several difficulties that do not occur in the undirected case.

It is well known that $\nu^*_c(G,w)$ can be computed in polynomial time by considering the dual problem whose solution is known to be computable in polynomial time [13]. This follows from the strong duality theorem. It also follows from the same method used in [11] that, using the ellipsoid method and a separation oracle which exists for the dual problem, an optimal fractional dicycle packing ψ yielding $\nu^*_c(G,w)$ can also be generated in polynomial time. However, we present a much simpler algorithm which avoids using the ellipsoid method and merely consists of solving some related simple linear program and slightly modifying the solution. In particular, we prove the following result.

Theorem 1.3 *If G is an n-vertex digraph associated with a nonnegative arc-weight function w, then a maximum fractional dicycle packing yielding $\nu^*_c(G,w)$ can be computed in polynomial time. Furthermore, a maximum fractional dicycle packing with at most $O(n^2)$ (resp. $O(n^3)$) dicycles receiving nonzero weight can be found in (resp. strongly) polynomial time.*

In the next two sections we prove our results.

2 Proofs of Theorem 1.1 and Theorem 1.2

2.1 Reducing the first part of Theorem 1.1 to a special case of Theorem 1.2

The following simple lemma shows that the problem of finding a maximum dicycle packing in a digraph G is equivalent to the problem of finding a maximum dicycle packing in the spanning subgraph G' of G obtained from G by deleting all 2-cycles. In particular, it shows that if $\nu^*_c(G') - \nu_c(G') = o(n^2)$ then $\nu^*_c(G) - \nu_c(G) = o(n^2)$.

Lemma 2.1 *If G is a digraph then there is always a maximum dicycle packing of G that contains all the 2-cycles.* ■

Proof: Assume that L is the set of dicycles of a maximal dicycle packing of G, containing the maximum possible number of 2-cycles. We claim that L contains all 2-cycles. Assume that some 2-cycle $\{(u, v), (v, u)\}$ is missing from L. Clearly, some $C \in L$ contains one of (u, v) or (v, u). Assume $(u, v) \in C$. Clearly, C is not a 2-cycle. If no element of L contains (v, u) we can replace C with the 2-cycle and obtain a maximum dicycle packing with more 2-cycles than there are in L, a contradiction. If some $C' \in L$ contains (v, u) we can replace C and C' with the 2-cycle and with a dicycle in the closed walk $(C - (u, v)) \cup (C' - (v, u))$ and obtain a maximum dicycle packing with more 2-cycles than there are in L, a contradiction. ■

Notice that Lemma 2.1 together with Theorem 1.2 applied to the family of all directed cycles of length at least 3 yields the first part of Theorem 1.1.

2.2 Tools used in the proof of Theorem 1.2

An important tool used in the proof of Theorem 1.2 is the following directed version of Szemerédi's regularity lemma. The proof, which is a modified version of the proof of the standard regularity lemma given in [15], can be found in [2]. We now give the definitions necessary in order to state the directed regularity lemma.

Let $G = (V, E)$ be a digraph, and let A and B be two disjoint subsets of $V(G)$. If A and B are non-empty and $e(A, B)$ is the number of arcs from A to B, the *density of arcs from A to B* is

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

For $\gamma > 0$ the pair (A, B) is called *γ-regular* if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \gamma|A|$ and $|Y| > \gamma|B|$ we have

$$|d(X, Y) - d(A, B)| < \gamma \qquad\qquad |d(Y, X) - d(B, A)| < \gamma.$$

An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \ldots, V_m$ whose sizes are as equal as possible. An equitable partition of the set of vertices V of a digraph G into the classes $V_1, \ldots, V_m$ is called *γ-regular* if $|V_i| \le \gamma|V|$ for every i and all but at most $\gamma\binom{m}{2}$ of the pairs (V_i, V_j) are γ-regular. The directed regularity lemma states the following:

Lemma 2.2 *For every $\gamma > 0$, there is an integer $M(\gamma) > 0$ such that every digraph G with $n > M$ vertices has a γ-regular partition of the vertex set into m classes, for some $1/\gamma \le m \le M$.* ■

Let H_0 be a fixed oriented graph with the vertices $\{1, \ldots, k\}$, $k \ge 3$. Let W be a k-partite oriented graph with vertex classes $V_1, \ldots, V_k$. A subgraph J of W with ordered vertex set $v_1, \ldots, v_k$ is *partite-isomorphic* to H_0 if $v_i \in V_i$ and the map $v_i \to i$ is an isomorphism from J to H_0. The following lemma is almost identical to the proof of Lemma 15 in [10] and hence the proof is omitted.

Lemma 2.3 *Let δ and ζ be positive reals. There exist $\gamma = \gamma(\delta, \zeta, k)$ and $T = T(\delta, \zeta, k)$ such that the following holds. Let W be a k-partite oriented graph with vertex classes $V_1, \ldots, V_k$ and $|V_i| = t > T$ for $i = 1, \ldots, k$. Furthermore, for each arc $(i,j) \in E(H_0)$, (V_i, V_j) is a γ-regular pair with density $d(i,j) \geq \delta$ and for each arc $(i,j) \notin E(H_0)$, $E(V_i, V_j) = \emptyset$. Then, there exists a spanning subgraph W' of W, consisting of at least $(1-\zeta)|E(W)|$ arcs such that the following holds. For an arc $e \in E(W')$, let $c(e)$ denote the number of subgraphs of W' that are partite isomorphic to H_0 and that contain e. Then, for all $e \in E(W')$, if $e \in E(V_i, V_j)$ then*

$$\left| c(e) - t^{k-2} \frac{\prod_{(s,p) \in E(H_0)} d(s,p)}{d(i,j)} \right| < \zeta t^{k-2}.$$

■

Finally, we need to state the seminal result of Frankl and Rödl [7] on near perfect coverings and matchings of uniform hypergraphs. Recall that if x, y are two vertices of a hypergraph then $deg(x)$ denotes the degree of x and $deg(x,y)$ denotes the number of hyperedges that contain both x and y (their *co-degree*). We use the stronger version of the Frankl and Rödl Theorem due to Pippenger (see, e.g., [8]).

Lemma 2.4 *For an integer $r \geq 2$ and a real $\beta > 0$ there exists a real $\mu > 0$ so that: If the r-uniform hypergraph L on q vertices has the following properties for some d:*
(i) $(1-\mu)d < deg(x) < (1+\mu)d$ holds for all vertices,
(ii) $deg(x,y) < \mu d$ for all distinct x and y,
then L has a matching of size at least $(q/r)(1-\beta)$. ■

2.3 Proof of Theorem 1.2

Let $\mathcal{F}$ be a family of oriented graphs, and let $\epsilon > 0$. To avoid the trivial case we assume that each element of $\mathcal{F}$ has at least three vertices. We shall prove there exists $N = N(\mathcal{F}, \epsilon)$ such that for all $n > N$, if G is an n-vertex oriented graph then $\nu^*_{\mathcal{F}}(G) - \nu_{\mathcal{F}}(G) < \epsilon n^2$.

The idea of the proof is as follows. Given an n-vertex graph G and a maximum fractional $\mathcal{F}$-packing ψ of G, we apply Lemma 2.2 to G and define a fractional $\mathcal{F}$-packing ψ' on the resulting m-vertex cluster graph of the partition (the graph whose vertices are the vertex classes of the partition and whose arcs connect appropriately defined dense pairs of vertex classes). We show that $|\psi'|$ is very close to $|\psi| m^2/n^2$. We then show how each fractional copy H of ψ' can be translated back into an appropriate number of (integral) edge-disjoint copies of H in G, such that the total number of copies constructed in G in this way is very close to $|\psi'| n^2/m^2$, and hence to $|\psi|$.

Let k_∞ be the maximal order of a graph in $\mathcal{F}$ (possibly $k_\infty = \infty$). Let $k_0 = \min\{k_\infty, \lceil 20/\epsilon \rceil\}$. Let $\delta = \beta = \epsilon/4$. For all $r = 2, \ldots, {k_0}^2$, let $\mu_r = \mu(\beta, r)$ be as in Lemma 2.4, and put $\mu = \min_{r=2}^{k_0^2}\{\mu_r\}$. Let $\zeta = \mu\delta^{{k_0}^2}/2$. For $k = 3, \ldots, k_0$, let $\gamma_k = \gamma(\delta, \zeta, k)$ and $T_k = T(\delta, \zeta, k)$ be as in Lemma 2.3. Let $\gamma = \min_{k=3}^{k_0}\{\gamma_k\}$. Let $M = M(\gamma\epsilon/(25{k_0}^2))$ be as in Lemma 2.2. Finally, we shall define N to be a sufficiently large constant, depending on the above chosen parameters, and for which various conditions stated in the proof below hold. Thus, indeed, $N = N(\mathcal{F}, \epsilon)$.

Fix an n-vertex oriented graph G with $n > N$ vertices. Fix a fractional $\mathcal{F}$-packing ψ with $|\psi| = \nu^*_{\mathcal{F}}(G)$. We may assume that ψ assigns a value to each *labeled* copy of an element of $\mathcal{F}$ simply by dividing the value of ψ on each nonlabeled copy by the size of the automorphism group of that element. If $\nu^*_{\mathcal{F}}(G) < \epsilon n^2$ we are done. Hence, we assume $\nu^*_{\mathcal{F}}(G) = \alpha n^2 \geq \epsilon n^2$.

We apply Lemma 2.2 to G and obtain a γ'-regular partition with m' parts, where $\gamma' = \gamma\epsilon/(25{k_0}^2)$ and $1/\gamma' < m' < M(\gamma')$. Denote the parts by $U_1, \ldots, U_{m'}$. Notice that the size of each part is either $\lfloor n/m' \rfloor$ or $\lceil n/m' \rceil$. For simplicity we may and will assume that n/m' is an integer, as this assumption does not affect the asymptotic nature of our result. For the same reason we may and will assume that $n/(25m'{k_0}^2/\epsilon)$ is an integer.

We randomly partition each U_i into $25{k_0}^2/\epsilon$ equal parts of size $n/(25m'{k_0}^2/\epsilon)$ each. All m' partitions are independent. We now have $m = 25m'{k_0}^2/\epsilon$ *refined* vertex classes, denoted $V_1, \ldots, V_m$. Suppose $V_i \subset U_s$ and $V_j \subset U_t$ where $s \neq t$. We claim that if (U_s, U_t) is a γ'-regular pair then (V_i, V_j) is a γ-regular pair. Indeed, if $X \subset V_i$ and $Y \subset V_j$ have $|X|, |Y| > \gamma n/(25m'{k_0}^2/\epsilon)$ then $|X|, |Y| > \gamma' n/m'$ and so $|d(X,Y) - d(U_s,U_t)| < \gamma'$ and $|d(Y,X) - d(U_t,U_s)| < \gamma'$. Also $|d(V_i,V_j) - d(U_s,U_t)| < \gamma'$ and $|d(V_j,V_i) - d(U_t,U_s)| < \gamma'$. Thus, $|d(X,Y) - d(V_i,V_j)| < 2\gamma' < \gamma$ and $|d(Y,X) - d(V_j,V_i)| < 2\gamma' < \gamma$.

Let H be a labeled copy of some $H_0 \in \mathcal{F}$ in G. If H has k vertices and $k \leq k_0$ then the expectation of the number of pairs of vertices of H that belong to the same vertex class in the refined partition is clearly at most $\binom{k}{2}\epsilon/(25{k_0}^2) < \epsilon/50$. Thus, the probability that H has two vertices in the same vertex class is also at most $\epsilon/50$. We call H *good* if it has $k \leq k_0$ vertices and its k vertices belong to k distinct vertex classes of the refined partition. By the definition of k_0, if H has $k > k_0$ vertices and $\psi(H) > 0$ then we must have $k > 20/\epsilon$. Since oriented graphs with k vertices have at least $k/2$ arcs, the contribution of graphs with $k > k_0$ vertices to $\nu^*_{\mathcal{F}}(G)$ is at most $\binom{n}{2}/(10/\epsilon) < \epsilon n^2/20$. Hence, if ψ^{**} is the restriction of ψ to good copies (the bad copies having $\psi^{**}(H) = 0$) then the expectation of $|\psi^{**}|$ is at least $(\alpha - \epsilon/50 - \epsilon/20)n^2$. We therefore *fix* a partition $V_1, \ldots, V_m$ for which $|\psi^{**}| \geq (\alpha - 0.07\epsilon)n^2$.

We say that the set of arcs $E(V_i, V_j)$ is *good* if (V_i, V_j) is a γ-regular pair and also $d(V_i, V_j) \geq \delta$. Notice that it is possible that $E(V_i, V_j)$ is good while $E(V_j, V_i)$ is not good (because of sparseness). Let G^* be the spanning subgraph of G consisting of the union of the good sets of arcs (thus, we discard arcs inside classes, between non regular pairs, between sparse pairs, or one-sided sparse pairs). Let ψ^* be the restriction of ψ^{**} to the labeled copies of elements of $\mathcal{F}$ in G^*. We claim that

$\nu^*_{\mathcal{F}}(G^*) \geq |\psi^*| > |\psi^{**}| - 1.02\delta n^2 \geq (\alpha - 0.07\epsilon - 1.02\delta)n^2 = (\alpha - 1.3\delta)n^2$. Indeed, by considering the number of discarded arcs we get (using $m' > 1/\gamma'$ and $\delta >> \gamma'$)

$$|\psi^{**}| - |\psi^*| \leq |E(G) - E(G^*)| <$$

$$\gamma' \binom{m'}{2} \frac{2n^2}{m'^2} + m'(m'-1)(\delta + \gamma') \frac{n^2}{m'^2} + m' \frac{n^2}{m'^2} < 1.02\delta n^2.$$

Let R denote the m-vertex digraph whose vertices are $\{1, \ldots, m\}$ and $(i,j) \in E(R)$ if and only if $E(V_i, V_j)$ is good. We define a (labeled) fractional dicycle packing ψ' of R as follows. Let H be a labeled copy of some $H_0 \in \mathcal{F}$ in R and assume that the vertices of H are $\{u_1, \ldots, u_k\}$ where u_i plays the role of vertex i in H_0. We define $\psi'(H)$ to be the sum of the values of ψ^* taken over all subgraphs of $G^*[V_{u_1}, \ldots, V_{u_k}]$ which are partite isomorphic to H_0, divided by n^2/m^2. Notice that by normalizing with n^2/m^2 we guarantee that ψ' is a proper fractional $\mathcal{F}$-packing of R and that $\nu^*_{\mathcal{F}}(R) \geq |\psi'| = m^2|\psi^*|/n^2 \geq m^2(\alpha - 1.3\delta)$. Notice also that although R may contain 2-cycles, they receive no weight in ψ' as G has no 2-cycles.

We use ψ' to define a random coloring of the arcs of G^*. Our "colors" are the labeled copies of elements of $\mathcal{F}$ in R. Let $d(i,j)$ denote the density from V_i to V_j and notice that $|E_{G^*}(V_i, V_j)| = d(i,j)n^2/m^2$. Let H be a labeled copy of some $H_0 \in \mathcal{F}$ in R, and assume that H contains the arc (i,j). Each $e \in E(V_i, V_j)$ is chosen to have the "color" H with probability $\psi'(H)/d(i,j)$. The choices made by distinct arcs of G^* are independent. Notice that this random coloring is legal (in the sense that the sum of probabilities is at most one) since the sum of $\psi'(H)$ taken over all labeled copies of elements of $\mathcal{F}$ containing (i,j) is at most $d(i,j)$. Notice also that some arcs might stay uncolored.

Let H be a labeled copy of some $H_0 \in \mathcal{F}$ in R, and assume that $\psi'(H) > m^{1-k_0}$. Without loss of generality, assume that the vertices of H are $\{1, \ldots, k\}$ where $i \in V(H)$ plays the role of $i \in V(H_0)$. Let r denote the number of arcs of H. Notice that $r < k_0^2$. Let W_H be the k-partite subgraph of G^* with vertex classes $V_1, \ldots, V_k$ and with the arcs $\cup_{(i,j) \in H} E(V_i, V_j)$. Notice that W_H satisfies the conditions in Lemma 2.3, since $t = n/m > N\epsilon/(25{k_0}^2 M) > T_k$ (here we assume $N > 25{k_0}^2 M T_k/\epsilon$). Let W'_H be the spanning subgraph of W_H whose existence is guaranteed in Lemma 2.3. Let X_H denote the spanning subgraph of W'_H consisting only of the arcs whose color is H. Notice that X_H is a random subgraph of W'_H. For an arc $e \in E(X_H)$, let $C_H(e)$ denote the set of subgraphs of X_H that contain e and that are partite isomorphic to H_0. Put $c_H(e) = |C_H(e)|$. The following lemma, shows that for all $e \in E(X_C)$, $c_H(e)$ can be tightly approximated with high probability.

Lemma 2.5 *With probability at least* $1 - m^3/n$, *for all* $e \in E(X_H)$,

$$\left| c_H(e) - t^{k-2}\psi'(H)^{r-1} \right| < \mu\psi'(H)^{r-1}t^{k-2}. \tag{1}$$

Proof: Let $C(e)$ denote the set of subgraphs of W'_H that contain e and that are partite isomorphic to H_0. Put $c(e) = |C(e)|$. According to Lemma 2.3, if $e \in E(V_i, V_j)$ then

$$\left| c(e) - t^{k-2} \frac{\prod_{(s,p)\in E(H_0)} d(s,p)}{d(i,j)} \right| < \zeta t^{k-2}. \tag{2}$$

Fix an arc $e \in E(X_H)$ belonging to $E(V_i, V_j)$. The probability that an element of $C(e)$ also belongs to $C_H(e)$ is precisely

$$\rho = \psi'(H)^{r-1} \cdot \frac{d(i,j)}{\prod_{(s,p)\in E(H_0)} d(s,p)}.$$

We say that two distinct elements $Y, Z \in C(e)$ are *dependent* if they share at least one arc other than e. Consider the dependency graph B whose vertex set is $C(e)$ and the edges connect dependent pairs. Since two dependent elements share at least three vertices (including the two endpoints of e), we have $\Delta(B) = O(t^{k-3})$ where $\Delta(B)$ is the maximum degree of B. Hence, $\chi(B) = O(t^{k-3})$ where $\chi(B)$ is the chromatic number of B. Put $s = \chi(B)$. Let $C^1(e), \ldots, C^s(e)$ denote a partition of $C(e)$ to independent sets. Let $C^q_H(e) = C^q(e) \cap C_H(e)$, $c^q(e) = |C^q(e)|$ and $c^q_H(e) = |C^q_H(e)|$. Clearly, $c^1(e) + \cdots + c^s(e) = c(e)$ and $c^1_H(e) + \cdots + c^s_H(e) = c_H(e)$. The expectation of $c^q_H(e)$ is $\rho c^q(e)$. Consider some $C^q(e)$ with $c^q(e) > \sqrt{t}$. According to a large deviation inequality of Chernoff (cf. [3] Appendix A), for every $\eta > 0$, and in particular for $\eta = \mu/8$, if n (and hence t and hence $c^q(e)$) is sufficiently large,

$$\Pr[|c^q_H(e) - \rho c^q(e)| > \eta \rho c^q(e)] < e^{-\frac{2(\eta \rho c^q(e))^2}{c^q(e)}} = e^{-2\eta^2\rho^2 c^q(e)} << t^{-k-1}.$$

It follows that with probability at least $1 - st^{-k-1} > 1 - t^{-3}$, for all $C^q(e)$ with $c^q(e) > \sqrt{t}$, $(1-\eta)\rho c^q(e) \le c^q_H(e) \le (1+\eta)\rho c^q(e)$ holds. Since the sum of $c^q(e)$ having $c^q(e) \le \sqrt{t}$ is $O(t^{k-2.5})$ and since $c(e) = \Theta(t^{k-2})$ we have that this sum is much less than $\rho\eta c(e)$. Thus, together with (2) and the fact that $\rho < \psi'(H)^{r-1}\delta^{-r}$ we have

$$c_H(e) = \sum_{q=1}^{s} c^q_H(e) \le \rho(1+\eta)(\sum_{q=1}^{s} c^q(e)) + \rho\eta c(e) = \rho(1+2\eta)c(e) \le \tag{3}$$

$$\rho(1+2\eta)t^{k-2}(\zeta + \frac{\prod_{(s,p)\in E(H_0)} d(s,p)}{d(i,j)}) = (1+2\eta)t^{k-2}(\psi'(H)^{r-1} + \zeta\rho) \le$$

$$t^{k-2}\psi'(H)^{r-1}(1+2\eta)(1+\zeta\delta^{-r}) \le$$

$$t^{k-2}\psi'(H)^{r-1}(1+\mu/4)(1+\mu/2) \le (1+\mu)t^{k-2}\psi'(H)^{r-1}.$$

Similarly,

$$c_H(e) \ge \rho(1-\eta)c(e) - \rho\eta c(e) = \rho(1-2\eta)c(e) \ge \tag{4}$$

$$\rho(1-2\eta)t^{k-2}(\frac{\prod_{(s,p)\in E(H_0)} d(s,p)}{d(i,j)} - \zeta) = (1-2\eta)t^{k-2}(\psi'(H)^{r-1} - \zeta\rho) \ge$$

$$t^{k-2}\psi'(H)^{r-1}(1-2\eta)(1-\zeta\delta^{-r}) \geq$$
$$t^{k-2}\psi'(H)^{r-1}(1-\mu/4)(1-\mu/2) \geq (1-\mu)t^{k-2}\psi'(H)^{r-1}.$$

Combining (3) and (4) we have that (1) holds for a fixed $e \in E(X_H)$ with probability at least $1-t^{-3}$. As $|E(X_H)| < n^2$ we have that (1) holds for all $e \in E(X_H)$ with probability at least $1-n^2/t^3 = 1-m^3/n$. ■

We also need the following simple lemma that gives a lower bound for the number of arcs of X_H.

Lemma 2.6 *With probability at least* $1-1/n$,

$$|E(X_H)| > (1-2\zeta)r\frac{n^2}{m^2}\psi'(H).$$

Proof: For $(i,j) \in E(H_0)$, the expected number of arcs of $E(V_i, V_j)$ that received the color H is precisely $d(i,j)\frac{n^2}{m^2}\frac{\psi'(H)}{d(i,j)} = \frac{n^2}{m^2}\psi'(H)$. Summing over all r arcs of H_0, the expected number of arcs of W_H that received the color H is precisely $r\frac{n^2}{m^2}\psi'(H)$. As at most $\zeta|E(W_H)|$ arcs belong to W_H and do not belong to W'_H we have that the expectation of $|E(X_H)|$ is at least $(1-\zeta)r\frac{n^2}{m^2}\psi'(H)$. As ζ, r, m are constants and as $\psi'(H)$ is bounded from below by the constant m^{1-k_0}, we have, by the common large deviation inequality of Chernoff (cf. [3] Appendix A), that for $n > N$ sufficiently large, the probability that $|E(X_H)|$ deviates from its mean by more than $\zeta r\frac{n^2}{m^2}\psi'(H)$ is exponentially small in n. In particular, the lemma follows. ■

Since R contains at most $O(m^{k_0})$ labeled copies of elements of $\mathcal{F}$ with at most k_0 vertices, we have that with probability at least $1-O(m^{k_0}/n)-O(m^{k_0+3}/n) > 0$ (here we assume again that N is sufficiently large) *all* labeled copies H of elements of $\mathcal{F}$ in R with $\psi'(H) > m^{1-k_0}$ satisfy the statements of Lemma 2.5 and Lemma 2.6. We therefore fix a coloring for which Lemma 2.5 and Lemma 2.6 hold for all labeled copies H of elements of $\mathcal{F}$ in R having $\psi'(H) > m^{1-k_0}$.

Let H be a labeled copy of some $H_0 \in \mathcal{F}$ in R with $\psi'(H) > m^{1-k_0}$, and let r denote the number of arcs of H. We construct an r-uniform hypergraph L_H as follows. The vertices of L_H are the arcs of the corresponding X_H from Lemma 2.5. The hyperedges of L_H correspond to the arc sets of the subgraphs of X_H that are partite isomorphic to H_0. We claim that our hypergraph satisfies the conditions of Lemma 2.4. Indeed, let q denote he number of vertices of L_H. Notice that Lemma 2.6 provides a lower bound for q. Let $d = t^{k-2}\psi'(H)^{r-1}$. Notice that by Lemma 2.5 *all* vertices of L_H have their degrees between $(1-\mu)d$ and $(1+\mu)d$. Also notice that the co-degree of any two vertices of L_H is at most t^{k-3} as two arcs cannot belong, together, to more than t^{k-3} subgraphs of X_H that are partite isomorphic to H_0. In particular, for N sufficiently large, $\mu d > t^{k-3}$. By Lemma 2.4 we have at least $(q/r)(1-\beta)$ arc-disjoint copies of H_0 in X_H. In particular, we have at least

$$(1-\beta)(1-2\zeta)\frac{n^2}{m^2}\psi'(H) > (1-2\beta)\psi'(H)\frac{n^2}{m^2}$$

such copies. Recall that $|\psi'| \geq m^2(\alpha - 1.3\delta)$. Since there are $O(m^{k_0})$ labeled copies H of elements of $\mathcal{F}$ in R with $0 < \psi'(H) \leq m^{1-k_0}$, their total contribution to $|\psi'|$ is $O(m)$. Hence, summing the last inequality over all H with $\psi'(H) > m^{1-k_0}$ we have at least

$$(1-2\beta)m^2(\alpha - 1.3\delta - O(\frac{1}{m}))\frac{n^2}{m^2} > n^2(\alpha - \epsilon)$$

arc-disjoint copies of elements of $\mathcal{F}$ in G. It follows that $\nu_{\mathcal{F}}(G) \geq n^2(\alpha - \epsilon)$. As $\nu^*_{\mathcal{F}}(G) = \alpha n^2$, Theorem 1.2 follows. ■

The proof of Theorem 1.2 implies a polynomial (in n) time algorithm that produces a set of $n^2(\alpha - \epsilon)$ arc-disjoint elements of $\mathcal{F}$ in G with probability at least, say, 0.99. Indeed, Lemma 2.2 can be implemented in $o(n^3)$ time using the algorithm of Alon et. al. [1] applied to the directed regularity lemma. Lemma 2.3 can be implemented using a simple greedy algorithm following the proof in [10]. Lemma 2.4 has a polynomial running time implementation due to Grable [9]. By Theorem 1.3, computing ψ and ψ^{**} can be done in $O(n^3)$ time. The other ingredients of the proof, namely, computing ψ' and the random coloring are easily implemented in polynomial time.

2.4 A lower bound for the error term in Theorem 1.1

We now prove the second part of Theorem 1.1. We must show that there exists a digraph G for which $\nu^*_c(G) - \nu_c(G) = \Omega(n^{3/2})$. Let T be the following Eulerian orientation of K_{11} with vertex set $\{0, \ldots, 10\}$. Each vertex i has an outgoing arc towards $i+2, i+6, i+7, i+8, i+10$ (indices module 11). It is easy to check that each arc lies on three directed triangles, and that there are 55 directed triangles. Thus, by assigning 1/3 to each triangle we get $\nu^*_c(T) = 55/3$. Trivially, however, $\nu_c(T) \leq \lfloor 55/3 \rfloor = 18 = 54/3$.

It is well known (by applying the Rödl nibble method or the result from [7]) that for all sufficiently large n, K_n contains more than, say, $n^2/111$ edge-disjoint copies of K_{11}. Hence, let G^* be an n-vertex oriented graph with $55r$ arcs where $r > n^2/111$ and which consists of a set R^* of r arc-disjoint copies of T. Consider a directed triangle S in G^*. We say that S is *good* if it is contained entirely in some T element of R^*. Notice that a bad directed triangle must have each arc in a distinct element of R^*. We construct a random spanning subgraph G' of G^* by independently choosing each element of R^* with probability $n^{-1/2}/11$. Let $R' \subset R^*$ denote the random subset chosen. Clearly, $E[|R'|] = rn^{-1/2}/11 > n^{3/2}/1221$. The maximum number of arc-disjoint good directed triangles in G' is at most $18|R'|$. The probability that a bad directed triangle appears in G' is precisely $n^{-3/2}/11^3$. As there are less than n^3 bad directed triangles in G^*, their expected number in G' is at most $n^{3/2}/1331$. It follows that there exists an n-vertex oriented graph G, which is composed of a set of t arc-disjoint copies of T, where $t \geq n^{3/2}(1/1221 - 1/1331)$, and with no bad directed triangle. Clearly, $\nu^*_c(G) = 55t/3$. Consider an integral dicycle packing of G. It may contain at most $54t/3$ directed triangles. The other cycles must have length at least 4. Thus, $\nu_c(G) \leq 54.75t/3$. It follows that $\nu_c^*(G) - \nu_c(G) \geq t/12 = \Omega(n^{3/2})$.

A similar argument shows that for every $\epsilon > 0$, there exists $k = k(\epsilon)$ such that if T is a tournament with k vertices and $\mathcal{F} = \{T\}$, there are oriented graphs G for which $\nu_{\mathcal{F}}^*(G) - \nu_{\mathcal{F}}(G) = \Omega(n^{2-\epsilon})$. Thus, in general, the $o(n^2)$ error term in Theorem 1.2 cannot be improved to $o(n^{2-\epsilon})$ for any positive ϵ. ■

3 Computing a Fractional Dicycle Packing in Polynomial Time

The main difficulty in proving Theorem 1.3 stems from the fact that the LP-formulation of the problem might have an exponential number of variables since, in general, the number of dicycles in a digraph might be exponential in the size of the digraph.

Let ψ be a fractional dicycle packing of a weighted digraph (G, w). Notice that it is sufficient to specify the values of ψ only for dicycles with $\psi(C) > 0$. Thus, if $\mathcal{C}(G)$ denotes the set of dicycles in G, let $\chi(\psi) = \{C \in \mathcal{C}(G) \ : \ \psi(C) > 0\}$ denote the *characteristic set* of ψ. An algorithm for the maximum dicycle packing problem is *polynomial* if it runs in time polynomial in the size of the input digraph (G, w), and delivers a polynomial size set of dicycles $\mathcal{C} \subseteq \mathcal{C}(G)$ and nonnegative numbers $\psi(C)$ for $C \in \mathcal{C}$, such that by assigning $\psi(C) = 0$ to all $C \in \mathcal{C}(G) - \mathcal{C}$ we obtain an optimal fractional dicycle packing of (G, w), namely $|\psi| = \nu_c^*(G, w)$.

A *fractional dicycle arc cover* of (G, w) is a function $x : E(G) \to [0, 1]$, such that $\sum_{e \in C} x(e) \geq 1$ for every $C \in \mathcal{C}(G)$. The *value* of a fractional dicycle arc cover x is $|x| = \sum_{e \in E(G)} w(e)x(e)$. Let $\tau_c^*(G, w)$ denote the value of a minimum fractional dicycle arc cover of (G, w). The *minimum fractional dicycle arc cover problem* is a dual to the maximum fractional dicycle packing problem.

For the rest of this section we will assume that G is the complete digraph (the digraph in which any pair of distinct vertices are connected with arcs in both directions). We may assume this since if G is not complete, we may add the nonexistent arcs and assign them zero weight. Notice that $\nu_c^*(G, w)$ and $\tau_c^*(G, w)$ remain intact after this modification.

It was shown in [13] that the minimum fractional dicycle arc cover problem is reduced to solving a linear program with a polynomial number of constraints. Let $\mathcal{C}_k(G)$ denote the set of all the dicycles in G of length k. The result in [13] shows that in order to solve the fractional dicycle arc cover problem in a complete digraph, it is sufficient to consider only dicycles of length 2 and 3.

Theorem 3.1 ([13],Theorem 4.1) *If* (G, w) *is a complete weighted digraph then*

$$\begin{array}{rlll} \tau_c^*(G,w) = & \min \sum_{e \in E(G)} w(e)x(e) & & \\ s.t. & \sum_{e \in C} x(e) \geq 1 & \text{for all } C \in \mathcal{C}_3(G) & \\ & \sum_{e \in C} x(e) = 1 & \text{for all } C \in \mathcal{C}_2(G) & \\ & x(e) \geq 0 & \text{for all } e \in E(G). & \end{array} \tag{5}$$

Proof of Theorem 1.3: Using Theorem 3.1 we will show that a maximum fractional dicycle packing with $O(n^2)$ (resp. $O(n^3)$) dicycles in its characteristic set, can be found in (resp., strongly) polynomial time. Consider the dual linear program to (5). By the strong duality theorem:

$$\begin{array}{rlll} \nu_c^*(G,w) = & \max \sum_{C \in \mathcal{C}_2(G) \cup \mathcal{C}_3(G)} \psi(C) & & \\ s.t. & \sum_{e \in C \in \mathcal{C}_2(G) \cup \mathcal{C}_3(G)} \psi(C) \leq w(e) & \text{for all } e \in E(G) & \\ & \psi(C) \geq 0 & \text{for all } C \in \mathcal{C}_3(G). & \end{array} \tag{6}$$

Note that a feasible solution ψ to (6) might not be a fractional dicycle packing, since there might be a dicycle $C \in \mathcal{C}_2(G)$ for which $\psi(C) < 0$ holds. A *fractional dicycle pseudopacking* is a fractional dicycle packing in which 2-cycles may receive negative values. In particular, a feasible solution of (6) is a fractional dicycle pseudopacking. We will show a polynomial time iterative procedure that converts an *optimal* solution ψ of (6) into a fractional dicycle packing of value $|\psi| = \nu_c^*(G, w)$, and such that the cardinality of the characteristic set is at most $|\chi(\psi)|$.

For a pair of vertices u, v, let $C_{uv} = C_{vu}$ denote the 2-cycle consisting of the two arcs $(u, v), (v, u)$. For a fractional dicycle pseudopacking ψ, let $\Delta_\psi(u, v) = w(u, v) - \sum_{(u,v) \in C} \psi(C)$. Notice that $\Delta_\psi(u, v) \geq 0$ for each $(u, v) \in E(G)$. Notice also that if ψ is an optimal solution of (6) then $\min\{\Delta_\psi(u, v)\ ,\ \Delta_\psi(v, u)\} = 0$. Our algorithm consists of repeated applications of one of the following two basic operations.

Operation A**:** Let ψ be a fractional dicycle pseudopacking with $|\psi| = \nu_c^*(G, w)$. Suppose that for $s, t \in V(G)$ holds: $\psi(C_{st}) < 0$, $\Delta_\psi(t, s) = 0$ and $\Delta_\psi(s, t) > 0$. Let C_1 be a dicycle such that $(t, s) \in C_1$ and $\psi(C_1) > 0$. Notice that C_1 must exist. Let $\delta = \min\{\Delta_\psi(s, t)\ ,\ \psi(C_1)\}$. Let $\hat{\psi}$ be the same as ψ except for $\hat{\psi}(C_1) = \psi(C_1) - \delta$ and $\hat{\psi}(C_{st}) = \psi(C_{st}) + \delta$. Notice that $\hat{\psi}$ is a fractional dicycle pseudopacking with $|\hat{\psi}| = |\psi| = \nu_c^*(G, w)$. Furthermore $|\chi(\hat{\psi})| \leq |\chi(\psi)|$ and if $\Delta_{\hat{\psi}}(s, t) > 0$, then $|\chi(\hat{\psi})| \leq |\chi(\psi)| - 1$.

Operation B**:** Let ψ be a fractional dicycle pseudopacking with $|\psi| = \nu_c^*(G, w)$. Suppose that for $u, v \in V(G)$ holds: $\psi(C_{uv}) < 0$ and $\Delta_\psi(u, v) = \Delta_\psi(v, u) = 0$. Let C_1, C_2 be dicycles such that $(u, v) \in C_1$, $(v, u) \in C_2$, and let $\mu = \min\{\psi(C_1)\ ,\ \psi(C_2)\} > 0$. Notice that C_1 and C_2 exist. Let $\hat{C}$ be a dicycle in $(C_1 \cup C_2) - C_{uv}$. Let $\hat{\psi}$ be the same as ψ except for $\hat{\psi}(C_i) = \psi(C_i) - \mu$ for $i = 1, 2$, $\hat{\psi}(\hat{C}) = \psi(\hat{C}) + \mu$, $\hat{\psi}(C_{uv}) = \psi(C_{uv}) + \mu$. Notice that $\hat{\psi}$ is a fractional dicycle pseudopacking with $|\hat{\psi}| = |\psi| = \nu_c^*(G, w)$. Furthermore, $\Delta_{\hat{\psi}}(u, v) = \Delta_{\hat{\psi}}(v, u) = 0$, $|\chi(\hat{\psi})| \leq |\chi(\psi)|$, and $|\{C \in \chi(\hat{\psi})\ :\ (u, v) \in C \text{ or } (v, u) \in C\}| \leq |\{C \in \chi(\psi)\ :\ (u, v) \in C \text{ or } (v, u) \in C\}| - 1$.

Our algorithm proceeds as follows. We begin with a fractional dicycle pseudopacking ψ which is an optimal solution of (6). At each stage we apply either Operation A or Operation B. Thus, at any stage we have a fractional dicycle pseudopacking $\hat{\psi}$ with $|\chi(\hat{\psi})| \leq |\chi(\psi)|$ and with $|\hat{\psi}| = |\psi| = \nu_c^*(G,w)$. At any stage, let $\mathcal{S}'$ denote the set of dicycles C with $\hat{\psi}(C) < 0$ and let $\mathcal{S}$ denote the set of cycles with $\hat{\psi}(C) > 0$. Notice that operations A and B guarantee that $\mathcal{S}'$ contains only 2-cycles. In the beginning, $\mathcal{S} \subset \mathcal{C}_2(G) \cup \mathcal{C}_3(G)$ but Operation B may add longer cycles to $\mathcal{S}$. Notice that in operations A and B we can increase the weight of a 2-cycle without decreasing the weight of any other 2-cycle. In the main loop, as long as $\mathcal{S}' \neq \emptyset$, the algorithm chooses $C_{uv} \in \mathcal{S}'$ and increases $\hat{\psi}(C_{uv})$ by $\min\{\Delta_{\hat{\psi}}(u,v)\ ,\ \Delta_{\hat{\psi}}(v,u)\}$. In case $\Delta_{\hat{\psi}}(u,v) = \Delta_{\hat{\psi}}(v,u) = 0$ holds, the algorithm iteratively applies Operation B until $\hat{\psi}(C_{uv}) \geq 0$ holds. Otherwise, for $(s,t) = (u,v)$ or for $(s,t) = (v,u)$, $\Delta_{\hat{\psi}}(s,t) > 0$ and $\Delta_{\hat{\psi}}(t,s) = 0$ holds. Then, the algorithm iteratively applies Operation A for an appropriate choice among $(s,t) = (u,v)$ or $(s,t) = (v,u)$, until $\hat{\psi}(C_{uv}) \geq 0$, or until $\Delta_{\hat{\psi}}(u,v) = \Delta_{\hat{\psi}}(v,u) = 0$ holds. During the operations, the algorithm updates the sets $\mathcal{S}'$ and $\mathcal{S}$. It is not hard to see, that every iteration of the main loop decreases $|\mathcal{S}'|$ by at least 1. Thus, at the end of the algorithm, $\hat{\psi}$ is an optimal fractional dicycle packing. As each inner loop is repeated at most $|\mathcal{S}|$ times, the algorithm is polynomial in the initial values of $|\mathcal{S}'|$, $|\mathcal{S}|$, and in $n = |V(G)|$.

It is known that any linear program of the form $\max\{y \in R^m\ :\ Qy \leq q\}$ which has an optimal solution, has an optimal solution $\tilde{y}$ which is *basic*. That is, there exists a set of m equations and tight inequalities, such that $\tilde{y}$ is the unique solution to the corresponding equation system. Let $\tilde{\psi}$ be a basic solution to (6). The linear program (6) has $O(n^3)$ variables and $O(n^3)$ constraints. Note, however, that only $O(n^2)$ of the variables do *not* have constraints of the form $\psi(C) \geq 0$. Thus, $\tilde{\psi}$ has only $O(n^2)$ nonzero entries, i.e., $|\chi(\tilde{\psi})| = O(n^2)$. One can compute an optimal basic solution to (6) in polynomial time using the interior point method.

Tardos [16] showed that there exists a strongly polynomial time algorithm for solving linear programs with $\{0, \pm 1\}$ constraint matrices, assuming all the other input numbers are rational. Thus, the linear program (6) can be solved in strongly polynomial time. However, the algorithm in [16] might produce an optimal solution $\tilde{\psi}$ which is not basic. In this case, we can guarantee only that $|\chi(\tilde{\psi})| = O(n^3)$. ■

Similar techniques can be used to show that there always exists a compact maximum *integral* dicycle packing. Let $\nu_c(G,w)$ denote the value of a maximum integral dicycle packing in (G,w).

Lemma 3.2 *Let (G,w) be a complete weighted digraph, and let $\tilde{\nu}_c(G,w)$ be the value of an optimal integral solution to (6). Then $\tilde{\nu}_c(G,w) = \nu_c(G,w)$.*

Proof: We can assume, w.l.o.g., that the weights are integral. Then the algorithm described above preserves integrality. Thus $\tilde{\nu}_c(G,w) \leq \nu_c(G,w)$. We now prove the reverse inequality. Let ψ be an optimal integral solution to (6). Among all integral dicycle pseudopackings in (G,w), let $\tilde{\psi}$ be one for which $\sum_{C \in \mathcal{C}_3(G) \cap \chi(\psi)} \psi(C)$ is maximal. We claim that $\chi(\tilde{\psi}) \subseteq \mathcal{C}_2(G) \cup \mathcal{C}_3(G)$. If not, then there is a dicycle $\hat{C} = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\}$, such that $k \geq 4$, and $\tilde{\psi}(\hat{C}) \geq 1$. Let $C_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$ and let $C_2 = \{(v_3, v_4), \ldots, (v_k, v_1), (v_1, v_3)\}$. Define $\hat{\psi}$ on $\chi(\tilde{\psi}) \cup \{C_1, C_2, C_{v_1 v_3}\}$ as follows: $\hat{\psi}(C) = \tilde{\psi}(C) - 1$ if $C = C_{v_1 v_3}$ or if $C = \hat{C}$, $\hat{\psi}(C) = \tilde{\psi}(C) + 1$ if $C = C_1$ or if $C = C_2$ and $\hat{\psi}(C) = \tilde{\psi}(C)$ otherwise. It is easy to see that $\hat{\psi}$ is a dicycle pseudopacking such that $\sum_{C \in \mathcal{C}_3(G) \cap \chi(\hat{\psi})} \psi(C) \geq 1 + \sum_{C \in \mathcal{C}_3(G) \cap \chi(\tilde{\psi})} \psi(C)$ contradicting our assumption. ■

Corollary 3.3 *For any integrally weighted digraph (G, w) on n vertices, there exists a maximum integral dicycle packing in G of value $\nu_c(G, w)$ whose characteristic set contains $O(n^3)$ dicycles.*

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